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ASYMPTOTIC NORMALITY OF STATISTICS BASED ON THE CONVEX MINORANTS OF EMPIRICAL DISTRIBUTION FUNCTIONS

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Let F_n be the Uniform empirical distribution function. Write $\hat{F}_n$ for the (least) concave majorant of F_n, and let $\hat{f}_n$ denote the corresponding density. It is shown that $n \int_0^1 (\hat{f}_n(t) - 1)^2 \, dt$ is asymptotically standard normal when centered at $\log n$ and normalized by $(3 \log n)^{1/2}$. A similar result is obtained in the 2-sample case in which $\hat{f}_n$ is replaced by the slope of the convex minorant of $\bar{F}_m = F_m \circ H_N^{-1}$.

1. Introduction. In 1956, Grenander introduced ideas of "non-parametric" maximum likelihood estimation. In one of the examples, he found the maximum likelihood estimate (M.L.E.) within the class of all distribution functions that are concave over $[0, \infty)$, or equivalently, the class of all monotone decreasing densities supported on $[0, \infty)$. The M.L.E. in this example is the concave majorant of the ordinary empirical distribution function. For a formal definition of maximum likelihood which covers these "non-parametric" cases, see Scholz (1980). For this and other examples of "non-parametric" estimation, see Barlow et al (1972).

Statistics based on either concave majorants or convex minorants of empirical distribution functions have arisen independently in at least two other very different contexts. In 1975, Behnen proposed a 2-sample rank statistic defined as "the supremum of all standardized and centered simple linear rank statistics having non-decreasing scores." This statistic was shown to have comparable performance to an adaptive statistic proposed by Randles and Hogg (1973) when used against shift alternatives, and to have a much superior performance against stochastically ordered alternatives. Although not mentioned in Behnen (1974, 1975), it was known independently to both Behnen and Scholz (personal correspondence, June and July, 1975) that this statistic is expressible in terms of the L_2-norm of the density function of the concave majorant of the usual 2-sample empirical distribution function. The asymptotic distribution of the statistic was left as an open question, although Behnen (1974) provided extensive simulations for selected sample sizes up to $m = n = 100$ which suggested to us the asymptotic normality of the statistic.

In a completely different context, Scholz (1982) proposed a procedure for the combination of p-values from independent tests of significance. The procedure, utilizing Roy's union-intersection principle, results in a statistic that is expressible similarly in terms of the L_2-norm of the density function of the concave majorant of the 1-sample Uniform empirical distribution function. Exact distributions are obtained by Scholz for the very small sample sizes which are important for this context. Simulations were also carried out by Scholz (personal communication) for moderate sample sizes to evaluate the feasibility of approximations using asymptotic theory.

In this paper, the asymptotic normality of the 1-sample statistic of Scholz is derived in Section 3 and that of the 2-sample statistic of Behnen is obtained in Section 5. The method used in Section 3 utilizes a conditional representation of the concave majorant of the

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Uniform empirical distribution function in terms of a sequence of Poisson and Gamma random variables. This representation is detailed in Section 2. This method is an extension of that used in Pyke (1965) and due originally to LeCam (1958). The 2-sample case is proved in Section 5 using a strong invariance principle together with the asymptotic normality of the L_2-norms of the slope processes of the convex minorants of a sequence of truncated Brownian Bridges. The latter is derived in Section 4.

As is pointed out in the remarks in Section 6, it is possible to prove the 2-sample result by an analogue of the method presented in Sections 2 and 3. On the other hand, it is possible to prove the 1-sample result by a method analogous to that of Sections 4 and 5. This is done in Groeneboom (1981, Theorem 3.2) where a detailed study of the concave majorant of Brownian Motion is presented.

2. The representation theorem. In this section, we describe the specific construction that is used to provide a tractable representation for the concave majorant of the Uniform empirical process. To do this, the following notation is needed. Let F_n denote the Uniform empirical distribution function, and write $\hat{F}_n$ for its concave majorant, the function on $[0, 1]$ formed as it were by stretching a rubber band over the top of F_n. Let $\eta_n + 1$ be the number of vertices of $\hat{F}_n$, including the end-points, $(0, 0)$ and $(1, 1)$. Let $\xi_{n0} = 0 < \xi_{n,1} < \cdots < \xi_{n,\eta_n} = 1$ be the x-coordinates of these vertices. For $1 \le i \le \eta_n$ and $1 \le j \le n$, define

$$\begin{aligned} & D_{ni} = \xi_{n,i} - \xi_{n,i-1}, \\ (2.1) \qquad & J_{n,i} = n[F_n(\xi_{n,i}) - F_n(\xi_{n,i-1})] = n[\hat{F}_n(\xi_{n,i}) - \hat{F}_n(\xi_{n,i-1})], \qquad \text{and} \\ & Q_{n,j} = \#\{i : J_{n,i} = j\} \end{aligned}$$

to be respectively the horizontal "width" and vertical "number of steps" associated with each of the segments of $\hat{F}_n$, and the frequency of segments of a given number of steps (cf. Figure 1). Notice that $Q_{n,0} = 1$ in view of the flat section of $\hat{F}_n$ that always occurs to the right of the largest order statistic. Set $\mathbf{D}^{(n)} = (D_{n,1}, \cdots, D_{n,\eta_n})$, $\mathbf{J}^{(n)} = (J_{n,1}, \cdots, J_{n,\eta_n})$ and $\mathbf{Q}^{(n)} = (Q_{n,1}, \cdots, Q_{n,n})$. If $\hat{f}_n$ denotes the density (slope) of $\hat{F}_n$, then the statistic that motivated this paper is

$$L_n := \| \hat{f}_n - 1 \|_2^2$$

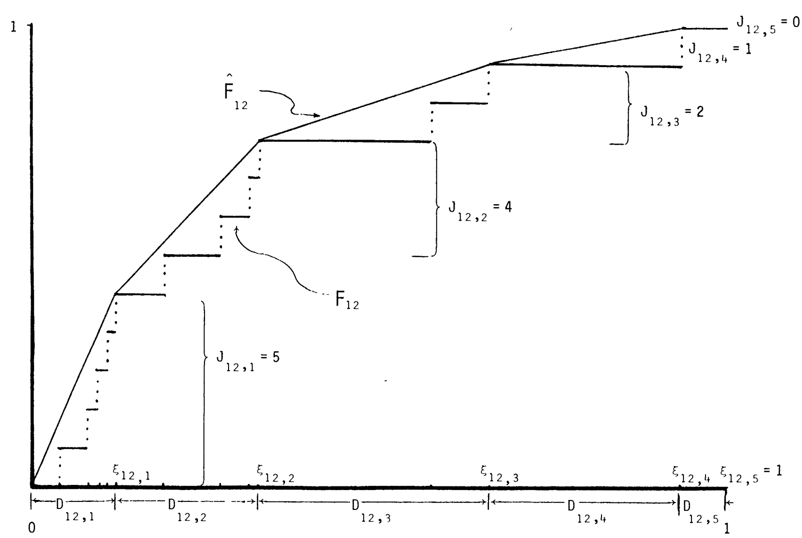

FIG. 1. *Sample realization of the Uniform empirical distribution function F_n and its concave majorant $\hat{F}_n$ for $n = 12$. Here $\eta_{12} = 5$, $\mathbf{J}^{(12)} = (5, 4, 2, 1)$ and $\mathbf{Q}^{(12)} = (1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0)$.*

which can be written in terms of the above notation as

(2.1a) $$L_n = n^{-1} \sum_{i=1}^{\eta_n} (J_{ni})^2/nD_{ni} - 1$$

in which J_{ni}/nD_{ni} is the slope of the ith segment of the concave majorant. The concave majorant of a sequence of partial sums of interchangeable r.v.'s was studied by Sparre Andersen (1954) who derived in particular the distribution of the number of vertices. Implicit in that paper is the following result for our problem where the interchangeable r.v.'s are the $n + 1$ spacings formed from the n independent Uniform (0, 1) observations. (For any r.v.'s X, Y, we write f_X and $f_{X|Y}$ for marginal and conditional density functions when well defined.)

LEMMA 2.1. *For non-negative integers* $q_1, \cdots, q_n$ *with* $\sum_{j=1}^{n} jq_j = n$,

(2.2) $$f_{\mathbf{Q}^{(n)}}(q_1, \cdots, q_n) = \prod_{j=1}^{n} j^{-q_j}/q_j!$$

and

(2.3) $$f_{\eta_n}(k) = \sum \{f_{\mathbf{Q}^{(n)}}(\mathbf{q}^{(n)}) : q_1 + \cdots + q_n = k\}.$$

PROOF. Conditionally given the ordered set of Uniform spacings, all permutations thereof are equally likely. With probability 1, all spacings and partial sums thereof are distinct. Partition the spacings into q_1 subsets of size 1, q_2 subsets of size 2, and so on, with $\sum_{j=1}^{n} jq_j = n$. The remaining spacing forms a subset of its own. Within each subset of size j, the probability is $1/j$ of choosing a permutation whose partial sums lie below the line segment joining the end points of that subset. (Spitzer's Lemma; Spitzer (1956), cf. Feller (1968), page 423.) The q_j subsets of size j can be permuted $q_j!$ times. Finally, the slopes of the line segments determined by each of the subsets can be ordered in exactly one way by decreasing slopes to form a concave majorant with the required $\mathbf{Q}^{(n)} = \mathbf{q}^{(n)}$. (2.3) is immediate. (Cf. the last paragraph of Hobby and Pyke, 1964.) □

If we let $\{N_j : j \geq 1\}$ be independent Poisson r.v.'s with $E(N_j) = 1/j$, it is clear from the form of (2.2) that if $T_n = \sum_{j=1}^{n} jN_j$ and $N_0 = 1$ a.s.,

(2.4) $$\mathbf{Q}^{(n)} =_L \mathbf{N}^{(n)} \mid [T_n = n].$$

Note that from a comparison of the form of the conditional and unconditional probabilities,

(2.5) $$P[T_n = n] = \exp\{-\sum_{j=1}^{n}(1/j)\}.$$

(Cf. Lemma 3.2 below.) Also, it is well known that if $\{Y_j : j \geq 1\}$ are independent Exp(1) r.v.'s then the conditional distribution of $n^{-1}(Y_1, \cdots, Y_{n+1})$ given $Y_1 + \cdots + Y_{n+1} = n$ is the same as that of the $n + 1$ Uniform spacings. We now build upon this as follows to provide a suitable conditional representation for the concave majorant. Let $\{S_{ji} : i \geq 1, j \geq 0\}$ be independent r.v.'s with S_{ji} being a $\Gamma(j, 1)$ r.v. for $j \geq 1$ and S_{oi} being a $\Gamma(1, 1)$ or Exp(1) r.v. That is, each S_{ji}, $j \geq 1$, is equal to a sum of j independent Exp(1) r.v.'s. Assume $\{N_j\}$ and $\{S_{ji}\}$ are independent.

Rewrite the spacings $\{D_{ni}\}$ of the concave majorant in a specific order as follows. First, write $\bar{D}_{no}$ for $D_{n\eta_n}$, the only zero-step spacing; then write all of the one-step spacings in the order of increasing magnitude (which, since the slopes are nonincreasing, is the same as the order of appearance in the concave majorant); then write all of the two-step spacings, and so on. In this order, denote them by $\bar{\mathbf{D}}^{(n)} = (\bar{D}_{no}; \bar{D}_{n11}, \cdots, \bar{D}_{n1Q_{n1}}; \bar{D}_{n21}, \cdots, \bar{D}_{n2Q_{n2}}; \cdots; \bar{D}_{nnQ_{nn}})$. Analogously, write $\bar{S}_{j1} \leq \bar{S}_{j2} \leq \cdots \leq \bar{S}_{jN_j}$ for the ordered values of $S_{j1}, \cdots, S_{jN_j}$ and set

$$\bar{\mathbf{S}}^{(n)} = (S_{01}; \bar{S}_{11}, \cdots, \bar{S}_{1N_1}; \bar{S}_{21}, \cdots, \bar{S}_{2N_2}; \cdots; \bar{S}_{n1}, \cdots, \bar{S}_{nN_n}).$$

Then the following representation holds, where $\mathbf{N}^{(n)} = (N_1, \cdots, N_n)$.

NOTE. We delete the parameter n from the notation whenever it is unlikely to cause confusion.

THEOREM 2.1. *For* $n \geq 1$

$$(\mathbf{Q}^{(n)}, n\bar{\mathbf{D}}^{(n)}) =_L (\mathbf{N}^{(n)}, \bar{\mathbf{S}}^{(n)}) \mid (T_n = n, S_n = n) \tag{2.6}$$

where

$$T_n = \textstyle\sum_{j=1}^{n} jN_j, \qquad S_n = \sum_{j=1}^{n} \sum_{i=1}^{N_j} S_{ji}. \tag{2.7}$$

PROOF. Let $\{Y(t): t \geq 0\}$ denote a Poisson process with $EY(t) = t$. It is well known that conditional on the $(n+1)$th jump occurring at $t = n$, $n^{-1}Y(n\cdot)$ on $[0, 1)$ is equal in law to F_n. The relationship given in (2.6) is closely related. First of all, by (2.4), the marginal distribution of $\mathbf{Q}$ is the same as the conditional distribution of $\mathbf{N}^{(n)}$ given $T_n = n$. Now $f_{\mathbf{Q},\bar{\mathbf{D}}}$ is obtainable by standard techniques starting from the Uniform distribution over $[0, 1]^n$. The main thing to observe is that as is implied in the proof of (2.4), the distribution of $\mathbf{Q}$ is the same for almost all values of the original ordered Uniform spacings. The latter when multiplied by n, can be represented as the ordered values of $n + 1$ independent Exp(1) r.v.'s $Y_1, \cdots, Y_{n+1}$ given $Y_1 + \cdots + Y_{n+1} = n$. Thus the same techniques that will yield $f_{\mathbf{Q},n\bar{\mathbf{D}}}(\mathbf{q}, \mathbf{d})$ from the joint density of ordered Uniform spacings will yield $f_{N^{(n)},\bar{S}^{(n)} \mid T_n, S_n}(\mathbf{q}, \mathbf{d}, n, n)$ from ordered exponentials. □

3. The one-sample limit theorem. The statistic L_n is shown to be asymptotically normal by studying a related statistic suggested by Theorem 2.1. First of all, notice that

$$L_n = n^{-1} \textstyle\sum_{j=1}^{n} j^2 \sum_{i=1}^{Q_j} (1/n\bar{D}_{nji}), \tag{3.1}$$

which suggests that one might study the conditional limiting distribution of

$$L_n^* = n^{-1} \textstyle\sum_{j=1}^{n} j^2 \sum_{i=1}^{N_j} (1/S_{ji})$$

under the conditions that $T_n = n$ and $S_n = n$. To this end, we introduce three suitably normalized r.v.'s, namely

$$U_n = (3 \log n)^{-1/2} \left(\textstyle\sum_{j=1}^{n} \{j^2 \sum_{i=1}^{N_j} (1/S_{ji} - 1/j) + \sum_{i=1}^{N_j} (S_{ji} - j)\} - \log n\right) \tag{3.2}$$

$$V_n = n^{-1/2} \textstyle\sum_{j=1}^{n} \sum_{i=1}^{N_j} (S_{ji} - j) \tag{3.3}$$

and

$$W_n = n^{-1} \textstyle\sum_{j=1}^{n} jN_j. \tag{3.4}$$

Observe first that the conditions $T_n = n$ and $S_n = n$ are equivalent to $W_n = 1$ and $V_n = 0$. Secondly, observe that under these conditions, U_n reduces to

$$\begin{aligned} U_n &= (3 \log n)^{-1/2} \left(\textstyle\sum_{j=1}^{n} j^2 \sum_{i=1}^{N_j} (1/S_{ji}) - n - \log n\right), \\ &= (3 \log n)^{-1/2} (nL_n^* - n - \log n), \end{aligned} \tag{3.5}$$

which is conditionally equal in law to the same expression with L_n replaced by L_n^*. The particular choice of U_n in (3.2) was not easy to obtain. Its specific combination of terms is essential in order to provide the desired asymptotic normality. The form given by (3.2) makes it easy to see the effect of the conditions $W_n = 1$ and $V_n = 0$. It may, however, be simplified as

$$U_n = (3 \log n)^{-1/2} \left(\textstyle\sum_{j=1}^{n} \sum_{i=1}^{N_j} (S_{ji} - j)^2/S_{ji} - \log n\right). \tag{3.5a}$$

One may also write

$$\begin{aligned} U_n &= (3 \log n)^{-1/2} \left(\textstyle\sum_{j=1}^{n} \sum_{i=1}^{N_j} (S_{ji} - j)^2/j - \log n\right) \\ &\quad - (3 \log n)^{-1/2} \textstyle\sum_{j=1}^{n} \sum_{i=1}^{N_j} (S_{ji} - j)^3/jS_{ji} \end{aligned} \tag{3.5b}$$

in which the randomness has been removed from the denominator in the first term while the second term is of smaller order.

LEMMA 3.1. *$(U_n, V_n, W_n) \to_L (U, V, W)$ where U and (V, W) are independent, U is $N(0, 1)$ and (V, W) has the infinitely divisible characteristic function*

$$\phi_{(V,W)}(s, t) = \exp\left\{\int_0^1 (e^{itu - s^2u^2/2} - 1)u^{-1}\, du\right\}.$$

Equivalently, $V =_L ZW^{1/2}$ where Z is a $N(0, 1)$ r.v. *independent of W.*

PROOF. Clearly $E(V_n) = 0$ and $E(W_n) = 1$. Also

$$\operatorname{var}(V_n) = n^{-1}\textstyle\sum_{j=1}^n E(N_j)\operatorname{var}(S_{ji}) = n^{-1}\sum_{j=1}^n (1/j)j = 1.$$

Since $E(S_{ji}^{-k}) = \Gamma(j-k)/\Gamma(j)$ for $j > k$, one obtains for $j > 2$ $E(S_{ji}^{-1}) = (j-1)^{-1}$, $E(S_{ji}^{-2}) = 1/(j-1)(j-2)$ and $\operatorname{var}(S_{ji}^{-1}) = (j-1)^{-2}(j-2)^{-1}$. Write U_n^* to denote U_n less its first 2 terms. Then $U_n^* - U_n = o_p(1)$. Hence, as $n \to \infty$,

$$\begin{aligned} E(U_n^*) &= (3\log n)^{-1/2}\left(\textstyle\sum_{j=3}^n j^2 \cdot j^{-1} \cdot j^{-1}(j-1)^{-1} - \log n\right) \\ &= (3\log n)^{-1/2}\left(\textstyle\sum_{j=2}^{n-1} j^{-1} - \log n\right) = O((\log n)^{-1/2}) \end{aligned}$$

and

$$\begin{aligned} \operatorname{var}(U_n^*) &= (3\log n)^{-1}\left(\textstyle\sum_{j=3}^n j^{-1}\{j^4 \operatorname{var}(S_{ji}^{-1}) + \operatorname{var}(S_{ji}) + 2j^2 \operatorname{covar}(S_{ji}^{-1}, S_{ji})\}\right. \\ &\qquad \left. + \textstyle\sum_{j=3}^n j^2 \operatorname{var}(N_j)E(S_{ji}^{-1} - j^{-1})\right) + o(1) \\ &= (3\log n)^{-1}\textstyle\sum_{j=3}^n\{j^3(j-1)^{-2}(j-2)^{-1} + 1 - 2j(j-1)^{-1} \\ &\qquad + (j-1)^{-1}\} + o(1) \\ &= (3\log n)^{-1}\textstyle\sum_{k=1}^{n-2}\{3/k + 4/k(k+1) + 1/k(k+1)^2\} + o(1) \to 1. \end{aligned} \tag{3.6}$$

The orders of the asymptotic variances are thereby established.

To determine the limiting distribution of (U_n, V_n, W_n), it suffices to show that all linear combinations, $aU_n^* + bV_n + cW_n$, converge in law and to specify the limiting distribution. Since the variances converge, it suffices to proceed as follows. In view of (3.5a), write $U_n^* = \sum_{j=3}^n (X_{nj} + Y_{nj}) + \varepsilon_n$, where for $j \ge 3$

$$\begin{aligned} X_{nj} &= (3\log n)^{-1/2}\textstyle\sum_{i=1}^{N_j}\{(S_{ji} - j)^2/S_{ji} - j/(j-1)\} \\ Y_{nj} &= (3\log n)^{-1/2}(jN_j - 1)/(j-1) \\ \varepsilon_n &= (3\log n)^{-1/2}\left(\textstyle\sum_{j=3}^n (j-1)^{-1} - \log n\right) = O((\log n)^{-1/2}). \end{aligned} \tag{3.7}$$

We note that $EX_{nj} = 0 = EY_{nj}$.

To establish the asymptotic normality of U_n^*, it suffices, since $\varepsilon_n \to 0$ and all variances converge, to show that $\sum_j P[|X_{nj} + Y_{nj}| > \varepsilon] \to 0$. (Cf. Loève, 1963, page 316). For this it suffices to compute fourth moments and use Markov's inequality. To this end, we compute

$$\begin{aligned} E|X_{nj}|^4 &= (3\log n)^{-2}E\left\{N_jE[(S_{j1} - j)^2/S_{j1} - j/(j-1)]^4\right. \\ &\qquad \left. + \binom{N_j}{2}(E[(S_{j1} - j)^2/S_{j1} - j/(j-1)]^2)^2\right\} \\ &= (3\log n)^{-2}\{j^{-1}A_j + (1/2j^2)B_j^2\} \end{aligned}$$

where A_j and B_j are the fourth and second central moments of $(S_{j1} - j)^2/S_{j1}$, respectively. It is easy to show that both A_j and B_j are uniformly bounded in $j > 4$, since for $j > 4$

$$E[(S_{j1} - j)^2/S_{j1}]^4 = [(j-1)(j-2)(j-3)(j-4)]^{-1}E(S_{j-4,1} - j)^8 \tag{3.8}$$

and by the c_r-inequality (cf. Loève, 1963, page 155)

$$E(S_{j-4,1} - j)^8 \le 2^7\{E[\textstyle\sum_{i=1}^{j-4}(Y_i - 1)]^8 + 4^8\}$$

where $Y_1, Y_2, \cdots$ are independent Exponential r.v.'s with mean 1. Straightforward computations show that $E(X_1 + \cdots + X_m)^8 \le Cm^4$ for any independent r.v.'s with means zero. Therefore,

$$\sum_{j>4} P[|X_{nj}| > \varepsilon] \le \varepsilon^{-4} \sum_{j>4} E|X_{nj}|^4 < C(\log n)^{-2} \sum_j j^{-1} \to 0.$$

Similarly,

$$\begin{aligned} E|Y_{nj}|^4 &= (3\log n)^{-2} j^4 (j-1)^{-4} E(N_j - 1/j)^4 \\ &= (3\log n)^{-2} j^4 (j-1)^{-4} (j^{-1} + 6j^{-2}) \le C(\log n)^{-2} j^{-1}. \end{aligned}$$

Hence

$$\sum_j P[|Y_{nj}| > \varepsilon] = O((\log n)^{-1}) \to 0.$$

We have thus established that $U_n \to_L U$, a $N(0, 1)$ r.v.

To determine the limiting joint distribution of (V_n, W_n), we compute the characteristic function

$$\tag{3.9} \begin{aligned} E(e^{isV_n + itW_n}) &= E\{\textstyle\prod_{j=1}^n ([\phi_{S_{j1}-j}(sn^{-1/2})]^{N_j} e^{it(j/n)N_j})\} \\ &= \exp\{\textstyle\sum_{j=1}^n j^{-1}(e^{it(j/n)}\phi_{S_{j1}-j}(sn^{-1/2}) - 1)\}. \end{aligned}$$

Now, as $j \to \infty$ while $j/n \to u \in (0, 1)$, we have by the Central Limit Theorem for $\{j^{-1/2}(S_{j1} - j)\}$ that

$$\phi_{S_{j1}-j}(sn^{-1/2}) \to e^{-s^2u^2/2}.$$

Recognizing a Riemann sum in the exponent of (3.9), the limit becomes

$$\tag{3.10} E(e^{isV + itW}) = \exp\left\{\int_0^1 (e^{itu - s^2u^2/2} - 1)u^{-1}\,du\right\}$$

as desired. It is straightforwardly checked by direct integration that the exponent in (3.10) may be written as

$$\tag{3.11} \int_0^1 \int_{-\infty}^{\infty} (e^{isv + itw} - 1)\phi(v/w)w^{-2}\,dv\,dw$$

where ϕ is the standard Normal density, so that the Lévy measure of the 2-dimensional infinitely divisible r.v. (V, W) is absolutely continuous with respect to Lebesgue measure on $((-\infty, 0) \cup (0, \infty)) \times (0, 1)$ with "density" $\phi(v/w)w^{-2}$. There is therefore no Normal part to the distribution of (V, W). This fact completes the proof since it is easily checked that (U_n, V_n, W_n) is in the domain of attraction of an infinitely divisible distribution and that if $\{Y_{ni}\}$ and $\{Z_{ni}\}$ are two triangular arrays which are jointly in the domain of attraction of an infinitely divisible distribution, and if the marginal limiting law of one is Normal and the other has no Normal component, then they are asymptotically independent. □

The above result gives the limiting joint distribution of (U_n, V_n, W_n), whereas the result being sought is the limiting conditional distribution of U_n given $V_n = 0$ and $W_n = 1$. To obtain the conditional result from the joint, we follow an idea used originally by LeCam (1958) to obtain limit laws for sums of a function of Uniform spacings. The method was used by Pyke (1965) to obtain limit laws of more general functions as well as the weak convergence of related processes. It can be shown that

$$\tag{3.12} \begin{aligned} \phi_{nm}(t) :&= E[e^{itU_m} \mid V_n = 0, W_n = 1] = E\{E[e^{itU_m} \mid V_m, W_m] \mid V_n = 0, W_n = 1\} \\ &= E[e^{itU_m}\rho_{nm}(W_m)r_{nm}(V_m, W_m)] \end{aligned}$$

where

$$\rho_{nm}(k/n) = P[W_m = k/m \mid W_n = 1]/P[W_m = k/m]$$

and

$$r_{nm}(v, k/m) = f_{V_m|W_m,V_n}(v, k/m, 0)/f_{V_m|W_m}(v, k/m)$$

for $k = 0, 1, \cdots$ and real v. To verify (3.12), use the fact that $f_{V_m|W_m,V_n} = f_{V_m|W_m,V_n,W_n}$ and $f_{W_m|V_n} = f_{W_m|V_n,W_n}$ to write

$$\phi_{nm}(t) = \sum_{k=0}^{n} P[W_m = k/m \mid W_n = 1] \int_{-\infty}^{\infty} E[e^{itU_m} \mid W_m = k/m, V_m = v]$$

$$\cdot f_{V_m|W_m,V_n}(v, k/m, 0)\, dv$$

$$= \sum_{k=0}^{n} P[W_m = k/m]\rho_{nm}(k/m) \int_{-\infty}^{\infty} E[e^{itU_m} \mid W_m = k/m, V_m = v]$$

$$\cdot r_{nm}(v, k/n) f_{V_m|W_m}(v, k/m)\, dv$$

which is then the unconditional expectation given in (3.12).

Since the N_j's are independent

$$\rho_{nm}(k/n) = P[T_n - T_m = n - k]/P[T_n = n] \tag{3.13}$$

where $T_n = \sum_{j=1}^{n} jN_j = nW_n$. Moreover, under the condition that $W_m = k/m$ (equivalently, $T_m = k$) and $V_n = 0$ (equivalently, $S_n := \sum_{j=1}^{n} \sum_{i=1}^{N_j} S_{ji} = n$), it is well known that $n^{-1}S_m$ is a Beta $(k, n - k)$ r.v., while under the single condition $V_n = 0$, then S_m is a Gamma $(k, 1)$ r.v. Therefore, for $k = 1, 2, \cdots, n$ and $0 < m^{1/2}v + k < n$,

$$f_{V_m|W_m,V_n}(v, k/m, 0) = \frac{m^{1/2}\Gamma(n)}{n\Gamma(k)\Gamma(n-k)} \left(\frac{m^{1/2}v + k}{n}\right)^{k-1} \left(1 - \frac{m^{1/2}v + k}{n}\right)^{n-k-1}$$

and

$$f_{V_m|W_m}(v, k/m) = m^{1/2}\left(\frac{m^{1/2}v + k}{\Gamma(k)}\right)^{k-1} e^{-m^{1/2}v-k}.$$

The densities are zero otherwise. Hence

$$r_{nm}(v, k/m) = \{\Gamma(n)/n^k\Gamma(n-k)\}\left(1 - \frac{m^{1/2}v + k}{n}\right)^{n-k-1} e^{m^{1/2}v+k}. \tag{3.14}$$

Upon substituting (3.13) and (3.14) into (3.12), one obtains a tractable unconditional form for the conditional characteristic function of U_m which can be used to prove the desired result. A significant step in the proof of this result will be the determination of the limiting behavior of the functions ρ_{nm} and r_{nm} in order to permit the use of the dominated convergence theorem in (3.12). To this end, we prove the following lemmas.

LEMMA 3.2. *With $T_n := \sum_{j=1}^{n} jN_j = nW_n$,*

$$P[T_n = k] = p_n := \exp(-\textstyle\sum_{j=1}^{n} j^{-1}), \quad \text{for } 0 \le k \le n, \tag{3.15}$$
$$\le p_n \quad \text{for } k > n$$

and $np_n \to e^{-\gamma} = .561\cdots$, where $\gamma = .5772\cdots$ is Euler's constant.

PROOF. The generating function of T_n is computed directly to be

$$p_n \exp(\textstyle\sum_{j=1}^{n} j^{-1}s^j). \tag{3.16}$$

For $k \le n$, the coefficient of s^k in (3.16) will be the same as the coefficient of s^k in $p_n(1 - s)^{-1}$ since the two functions differ in the exponent by powers of s higher than n. For $k > n$, the latter would be greater. This proves (3.15). The rest of the Lemma is clear. □

LEMMA 3.3. *For $m = o(n)$, $\rho_{nm}(k/m) \to 1$, as $n \to \infty$, uniformly for $k \le cn$ for some $c < 1$.*

PROOF. By (3.13) and Lemma 3.2, it only remains to show that the numerator of ρ_{nm} satisfies

$$\lim_{n\to\infty} nP[T_n - T_m = n - k] = e^{-\gamma}. \tag{3.17}$$

By the Fourier inversion formula for the characteristic function of $T_n - T_m = \sum_{j=m+1}^{n} jN_j$,

$$P[T_n - T_m = n - k] = (2\pi)^{-1} \int_0^{2\pi} f_{nm}(t)e^{-(n-k)it}\, dt \tag{3.18}$$

where

$$f_{nm}(t) = \exp\{-\textstyle\sum_{j=1}^{n} j^{-1}(1 - e^{itj}) + \sum_{j=1}^{m} j^{-1}(1 - e^{itj})\}$$

is the characteristic function of $T_n - T_m$. Integration by parts shows that the right hand side of (3.18) is equal to

$$(2(n-k)\pi)^{-1} \int_0^{2\pi} \{\textstyle\sum_{j=1}^{n} e^{itj} - \sum_{j=1}^{m} e^{itj}\} f_{nm}(t)e^{-(n-k)it}\, dt.$$

Since $p_n = P[T_n = n] = P[T_n = n - k]$ for all $k \le n$, then similarly one obtains

$$(n-k)p_n = (2\pi)^{-1} \int_0^{2\pi} \textstyle\sum_{j=1}^{n} e^{itj} g_n(t)e^{-(n-k)it}\, dt$$

where

$$g_n(t) = \exp\{-\textstyle\sum_{j=1}^{n} j^{-1}(1 - e^{itj})\}$$

is the characteristic function of T_n. Hence

$$\begin{aligned}(n-k)\{P[T_n - T_m = n-k] - p_n\} &= (2\pi)^{-1} \int_0^{2\pi} \textstyle\sum_{j=1}^{n} e^{itj} \cdot \{f_{nm}(t) - g_n(t)\}e^{-(n-k)it}\, dt \\ &\quad - (2\pi)^{-1} \int_0^{2\pi} \textstyle\sum_{j=1}^{m} e^{itj} f_{nm}(t)e^{-(n-k)it}\, dt \\ &= P[T_n - T_m \le n-k-1] - P[T_n \le n-k-1] \\ &\quad - P[n-m-k \le T_n - T_m \le n-k-1] \\ &= P[T_n - T_m \le n-m-k-1] \\ &\quad - P[T_n \le n-k-1].\end{aligned} \tag{3.19}$$

Thus if $m = o(n)$ and $k = o(n)$, it follows from Lemma 3.1 that $W_n = n^{-1}T_n \to_L W$, a continuous r.v., and so the right hand side of (3.19), which is bounded for k bounded away from n, converges to 0 as $n \to \infty$. Thus

$$\lim_{n\to\infty} nP[T_n - T_m = n - k] = \lim_{n\to\infty} np_n = e^{-\gamma}$$

by Lemma 3.2. The convergence is uniform for $k \le cn$ when $0 < c < 1$. This completes the proof. □

LEMMA 3.4. *For $m/n \to b \ge 0$ and $k/m \to x > 0$ with $bx < 1$,*

$$\lim_{n\to\infty} r_{nm}(v, k/m) = (1 - bx)^{-1/2} \exp\{-v^2 b/2(1 - bx)\}$$

uniformly over $|v|(m/n)^{1/2} \le v_0$ and $k \le cn$ for $0 < v_0 < \infty$ and $c < 1$.

PROOF. By (3.14) and Stirling's formula,

$$r_{nm}(v, k/m) = n^{n-k-1/2}(n-k)^{-n+k+1/2}\left(1 - \frac{m^{1/2}v + k}{n}\right)^{n-k-1} e^{m^{1/2}v}$$
$$= (1 - k/n)^{-1/2}\left(1 - \frac{m^{1/2}v}{n-k}\right)^{n-k-1} e^{m^{1/2}v}.$$

Now, when $0 < m^{1/2}v + k < n$,

$$\log\left\{\left(1 - \frac{vm^{1/2}}{n-k}\right)^{n-k} e^{m^{1/2}v}\right\} = m^{1/2}v + (n-k)\log\left(1 - \frac{vm^{1/2}}{n-k}\right)$$
$$= m^{1/2}v + (n-k)\left\{-\frac{vm^{1/2}}{n-k} - \frac{v^2 m}{2(n-k)^2} - R\left(\frac{vm^{1/2}}{n-k}\right)\right\}$$

where we have written $R(x) = -\log(1-x) - x - x^2/2$ for $|x| < 1$. Since

$$|R(x)| \le |x|^3/3\,(1 - |x|) \le |x|^3 \quad \text{if} \quad |x| \le 2/3$$

one has

$$(n-k)\left|R\left(\frac{vm^{1/2}}{n-k}\right)\right| \le \frac{|v|^3 m^{3/2}}{(n-k)^2} = n^{-1/2}\frac{(|v|(m/n)^{1/2})^3}{(1-k/n)^2}$$

if $|v|\,(m/n)^{1/2}$ is bounded. The result follows. □

This result, when $k = m$, was used in Le Cam (1958) and Pyke (1965, page 410). In the present application, we have $b = 0$. In all cases, the result simply represents the known asymptotic behaviors of Beta and Gamma densities.

In both of the above results, Lemmas 3.3 and 3.4, the uniformity requires that $k < cn$ for some $c < 1$. Since k is a sample value for T_m, the application of these results will require that for $m = o(n)$,

$$P[T_m > cn \mid T_n = n] \to 0$$

as $n \to \infty$. Now

$$P[T_m > cn \mid T_n = n] \le P[T_m > cn]/P[T_n = n].$$

Since $nP[T_n = n] = np_n \to e^{-\gamma}$, it suffices to show that $nP[T_m > cn] \to 0$. But by Markov's inequality,

$$nP[T_m > cn] \le nE(e^{tT_m})e^{-cnt} = ne^{-cnt}\exp\{\textstyle\sum_{j=1}^m j^{-1}(e^{tj} - 1)\}$$
$$\le ne^{-cn/m}\exp\left\{\int_0^1 u^{-1}(e^u - 1)\,du\right\}$$

if one chooses $t = 1/m$. Thus we have proved

LEMMA 3.5. *If* $ne^{-cn/m} = o(1)$, *then*

$$\lim_{n\to\infty} P[T_m > cn \mid T_n = n] = 0. \tag{3.20}$$

We can now state and prove the main result.

THEOREM 3.1. *As* $n \to \infty$,

$$K_n := (3\log n)^{-1/2}(nL_n - n - \log n) \to_L U, \tag{3.21}$$

a $N(0, 1)$ *random variable.*

PROOF. In view of (3.5) and the discussion leading up to it, K_n is equal in law to the conditional law of U_n given $V_n = 0$ and $W_n = 1$. But for $0 < m < n$

$$E[e^{itU_n} \mid V_n = 0,\ W_n = 1] = \phi_{nm}(t) + R_{nm}(t)$$

where

$$R_{nm}(t) = E[e^{itU_m}(e^{it(U_n - U_m)} - 1) \mid V_n = 0,\ W_n = 1]. \tag{3.22}$$

The object of the proof will be to show that $\phi_{nm}(t) \to \exp(-t^2/2)$ and $R_{nm}(t) \to 0$ as $n, m \to \infty$. We first study $\phi_{nm}(t)$.

By the form for ϕ_{nm} in (3.12), by the convergence in law of (U_m, V_m, W_m) given in Lemma 3.1, and by the convergence of the ratios ρ_{nm} and r_{nm} given in Lemmas 3.3 and 3.4, it follows that if $ne^{-cn/m} = o(1)$, then

$$\begin{aligned} \lim_{n\to\infty} \phi_{nm}(t) &= \lim_{n\to\infty} E[e^{itU_m}\rho_{nm}(W_m) r_{nm}(V_m, W_m)] \\ &= E[e^{itU} \cdot 1 \cdot 1] = e^{-t^2/2}. \end{aligned} \tag{3.23}$$

In this, the result of Lemma 3.5 is used to show that $E[e^{itU_m} I_{[T_m > cn]} \mid T_n = n,\ V_n = 0] \to 0$, while restricting the other computations to the events $[T_m \le cn]$ and $[(m/n)^{1/2} | V_m | \le v_0]$ for constants c and v_0 on which the convergences in Lemmas 3.3 and 3.4 are uniform. Note also that $P[(m/n)^{1/2} | V_m | > v_0 \mid T_n = n,\ V_n = 0]$ is dominated by $P[n^{1/2} | X_{(k)} - k/n | > v_0]$ where $\{X_{(k)}\}$ are the Uniform order statistics of a sample size n.

Consider now the second term $R_{nm}(t)$. In analogy to (3.12), the conditional defining relation in (3.16) is equivalent to

$$R_{nm}(t) = E[e^{itU_m}(e^{it(U_n - U_m)} - 1)\rho_{nm}(W_m) r_{nm}(V_m, W_m)].$$

In view of the above derivation of (3.23), the convergence of $R_{nm}(t)$ to zero will be complete if we can show that $U_n - U_m \to_p 0$. For this, it will be necessary to be more specific about the choice of m. The only condition so far has been that of Lemma 3.5. In what follows we will need further to assume that $(\log m)/\log n \to 1$. (To see that this is possible, consider $\log m/\log n = 1 - 1/\log \log n$, for which $ne^{-cn/m} = o(1)$. $\log m/\log n \to 1$). To see that this suffices, set

$$X_j = \textstyle\sum_{i=1}^{N_j} (S_{ji} - j)^2/S_{ji}, \quad b_n = (3 \log n)^{-1/2}$$

so that by (3.5a)

$$U_n - U_m = (b_n/b_m - 1)U_m + b_n \textstyle\sum_{j=m+1}^{n} X_j - b_n \log(n/m).$$

By (3.7), $EX_j = 1/(j-1)$. Therefore, write

$$U_n - U_m = (b_n/b_m - 1)U_m + b_n \textstyle\sum_{j=m+1}^{n} (X_j - (j-1)^{-1}) - b_n(\log(n/m) - \sum_{j=m}^{n-1} j^{-1}).$$

Clearly the last term converges to 0. Since $U_m \to_L$, the first term converges to 0 because $(b_n/b_m)^2 = (\log n)/\log m \to 1$. For the middle term, we use (3.6) to compute its variance to be

$$\begin{aligned} b_n^2 \textstyle\sum_{k=m-1}^{n-2} 3/k + O(1) &= (\log n)^{-1} \log(n/m) + o(1) \\ &\doteq 1 - (\log m)/\log n = o(1). \end{aligned}$$

This shows that $U_n - U_m \to_p 0$ as desired. The proof is complete. □

4. L_2-norm of slopes of convex minorants of truncated Brownian bridges.

We shall prove the following result.

THEOREM 4.1. *Let $B = \{B(t) : t \in [0, 1]\}$ be (standard) Brownian bridge on $[0, 1]$, let $B_{t,u} = B.1_{[t,u]}$, where $1_{[t,u]}$ is the indicator of the interval $[t, u]$ and let $g_{t,u}$ be a version of* *the slope of the convex minorant of* $B_{t,u}$ *on* (0, 1). *Then*

$$\left\{\int_0^1 g_{1/n,1-1/n}^2(u)\,du - \log n\right\} \Big/ \sqrt{3\log n} \to_L Z, \tag{4.1}$$

where Z is a standard normal random variable.

Theorem 4.1 will be used in Section 5 to derive the asymptotic normality of a statistic proposed by Behnen (1975). We shall derive Theorem 4.1 from Theorem 3.1 by using strong approximation of the empirical process by versions of Brownian bridges in Komlós et al (1975). For this application of Theorem 3.1, note that $L_n = \|\hat{f}_n - 1\|_2^2$, in which $\hat{f}_n$ is the density of the *concave majorant* of the 1-sample Uniform empirical distribution function F_n, is unchanged in distribution if $\hat{f}_n$ were changed to the density of the *convex minorant* of F_n. This is due to the fact that $U =_L 1 - U$ when U is Uniform (0, 1).

The following class of functions will play a fundamental role in the sequel.

DEFINITION 4.1. $\mathcal{M}$ is the set of right-continuous and nondecreasing step-functions $J:[0, 1] \to \mathbb{R}$, which have only finitely many jumps and satisfy $\int_0^1 J(u)\,du = 0$ and $\int_0^1 J^2(u)\,du = 1$.

Notice that all functions $J \in \mathcal{M}$ satisfy the inequalities

$$\int_{[a,b]} u^{\alpha}\,dJ(u) \le a^{\alpha-1/2} + b^{\alpha-1/2} + \alpha\int_a^b u^{\alpha-3/2}\,du, \tag{4.2}$$

for $\alpha \ge 0$ and $0 < a < b \le 1/2$. This follows by integration by parts and the fact that $|J(u)| \le u^{-1/2}$ for $0 < u \le 1/2$. Similarly,

$$\int_{[a,b]} (1-u)^{\alpha}\,dJ(u) \le (1-a)^{\alpha-1/2} + (1-b)^{\alpha-1/2} + \int_a^b (1-u)^{\alpha-3/2}\,du,$$

for $\alpha \ge 0$ and $1/2 \le a < b < 1$.

LEMMA 4.1. *Let* $G:[0, 1] \to \mathbb{R}$ *be a bounded function such that* $G(0) = G(1) = 0$ *and*

$$\lim_{t\downarrow 0} t^{-1/2-\delta}G(t) = \lim_{t\downarrow 0} t^{-1/2-\delta}G(1-t) = 0, \tag{4.3}$$

for some $\delta > 0$. *Then* $\|\tilde{g}\|_2 < \infty$ *and*

$$\|\tilde{g}\|_2 = -\inf_{J\in\mathcal{M}} \int_{(0,1)} G(u)\,dJ(u), \tag{4.4}$$

where $\tilde{g}$ *is a version of the slope of the convex minorant* $\tilde{G}$ *of* G.

PROOF. First suppose that G is a step-function which only has jumps at the points $t_1 < \cdots < t_n$, where $t_1 > 0$ and $t_n < 1$. It follows from (4.3) that in this case $G(t) = 0$, if $t < t_1$ or $t > t_n$. Since $\tilde{G} \le G$, we have

$$\int_{(0,1)} \tilde{G}\,dJ \le \int_{(0,1)} G\,dJ, \quad J \in \mathcal{M}. \tag{4.5}$$

Integration by parts and the Cauchy-Schwarz inequality give

$$-\int_{(0,1)} \tilde{G}\,dJ = \int_0^1 \tilde{g}(u)J(u)\,du \le \|\tilde{g}\|_2\|J\|_2 = \|\tilde{g}\|_2. \tag{4.6}$$

Since $G(0) = G(1) = 0$, we also have $\tilde{G}(0) = \tilde{G}(1) = 0$ and hence $\int_0^1 \tilde{g}(u)\,du = 0$. Suppose $\|\tilde{g}\|_2 > 0$. Without loss of generality we may take a right-continuous version $\tilde{g}$ of the slope of $\tilde{G}$ and in this case the function $\tilde{J} = \tilde{g}/\|\tilde{g}\|_2$ belongs to $\mathcal{M}$. Hence the upper bound in (4.6) is attained for $\tilde{J} = \tilde{g}/\|\tilde{g}\|_2$. Combining (4.5) and (4.6) we get

$$-\inf_{J\in\mathscr{M}}\int_{(0,1)} G\,dJ \le -\int_{(0,1)} \tilde{G}\,d\tilde{J} = \|\tilde{g}\|_2. \tag{4.7}$$

Let D be the set of discontinuity points of $\tilde{J}$, then D is not empty, since otherwise $\tilde{G} \equiv 0$ and hence $\|\tilde{g}\|_2 = 0$. The set D is a subset of the set $\{t_1, \cdots, t_n\}$ of discontinuity points of G. Let $H:[0, 1] \to \mathbb{R}$ be the function defined by

$$H(t) = \begin{cases} G(t-) \wedge G(t+), \text{ if } t \in D \text{ and } G(t) > G(t-) \wedge G(t+) \\ G(t), \text{ otherwise.} \end{cases}$$

Then $H(t) = \tilde{G}(t)$ if $t \in D$ and hence, since $\tilde{J}$ is a step-function which only has jumps in D,

$$\int_{(0,1)} H\,d\tilde{J} = \int_{(0,1)} \tilde{G}\,d\tilde{J}. \tag{4.8}$$

The class $\mathscr{M}$ is also considered in Behnen (1975) and Scholz (1981) (with slight modifications). It can be used to give a convenient representation of the L_2-norm of the slope of the convex minorant of bounded real-valued functions on [0, 1], which satisfy certain regularity conditions near the boundary of the interval [0, 1]. The representation of the L_2-norm of the slope of the convex minorant by means of functions in $\mathscr{M}$ has been studied by F. Scholz, and the following lemma is a generalization of results in Scholz (1981).

It is clear that the integral $\int_{(0,1)} H\,dJ$ can be approximated arbitrarily close by integrals $\int_{(0,1)} G\,dJ$, with $J \in \mathscr{M}$ (move the points $t \in D$, where $G(t) > G(t-) \wedge G(t+)$ a bit to the right or left and consider functions $J \in \mathscr{M}$ which have jumps of approximately the same height as J at the shifted points instead of the original points). Relation (4.4) now follows from (4.7) and (4.8).

If $\|\tilde{g}\|_2 = 0$, then $\tilde{G} \equiv 0$, and hence $G \ge 0$. In this case (4.4) also holds, since $\int_{(0,1)} G\,dJ = 0$ for any function $J \in \mathscr{M}$ such that J is constant on the intervals $[0, t)$ and $[t, 1)$, which $t \in (0, t_1)$.

Now consider an arbitrary bounded function $G:[0, 1] \to \mathbb{R}$ such that $G(t) = 0$, if $t \le a$ or $t \ge 1 - a$, where $a \in (0, \frac{1}{2})$. Define for each n the intervals $I_{k,n}$ by $I_{k,n} = [k2^{-n}, (k+1)2^{-n})$, $k = 0, 1, \cdots, 2^n - 2$, $I_{k,n} = [k2^{-n}, 1]$, if $k = 2^n - 1$, and let G_n be the step-function defined by

$$G_n(t) = \inf_{u\in I_{k,n}} G(u), \quad \text{if } t \in I_{k,n},\ k = 0, 1, \cdots, 2^n - 1.$$

Fix $\varepsilon > 0$. Let $\mathscr{P}$ be the set of finitely discrete probability measures on [0, 1]. Then, if $\tilde{H}$ is the convex minorant of a function $H:[0, 1] \to \mathbb{R}$, we have for each $t \in [0, 1]$,

$$\tilde{H}(t) = \inf\left\{\int_{[0,1]} H(u)\,dP : \int_{[0,1]} u\,dP = t, \quad P \in \mathscr{P}\right\}$$

(see e.g. Rockafellar, 1970, page 36). Thus there exist positive constants $c_{1,n}, \cdots, c_{m(n),n}$ and points $t_{1,n}, \cdots, t_{m(n),n}$ belonging to $m(n)$ disjoint intervals $I_{k,n}$ such that, for each n and fixed $t \in [0, 1]$,

$$\textstyle\sum_{i=1}^{m(n)} c_{i,n} = 1, \quad \sum_{i=1}^{m(n)} c_{i,n} t_{i,n} = t \quad \text{and} \quad \tilde{G}_n(t) > \sum_{i=1}^{m(n)} c_{i,n} G_n(t_{i,n}) - \varepsilon.$$

This implies that there are points $t'_{i,n}$, with $|t'_{i,n} - t_{i,n}| \le 2^{-n}$, such that

$$|t - \textstyle\sum_{i=1}^{m(n)} c_{i,n} t'_{i,n}| \le 2^{-n} \quad \text{and} \quad \tilde{G}_n(t) > \sum_{i=1}^{m(n)} c_{i,n} G(t'_{i,n}) - 2\varepsilon \tag{4.9}$$

(let $t_{i,n}$ and $t'_{i,n}$ belong to the same interval $I_{k,n}$, and use the definition of G_n). The sequence $\{\tilde{G}_n\}$ is increasing and hence $\lim_{n\to\infty}\tilde{G}_n(t)$ exists (and is ≤ 0). The convex minorant $\tilde{G}$ of G is continuous on [0, 1], since G is bounded on $[a, 1 - a]$ and zero outside this interval. Hence by (4.9), $\tilde{G}(t) \le \lim_{n\to\infty}\tilde{G}_n(t) + 2\varepsilon$. We also have $\tilde{G}(t) \ge \tilde{G}_n(t)$, for all n, and thus $\lim_{n\to\infty}\tilde{G}_n(t) = \tilde{G}(t)$.

Since the sequence $\{\tilde{G}_n\}$ converges pointwise to $\tilde{G}$, the right-continuous slopes $\tilde{g}_n$ of $\tilde{G}_n$ converge to the right-continuous slope $\tilde{g}$ of $\tilde{G}$, except possibly at countably many points of $[0, 1]$ (see e.g. Roberts and Varberg, 1973, Problem C(9), page 20). The functions G_n and G are uniformly bounded below on $(a, 1-a)$ and zero outside this interval. This implies that the slopes g_n and g are uniformly bounded on $(0, 1)$. Hence, by dominated convergence, $\lim_{n\to\infty}\|\tilde{g}_n - \tilde{g}\|_2 = 0$.

Choose n_0 such that $\|\tilde{g}_n\|_2 > \|\tilde{g}\|_2 - \varepsilon$, for $n \geq n_0$. By the first part of the proof there exists for each n a step-function $J_n \in \mathscr{M}$, such that $-\int_{(0,1)} G_n\, dJ > \|\tilde{g}_n\|_2 - \varepsilon$, where the points of jump of J_n, say $u_{1,n}, \cdots, u_{p(n),n}$, belong to disjoint intervals $I_{k,n}$ and are contained in $[a, 1-a]$. By the definition of G_n there exist points $u'_{1,n}, \cdots, u'_{p(n),n}$ such that $G(u'_{i,n}) < G_n(u_{i,n}) + \varepsilon$ and $u'_{i,n}$ and $u_{i,n}$ belong to the same interval $I_{k,n}$. Furthermore, let J'_n be the right-continuous step-function which has the same jumps as J_n, but at the points $u'_{i,n}$ instead of $u_{i,n}$ (note that in general $J'_n \notin \mathscr{M}$). Then, by (4.2), we have for $n \geq n_0$,

$$-\int_{(0,1)} G\, dJ'_n > -\int_{(0,1)} G_n\, dJ_n - 2\varepsilon a^{-1/2} > \|\tilde{g}\|_2 - 2\varepsilon - 2\varepsilon a^{-1/2}.$$

It is also clear from (4.2) and the definition of the points $u'_{i,n}$ that

$$\left| \int_0^1 J'_n(u)\, du \right| = \left| \int_0^1 J'_n(u)\, du - \int_0^1 J_n(u)\, du \right| \leq 2^{-n-1} a^{-1/2}$$

and

$$\left| \int_0^1 (J'_n(u))^2\, du - 1 \right| = \left| \int_0^1 (J'_n(u))^2\, du - \int_0^1 J_n^2(u)\, du \right| \leq 2^{-n-1} a^{-1}.$$

Thus, for n sufficiently large we can find a $J''_n \in \mathscr{M}$, obtained from J'_n by making slight adjustments of mass, which satisfies

$$-\int_{(0,1)} G\, dJ''_n > \|\tilde{g}\|_2 - 3\varepsilon - 2\varepsilon a^{-1/2}.$$

Therefore $-\inf_{J\in\mathscr{M}} \int_{(0,1)} G\, dJ \geq \|\tilde{g}\|_2$. Since $-\inf_{J\in\mathscr{M}} \int_{(0,1)} G\, dJ \leq -\inf_{J\in\mathscr{M}} \int_{(0,1)} G\, dJ = \|\tilde{g}_n\|_2$, for each n, relation (4.4) now follows.

Finally, let G be an arbitrary bounded function, such that $G(0) = G(1) = 0$ and (4.3) is satisfied. By (4.3) and the boundedness of G there exists a constant $c > 0$, such that

$$|G(t)| \leq c.\min\{t^{1/2+\delta}, (1-t)^{1/2+\delta}\}, \quad t \in [0, 1]. \tag{4.10}$$

Thus, if $\tilde{g}$ is the right-continuous slope of the convex minorant $\tilde{G}$ of G we have

$$|\tilde{g}(t)| \leq c \cdot \min\{t^{-1/2+\delta}, (1-t)^{-1/2+\delta}\},\ t \in (0, 1). \tag{4.11}$$

This implies

$$|\tilde{G}(t)| \leq (2c/(1+2\delta))\min\{t^{1/2+\delta}, (1-t)^{1/2+\delta}\}, \quad t \in [0, 1]. \tag{4.12}$$

Define for each $t \in (0, ½)$ the function G_t by

$$G_t(u) = \begin{cases} G(u), & \text{if } u \in [t, 1-t], \\ 0, & \text{otherwise.} \end{cases}$$

By (4.10), (4.2) and integration by parts, we have for all $J \in \mathscr{M}$

$$\left| \int_{(0,1)} G\, dJ - \int_{(0,1)} G_t\, dJ \right| \leq c \int_{(0,t)} u^{1/2+\delta}\, dJ + c \int_{(1-t,1)} (1-u)^{1/2+\delta}\, dJ \leq c.t^{\delta}/\delta. \tag{4.13}$$

Let $H_n = G_{1/n}$ and let $\tilde{H}_n$ be the convex minorant of H_n. The sequence $\{H_n\}$ converges uniformly to G, as $n \to \infty$, and hence, by the same argument as used above, $\{\tilde{H}_n\}$ converges pointwise to $\tilde{G}$. By (4.11) and (4.12) the right-continuous slopes $\tilde{h}_n$ of $\tilde{H}_n$ are uniformly bounded (in absolute value) by an L_2-function f of the form $f(u) = k \cdot \min\{u^{-1/2+\delta}, (1 - u)^{-1/2+\delta}\}$, $u \in (0, 1)$, where k is some positive constant. Since a similar bound holds for $\tilde{g}$, we have by dominated convergence

$$\lim_{n\to\infty} \| \tilde{h}_n - \tilde{g} \|_2 = 0. \tag{4.14}$$

Thus $\lim_{n\to\infty} \| \tilde{h}_n \|_2 = \| \tilde{g} \|_2$ and by (4.13),

$$\lim_{n\to\infty} \| \tilde{h}_n \|_2 = \lim_{n\to\infty}\{-\inf_{J\in\mathcal{M}} \int_{(0,1)} G_{1/n}\, dJ\} = -\inf_{J\in\mathcal{M}} \int_{(0,1)} G\, dJ. \tag{4.15}$$

The result now follows from (4.14) and (4.15). □

REMARK 4.1. It is clear that condition (4.3) can be somewhat weakened and we mainly chose (4.3) for convenience.

PROOF OF THEOREM 4.1. Let U_n be the empirical process defined by $U_n(t) = \sqrt{n}\,(F_n(t) - t)$, $t \in [0, 1]$, where F_n is the empirical df of a uniform distribution on [0, 1]. With probability one all observations are contained in the open interval (0, 1) and hence U_n satisfies almost surely the conditions of Lemma 4.1. Let $\tilde{u}_n$ be a version of the slope of the convex minorant $\tilde{U}_n$ of U_n. Then, by Lemma 4.1,

$$\| \tilde{u}_n \|_2 = -\inf_{J\in\mathcal{M}} \int_{(0,1)} U_n\, dJ.$$

Fix $\varepsilon > 0$ and let $a_n = (\log n)^4/n$, $b_n = 1 - a_n$. There exists $M > 0$ such that $P[\sup_{t\in(0,1)} | U_n(t) | / \sqrt{t(1-t)} \geq M \sqrt{\log_2 n}] < \varepsilon$ for all $n \geq 3$, where $\log_2 n = \log\log n$ since $\sup_{0<t<1} | U_n(t) | / \sqrt{t(1-t)\log_2 n} \to \sqrt{2}$ in probability, as $n \to \infty$. If $| U_n(t) | \leq M\sqrt{t \log_2 n}$ and $J \in \mathcal{M}$, we have by (4.2)

$$\int_{[\delta/n, a_n]} | U_n |\, dJ \leq M\sqrt{\log_2 N} \int_{[\delta/n, a_n]} \sqrt{t}\, dJ(t) \leq M\{(\log_2 n)(\log(na_n/\delta))\}^{1/2} \leq c \cdot \log_2 n,$$

for some constant c independent of n. A similar upper bound holds for $\int_{[b_n, 1-\delta/n]} | U_n |\, dJ$.

Since $\sup_{J\in\mathcal{M}} \int_{(0,\delta/n]} t\, dJ(t) \to 0$, and similarly $\sup_{J\in\mathcal{M}} \int_{[1-\delta/n,1)} (1 - t)\, dJ(t) \to 0$, as $n \to \infty$, there exists a constant k such that, for all large n, $P[\sup_{J\in\mathcal{M}} \int_{(0,a_n]\cup[b_n,1)} | U_n |\, dJ \geq k \log_2 n] < 2\varepsilon$. By Theorem 3.1 and Lemma 4.1,

$$\left(\left\{\inf_{J\in\mathcal{M}} \int_{(0,1)} U_n\, dJ\right\}^2 - \log n\right) / \sqrt{3 \log n} \to_L Z,$$

where Z is a standard normal random variable. Furthermore, since $\inf_{J\in\mathcal{M}} \int_{(0,1)} U_n\, dJ - \inf_{J\in\mathcal{M}} \int_{(a_n,1-a_n)} U_n\, dJ = O(\log_2 n)$ on a set of probability $> 1 - 2\varepsilon$, and since $\varepsilon > 0$ was arbitrarily chosen, we have

$$\left(\left\{\inf_{J\in\mathcal{M}} \int_{(a_n,b_n)} U_n\, dJ\right\}^2 - \log n\right) / \sqrt{3 \log n} \to_L Z. \tag{4.16}$$

By Komlós et al (1975), there are versions of Brownian bridges B_n such that $\sup_{t\in(0,1)}| U_n(t) - B_n(t)| = O_p((\log n)/\sqrt{n})$ with probability one. Hence, by (4.2),

$$\left| \inf_{J\in\mathcal{M}} \int_{(a_n,b_n)} U_n\, dJ - \inf_{J\in\mathcal{M}} \int_{(a_n,b_n)} B_n\, dJ \right| = O_p\left(\sup_{J\in\mathcal{M}} \int_{(a_n,b_n)} n^{-1/2}\log n\, dJ\right)$$

$$= O_p(1/\log n),$$

This implies

$$(4.17)\qquad \left(\left\{\inf_{J\in\mathcal{M}}\int_{(a_n,b_n)} B_n\,dJ\right\}^2 - \log n\right)\Big/\sqrt{3\log n} \to_L Z.$$

Since $\limsup_{h\downarrow 0}\sup_{0<t<h} |B(t)|/\sqrt{2t\log_2 t} = 1$ a.s., we have by (4.2),

$$\sup_{J\in\mathcal{M}}\int_{[1/n,a_n]} |B_n|\,dJ = O_p((\log_2 n)^{1/2})\int_{[1/n,a_n]} u^{1/2}\,dJ(u)$$

$$= O_p((\log_2 n)^{3/2}).$$

Thus we can replace a_n by $1/n$ and b_n by $1 - 1/n$ in (4.17). By Lemma 4.1 we have $-\inf_{J\in\mathcal{M}}\int_{[1/n,1-1/n]} B_n\,dJ = \|g_{1/n,1-1/n}\|_2$, where $g_{1/n,1-1/n}$ is a version of the slope of the convex minorant of $B_n\cdot 1_{[1/n,1-1/n]}$. Since the distribution of $\|g_{1/n,1-1/n}\|_2$ will be the same for any version of the Brownian bridge B_n, the result now follows. □

5. Asymptotic normality of a statistic proposed by Behnen. Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be two independent samples from a uniform distribution on [0, 1], let $F_m(G_n)$ be the empirical df of the first (second) sample and let H_n be the empirical df of the combined sample. With probability one, all observations in the combined sample are different and contained in the open interval (0, 1). Thus, on a set of probability one, we can define the inverse H_N^{-1} of H_N as the right-continuous df such that $H_N(H_N^{-1}(k/N)) = k/N$ and $H_N^{-1}(u) = H_N^{-1}(k/N)$, $k/(N+1) \le u < (k+1)/(N+1)$, $k = 0, \cdots, N$. In the sequel we will restrict our attention to the set where H_N^{-1} is well-defined and we shall omit the expression "with probability one". We define the (random) dfs $\bar{F}_m$ and $\bar{G}_n$ by

$$\bar{F}_m = F_m\circ H_N^{-1} \quad\text{and}\quad \bar{G}_n = G_n\circ H_N^{-1}.$$

Note that by our definition of H_N^{-1} these dfs are right-continuous.

Behnen (1975) considered the statistic

$$(5.1)\qquad T_N = \sup_{J\in\mathcal{M}}\int_{(0,1)} J(u)\,d\bar{F}_m(u)$$

(actually he considered slightly different versions, but this will make no difference for the limiting behavior). By integration by parts and Lemma 4.1 it is seen that

$$(5.2)\qquad T_N = -\inf_{J\in\mathcal{M}}\int_{(0,1)} (\bar{F}_m(u) - u)\,dJ(u) = \|\tilde{f}_{m,N} - 1\|_2,$$

where $\tilde{f}_{m,N}$ is a version of the slope of the convex minorant of $\bar{F}_m$. Let

$$(5.3)\qquad L_N(t) = (1-\lambda_N)\{\lambda_N^{-1/2}U_m(H_N^{-1}(t)) - (1-\lambda_N)^{-1/2}V_n(H_N^{-1}(t))\},$$

for all $t \in [0, 1]$, where $U_m(u) = \sqrt{m}\,(F_m(u) - u)$, $V_n(u) = \sqrt{n}\,(G_n(u) - u)$ and $\lambda_N = m/N$. Then

$$\inf_{J\in\mathcal{M}}\int_{(0,1)} \sqrt{N}(\bar{F}_m(t) - t)\,dJ(t) = \inf_{J\in\mathcal{M}}\int_{(0,1)} L_N\,dJ + O(1),$$

since $|\sqrt{N}(\bar{F}_m(t) - t) - L_N(t)| = \sqrt{N}\,|H_N(H_N^{-1}(t)) - t| \le N^{-1/2}$, $t \in [0, 1]$, (cf. Pyke and Shorack, 1968, Lemma 3.1, page 762, but note that our definition of H_N^{-1} is different; in particular $|H_N(H_N^{-1}(t)) - t| = t \wedge (1-t)$, if $t \wedge (1-t) < (N+1)^{-1}$).

To obtain the limiting behavior of $\|\tilde{f}_{m,N}\|_2$, with $\tilde{f}_{m,N}$ as in (5.2), we compare L_N with the corresponding functional for Brownian bridges B_m and B_n':

$$(5.4)\qquad \bar{L}_N(t) = (1-\lambda_N)\{\lambda_N^{-1/2}B_m(H_N^{-1}(t)) - (1-\lambda_N)^{-1/2}B_n'(H_N^{-1}(t))\}.$$

LEMMA 5.1. *Let $a_N = (\log N)^4/N$, $b_N = 1 - a_N$ and let U_m, V_n, B_m and B'_n be independent versions of empirical processes and Brownian bridges respectively, such that*

$$\sup_{t\in(0,1)}|U_m(t) - B_m(t)| = O_p(\log m/\sqrt{m})$$

and

$$\sup_{t\in(0,1)}|V_n(t) - B'_n(t)| = O_p(\log n/\sqrt{n}),$$

as $m, n \to \infty$. Then, if λ_N is bounded away from 0 and 1, as $N \to \infty$, we have

$$\inf_{J\in\mathcal{M}}\int_{[a_N,b_N]} L_N\,dJ - \inf_{J\in\mathcal{M}}\int_{[a_N,b_N]} \bar{L}_N\,dJ = O(1/\log N), \tag{5.5}$$

with probability one, as $N \to \infty$.

PROOF. Note that $\sup_{t\in(0,1)}|U_m(H_N^{-1}(t)) - B_m(H_N^{-1}(t))| \le \sup_{t\in(0,1)}|U_m(t) - B_m(t)|$, with a similar relation for $V_n\circ H_N^{-1} - B'_n\circ H_N^{-1}$. The rest of the proof follows exactly the same pattern as the argument in the proof of Theorem 4.1. □

The next lemma shows that we can replace $B_m\circ H_N^{-1}$ by B_m and $B'_n\circ H_N^{-1}$ by B'_n in the statistic $\inf_{J\in\mathcal{M}}\int_{[a_N,b_N]}\bar{L}_N\,dJ$.

LEMMA 5.2. *Let $a_N = (\log N)^4/n$ and $b_N = 1 - a_N$. Then, if B is a Brownian bridge on* [0, 1], *we have*

$$\sup_{J\in\mathcal{M}}\int_{[a_N,b_N]} |B(H_N^{-1}(t)) - B(t)|\,dJ(t) \to 0,$$

in probability, as $N \to \infty$.

PROOF. Fix $\varepsilon > 0$. There exist $\delta > 0$ and $M_1 > 1$, such that

$$P[\sup_{0<s<t<1}|B(t) - B(s)|/\sqrt{2(t-s)\log(1/(t-s))} \ge M_1] < \varepsilon.$$

(This follows from Itô and McKean 1974, page 36, formula 1.) Since $\sup_{t\in(0,1)}|H_N^{-1}(t) - t|\,\sqrt{t(1-t)} = O_p(\{(\log_2 N)/N\}^{1/2})$, (see e.g. Eicker, 1979, (1.9), page 119), there exist $M_2 > 0$ and $N_0(\varepsilon)$ such that

$$P[\sup_{t\in(0,1)}|H_N^{-1}(t) - t|/\sqrt{t(1-t)} \ge M_2((\log_2 N)/N)^{1/2}] < \varepsilon.$$

Thus there exists $M_3 > 0$, such that

$$P[\sup_{t\in[a_N,b_N]}|B(H_N^{-1}(t)) - B(t)|/(t(1-t))^{1/4} \ge M_3(\log N)^{1/2}(N^{-1}\log_2 N)^{1/4}] < \varepsilon.$$

By (4.2) and its version on [½, 1), it is seen that

$$\sup_{J\in\mathcal{M}}\int_{[a_N,b_N]} (t(1-t))^{1/4}\,dJ(t) \le k_1 N^{-1/4}\log N\int_{a_N}^{b_N} (t(1-t))^{-3/2}\,dt$$

$$\le k_2 N^{-1/4}(\log N)N^{1/2}/\log^2 N = k_2 N^{1/4}/\log N,$$

for some positive constants k_1 and k_2. Thus there exists an $M_4 > 0$, such that

$$P[\sup_{J\in\mathcal{M}}\int_{[a_N,b_N]} |B(H_N^{-1}(t)) - B(t)|\,dJ(t) \ge M_4(\log N)^{-1/2}(\log_2 N)^{1/4}] < 2\varepsilon,$$

and the result follows. □

Now, using the same notation as in Lemma 5.1, we define

$$\bar{L}_{NO}(t) = (1-\lambda_N)\{\lambda_N^{-1/2}B_m(t) - (1-\lambda_N)^{-1/2}B_n'(t)\}. \tag{5.6}$$

By Theorem 4.1 we have

$$\left\{\left(\inf_{J\in\mathcal{M}}(\lambda_N/(1-\lambda_N))^{1/2}\int_{[a_N,b_N]} \bar{L}_{NO}\,dJ\right)^2 - \log m\right\}/\sqrt{3\log m} \to_L Z, \tag{5.7}$$

with Z standard normal, if λ_N stays bounded away from 0 and 1, as $N\to\infty$. To see this, note that $\bar{L}_{NO}$ again represents a Brownian bridge (as a sum of two independent Brownian bridges), but that the variance is $(1-\lambda_N)/\lambda_N$ times the variance of the standard Brownian bridge on [0, 1]. Furthermore, it was shown in the proof of Theorem 4.1 that replacing $[a_N, b_N]$ by $[1/N, 1-1/N]$ leads to the same limiting (normal) distribution.

The asymptotic (standard) normality of the statistic

$$\{(N\lambda_N/(1-\lambda_N))T_N^2 - \log m\}/\sqrt{3\log m},$$

with T_N defined by (5.2) (or, equivalently, (5.1)), will now follow if we can show that $\sup_{J\in\mathcal{M}}\int_{(0,a_N]}\bar{F}_m\,dJ/(\log m)^{1/4}$ and $\sup_{J\in\mathcal{M}}\int_{[b_N,1)}(1-\bar{F}_m)\,dJ/(\log m)^{1/4}$ tend to zero in probability (with a similar statement for the functional with $\bar{F}_m$ replaced by $\bar{G}_n$). First, by our definition of H_N^{-1}, we have $\bar{F}_m(t)=0$, if $t<(N+1)^{-1}$. Second, for fixed $\varepsilon>0$, there exists $b=b(\varepsilon)$ such that

$$P[\bar{F}_m(t)\le F_m(bt),\quad \text{all}\quad t\in[0,1]]\ge 1-\varepsilon$$

(see Lemma 2.5, page 761, Pyke and Shorack, 1968; our interval for t is [0, 1] rather than [1/N, 1], because of our definition of H_N^{-1}). There exists $M>0$ such that $P[\sup_{t\in(0,1)}|U_m(bt)|/\sqrt{t}\ge M\sqrt{\log_2 m}]<\varepsilon$, for all large m. Thus,

$$P\left[\sup_{J\in\mathcal{M}}\int_{[1/(N+1),a_N]}\sqrt{m}\,\bar{F}_m\,dJ\ge k\log_2 m\right]<\varepsilon, \text{ if } m \text{ is large,}$$

for some constant $k>0$ (see the proof of Theorem 4.1). Similar arguments hold for $\int_{[b_N,1)}\sqrt{m}(1-F_m)\,dJ$. We have proved

THEOREM 5.1. *Let* $T_N=\sup_{J\in\mathcal{M}}\int_{(0,1)}J(u)\,d\bar{F}_m(u)$. *Then* $T_N=\|\tilde{f}_{m,N}-1\|_2$, *where* $\tilde{f}_{m,N}$ *is a version of the slope of the convex minorant of* $F_m\circ H_N^{-1}$, *and the statistic* $\{(N\lambda_N/(1-\lambda_N))T_N^2-\log m\}/\sqrt{3\log m}$ *tends in law to a standard normal distribution, if* λ_N *stays bounded away from* 0 *and* 1, *as* $N\to\infty$.

6. Concluding remarks. Both limit theorems involve non-negative random variables, namely, square L_2-norms. As such, one possible guide to the rate of convergence is the sample size required before zero is 3 standard deviations from the mean under the approximating Normal distribution. In the one-sample case, this requires $\log n = 3(3\log n)^{1/2}$ or $n>5\times10^{11}$. For 2 standard deviations, one requires $n\ge 162{,}755$. The results are similar for the 2-sample statistic. By this, one sees the extreme slowness of the convergence for the squared norms. However, by simple transformations, one can find functions of the statistics for which the convergence is much improved. Behnen (1974) used the L_2-norm itself, that is, the square-root transformation, for his Monte Carlo simulations. Here, the asymptotic variance is constant and the corresponding sample sizes are 854 and 20, respectively.

Monte Carlo simulations of sample sizes $n=4(1)10$ (20,000 replications) and 50 (5,000 replications) for the log transformation have been carried out by Scholz (personal communication). They indicate great improvement but the tails are still heavy for $n=50$. Behnen (1974) had earlier provided simulations for the two-sample statistic for selected sample sizes up to $m=n=100$. Although the convergence is slow, the fit was sufficiently close to suggest the asymptotic normality of the statistic.

It is possible to generalize the representation approach used for Theorem 3.1 to obtain an alternate proof of the two-sample result, Theorem 5.1. The only difficulty is in defining a suitable "randomization" of the coincidences that can now occur in order that the resultant distribution of heights remain the same as in (2.2). The coincidences enter because $\bar{F}_m$, unlike F_m, has its jumps occurring at the equi-distant points $\{i/N\}$. One approach is to affix small (continuous) random perturbations to these points to prevent ties among the slopes of the segments of the concave majorant without changing significantly the value of the statistic. Once this is done, one uses Negative Binomial rather than Gamma random variables for the $\{S_{ji}\}$.

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