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THE RELIABILITY OF *K* OUT OF *N* SYSTEMS

BY PHILIP J. BOLAND AND FRANK PROSCHAN[1]

University College Dublin and Florida State University

A system with n independent components which functions if and only if at least k of the components function is a k out of n system. Parallel systems are 1 out of n systems and series systems are n out of n systems. If $\mathbf{p} = (p_1, \cdots, p_n)$ is the vector of component reliabilities for the n components, then $h_k(\mathbf{p})$ is the reliability function of the system. It is shown that $h_k(\mathbf{p})$ is Schur-convex in $[(k-1)/(n-1), 1]^n$ and Schur-concave in $[0, (k-1)/(n-1)]^n$. More particularly if $\prod$ is an $n \times n$ doubly stochastic matrix, then $h_k(\mathbf{p}) \geq (\leq) h_k(\mathbf{p}\prod)$ whenever $\mathbf{p} \in [(k-1)/(n-1), 1]^n([0, (k-1)/(n-1)]^n)$. This Theorem is compared with a result on Schur-convexity and -concavity by Gleser [2] which in turn extends work of Hoeffding [4].

1. Introduction and summary. A k out of n system is a system with n components which functions if and only if k or more of the components function. Herein we assume that the n components of the system function independently. If $\mathbf{p} = (p_1, \cdots, p_n)$ is the vector of component reliabilities (functioning probabilities) and $\boldsymbol{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n)$ represents any vector with components equal to zeroes or one, then

$$h_k(\mathbf{p}) = h_k(p_1, \cdots, p_n) = \textstyle\sum_{\varepsilon, \varepsilon_1 + \cdots + \varepsilon_n \geq k} p_1^{\varepsilon_1} \cdots p_n^{\varepsilon_n}(1-p_1)^{1-\varepsilon_1} \cdots (1-p_n)^{1-\varepsilon_n} \tag{1.1}$$

is the probability that k or more of the components function. This function $h_k: [0, 1]^n \to [0, 1]$ is called the reliability function for a k out of n system with independent components. A one out of n system is a parallel system, an $n-1$ out of n system is a "fail-safe" system (see Barlow-Proschan [1]), and an n out of n system is a series system. For these systems it is easy to see that

$$h_1(\mathbf{p}) = 1 - \textstyle\prod_{i=1}^n (1-p_i), \quad h_{n-1}(\mathbf{p}) = \prod_{i=1}^n p_i + \sum_{i=1}^n ((1-p_i) \prod_{j \neq i} p_j)$$

and

$$h_n(\mathbf{p}) = \textstyle\prod_{i=1}^n p_i.$$

Note that if S is the number of successes in n independent Bernoulli trials, where for $i = 1, \cdots, n$, p_i is the probability of success on the ith trial, then $P(S \geq k) = h_k(\mathbf{p}) = h_k(p_1, \cdots, p_n)$. Hence the results in this paper on k out of n systems have interpretations in terms of the number of successes in Bernoulli trials.

If $\mathbf{p} = (p_1, \cdots, p_n)$, then let $\bar{\mathbf{p}} = (\sum_1^n p_i/n, \cdots, \sum_1^n p_i/n)$.

Hoeffding ([4], 1956) showed that

$$h_k(p_1, \cdots, p_n) \geq h_k(\bar{p}, \cdots, \bar{p}) \quad \text{if} \quad \textstyle\sum_1^n p_i \geq k$$

and

$$h_k(p_1, \cdots, p_n) \leq h_k(\bar{p}, \cdots, \bar{p}) \quad \text{if} \quad \textstyle\sum_1^n p_i \leq k-1.$$

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Gleser [2], using the theory of majorization and Schur functions (see Marshall and Olkin [5], Theorem 12. K.1), extended Hoeffding's result and showed that $h_k(\mathbf{p})$ is Schur-convex in the region where $\sum_1^n p_i \geq k+1$ and Schur-concave in the region where $\sum_1^n p_i \leq k-2$. More particularly Gleser showed that if $\mathbf{p} = (p_1, \cdots, p_n) \in [0, 1]^n$ and Π is a doubly stochastic matrix, then

$$h_k(\mathbf{p}) \geq h_k(\mathbf{p}\Pi) \quad \text{whenever} \quad \textstyle\sum_1^n p_i \geq k+1$$

and

$$h_k(\mathbf{p}) \leq h_k(\mathbf{p}\Pi) \quad \text{whenever} \quad \textstyle\sum_1^n p_i \leq k-2.$$

This result allows one to make more general comparisons than one could with Hoeffding's result. The major result of the present paper enables one to extend the regions of Schur-convexity and -concavity of the function $h_k(\mathbf{p})$. The result is as follows:

THEOREM 1.1. *$h_k(\mathbf{p})$ is Schur-concave in the region* $[0, (k-1)/(n-1)]^n$ *and Schur-convex in the region* $[(k-1)/(n-1), 1]^n$.

2. Majorization, Schur-convexity, and Schur-concavity. A vector $\mathbf{x} = (x_1, \cdots, x_n)$ is said to majorize a vector $\mathbf{y} = (y_1, \cdots, y_n)(\mathbf{x} \overset{m}{>} \mathbf{y})$ if $x_{[1]} \geq y_{[1]}$, $x_{[1]} + x_{[2]} \geq y_{[1]} + y_{[2]}$, $\cdots, \sum_1^{n-1} x_{[i]} \geq \sum_1^{n-1} y_{[i]}$, and $\sum_1^n x_{[i]} = \sum_1^n y_{[i]}$, where the $x_{[i]}$'s and $y_{[i]}$'s are components of $\mathbf{x}$ and $\mathbf{y}$ respectively arranged in descending order. The following lemma characterizes majorization and is due to Hardy, Littlewood and Pólya ([3], 1934) (see also Marshall and Olkin [5], Theorem 2.8.2).

LEMMA 2.1. *The vector* $\mathbf{x}$ *majorizes the vector* $\mathbf{y}$ *if and only if there exists an* $n \times n$ *doubly stochastic matrix* Π *such that* $\mathbf{y} = \mathbf{x}\Pi$.

A real valued function h defined on a set $A \subset R^n$ is Schur-convex (Schur-concave) if $h(\mathbf{x}) \geq (\leq) h(\mathbf{y})$ whenever $\mathbf{x} \overset{m}{>} \mathbf{y}$ and $\mathbf{x}, \mathbf{y} \in A$. Now assume that $A \subset R^n$ is a permutation symmetric convex set with nonempty interior. If h is continuously differentiable on the interior A° of A and continuous on A then h is Schur-convex (-concave) on A if and only if for all $i \neq j$ and $\mathbf{x} \in A^\circ$,

$$(x_i - x_j)\left(\frac{\partial h}{\partial x_i}(\mathbf{x}) - \frac{\partial h}{\partial x_j}(\mathbf{x})\right) \geq (\leq) 0. \tag{2.1}$$

This characterization of Schur-convexity (-concavity) is known as the Schur-Ostrowski condition (see Marshall and Olkin [5]).

In investigating regions of Schur-convexity of the reliability function $h_k: [0, 1]^n \to [0, 1]$, we use the following notation. If $\mathbf{r} \in [0, 1]^m$ and ℓ is any integer then $h_\ell(\mathbf{r})(h_\ell^*(\mathbf{r}))$ will denote the probability that ℓ or more (exactly ℓ) independent components with respective probabilities $r_1, \cdots, r_m$ function. Note for example that with this notation $h_{-1}(\mathbf{r}) = 1$ and $h_{-1}^*(\mathbf{r}) = 0$. We assume here that $\mathbf{p} = (p_1, \cdots, p_n) \in [0, 1]^n$ and let $\mathbf{p}^i(\mathbf{p}^{ij})$ be the vector in $[0, 1]^{n-1}([0,1]^{n-2})$ obtained from $\mathbf{p}$ by deleting its ith coordinate (ith and jth coordinates).

LEMMA 2.2. *For* $1 \leq k < n$, $\mathbf{p} \in (0, 1)^n$ *and* $n > 2$,

$$\left(\frac{\partial h_k}{\partial p_i}(\mathbf{p}) - \frac{\partial h_k}{\partial p_j}(\mathbf{p})\right)(p_i - p_j) = -(p_i - p_j)^2[h_{k-2}^*(\mathbf{p}^{ij}) - h_{k-1}^*(\mathbf{p}^{ij})]$$

for all i, j, $i \neq j$.

PROOF. For any index i, $1 \leq i \leq n$, we have that

$$h_k(\mathbf{p}) = p_i h_{k-1}(\mathbf{p}^i) + (1 - p_i) h_k(\mathbf{p}^i).$$

Hence for $j \neq i$,

$$\frac{\partial h_k}{\partial p_i}(\mathbf{p}) = h_{k-1}(\mathbf{p}^i) - h_k(\mathbf{p}^i)$$
$$= p_j h_{k-2}(\mathbf{p}^{ij}) + (1-p_j)h_{k-1}(\mathbf{p}^{ij}) - (p_j h_{k-1}(\mathbf{p}^{ij}) + (1-p_j)h_k(\mathbf{p}^{ij})).$$

Therefore

$$\left(\frac{\partial h_k}{\partial p_i}(\mathbf{p}) - \frac{\partial h_k}{\partial p_j}(\mathbf{p})\right)(p_i - p_j)$$
$$= -(p_i-p_j)^2\,[(h_{k-2}(\mathbf{p}^{ij}) - h_{k-1}(\mathbf{p}^{ij})) - (h_{k-1}(\mathbf{p}^{ij}) - h_k(\mathbf{p}^{ij}))]$$
$$= -(p_i-p_j)^2[h^*_{k-2}(\mathbf{p}^{ij}) - h^*_{k-1}(\mathbf{p}^{ij})].$$

We see then by the Schur-Ostrowski condition (2.1) that $h_k(\mathbf{p})$ is Schur-convex (-concave) in regions where for all i, j, $i \neq j$, $h^*_{k-2}(\mathbf{p}^{ij}) \leq (\geq) h^*_{k-1}(\mathbf{p}^{ij})$. In particular note that if $k = 1$ (respectively n), i.e. for a parallel (series) system, $h^*_{k-2}(\mathbf{p}^{ij}) = 0$ ($h^*_{k-1}(\mathbf{p}^{ij}) = 0$), and hence $h_1(\mathbf{p})(h_n(\mathbf{p}))$ is Schur-convex (-concave) in $[0, 1]^n$.

LEMMA 2.3. *Let $n > 2$ and k be such that $2 \leq k \leq n$. Then for any $i \neq j$,*

$$h^*_{k-2}(\mathbf{p}^{ij}) \leq (\geq) h^*_{k-1}(\mathbf{p}^{ij})$$

whenever $p_\ell \geq (\leq)(k-1)/(n-1)$ for all $\ell \neq i, j$.

PROOF. Due to the symmetry of the situation, we need only show that $h^*_{k-2}(\mathbf{p}^{12}) \leq h^*_{k-1}(\mathbf{p}^{12})$ whenever $p_\ell \geq (k-1)/(n-1)$ for $\ell = 3, \cdots, n$. In this proof, $\boldsymbol{\varepsilon} = (\varepsilon_3, \cdots, \varepsilon_n)$ and $\boldsymbol{\varepsilon}' = (\varepsilon'_3, \cdots, \varepsilon'_n)$ will denote $n-2$ dimensional vectors whose components are zeroes or ones such that $\varepsilon_3 + \cdots + \varepsilon_n = k-2$ and $\varepsilon'_3 + \cdots + \varepsilon'_n = k-1$. Now

$$h^*_{k-1}(\mathbf{p}^{12}) = \textstyle\sum_{\boldsymbol{\varepsilon}} p_3^{\varepsilon_3} \cdots p_n^{\varepsilon_n}(1-p_3)^{1-\varepsilon_3} \cdots (1-p_n)^{1-\varepsilon_n} \tag{2.2a}$$

and

$$h^*_{k-1}(\mathbf{p}^{12}) = \textstyle\sum_{\boldsymbol{\varepsilon}'} p_3^{\varepsilon'_3} \cdots p_n^{\varepsilon'_n}(1-p_3)^{\varepsilon'_3} \cdots (1-p_n)^{\varepsilon'_n}. \tag{2.2b}$$

Consider a term in the expansion of (2.2a), say for simplicity of notation, the term $p_3 \cdots p_k(1-p_{k+1}) \cdots (1-p_n)$. Then

$$p_3 \cdots p_k(1-p_{k+1}) \cdots (1-p_n)$$
$$= \frac{1}{n-k}[\underbrace{p_3 \cdots p_k(1-p_{k+1}) \cdots (1-p_n) + \cdots + p_3 \cdots p_k(1-p_{k+1}) \cdots (1-p_n)}_{n-k \text{ times}}]$$
$$\leq \frac{1}{n-k}\left[p_3 \cdots p_k\left(\frac{n-k}{n-1}\right)(1-p_{k+2}) \cdots (1-p_n) + \cdots + p_3 \cdots p_k(1-p_{k+1}) \cdots (1-p_{n-1})\left(\frac{n-k}{n-1}\right)\right]$$
$$\left(\text{since } 1 - p_\ell \leq 1 - \frac{k-1}{n-1} = \frac{n-k}{n-1} \text{ for all } \ell = 3, \cdots, n\right)$$
$$= \frac{1}{k-1}\left[p_3 \cdots p_k\left(\frac{k-1}{n-1}\right)(1-p_{k+2}) \cdots (1-p_n) + \cdots + p_3 \cdots p_k(1-p_{k+1}) \cdots (1-p_{n-1})\frac{k-1}{n-1}\right]$$

$$\leq \frac{1}{k-1}[p_3 \cdots p_k p_{k+1}(1-p_{k+2}) \cdots (1-p_n)$$

$$+ \cdots + p_3 \cdots p_k(1-p_{k+1}) \cdots (1-p_{n-1})p_n].$$

Hence similarly for a general term in the expansion of (2.2a), we have

$$p_3^{\varepsilon_3} \cdots p_n^{\varepsilon_n}(1-p_3)^{1-\varepsilon_3} \cdots (1-p_n)^{1-\varepsilon_n}$$

$$\leq \frac{1}{k-1} \sum_{\varepsilon', \varepsilon'_\ell \geq \varepsilon_\ell, \ell=3,\cdots,n} p_3^{\varepsilon'_3} \cdots p_n^{\varepsilon'_n}(1-p_3)^{1-\varepsilon'_3} \cdots (1-p_n)^{1-\varepsilon'_n}.$$

Therefore

$$h_{k-2}^*(\mathbf{p}^{12}) = \sum_\varepsilon p_3^{\varepsilon_3} \cdots p_n^{\varepsilon_n}(1-p_3)^{1-\varepsilon_3} \cdots (1-p_n)^{1-\varepsilon_n}$$

$$\leq \sum_\varepsilon \frac{1}{k-1} \sum_{\varepsilon', \varepsilon'_\ell \geq \varepsilon_\ell, \ell=3,\cdots,n} p_3^{\varepsilon'_3} \cdots p_n^{\varepsilon'_n}(1-p_3)^{1-\varepsilon'_3} \cdots (1-p_n)^{1-\varepsilon'_n}$$

$$\leq \sum_{\varepsilon'} \frac{1}{k-1}(k-1) p_3^{\varepsilon'_3} \cdots p_n^{\varepsilon'_n}(1-p_3)^{1-\varepsilon'_3} \cdots (1-p_n)^{1-\varepsilon'_n}$$

(since for each ε' there exist $k-1$ distinct

ε where $\varepsilon'_\ell \geq \varepsilon_\ell$ for $\ell = 3, \cdots, n$)

$$= h_{k-1}^*(\mathbf{p}^{12}). \quad \square$$

3. Proof of Theorem 1.1. Using the Schur-Ostrowski condition (2.1) it is easy to verify Theorem 1.1 when $n = 2$. For $n > 2$, we have already noted that when $k = 1$ (parallel system), h_k is Schur-convex in $[0, 1]^n$. For $n > 2$ and $k \geq 2$, it follows from Lemmas 2.2 and 2.3 that h_k satisfies the Schur-Ostrowski condition for Schur-convexity (-concavity) on $[(k-1)/(n-1), 1]^n([0, (k-1)/(n-1)]^n)$.

REMARK 3.1. Gleser's result [2] shows that $h_k(\mathbf{p})$ is Schur-convex (-concave) on the set $\{\mathbf{p} : p_1 + \cdots + p_n \geq k+1\}(\{\mathbf{p} : p_1 + \cdots + p_n \leq k-2\})$. This result and Theorem 1.1 enable one to make various comparisons of system reliability, and neither result encompasses the other.

For example let us consider a 3 out of 4 system. Then Theorem 1.1 implies that a system with component reliabilities (1.0, .9, .8, .7) is superior (has higher reliability) than a system with component reliabilities (.95, .95, .75, .75) which in turn is superior to one with component reliabilities (.85, .85, .85, .85). On the other hand Theorem 1.1 also implies that a system with component reliabilities (.6, .5, .3, .2) is inferior to one with component reliabilities (.6, .4, .4, .2) which in turn is inferior to one with component reliabilities (.4, .4, .4, .4). These comparisons are not implied by Gleser's result.

REMARK 3.2. If $h(\mathbf{p})$ is the reliability function of a coherent system, then the reliability importance of component j is $I_h(j) = (\partial h/\partial p_j)(\mathbf{p})$ (see Barlow-Proschan [1]).

Assume that the components of a k out of n system have been labeled so that component reliabilities are ordered as follows: $p_1 \leq p_2 \leq p_3 \leq \cdots \leq p_n$. From Lemma 2.2 we see that

$$I_{h_k}(i) - I_{h_k}(j) = -(p_i - p_j)[h_{k-2}^*(\mathbf{p}^{ij}) - h_{k-1}^*(\mathbf{p}^{ij})]$$

for all i, j where $1 \leq k < n$ and $n > 2$. It then follows from our results that whenever $\mathbf{p} \in [(k-1)/(n-1), 1]^n$ the most reliable component (component n) is the most important to the system, while whenever $\mathbf{p} \in [0, (k-1)/(n-1)]^n$ the least reliable component (component 1) is the most important to the system.

REMARK 3.3. Let S be the number of successes in n independent Bernoulli trials where p_i is the probability of success on the ith trial. Then by Theorem 1.1, $P(S \geq k)$ is Schur convex (concave) in $[(k-1)/(n-1), 1]^n([0, (k-1)/(n-1)]^n)$. Suppose now that $1 \leq k < k' \leq n$. Then it follows by the above that $P(k' > S \geq k)$ is Schur-convex in $[(k-1)/(n-1), (k'-1)/(n-1)]^n$.

REFERENCES

[1] BARLOW, R. E. and PROSCHAN, F. (1965). *Mathematical Theory of Reliability*, Wiley, New York.
[2] GLESER, L. (1975). On the distribution of the number of successes in independent trials. *Ann. Probab.* **3** 182–188.
[3] HARDY, G. H., LITTLEWOOD, J. E., and PÓLYA, G. (1959). *Inequalities*, 2nd ed. Cambridge Univ. Press.
[4] HOEFFDING, W. (1956). On the distribution of the number of successes in independent trials. *Ann. Math. Statist.* **27** 713–721.
[5] MARSHALL, A. W. and OLKIN, I. (1979). *Inequalities: Theory of Majorization and its Applications.* Academic Press, New York.

UNIVERSITY COLLEGE DUBLIN
DEPARTMENT OF MATHEMATICS
BELFIELD, DUBLIN 4, IRELAND

THE FLORIDA STATE UNIVERSITY
DEPARTMENT OF STATISTICS
TALLAHASSEE, FLORIDA 32306