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WEAK CONVERGENCE TO BROWNIAN EXCURSION

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We give a concise proof, using rescaling, random time change and simple infinitesimal generator computations of the fact that Brownian excursion is the weak limit as $\varepsilon \to 0$ of tied down Brownian motion conditioned never to fall below $-\varepsilon$.

1. Introduction. In [2] Durrett, Iglehart and Miller prove that Brownian excursion is the weak limit as $\varepsilon \to 0$ of tied down Brownian motion conditioned never to fall below $-\varepsilon$. The purpose of this note is to give another proof which yields also a natural "almost everywhere" conclusion and makes the necessary computations all at the same time, and avoids considerations of tightness. The method is to transform the relevant processes into familiar ones defined for all $t \geq 0$ and then use rescaling and a random time change.

2. Definitions and notation. Let C and C_u denote the continuous functions, $t \to w(t)$, on $[0, \infty)$ and $[0, u]$ respectively and let $\mathscr{C}$ and $\mathscr{C}_u$ denote the σ-algebras generated by the coordinate functions. Let $\{W_0^+(t); 0 \leq t \leq 1\}$ denote the Brownian excursion process. This is by definition a Markov process starting at 0, with continous paths and transition and absolute densities given by

$$g(t-s, x, y)\left(\frac{1-s}{1-t}\right)^{3/2} \frac{y\exp(-y^2/2(1-t))}{x\exp(-x^2/2(1-s))}$$

$$2y^2\exp(-y^2/2t(1-t))/[2\pi t^3(1-t)^3]^{1/2}$$

respectively with $0 < s < t < 1$, $0 < x$, $0 < y$, where g is the usual Gaussian kernel $g(t, x, y) = (2\pi t)^{-1/2}\exp[-(x-y)^2/2t]$. There are other definitions of W_0^+ reflecting its interpretation as a suitably scaled excursion of reflecting Brownian motion between two successive zeroes. Proper definitions and the equivalence of these are given in [3]. Let $\{B(t); t \geq 0\}$ denote Brownian motion with continuous paths and $B(0) = 0$ and let $\{W_0(t); 0 \leq t \leq 1\}$ denote tied down Brownian motion. This is by definition a process equivalent to $B(t) - tB(1)$, which is the same as saying that it is Gaussian, with continuous paths, mean zero and $\text{Cov}(W_0(s), W_0(t)) = s(1-t)$ if $s \leq t$. Let $\{Q(t); t \geq 0\}$ be the three-dimensional Bessel process starting at 0. This is by definition the radial part of three dimensional Brownian motion. It is a time homogeneous Markov process with continuous paths. Its transition density is

$$(y/x)\{g(t, x, y) - g(t, -x, y)\}, \quad t > 0, x > 0, y > 0$$

with g the Gaussian kernel, and its absolute density is $2y^2(2\pi t^3)^{-1/2}\exp(-y^2/2t)$ for $y > 0$. It is well known that with probability one, $Q(t) > 0$ for all $t > 0$.

The processes W_0^+ and W_0 induce probability measures P^+ and P_0 on $\mathscr{C}_1$ relative to which the coordinate processes are equivalent to the originals; and likewise the Bessel process induces a measure q on $\mathscr{C}$. Let $W \subset C$ consist of those functions w with $w(0) = 0$ and $w(t) > 0$ for all $t > 0$, and let $\mathscr{W}$ be the trace of $\mathscr{C}$ on W. By our previous comment, $q(W) = 1$ so that over the probability space $(W, \mathscr{W}, q)$ the coordinate process $\{w(t); t \geq 0\}$ is equivalent to $\{Q(t); t \geq 0\}$.

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If $w \in \mathscr{C}$, define a continuous function ϕw on $[0, 1)$ by $\phi w(t) = (1 - t)w(t/1 - t)$. Note ϕw is defined only on $[0, 1)$ and hence is not an element of C_1; this annoyance requires that we say a few extra words later on. Of course ϕ is defined also on C_u and for $w \in C_u$, $\phi w \in C_{u/1+u}$. On C_u the mapping ϕ is continuous in that $\| w - w' \|_u \geq \| \phi w - \phi w' \|_{u/1+u}$ where $\| w \|_r$ denotes $\sup\{| w(t) | : 0 \leq t \leq r\}$. The inverse to ϕ is the function $\phi^{-1}w(t) = (t + 1)w(t/t + 1)$ for w a continuous function on $[0, 1)$.

Finally, given ε strictly positive set $\Lambda_\varepsilon = \{w \in C_1 \,|\, w(t) > -\varepsilon \text{ for all } t \in [0, 1]\}$. Define the measure P_ε on $\mathscr{C}_1$ by $P_\varepsilon(A) = P_0(A \cap \Lambda_\varepsilon)/P_0(\Lambda_\varepsilon)$. Then relative to P_ε the coordinate process is tied down Brownian motion conditioned never to reach $-\varepsilon$, and so the theorem of [2] states that as $\varepsilon \to 0$, $P_\varepsilon \to P^+$ in the sense of weak convergence on C_1.

3. Main results. We will start with a simple observation.

(3.1) THEOREM. (a) *The process* $\{(t + 1)W_0(t/(t + 1)); 0 \leq t < \infty\}$ *is equivalent to Brownian motion.* (b) *the process* $\{(t + 1)W_0^+(t/(t + 1)); 0 \leq t < \infty\}$ *is equivalent to the three-dimensional Bessel process.*

PROOF. The process in (a) is a Gaussian process starting at 0, with 0 mean and continuous paths; its covariance function is easily computed to be $\min(s, t)$, and these properties characterize Brownian motion. The process in (b) certainly is Markovian with continuous paths and initial position 0, so we need check only that the absolute densities and joint densities at pairs of distinct times are those of the Bessel process. This follows from the change of variable rules and a little algebra starting from the formulas displayed at the beginning of Section 2.

Set $\Gamma_\varepsilon = \phi^{-1}(\Lambda_\varepsilon)$. (Obviously the inclusion of the time parameter $t = 1$ in defining Λ_ε is inconsequential since $P_0(w(1) = 0) = 1$). Clearly $\Gamma_\varepsilon = \{w \in C \,|\, w(t) > -\varepsilon t - \varepsilon \text{ for all } t \geq 0\}$. The mapping ϕ^{-1} as a direct mapping carries the measure P_ε to a measure $\bar{P}_\varepsilon$ on $\mathscr{C}$. Theorem (3.1)(a) and the description of $\phi^{-1}(\Lambda_\varepsilon)$ says that relative to $\bar{P}_\varepsilon$ the coordinate process $y_\varepsilon(t, w) = w(t)$ is Brownian motion starting at 0 and conditioned to lie always above the line $-\varepsilon t - \varepsilon$.

Our main theorem is as follows:

(3.2) THEOREM. *Over* $(W, \mathscr{W}, q)$ *there is a family* $\{Y_\varepsilon(t); t \geq 0\}$ *of stochastic processes with these properties*: (a) *the paths of* Y_ε *are continuous and* $Y_\varepsilon(0) = 0$, (b) *for each* ε *the process* Y_ε *is equivalent to Brownian motion starting at* 0 *and conditioned to exceed* $-\varepsilon t - \varepsilon$, (c) *for every* $w \in W$, $Y_\varepsilon(t, w) \to w(t)$ *as* $\varepsilon \to 0$ *uniformly in t restricted to compacts in* $[0, \infty)$.

REMARK. To carry out the proof, we need a bit of diffusion theory. The discussion in [1] is more than adequate. We will identify diffusion processes by their starting points and infinitesimal generators. The boundaries will be inaccessible so no delicate issue of boundary conditions arises.

PROOF. The path $t \to Y_\varepsilon(t)$ will be obtained from the path $t \to w(t)$ by a deterministic scale change and a random (that is, dependent on w) time change. Also a shift in initial position is needed. Coming to the formalities, for $\varepsilon > 0$ make the following definitions: (a) the scale change $\psi_\varepsilon(x) = (1/2\varepsilon)\log(1 + 2\varepsilon x)$ for $x \geq 0$; obviously $\psi_\varepsilon(x) \to x$ uniformly on compacts as $\varepsilon \to 0$. (b) ε' defined to be the solution of $\psi_\varepsilon(\varepsilon') = \varepsilon$; thus $\varepsilon' \to 0$ as $\varepsilon \to 0$. (c) the shifting parameter $\sigma_{\varepsilon'}(w) = \inf\{t \,|\, w(t) = \varepsilon'\}$ for $w \in W$; then for each w, $\sigma_{\varepsilon'}(w) \to 0$ as $\varepsilon \to 0$. (d) the shifted path $w_{\varepsilon'} \in W$ defined by $w_{\varepsilon'}(t) = w(t + \sigma_{\varepsilon'}(w))$; then uniformly in t bounded $w_{\varepsilon'}(t) \to w(t)$ as $\varepsilon \to 0$. (e) the random time change $\tau_\varepsilon(\cdot, w) = A_\varepsilon^{-1}(\cdot, w)$ where $A_\varepsilon(t, w) = \int_0^t g_\varepsilon(w(s))\, ds$ and $g_\varepsilon(x) = (1 + 2\varepsilon x)^{-2}$ for $x \geq 0$; of course τ_ε is the inverse of the classical additive functional A_ε, and clearly $\tau_\varepsilon(t, w) \to t$ uniformly in t bounded as $\varepsilon \to 0$. (f) we put these things together by setting $z_\varepsilon(t, w) = \psi_\varepsilon(w(\tau_\varepsilon(t, w)))$, $Z_\varepsilon(t, w) = z_\varepsilon(t, w_{\varepsilon'})$ and finally $Y_\varepsilon(t, w) = Z_\varepsilon(t, w) - \varepsilon t - \varepsilon$.

Now the coordinate process is the Bessel process, so it has generator $A_1f(x) = (1/2)f''(x) + (1/x)f'(x)$ and starts at 0. The process z_ε is obtained by a random time change τ_ε followed by a mapping by ψ_ε. The result of the time change is to replace the generator by $A_2f(x) = (1/g_\varepsilon(x))A_1f(x)$, and the result of the ensuing scale change is to replace the generator A_2 by $A_3f(x) = A_2(f\circ\psi_\varepsilon)(\psi_\varepsilon^{-1}x)$. The actual computation requires only a little calculus, and the result is $A_3f(x) = (1/2)f''(x) + (\varepsilon \coth \varepsilon x)f'(x)$. So z_ε is a diffusion with A_3 as generator, and it starts at 0. The process Z_ε is a diffusion with the same generator but the path is shifted so that its initial position is ε. From the properties of ψ_ε and τ_ε it is obvious that for each $w \in W$, $Z_\varepsilon(t, w) \to w(t)$ uniformly in t bounded, as $\varepsilon \to 0$.

Now shifting attention for a moment, consider the process $\{B(t) + \varepsilon t + \varepsilon, t \geq 0\}$ where B is Brownian motion starting at 0. This process has generator $B_1f(x) = (1/2)f''(x) + \varepsilon f'(x)$ and ε as starting point. Its scale function $s(x)$, that is a non-constant solution of $B_1s = 0$, is $e^{-2\varepsilon x}$ and so $s_1(x) = (s(0) - s(x))/s(0) = 1 - e^{-2\varepsilon x}$ is the probability that this process starting at $x > 0$ never reaches 0. Then the diffusion process whose generator is $B_2f = s_1^{-1}B_1(fs_1)$ is indeed $B(t) + \varepsilon t + \varepsilon$ conditioned never to reach 0. Carrying out the calculation, we find $B_2f(x) = A_3f(x)$. This identifies Z_ε as $B(t) + \varepsilon t + \varepsilon$ starting at ε and conditioned never to reach 0. Consequently, Y_ε satisfies requirement (b) of the theorem. Requirement (c) is obvious from the corresponding convergence of $Z_\varepsilon(t)$ so the proof is complete.

Now it requires only a few observations to obtain the weak convergence result of [2]. Specifically, over the probability space $(W, \mathscr{W}, q)$ the distribution of ϕw is P^+ and that of ϕY_ε is P_ε. Furthermore, if $u < 1$ then $\phi Y_\varepsilon(t) \to w(t)$ uniformly on $0 \leq t \leq u$ as $\varepsilon \to 0$. Consequently, if $\pi_u : C_1 \to C_u$ is the restriction mapping, $\pi_u w(t) = w(t)$, $0 \leq t \leq u$, then Theorem (3.2) implies that as $\varepsilon \to 0$

$$\int f\circ \pi_u \, dP_\varepsilon \to \int f \circ \pi_u \, dP_0 \tag{3.3}$$

for every bounded continuous f on C_u. This is a little less than weak convergence, which requires that (3.3) hold when $f\circ\pi_u$ is replaced by an arbitrary bounded continuous function on C_1. However, the validity of (3.3) for just one strictly positive u implies that the family $\{P_\varepsilon; \varepsilon > 0\}$ when restricted to C_u is tight and hence implies that given any $r > 0$, $P_\varepsilon(\{w \mid \| w_\delta \| > r\}) \to 0$, uniformly in ε, in $\delta \to 0$. This is because $w \to \| w \|_\delta$ is a continuous function on C_δ and P^+ and P_ε attribute all their mass to $\{w \mid w(0) = 0\}$. The measures P_ε are invariant under the transformation $w(t) \to w(1-t)$ and so uniformly in ε $P_\varepsilon(\{w \mid \max_{t>1-\delta} | w(t)| > r\}) \to 0$ as $\delta \to 0$. With this in mind, the reader will have no difficulty showing that (3.3) holds when $f\circ\pi_u$ is replaced by an arbitrary bounded continuous function on C_1. So the proof of weak convergence is complete.

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