The Annals of Probability
1982, Vol. 10, No. 4, 955–957

A LOWER BOUND OF THE ASYMPTOTIC BEHAVIOR OF SOME MARKOV PROCESSES

BY TZUU-SHUH CHIANG

University of Wisconsin-Milwaukee

Let $X_0, X_1 \cdots$ be a Markov process with transition function $p(x, dy)$. Let $L_n(\omega, \cdot)$ be its average occupation time measure, i.e., $L_n(\omega, A) = 1/n \cdot \sum_{i=0}^{n-1} \chi_A(x_i(\omega))$. A powerful theorem concerning the lower bound of the asymptotic behavior of $L_n(\omega, \cdot)$ was proved by Donsker and Varadhan when $p(x, dy)$ satisfies a homogeneity condition. This paper tries to extend their results to some cases where such a homogeneity condition is not satisfied. This particularly includes symmetric random walks and Harris' chains.

1. Introduction. Before stating the results, a brief description of the problem and some answers to the problem will be reviewed.

Let $X_0, X_1, X_2, \cdots$ be a sequence of random variables taking values in a complete separable metric space X. Each realization ω of the process is a sequence $x_0, x_1, x_2, \cdots$ where $x_i \in X$. If we denote Ω the space of all such sequences, then for each $\omega \in \Omega$, positive integer n and set $A \subseteq X$, let $L_n(\omega, A) = (1/n) \sum_{i=0}^{n-1} \chi_A(x_i)$ be the proportion of time the process spends in A during first n-steps. We note that $L_n(\omega, \cdot)$ maps Ω into $\mathscr{M}$, the space of all probability measures on X. The large deviations problem is to find a functional $I(\mu)$ from $\mathscr{M}$ to $[0, \infty]$ such that:

(I.1) I is lower semi-continuous

(I.2) For any closed set $C \subseteq \mathscr{M}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log p(L_n(\omega, \cdot) \in C) \le -\inf_{\mu \in C} I(\mu).$$

(I.3) For any open set $G \subseteq \mathscr{M}$,

$$\liminf_{n\to\infty} \frac{1}{n} \log p(L_n(\omega, \cdot) \in G) \ge -\inf_{\mu \in G} I(\mu).$$

Here we endow $\mathscr{M}$ with the weak topology.

The problem of studying the asymptotic behavior of $L_n(\omega, \cdot)$ was proposed by Sanov [6] when $X_0, X_1, X_2 \cdots$ is an i.i.d. sequence. His work has been extended by Hoadley [5], Bahadur and Zabell [1] and many others. Donsker and Varadhan generalized this problem assuming the Markovian property of the process. Let $X_0, X_1, X_2 \cdots$ be a Markov process with a Feller transition function $p(x, dy)$ and initial distribution δ_x. They define the I-functional as follows:

For any $\mu \in \mathscr{M}$,

$$I(\mu) = \sup_{f \in B^c(X)} \int \log \frac{f(x)}{(pf)(x)} \mu(dx) \tag{I.4}$$

where $(pf)(x) = \int f(y) p(x, dy)$ and $B^c(x)$ is the set of all positive functions for each of which there exist a, b such that $0 < a \le f(x) \le b < \infty$ for all $x \in X$. It has been proved that the I-functional defined by (1.4) is appropriate in many cases. For instance $I(\mu)$ is always lower semi-continuous and (1.2) is always true if we require the state space X to be a separable compact metric space (see [2]). (For the case that X is not compact, readers are

Received December 1980; revised March 1982.

AMS 1980 *subject classifications.* Primary 60F10; secondary 60J05.

Key words and phrases. Average occupation time, large deviations, Markov process, indecomposability, symmetric random walks, Harris' chains.

referred to [3] for another condition). The authors of [2] also proved in [3] that (I.3) is true (for general state space) if the following condition (called Hypothesis (H) in [3]) is satisfied.

Hypothesis H. For arbitrary two points x and x', the resolvents $\sum_{n=1}^{\infty} (1/2^n)p^n(x, dy)$ and $\sum_{n=1}^{\infty} (1/2^n)p^n(x', dy)$ are mutually absolutely continuous. Here $p^n(x, dy)$ denotes the nth transition function.

We agree with the authors of [3] that the I-functional defined by (I.4) is appropriate. Hypothesis (H), however, seems very stringent. Even when X is compact, it was used in a crucial way to obtain (I.3) in [2]. There are many interesting transition functions which do not satisfy (H). In fact, the random walks on the real line or unit circle do not always satisfy (H). The purpose of this paper, therefore, is to find various other conditions under which (I.3) is true. In Section 1, a condition which is weaker than (H) is considered. We call that hypothesis (gH), (generalized H).

(gH): There exists a $k \geq 1$ such that $\sum_{n=k}^{\infty} (1/2^n) p^n(x, dy)$ and $\sum_{n=k}^{\infty} (1/2^n) p^n(x', dy)$ are mutually absolutely continuous for every x and x'.

Note that (H) is the special case of (gH) when $k = 1$. Under hypothesis (gH), (I.3) is obtained in Theorem (1.7). We consider this generalization worthwhile because the proof of (1.7) is not entirely trivial and a new method of approximation of $P(L_n(\omega,\cdot) \in G)$ has to be used. One example which satisfies (gH) but not (H) is given in Section 1.

In Section 2, we consider two classes of transition functions which do not satisfy (gH). One is the class of symmetric random walks and the other is that of Harris' chains. In both cases, however, we only get results which are weaker than (I.3). Theorems (2.5) and (2.7) show that (I.3) is true when the infimum of the right-hand side of (I.3) is taken over $\mathscr{M}' \subseteq \mathscr{M}$ where $\mathscr{M}'$ is some suitable subset of $\mathscr{M}$, i.e.

$$\text{(I.5)} \qquad \liminf_{n\to\infty} \frac{1}{n} \log p_x(L_n(\omega, \cdot) G) \geq -\inf_{\mu\in\mathscr{M}'\cap G}, I(\mu).$$

In the case of symmetric random walks on the real line or on the unit circle, $\mathscr{M}'$ is taken to be the set all absolutely continuous probability measures. In the case that X_0, X_1, X_2, $\cdots$ is a Harris' chain, $\mathscr{M}'$ is taken to be the set of all probability measures which are absolutely continuous with respect to its invariant distribution. If $\mathscr{M}' = \mathscr{M}$ then (I.5) is the same as (I.3). One such example is given at the end of Section 2. For obvious reasons, however, the technique used there can't be generalized any further. Applications of (I.2) and (I.3) to Gaussian measures on Banach spaces can be found in [3]. Statistical applications can be found in [1].

Section 0. *Notations and preliminary results.* We first review and summarize some results of [3]. Let X be a complete separable metric space and $p(x, dy)$ a transition function defined on X. We always assume $p(x, dy)$ is a Feller transition function throughout this paper. Let $\mathscr{M}$ be the space of all Borel probability measures on X and $B(X)$ the space of real-valued Borel functions f defined on X for each of which there exist a, b such that $0 < a \leq f(x) \leq b < \infty$ for all $x \in X$. With $pf(x) = \int f(y)p(x, dy)$, define, for each $\mu \in \mathscr{M}$,

$$I(\mu) = \sup_{f\in B(X)} \int \log \frac{f(x)}{pf(x)} \mu(dx).$$

It is easy to see that $I(\cdot)$ is a convex functional. Let $B^c(X)$ be the subset of $B(X)$ consisting of functions f which are also continuous, then it is not hard to see that

$$I(\mu) = \sup_{f\in B^c(X)} \int \log \frac{f(x)}{pf(x)} \mu(dx).$$

Let $\mathscr{M}_{X\times X}$ be the space of all Borel probability measures defined on $X \times X$ and $\mathscr{M}_\mu$ the set of $\lambda \in \mathcal{M}_{X\times X}$ with both the first and second marginals μ. For $\lambda \in \mathcal{M}$, define:

$$\bar{I}_p(x, dy)\mu(dx)^{(\lambda)} = \sup_{f\in B^c(X\times X)}\left[\iint \log f(x,y)\lambda(dx,dy) - \log \iint f(x,y)p(x,dy)\mu(dx)\right]$$

where $B^c(X\times X)$ is the set of all real-valued continuous functions $f(x, y)$ on $X \times X$ and for each of which there exist a, b such that $0 < a \leq f(x, y) \leq b < \infty$ for every $(x, y) \in X\times X$. It is shown in [3] that

$$I(\mu) = \inf_{\lambda\in\mathcal{M}_\mu}\bar{I}_{p(x,dy)\mu(dx)}(\lambda). \tag{0.0}$$

The assumptions that $p(x, dy)$ is a Feller transition function and X is a complete separable metric space imply that $\bar{I}_{p(x,dy)\mu(dx)}(\cdot)$ is lower semi-continuous and $\mathcal{M}_\mu$ is compact respectively. Hence $\inf_{\lambda\in\mathcal{M}_\mu}\bar{I}_{p(x,dy)\mu(dx)}(\lambda)$ is actually attained at $\bar{\lambda} \in \mathcal{M}_\mu$ if $(\mu) < \infty$. If we let $p(x, E) = \bar{\lambda}(dx \times E)/\mu(dx)$; i.e., let $\bar{p}(x, E)$ be the Radon-Nikodym derivative of $\bar{\lambda}(dx \times E)$ with respect to μ, for each $E \subseteq X$, $\bar{p}(x, E)$ is defined for almost every x relative to $\mu(\mathrm{dx})$. However, a version of $\bar{p}(x, E)$ can be chosen so that $\bar{p}(x, dy)$ is a transition function since X is a complete separable metric space. We note that μ is invariant for $\bar{p}(x, dy)$ since both the first and second marginals for $\bar{\lambda}(dx, dy)$ are μ. In [2], $\bar{I}_{p(x,dy)\mu(dx)}(\lambda)$ was expressed in terms of the Radon-Nikodym derivative of $\lambda(dx, dy)$ with respect to $p(x, dy)\mu(dx)$, i.e.,

$$\bar{I}_{p(x,dy)\mu(dx)}(\lambda) = \begin{cases} \infty \text{ if } \lambda </< p(x,dy)\mu(dx) \\ \int \log f(x,y)\lambda(dx,dy) \text{ where } f(x,y) \text{ is the} \\ \text{Radon-Nikodym derivative of } \lambda(dx,dy) \text{ witħ} \\ \text{respect to } p(x,dy)\mu(dx). \end{cases} \tag{0.1}$$

Let's summarize the above as the following lemma:

LEMMA (0.1). *Let I, $\bar{I}$, $\mathcal{M}_\mu$ be defined as above and μ a Borel probability measure. If $I(\mu) < \infty$, then there exists a transition function $\bar{p}(x, dy) \ll p(x, dy)$ for almost every x relative to $\mu(dx)$ such that μ is an invariant measure for $\bar{p}(x, dy)$ and*

$$I(\mu) = \bar{I}_{p(x,dy)\mu(dx)}(\bar{p}(x,dy)\mu(dx)) = \int \log\frac{\bar{p}(x,dy)}{p(x,dy)}\bar{p}(x,dy)\mu(dx)$$

where $\dfrac{\bar{p}(x,dy)}{p(x,dy)}$ is the Radon-Nikodym derivative of $\bar{p}(x, dy)$ with respect to $p(x, dy)$.

Let $f_n \in B^c(X)$ be a sequence of functions such that $I(\mu) = lim_{n\to\infty}\int \log\,(f_n(x)/pf_n(x))\mu(dx)$. The following lemma (Lemma 2.4 in [3]) reveals important information about $\bar{p}(x, dy)/p(x, dy)$.

LEMMA (0.2). *Let $f_n(x)$ be as above and μ a Borel probability measure with $I(\mu) < \infty$. Then $(f_n(y)/pf_n(x))p(x, dy)\mu(dx) \to \bar{p}(x, dy)\mu(dx)$ in variational norm.*

Let $\bar{p}_\mu$ denote the Markov process with initial distribution μ and transition function $\bar{p}(x, dy)$. Since μ is invariant with respect to $\bar{p}(x, dy)$, $\bar{p}_\mu$ is a stationary process. Moreover, if $p(x, dy)$ and $p(x', dy)$ are mutually absolutely continuous for every x and x', then it is shown in [3] that $\bar{p}_\mu$ is actually ergodic. We state this fact in the following lemma.

LEMMA (0.3). *Let $p(x, dy)$ be a transition function such that for every x and x', $p(x, dy)$ and $p(x', dy)$ are mutually absolutely continuous and μ a Borel probability measure with $I(\mu) < \infty$. Then $\bar{P}_\mu$ is ergodic.*

The following lemma was proved as Theorem (3.1) in [3]:

LEMMA (0.4). *Let $p'(x, dy) \ll p(x, dy)$ for* a.e. *$x - \mu(dx)$ be an arbitrary transition function such that the Markov process p'_μ, with initial distribution μ and transition function $p'(x, dy)$, is ergodic. Then there exists a Borel set A with $\mu(A) = 1$ such that for every open set G in $\mathscr{M}$ containing μ,*

$$\text{(0.2)} \qquad \liminf_{n\to\infty} \frac{1}{n} \log P_x(\omega: L_n(\omega, \cdot) \in G) \ge - \int \log \frac{p'(x, dy)}{p(x, dy)} p'(x, dy)\mu(dx)$$

if $p(x, A) > 0$. In particular, if $p(x, dy)$ is not singular with respect to $\mu(dy)$, then (0.2) *holds.*

For any transition function function $p(x, dy)$ and any $0 < \xi < 1$, let's define:

$$p_\xi(x, dy) = (1 - \xi)(p(x, dy) + \xi p^2(x, dy) + \xi^2 p^3(x, dy) + \cdots)$$

$$p_\xi^{(k)}(x, dy) = (1 - \xi)(p^k(x, dy) + \xi p^{k+1}(x, dy) + \xi^2 p^{k+2}(x, dy) + \cdots)$$

$$\text{for} \quad k = 1, 2, 3, \cdots .$$

Here, $p^n(x, dy)$ is the nth transition function. It is easy to see that $p_\xi(x, dy)$ and $p_\xi^{(k)}(x, dy)$ are Feller transition functions if $p(x, dy)$ is. Also, note that $p_\xi(x, dy) = p_\xi^{(1)}(x, dy)$. For transition functions $p_\xi(x, dy)$ and $p_\xi^{(k)}$ (x, dy) we denote the corresponding I-functionals and transition functions $\bar{p}(x, dy)$ in Lemma (0.1) by $I_\xi, I_\xi^{(k)}$ and $\bar{p}_\xi(x, dy)$ respectively, i.e., for instance, $I_\xi(\mu) = \sup_{f \in B^c(X)} \int \log f(x)/p_\xi f(x)\, \mu\, (dx)$ and etc.

SECTION 1. We first state the hypothesis (gH):

(gH): There is a fixed measure β(dy) and some integer k such that $\sum_{n=k}^{\infty} p^n(x, dy)/2^n$ is equivalent to $\beta(dy)$ for every x. Here $p^n(x, dy)$ is the nth transition function. (Two measures are equivalent if they are mutually absolutely continuous.)

Note that the hypothesis (H) in [3] is the special case of (gH) when $k = 1$. It's also easy to see that $p_\xi^{(k)}$ (x, dy) is equivalent to $\beta(dy)$ for every x if (gH) is satisfied.

In this section we will show (Theorem (1.7)) that for every open (weakly) set G containing μ and every point $x \in X$,

$$\text{(1.1)} \qquad \liminf_{n\to\infty} \frac{1}{n} \log P_x(L_n(\omega, \cdot) \in G) \ge - I(\mu).$$

Here, $L_n(\omega, \cdot)$ denotes the average occupation time measure and P_x the Markov process with initial distribution δ_x and transition function $p(x, dy)$. Note that (1.1) is equivalent to (I.3) when we take the supremum over all μ's which belong to G.

With $p_\xi(x, dy), p_\xi^{(k)}(x, dy)$, I_ξ and $I_\xi^{(k)}$ defined as in the preceding section, we first prove some computational lemmas.

LEMMA (1.1). *Let $g(x, y)$ be the density of $p_\xi^{(k)}(x, dy)$ with respect to $p_\xi(x, dy)$. Then $g(x, y) \le (1/\xi^{k-1})$* a.e. *$-p_\xi(x, dy)\mu(dx)$ for $k = 1, 2, \cdots$.*

PROOF. For almost every $(x, y) - p_\xi(x, dy)\mu(dx)$,

$$g(x, y) = \frac{(1 - \xi)(p^k(x, dy) + \xi p^{k+1}(x, dy + \cdots))}{(1 - \xi)(p(x, dy) + \cdots + \xi^{k-1}p^k(x, dy) + \xi^k p^{k+1}(x, dy) + \cdots)}$$

$$\le \frac{p^k(x, dy) + \xi p^{k+1}(x, dy + \cdots)}{\xi^{k-1}(p^k(x, dy) + \xi p^{k+1}(x, ay) + \cdots)} = \frac{1}{\xi^{k-1}}.$$

LEMMA (1.2). *Let μ be a Borel probability measure such that $I(\mu) < \infty$. Then:* $I_\xi^{(k)}(\mu) < \infty$ *for* $0 < \xi < 1$ *and* $k = 1, 2, 3, \cdots$.

PROOF. By definition,

$$\begin{aligned} I_\xi^{(k)}(\mu) &= \sup_{f\in B(x)} \int \log \frac{f(x)}{(p_\xi^{(k)}f)(x)}\mu(dx) \\ &\le \sup_{f\in B(X)} \int \log \frac{f(x)}{(1-\xi)(p_\xi^k f)(x)}\mu(dx) \\ &\le \sup_{f\in B(x)} \int \log \frac{f(x)}{(p_\xi f)(x)}\mu(dx) + \sup_{f\in B(x)} \int \log \frac{(p_\xi f)(x)}{(P_\xi(p_\xi f))(x)}\mu(dx) + \cdots \\ &\quad + \sup_{f\in B(x)} \int \log \frac{(p_\xi^{k-1})(x)}{(p_\xi(p_\xi^{k-1}f))(x)}\mu(dx) - \log(1-\xi) \\ &\le k\cdot \sup_{f\in B(x)} \int \log \frac{f(x)}{(p_\xi f)(x)}\mu(dx) - \log(1-\xi) \\ &= k\cdot I(\mu) - \log(1-\xi) < \infty. \end{aligned}$$

Hence $I_\xi^{(k)}(\mu) < \infty$ if $I(\mu) < \infty$. This completes the proof.

LEMMA 1.3. *Let $p(x, dy)$ be a transition function which satisfies* (gH). *Then $I(\mu) < \infty$ only if $\mu \ll \beta$.*

PROOF. By Lemma (1.2), $I_\xi^{(k)}(\mu) < \infty$. Therefore there exists a transition function $\bar{p}_\xi(k)_{(x,dy)\ll p_\xi}$ for a.e. $x - \mu(dx)$ and $\mu(dx)$ is an invariant distribution relative to $\bar{p}_\xi(x, dy)$ by Lemma (0.1). If A is a set such that $\beta(A) = 0$, then $p_\xi^{(k)}(x, dy) = 0$ for every x. Thus $\bar{p}_\xi(x, A) = 0$ a.e. $x - \mu(dx)$. Therefore, $\mu(A) = \int \bar{p}_\xi^{(k)}(x, A)\mu(dx) = 0$. This completes the proof.

Let $p(x, dy)$ be a transition function which satisfies (gH) and μ a Borel probability measure with $I(\mu) < \infty$. Let

$$\bar{p}_{\xi,t}(x, dy) = (1-t)\bar{p}_\xi(x, dy) + t\bar{p}_\xi^k(x, dy) \quad \text{for} \quad 0 < t < 1.$$

Since $\bar{p}_\xi(, dy) \ll p_\xi(x, dy)$ and $\bar{p}_\xi^{(k)}(x, dy) \ll p_\xi^{(k)}(x, dy)$ for a.e. $x - \mu(dx)$, we have $\bar{p}_{\xi,t}(x, dy) \ll p_\xi(x, dy)$ a.e. $x - \mu(dx)$ for every $0 < t < 1$.

LEMMA 1.4. *Let $\bar{p}_{\xi,t}(x, dy)/p_\xi(x, dy)$ be the Radon-Nikodym derivative of $\bar{p}_{\xi,t}(x, dy)$ with respect to $p_\xi(x, dy)$. Then:*

$$\int \log \frac{\bar{p}_{\xi,t}(x, dy)}{p_\xi(x, dy)}\bar{p}_{\xi,t}(x, dy)\mu(dx) \to I_\xi(\mu) \quad as \quad t \to 0.$$

PROOF. $\bar{p}_{\xi,t}(x, dy)\mu(dx) \in \mathscr{M}_\mu$ since both $\bar{p}_\xi(x, dy)\mu(dx)$ and $\bar{p}_\xi^{(k)}(x, dy)\mu(dx)$ are in $\mathscr{M}_\mu$. Therefore by (0.0) and (0.1), we have

$$\int \log \frac{\bar{p}_{\xi,t}(x, dy)}{p_\xi(x, dy)}\bar{p}_{\xi,t}(x, dy)\mu(dx) \ge I_\xi(\mu).$$

Now,

$$\int \log \frac{\bar{p}_{\xi,t}(x, dy)}{p_\xi(x, dy)}\bar{p}_{\xi,t}(x, dy)\mu(dx)$$

$$= \int (1-t)\left(\frac{\bar{p}_\xi(x, dy)}{p_\xi(x, dy)} + t\frac{\bar{p}_\xi^{(k)}(x, dy)}{p_\xi(x, dy)}\right) \cdot \log\left(\frac{(1-t)\bar{p}_\xi(x, dy)}{p_\xi(x, dy)} + \frac{t\bar{p}_\xi^{(k)}(x, dy)}{p_\xi(x, dy)}\right) p_\xi(x, dy)\mu(dx)$$

$$\leq (1-t)\int \log\frac{\bar{p}_\xi(x, dy)}{p_\xi(x, dy)} \log\frac{\bar{p}_\xi(x, dy)}{p_\xi(x, dy)} p_\xi(x, dy)\mu(dx) + t\int \log\frac{\bar{p}_\xi^{(k)}(x, dy)}{p_\xi(x, dy)} \log\frac{\bar{p}_\xi^{(k)}(x, dy)}{p_\xi(x, dy)} p_\xi(x, dy)\mu(dx).$$

The last inequality is true because $x \cdot \log x$ is a convex function on $[0, \infty)$. Since:

$$\int \frac{\bar{p}_\xi^{(k)}(x, dy)}{p_\xi(x, dy)} \log\frac{\bar{p}_\xi^{(k)}(x, dy)}{p_\xi(x, dy)} p_\xi(x, dy)\mu(dx)$$

$$= \int \log\frac{\bar{p}_\xi^{(k)}(x, dy)}{p_\xi^{(k)}(x, dy)} \bar{p}_\xi^{(k)}(x, dy)\mu(dx) + \int \log\frac{p_\xi^{(k)}(x, dy)}{p_\xi(x, dy)} \bar{p}_\xi^{(k)}(x, dy)\mu(dx)$$

$$\leq I_\xi^{(k)}(\mu) + \log\frac{1}{\xi^{k-1}} \qquad \text{by Lemmas (0.1) and (1.1).}$$

Therefore

$$\int \log\frac{\bar{p}_{\xi,t}(x, dy)}{p_\xi(x, dy)} \bar{p}_{\xi,t}(x, dy)\mu(dx) \leq (1-t)\int \frac{\bar{p}_\xi(x, dy)}{p_\xi(x, dy)} \log\frac{\bar{p}_\xi(x, dy)}{p_\xi(x, dy)} \cdot p_\xi(x, dy)\mu(dx) + t\left(I_\xi^{(k)}(\mu) + \log\frac{1}{\xi^{k-1}}\right)$$

$$= (1-t)I_\xi(\mu) + t\left(I_\xi^{(k)}(\mu) + \log\frac{1}{\xi^{k-1}}\right) \qquad \text{by Lemma (0.1).}$$

We then complete the proof by letting $t \to 0$.

We now prove that (1.1) holds for the process $p_{\xi,x}$, the Markov process with initial distribution δ_x and transition function $p_\xi(x, dy)$.

PROPOSITION (1.5). *Let $p(x, dy)$ be a Feller transition function which satisfies* (gH) *and μ a Borel probability measure with $I(\mu) < \infty$. Then for every x and open set G containing μ,*

$$\liminf_{n\to\infty} \frac{1}{n} \log p_{\xi,x}(\omega: L_n(\omega, \cdot) \in G) \geq I_\xi(\mu). \tag{1.2}$$

PROOF. By Lemma (1.2), $I_\xi(\mu)$ and $I_\xi^{(k)}(\mu) < \infty$. Let $\bar{p}_\xi(x, dy)$ and $\bar{p}_\xi^{(k)}(x, dy)$ be the transition functions in Lemma (0.1) corresponding to $p_\xi(x, dy)$ and $p_\xi^{(k)}(x, dy)$ respectively. By (gH), $p_\xi^{(k)}(x, dy)$ is equivalent to $\beta(dy)$ for every x, therefore $\bar{p}_{\xi,\mu}^{(k)}$, the Markov process with initial distribution μ and transition function $p^{(k)}(x, dy)$ is ergodic by Lemma (0.3). As in Lemma (1.4), let $\bar{p}_{\xi,t}(x, dy) = (1-t)\bar{p}_\xi(x, dy) + t\bar{p}_\xi^{(k)}(x, dy)$, $0 < t < 1$. Since a stationary Markov process with initial distribution μ and transition function $\pi(x, dy)$ is ergodic if and only if there does not exist a set A with $0 < \mu(A) < 1$ such that $\int_A \pi(x, A^c)\mu(dx) = \int_{A^c} \pi(x, A)\mu(dx) = 0$. Therefore, it is not hard to see that $\bar{p}_{\xi,t,\mu}$, the Markov process with initial distribution μ and transition function $\bar{p}_{\xi,t}(x, dy)$ is ergodic. By Lemma (0.4), therefore,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_{\xi,x}(\omega: L_n(\omega, \cdot) \in G) \geq -\int \log\frac{\bar{p}_{\xi,t}(x, dy)}{p_\xi(x, dy)} \bar{p}_{\xi,t}(x, dy)\mu(dx)$$

if $p_\xi(x, dy)$ is not singular to $\mu(dy)$. But by Lemma (1.3) and (gH), we have $\mu(dy) \ll \beta(dy) \ll p_\xi(x, dy)$. We now use Lemma (1.4) and complete the proof by letting $t \to 0$.

The following lemma was actually proved in Theorem (3.3) [3].

LEMMA (1.6). *Let $p(x, dy)$ be a Feller transition function and μ a probability measure with $I(\mu) < \infty$. Then,* (1.1) *holds if* (1.2) *holds for every* $\xi \in (0, 1)$.

Combining Proposition (1.5) and Lemma (1.6), we have:

THEOREM (1.7). *Let $p(x, dy)$ be a Feller transition function satisfying* (gH). *Then for every Borel probability measure $\mu \in \mathscr{M}$ and $x \in X$, we have*

$$\liminf_{n\to\infty} \frac{1}{n} \log P_x(\omega: L_n(\omega, \cdot) \in G) \geq -I(\mu),$$

where G is an open set in $\mathscr{M}$ containing μ.

For the rest of this section, we'll consider an application of Theorem (1.7) and also provide an example which satisfies (gH) but not (H). Let $X_0, X_1, \cdots$ be a sequence of independent random variables with common distribution $\beta(dy)$. Each realization of ω of this process is in a sequence $x_0, x_1, \cdots$ where $x_i \in R$ for all i. Recall that $L_n(\omega, A)$ is the average occupation time of A when the sample is $x_0, x_1, x_2, \cdots x_{n-1}$. For a pair of sets A, B, we define the average of successive occupation time of A, B as follows:

$$\tilde{L}_n(\omega, A \times B) = \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(x_i)\chi_B(x_i). \tag{1.3}$$

Theorem (1.10) gives an estimate of the limiting distribution of $\tilde{L}_n(\omega, \cdot)$. We begin with two simple lemmas whose proof can be easily furnished by readers.

LEMMA (1.8). *Let $p(x, dy)$ be a transition function. If there exists a reference measure $\beta(dy)$ such that $p^2(x, dy)$ is equivalent to $\beta(dy)$ for every x, then* (gH) *is satisfied with $k = 2$.*

LEMMA (1.9). *For any two numbers x, y and Borel set $E \subseteq R^2$, the function $\tilde{p}(\cdot, \cdot)$ defined by $\tilde{p}((x, y), E) = \beta(E_y)$ where $E_y = \{z: (y, z) \in E\}$ is a Feller transition function.*

For the transition function $\tilde{p}(\cdot, \cdot)$ defined in Lemma (1.9), we define:

$$I(\mu) = \sup_{f\in B^c(R^2)} \int \log \frac{f(x, y)}{(\tilde{p}f)(x, y)} \mu(dx, y)$$

where μ is a Borel probability measure on $\mathbb{R}^2$ and

$$\tilde{L}_n(\omega, A) = \sum_{i=0}^{n-1} \chi_A(x_i, x_{i+1}) \quad \text{where} \quad A \subseteq R^2. \tag{1.4}$$

Note that (1.4) agrees with (1.3) if A is a rectangle.

THEOREM (1.10). *Let $X_0, X_1, \cdots$ be a sequence of* i.i.d. *random variables with a common distribution $\beta(dy)$. Let $\tilde{p}(\cdot, \cdot)$, $\tilde{I}(\cdot)$ and $\tilde{L}_n(\omega, \cdot)$ be defined as above. Then for any probability measure μ and open set G containing μ,*

$$\liminf_{n\to\infty} \frac{1}{n} \log P(\omega: \tilde{L}_n(\omega, \cdot) \in G) \geq -\tilde{I}(\mu).$$

PROOF. Define $Y_n = (X_n, X_{n+1})$ $n = 0, 1, \cdots$. Then it is easy to see that $Y_0, Y_1, \cdots$ is a Markov process with transition function $\tilde{p}(\cdot, \cdot)$. It is also easy to see that $\tilde{p}^2((x, y), \cdot)$ is equivalent to $\beta \times \beta$ for every $(x, y) \in R^2$. Therefore, $\tilde{p}(\cdot, \cdot)$ satisfies (gH) when $k = 2$ by Lemma (1.8). The conclusion now follows from Theorem (1.7).

REMARK. The process $Y_0, Y_1 \cdots$ defined in Theorem (1.10) is an example which satisfies (gH) but not (H).

SECTION 2. *Symmetric random walks and Harris' chains.* In this section, we'll discuss some symmetric random walks and Harris' chains which do not satisfy hypothesis (gH). In these cases, it is not always true that there exists a transition function $p'(x, dy) \ll p(x, dy)$ such that the Markov process with initial distribution μ and transition function $p'(x, dy)$ is ergodic. In Lemma (2.1) we impose some conditions on μ under which such an ergodic Markov process can be obtained.

Let $p(x, dy)$ be a transition function with state space X. We say $p(x, dy)$ satisfies Harris' recurrence condition if there exists a non-negative σ-finite measure $Q(dy)$ such that for every point x and every set A with $Q(A) > 0$, we have:

$$P_x(\cup_{n=1}^{\infty} X_n \in A) = 1 \tag{2.1}$$

where $X_0, X_1, \cdots$ is the Markov process with initial distribution δ_x and transition function $p(x, dy)$. It is shown in [4] that under this condition there exists a σ-finite invariant measure $\beta(dy)$ which is unique up to a constant and $Q(dy) \ll \beta(dy)$. We'll show in Theorem (2.7) that (1.1) holds if $\mu(dy) \ll \beta(dy)$. A Markov process with state space R or the unit circle is a random walk if the transition function $p(x, dy) = p(dy - x)$. It is symmetric if $p(dy - x) = p(x - dy)$. In Theorem (2.5) we'll show that with a necessary condition on $p(x, dy)$, (1.1) is true if μ is absolutely continuous.

We begin with a few definitions. For a set $E \subseteq X \times X$, let E^t denote the symmetric image of E, i.e., $E^t = \{(y, x) : (x, y) \in E\}$. Also, let $E_x = \{z : (x, z) \in E\}$. Define two measures on $X \times X$ as follows: for every Borel set E contained in $X \times X$,

$$(p\mu)_1(E) = \int p(x, E_x)\mu(dx)$$

$$(p\mu)_2(E) = \int p(x, E^t_x)\mu(dx).$$

Apparently, $(p\mu)_2(E) = (p\mu)_1(E^t)$. As in Section 1, let $p_\xi(x, dy) = (1 - \xi)(p(x, dy) + \xi p^2(x, dy) + \cdots)$ for $0 < \xi < 1$. It is not hard to see that $p^2_\xi(x, dy) \ll p_\xi(x, dy)$ for every $x \in X$.

LEMMA (2.1). *Let μ be a probability measure with $I(\mu) < \infty$. If* (i) *for every set A with $0 < \mu(A) < 1$, $\int_{A^c} p_\xi(x, A)\mu(dx) < 0$ and* (ii) *for every pair of Borel sets E, F contained in X with $E \cap F = \phi$ and $\mu(E) + \mu(F) = 1$, $(p_\xi\mu)_1$, and $(p_\xi\mu)_2$ are not mutually singular on $E \times F$. Then, $\bar{P}_{\xi,\mu}$ is ergodic for $0 < \xi < 1$.*

PROOF. By Lemmas (1.2) and (0.1), there exists a transition function $p_\xi(x, dy) \ll p_\xi(x, dy)$ a.e. $x - \mu(dx)$ and μ is an invariant measure relative to $\bar{p}_\xi(x, dy)$. Suppose $\bar{P}_{\xi,\mu}$, the Markov process with initial distribution μ and transition function $\bar{p}_\xi(x, dy)$, is not ergodic, then there exists a set A such that $0 < \mu(A) < 1$ and $\int_A \bar{p}_\xi(x, A^c)\mu(dx) = \int_{A^c} \bar{p}_\xi(x, A)\mu(dx) = 0$. Let f_n be a sequence of functions as in Lemma (0.2) with respect to $p_\xi(x, dy)$. Then:

$$\lim_{n\to\infty} \frac{f_n(y)}{(p_\xi f)(x)} = \frac{\bar{p}_\xi(x, dy)\mu(dx)}{p_\xi(x, dy)\mu(dx)} \quad \text{a.e.} \quad p_\xi(x, dy)\mu(dx). \tag{2.2}$$

Since $\bar{p}_\xi(x, A^c) = 0$ a.e. $x - \mu(dx)$ in A, we have

$$\lim_{n\to\infty} \frac{f_n(y)}{(p_\xi f_n)(x)} = 0 \quad \text{a.e.} \quad y - p_\xi(x, dy) \text{ in } A^c \quad \text{for a.e.} \quad x - \mu(dx) \text{ in } A. \tag{2.3}$$

Similarly:

$$\lim_{n\to\infty} \frac{f_n(y)}{(p_\xi f_n)(x)} = 0 \quad \text{a.e.} \quad y - p_\xi(x, dy) \text{ in } A \quad \text{for a.e.} \quad x - \mu(dx) \text{ in } A^c. \tag{2.4}$$

Let B_0 be the set of points which either (2.2), (2.3) or (2.4) is violated, i.e.,

$$B_0 = \left\{x \in A : \lim_{n\to\infty} \frac{f_n(y)}{(p_\xi f_n)(x)} \neq \frac{\bar{p}_\xi(x, dy)}{p_\xi(x, dy)} \quad \text{a.e.} \quad y - p_\xi(x, dy) \quad \text{or} \quad \bar{p}_\xi(x, A^c) > 0\right\}.$$

$$\cup \left\{x \in A^c : \lim_{n\to\infty} \frac{f_n(y)}{(p_\xi f_n)(x)} \neq \frac{\bar{p}_\xi(x, dy)}{p_\xi(x, dy)} \quad \text{a.e.} \quad y - p_\xi(x, dy) \quad \text{or} \quad \bar{p}_\xi(x, A) > 0\right\}.$$

Thus $\mu(B_0) = 0$ by (2.2), (2.3), and (2.4). Inductively, if B_{n-1} has been defined and $\mu(B_{n-1}) = 0$, define: $B_n = \{x : \bar{p}_\xi(x, B_{n-1}) > 0\}$. Then $\mu(B_n) = 0$ since μ is an invariant measure relative to $\bar{p}_\xi(x, dy)$. Therefore if $B = \cup_{i=0}^{\infty} B_i$, $\mu(B) = 0$. If $x \notin B$, then $x \notin B_n$ for every $n = 1, 2, \cdots$, so $\bar{p}_\xi(x, B_n) = 0$ for $n = 0, 1, 2, \cdots$, i.e., $\bar{p}_\xi(x, B) = 0$. Let $G = \{x \in A - B : \bar{p}_\xi(x, A^c - B) > 0\}$. By assumption (i), $\mu(G) > 0$. For each $x_0 \in G$, let $E(x_0) = \{y \in A^c - B : \lim_{n\to\infty} f_n(y)/(p_\xi f_n)(x_0) = 0\}$. By (2.3) $p_\xi(X_0, (A^c - B) - E(x_0)) = 0$. Let $F(X_0) = \{y \in A^c - B : p_\xi(y, (A^c - B) - E(x_0)) = 0\}$. Since $p_\xi^2(x_0, dy) \ll p_\xi(x_0, dy)$, so $p_\xi^2(x_0, (A^c - B) - E(x_0)) = 0$, i.e.,

$$p_\xi(y, (A^c - B) - E(x_0)) = 0 \quad \text{a.e.} \quad y - p_\xi(x_0\, dy).$$

i.e.,

$$p_\xi(x_0, (A^c - B) - F(x_0)) = 0.$$

Let $y \in F(x_0)$. Then $\lim_{n\to\infty}(f_n(z)/(p_\xi f)_n(y))$ exists a.e. $z - p_\xi(y, dz)$ and $\bar{p}_\xi(y, A^c - B) = 1$. So, there exists a $z \in A^c - B$ such that $\lim_{n\to\infty}(f_n(z)/(p_\xi f_n)(y)) > 0$. The fact that $y \in F(x_0)$ implies that such a z can be chosen from $E(x_0)$, so:

$$\lim_{n\to\infty} \frac{(p_\xi f_n)(y)}{(p_\xi f_n)(x_0)} = \left(\lim_{n\to\infty} \frac{(P_\xi f_n)(y)}{f_n(z)}\right) \times \left(\lim_{n\to\infty} \frac{f_n(x)}{(p_\xi f_n)(x_0)}\right) = 0.$$

If we define $H(x, y) = \lim_{n\to\infty}((p_\xi f_n)(x)/(p_\xi f_n)(y))$ whenever the limit exists, then $H(x, y)$ is well defined and equals 0 a.e. $p_\xi(x, dy)\mu(dx)$ on $(A - B) \times (A^c - B)$, i.e., $H(x, y) = 0$ a.e. $(p_\xi\mu)_1(dx, dy)$ on $(A - B) \times (A^c - B)$.

By a similar argument we have:

$$H(x, y) = 0 \quad \text{a.e.} \quad (p_\xi\mu)_1(dx, dy) \quad \text{on} \quad (A^c - B) \times (A - B). \tag{2.5}$$

Since $H(x, y) = 1/H(y, x)$, so if $H(x, y) = 0$ a.e. $(p_\xi\mu)_1(dx, dy)$ on $(A - B) \times (A^c - B)$, then:

$$H(x, y) = \infty \quad \text{a.e.} \quad (p_\xi\mu)_2(dx, dy) \quad \text{on} \quad (A^c - B) \times (A - B). \tag{2.6}$$

Now (2.6) contradicts (2.5) because of Assumption (ii). This proves that $\bar{P}_{\xi,\mu}$ is ergodic.

If $p(x, dy)$ is a symmetric random walk on the real line, i.e., $p(x, dy) = p(dy - x) = p(x - dy)$, and μ is absolutely continuous, then (ii) of the preceding lemma is always satisfied. First, we have the following:

LEMMA (2.2). *Let X be the real line and $m(dy)$ the Lebesgue measure and $p(x, dy)$ a symmetric random walk. Then $(pm)_1 = (pm)_2$.*

PROOF. It is clear that all we need to show is that $(pm)_1\ (A_1 \times A_2) = (pm)_2(A_1 \times A_2)$ for every rectangle $A_1 \times A_2$. Let $p(dy) = p(0, dy)$. Now:

$$(pm)_1(A_1 \times A_2) = \int_{A_1} p(x, A_2)m(dx)$$

$$= \int\int \chi_{A_1}(x)\chi_{A_2-x}^{(y)}p(dy)m(dx)$$

$$= \int\int \chi_{A_1}(x)\chi_{A_2}(x + y)p(dy)m(dx)$$

(by symmetry and Fubini's theorem)

$$= \int\int \chi_{A_1}(x)\chi_{A_2}(x - y)m(dx)p(dy)$$

(translation invariance of $m(dx)$

$$= \int\int \chi_{A_1}(x + y)\chi_{A_2}^{(x)}p(dy)m(dx)$$

$$= \int\int \chi_{A_2}(x)\chi_{A_1-x}p(dy)m(dx)$$

$$= \int_{A_2} p(A_1 - x)m(dx)$$

$$= (pm)_1(A_2 \times A_1)$$

$$= (pm)_2(A_1 \times A_2).$$

This completes the proof.
We omit the proof of the following easy corollary:

COROLLARY (2.3). *Let μ be an absolutely continuous probability measure on R and $p(x, dy)$ a symmetric random walk. Then for every pair of sets E, F such that $E \cap F = \phi$ and $\mu(E) + \mu(F) = 1$, $(p\mu)_1$ and $(p\mu)_2$ are equivalent on $E \times F$.*

We say that a measure μ is indecomposable with respect to a transition function $p(x, dy)$ if there does not exist a set A with $\mu(A)$, $\mu(A^c) > 0$ such that $\int_A p(x, A^c)\mu(dx) = \int_{A^c} p(x, A)\mu(dx) = 0$. With $p_\xi(x, dy) = (1 - \xi)(p(x, dy) + \xi p^2(x, dy) + \cdots)$ and $m(dx)$ the Lebesgue measure, we have:

LEMMA (2.4). *Let $p(x, dy)$ be a symmetric random walk with respect to which $m(dx)$ is indecomposable. Then for any set A with $m(A) > 0$, we have $p_\xi(x, A) > 0$ a.e. $x - m(dx)$.*

PROOF. Without loss of generality, we assume $m(A^c) > 0$. Let $E_n = \{x \in A^c : p^n(x, A) > 0\}$ and $E = \cup_{n=1}^\infty E_n$. We claim that $m(A^c - E) = 0$. For if not, let $\tilde{A} = E \cup A$. Then the set $\tilde{E} = \{x \in (E \cup A)^c : p(x, \tilde{A}) > 0\}$ has positive m-measure by the assumption $x \in \tilde{E} \Rightarrow p^n(x, A) = 0 \ \forall\ n$, but $p(x, \tilde{A}) > 0$, so $p(x, E^n) > 0$ for some n. Then: $p^{n+1}(x, A) \geq \int_{E_n} p^n(y, A)p(x, dy) > 0$ i.e., $x \in E_{n+1} \subseteq E$. This contradicts the assumption $x \in \tilde{E} \subseteq E^c$. Therefore, $m(A^c - E) = 0$, so,

$$p_\xi(x, A) > 0 \quad \text{a.e.} \quad x - m(dx) \quad \text{in} \quad A^c. \tag{2.7}$$

Similarly, (replace A by E in the above argument) we can prove $P_\xi(x, E) > 0$ a.e. $x - m(dx)$ in A. Therefore:

$$p_\xi^2(x, A) \geq \int_E p_\xi(y, A)p_\xi(x, dy) > 0 \quad \text{a.e.} \quad x - m(dx) \quad \text{in} \quad A.$$

Since $p_\xi^2(x, dy) \ll p_\xi(x, dy)$, we have:

$$p_\xi(x, A) > 0 \quad \text{a.e.} \quad x - m(dx) \quad \text{in} \quad A. \tag{2.8}$$

By (2.7), (2.8), we conclude $p_\xi(x, A) > 0$ a.e. $x - m(dx)$ if $m(A) > 0$.
We now state and prove the following theorem:

THEOREM (2.5). *Let $p(x, y)$ be a symmetric random walk on the real line with respect to which $m(dx)$ is indecomposable. Then for every probability measure $\mu \ll m$ and $x \in R$, we have:*

$$\liminf_{n\to\infty} \frac{1}{n} \log P_x(L_n(\omega, \cdot) \in G) \geq -I(\mu)$$

where G is an open set in $\mathcal{M}$ containing μ.

PROOF. By Corollary (2.3) and Lemma (2.4), Conditions (i) and (ii) of Lemma (2.1) are satisfied: Therefore, $P_{\xi,\mu}$ is ergodic for every $0 < \xi < 1$. By Lemma (0.4) there exists a set

A with $\mu(A) = 1$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \log P_{\xi,x}(\omega : L_n(\omega, \cdot) \in G) \geq -I(\mu) \quad \text{if} \quad p_\xi(x, A) > 0.$$

By Lemma (2.4), $p_\xi(x, A) > 0$ a.e. $x - m(dx)$. For any probability measure λ, let λ_x be the measure such that $\lambda_x(A) = \lambda(A - x)$. For an arbitrary point x, let x_n be a sequence of points such that $x_n \to x$ and $p_\xi(x_n, A) > 0$ for every n. Let $\varepsilon > 0$ be chosen and $G' \subset G$ be an open set containing μ such that $\lambda \in G'$ implies $\lambda_x \in G$ if $|x| < \varepsilon$. Then, $P_{\xi,x}(L_n(\omega, \cdot) \in G) \geq P_{\xi,X'}(L_n(\omega, \cdot) \in G')$ if $|x - x'| < \varepsilon$. Therefore,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_{\xi,x}(L_n(\omega, \cdot) \in G) \geq \liminf_{n\to\infty} P_{\xi,x_k}(L_n(\omega, \cdot) \in G') \geq -I_\xi(\mu)$$

when k is large. Thus,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_{\xi,x}(L_n(\omega, \cdot) \in G) \geq I_\xi(\mu)$$

for every x. This completes the proof because of Lemma (1.6).

Remark (1). The indecomposability of $m(dx)$ with espect to $p(x, dy)$ in Theorem (2.5) is just to rule out the lattice case. The theorem would be false if we consider, for instance, the transition function $p(x, x + 1) = p(x, x - 1) = ½$. One can easily get a counter-example of (1.1) for this process.

Remark (2). In general, Theorem (2.5) is true if $p(x, dy)$ is a symmetric random walk on a commutative locally compact group and μ is absolutely continuous with respect to its Harr measure.

Remark (3). (Due to the referee). One can extend Theorem (2.5) to general measures by convoluting it with smooth measures. To be precise, let μ be any measure and let ϕ_ε be smooth measures such that $\mu_\varepsilon \to \mu$ weakly as $\varepsilon \to 0$ where $\mu_\varepsilon = \mu * \phi_\varepsilon$. Since the I-functional is translation invariant (for random walks), convex and lower semi-continuous, $I(\mu_\varepsilon) \to I(\mu)$ as $\varepsilon \to 0$. This does it.

We next consider a transition function $p(x, dy)$ with the following property. For some fixed σ-finite measure $\beta(dy)$, the absolutely continuous part of $p_\xi(x, dy)$ with respect to $\beta(dy)$ is actually equivalent to $\beta(dy)$ for every x. Then as a simple application of Lemmas (2.1), (0.4) and (1.6), we have: (Detail of the proof is omitted).

Corollary 2.6. *Let $p(x, dy)$ be Feller transition function and $\beta(dy)$ a fixed σ-finite measure. Let $p_\xi(x, dy) = p_\xi^a(x, dy) + p_\xi^s(x, dy)$ be the Lebesgue decomposition of $p_\xi(x, dy)$ with respect to $\beta(dy)$. If $p_\xi^a(x, dy)$ is equivalent to $\beta(dy)$ for every x, then for every $\mu \ll \beta$ and $x \in X$, we have*:

$$\liminf_{n\to\infty} \frac{1}{n} \log P_x(L_n(\omega, \cdot) \in G) \geq -I(\mu)$$

where G is an open set in $\mathcal{M}$ containing μ.

We say that a transition function $p(x, dy)$ satisfies Harris' recurrence condition if there exists a σ-finite measure $Q(dy)$ such that for every point x and every set A with $Q(A) > 0$

$$P_x\left(\cup_{n=1}^{\infty} X_n \in a\right) = 1. \tag{2.9}$$

It is shown in [4] that under this condition there exists a σ-finite invariant measure $\beta(dy)$ which is unique up to a constant and $Q(dy) \ll \beta(dy)$. Also (2.9) is true for every set with $\beta(A) > 0$. It is easy to see that conditions of Corollary (2.6) are satisfied if $p(x, dy)$ satisfies

Harris' recurrence condition and β is taken to be the invariant measure. Therefore, we have the following:

THEOREM 2.7. *Let $p(x, dy)$ be a Feller transition function satisfying Harris' recurrence conditions and $\beta(dy)$ be the invariant measure relative to $p(x, dy)$. If $\mu \ll \beta$, then for every x and open set G in M containing μ we have*:

$$\liminf_{n\to\infty} \frac{1}{n} \log P_x(L_n(\omega, \cdot) \in G) \geq -I(\mu).$$

For the rest of this section, let's consider a transition function $p(x, dy)$ which satisfies Corollary (2.6) and has the singular part $p^s(x, dy) = I(x, dy)$ where $I(x, dy)$ is the deterministic transition function that $I(x, x) = 1$. This is an interesting case because (1.1) not only holds true for absolutely continuous measures but true for every probability measure. We will illustrate the technique by proving the special case when $X = R$ and $p(x, dy) = ½\, \tilde{p}(x, dy) + ½\, I(x, dy)$ where $\tilde{p}(x, dy)$ is equivalent to Lebesgue measure $m(dy)$ for every x. It should not be hard to formulate the general case from this special one.

THEOREM (2.8). *Let $p(x, dy) = ½\, \tilde{p}(x, dy) + ½\, I(x, dy)$ where $\tilde{p}(x, dy)$ is equivalent to $m(dy)$ for every x. Let μ be a probability measure. Then, for every x and open set G in $\mathscr{M}$ containing μ, we have*:

$$\liminf_{n\to\infty} \frac{1}{n} \log P_x(L_n(\omega, \cdot) \in G) \geq -I(\mu). \tag{2.10}$$

PROOF.

$$I(\mu) = \sup_{f\in B^c(X)} \int \log \frac{f^{(x)}}{(pf)(x)} \mu(dx) \leq \sup_{f\in B^c(X)} \int \log \frac{f^{(x)}}{½\, f(x)} \mu(dx) = \log 2.$$

If $\mu \ll \mathscr{M}$, then (2.10) is true by Corollory (2.6). On the other hand, if $\mu \not\ll \mathscr{M}$, we can write $\mu = \mu^a + \mu^s$ where $\mu^a \ll m$ and $\mu^s \perp m$. Let $\mu_1 = \mu^a \| \mu^a \|$ and $\mu_2 = \mu^s \| \mu^s \|$ so that μ_1 and μ_2 are probability measures. Let $\bar{p}(x, dy)$ be the transition function in Lemma (0.1) so that

$$I(\mu) = \int \log \frac{\bar{p}(x, dy)}{p(x, dy)} \bar{p}(x, dy)\mu(dx)$$

and μ is an invariant distribution relative to $\bar{p}(x, dy)$. Let A be a set in R such that $m(A) = 0$ but $\mu_2(A) = 1$. Let $\bar{p}(x, dy) = \bar{P}^a(x, dy) + \bar{p}^s(x, dy)$ be the Lebesgue decomposition of $\bar{p}(x, dy)$ with respect to $m(dy)$. Since $\mu^*(A) = 0$, we have:

$$\begin{aligned}\mu^s(A) = \mu^s(A) + \mu^a(A) = \mu(A) &= \int \bar{p}(x, A)\mu(dx)\\ &= \int_A \bar{p}^s(x, A)\mu(dx) + \int_{A^c} \bar{p}^s(x, A)\mu(dx).\end{aligned}$$

But $\bar{p}^s(x, dy) \ll I(x, dy)$, so $\int_{A^c} \bar{p}^s(x, A)\mu(dx) = 0$. Therefore, $\mu^s(A) = \int_A \bar{p}^s(x, A)\mu(dx)$. This implies $\bar{p}^s(x, A) = 1$ a.e. $x - \mu(dx)$ in A. i.e., $\bar{p}(x, dy) = \bar{p}^s(x, dy) = I(x, dy)$ a.e. $x - \mu(dx)$ in A. Thus μ_2 is an invariant distribution relative to $\bar{p}(x, dy)$. Hence μ, is also an invariant distribution relative to $\bar{p}(x, dy)$. So:

$$\begin{aligned}I(\mu) &= \int\int \log \frac{\bar{p}(x, dy)}{p(x, dy)} \bar{p}(x, dy)\mu(dx)\\ &= \int_A \int \log \frac{\bar{p}(x, dy)}{p(x, dy)} \bar{p}(x, dy)\mu(dx) + \int_{A^c} \int \log \frac{\bar{p}(x, dy)}{p(x, dy)} \bar{p}(x, dy)\mu(dx)\end{aligned}$$

$$= \int_A \int \log \frac{I(x, dy)}{p(x, dy)} \bar{p}(x, dy)\mu(dx) + \int_{A^c} \int \log \frac{\bar{p}(x, dy)}{p(x, dy)} \bar{p}(x, dy)\mu(dx)$$

$$= \log 2 \cdot \mu(A) + \| \mu^a \| \int \int \log \frac{\bar{p}(x, dy)}{p(x, dy)} \bar{p}(x, dy)\mu_1(dx)$$

$$\geq \log 2 \cdot \| \mu^s \| + \| \mu^a \| I(\mu_1).$$

The last inequality is true because $I(\mu_1) = \inf_{\lambda \in \mathscr{M}_{\mu_1}} \bar{I}_{p(x,dy)\mu(dx)}(\lambda)$ by (0.0). Let $\lambda_k(dy)$ be a sequence of measures converging to $\mu^s(dy)$ weakly such that $\lambda_k \ll m$ and $\| \lambda_k \| = \| \mu^s \|$ for every k. Thus, $\lambda_k + \mu^a \ll m$ for every k and $\lambda_k + \mu^a$ converges to μ weakly.

$$I(\lambda_k + \mu^a) = I\left(\| \lambda_k \| \cdot \frac{\lambda_k}{\| \lambda_k \|} + \| \mu^a \| \mu_1 \right) \leq \| \mu^s \| I\left(\frac{\lambda_k}{\| \mu^s \|} \right) + \| \mu^a \| I(\mu_1)$$

$$= \| \mu^s \| \log 2 + \| \mu^a \| I(\mu_1) \leq I(\mu).$$

Hence: $\limsup_{n\to\infty} I(\lambda_k + \mu^a) \leq I(\mu)$. But I is lower semi-continuous and $\lambda_k + \mu^a \to \mu$ weakly, $\liminf_{k\to\infty} I(\lambda_k + \mu^a) \geq I(\mu)$. Therefore: $\lim_{k\to\infty} I(\lambda_k + \mu^a) = I(\mu)$. Since $\lambda_k + \mu^a \in G$ when k is large, we have $\liminf_{n\to\infty} 1/n \log P_x(L_n(\omega, \cdot) \in G) \geq -I(\lambda_k + \mu^a)$ for large k. We then complete the proof by letting $k \to \infty$.

Acknowledgments. This paper is part of the author's dissertation written under the guidance of Professor Steven Orey. The author likes to take this opportunity to express his deep thanks to Professor Orey for his suggestions and many hours of mathematical conversation. The author also thanks the referee for a suggestion on Remark (3).

REFERENCES

[1] BAHADUR, R. R. and ZABELL, S. L. (1979). Large deviation of the sample mean in general vector spaces. *Ann. Probability* **7** 587–621.

[2] DONSKER, M. D. and VARADHAN, S. R. S. (1975). Asymptotic evaluation of certain Markov process expectations for large time—I. *Comm. Pure Appl. Math.* **27** 1–47.

[3] DONSKER, M. D. and VANADHAN, S. R. S. (1976). Asymptotic evaluation of certain Markov process expectations for large time—III. *Comm. Pure Appl. Math.* **29** 389–461.

[4] HARRIS, T. E. (1955). The existence of stationary measures for certain Markov processes. Third Berkeley Symposium.

[5] HOADLEY, A. B. (1967). On the probability of large deviations of functions of several empirical cdf's. *Ann. Math. Statist.* **38** 360–382.

[6] SANOV, I. N. (1953). On the probability of large deviations of random variables (in Russian). *Math. sb.* **42** 11–44, (English translation in *Selected Transl. Math. Statist. Probability* 213–244, 1961.

INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
NANKANG, TAIWAN
R. O. C.