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THE MINIMAL GROWTH RATE OF PARTIAL MAXIMA

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Let $X_1, X_2, \cdots$ be i.i.d. random variables and let $M_n = \max_{i \le j \le n} X_j$. For each real sequence $\{b_n\}$, a sequence $\{b_n^*\}$ and a sub-sequence of integers $\{n_k\}$ is explicitly constructed such that $P(M_n \le b_n \text{ i.o.}) = 1$ iff $\sum_k P(M_{n_k} \le b^*_{n_k}) = \infty$. This result gives a complete characterization of the upper and lower-class sequences (as introduced by Paul Lévy) for the a.s. minimal growth rate of $\{M_n\}$.

1. Introduction. Let $X, X_1, X_2, \cdots$ be a sequence of independent, identically distributed (i.i.d.) random variables and put $M_n = \max_{1 \le j \le n} X_j$. For each real sequence $\{b_n\}$, $P(M_n \le b_n \text{ i.o.})$ assumes a value of either zero or one. A series criterion is presented, identifying which is the case. The proof involves construction of blocks of certain sub-sequences of events. These are partitioned into sub-blocks and then split further. Thereby, probabilities of unions are approximated in terms of their constituent events. The technique employed is quite general. Indeed, it can be adapted to evaluate $\lim \inf_{n \to \infty} S_n/a_n$, where $S_n = Y_1 + \cdots + Y_n$ is a sum of any i.i.d. non-negative infinite mean random variables and $\{a_n\}$ any sequence of positive reals (see Klass, 1982).

Provided $\{b_n\}$ and $\{P(M_n \le b_n)\}$ are eventually non-decreasing and non-increasing, respectively, Barndorff-Nielson [1961] proved that

$$(1) \qquad P(M_n \le b_n \text{ i.o.}) = \begin{cases} 1 & \text{if } \sum_{j=3}^{\infty} P(M_j \le b_j) j^{-1} \log \log j = \infty \\ 0 & \text{if } \sum_{j=3}^{\infty} P(M_j \le b_j) j^{-1} \log \log j < \infty. \end{cases}$$

Independently, Robbins and Siegmund [1970] rediscovered an essentially equivalent form of (1). The requirement that b_n be non-decreasing causes no real problem. However, as noted by Barndorff-Nielson, (1) can fail if $P(M_n \le b_n)$ is not eventually monotonic. Seeking a result valid for all sequences $\{b_n\}$, we rewrite the series condition in (1), assuming $P(M_n \le b_n)$ is non-increasing for n large.

For $k \ge 3$ let $n_k = [\exp k/\log k]$, where $[x]$ denotes the greatest integer in x. (Beginning with Erdös [1942] this sequence has had a long history in connection with delicate laws of the iterated logarithm.) Take k sufficiently large. By monotonicity of $P(M_j \le b_j)$ for $j \ge n_k$,

$$P(M_{n_{k+1}} \le b_{n_{k+1}}) \textstyle\sum_{n_k \le j < n_{k+1}} j^{-1} \log \log j \le \sum_{n_k \le j < n_{k+1}} P(M_j \le b_j) j^{-1} \log \log j$$
$$\le P(M_{n_k} \le b_{n_k}) \textstyle\sum_{n_k \le j < n_{k+1}} j^{-1} \log \log j$$

since $\lim_{k \to \infty} \sum_{n_k \le j < n_{k+1}} j^{-1} \log \log j = 1$, the two series $\sum_{j=3}^{\infty} P(M_j \le b_j) j^{-} \log \log j$

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and $\sum_{k=3}^{\infty} P(M_{n_k} \le b_{n_k})$ converge or diverge together. Thus

$$(2) \qquad P(M_n \le b_n \text{ i.o.}) = \begin{cases} 1 & \text{if } \sum_{k=3}^{\infty} P(M_{n_k} \le b_{n_k}) = \infty \\ 0 & \text{if } \sum_{k=3}^{\infty} P(M_{n_k} \le b_{n_k}) < \infty \end{cases}$$

provided $\{b_n\}$ and $\{-P(M_n \le b_n)\}$ are both eventually non-decreasing.

(Barndorff-Nielson's proof of (1) used the divergence half of (2), whereas Robbins and Siegmund [1970] stated a result similar to (2), replacing $P(M_{n_k} \le b_{n_k})$ by $\exp\{-n_k P(X > b_{n_k})\}$ and assuming that

(i) X has a continuous distribution function,
(ii) b_n is eventually non-decreasing, and
(iii) $\lim_{n\to\infty}\inf n(\log\log n)^{-1}P(X \ge b_n) \ge 1$.)

The subsequence $\{n_k\}$ of $\{n\}$, while used to combat the extreme dependence existing among neighboring events in the sequence $\{M_n \le b_n\}_{n\ge1}$, is not always adequate. Suppose X has a continuous distribution and that b_n is chosen so that

$$P(X \le b_n) = \exp\left(\frac{2 \log k}{j_k}\right) \quad \text{for} \quad j_k \le n < j_{k+1},$$

where $j_k = 2^{2^k}$ for $k = 1, 2, \cdots$. Then $P(M_n \le b_n \text{ i.o.}) = P(M_{j_k} \le b_{j_k} \text{ i.o})$, which is zero by the Borel-Cantelli lemma. However, $\sum_{k=3}^{\infty} P(M_{n_k} \le b_{n_k}) = \infty$ and so (2) does not hold. (This is the Barndorff-Nielson counterexample.)

If (2) is to remain valid in general, the subsequence $\{n_k\}$ must be redefined. Clearly, it must depend on both X and $\{b_n\}$. Regard any subsequence $\{n_k\}$ of $\{n\}$ as a collection of check-points along which the events $\{M_n \le b_n\}_{n\ge1}$ are to be monitored. To ensure accurate detection of infinitely many such events, the subsequence $\{n_k\}$ must satisfy $P(M_{n_k} \le b_{n_k} \text{ i.o.}) = P(M_n \le b_n \text{ i.o.})$. This condition limits the growth of $\{n_k\}$. For purposes of easy evaluation, $P(M_{n_k} \le b_{n_k} \text{ i.o.})$ should be zero or one depending on convergence or divergence of $\sum_k P(M_{n_k} \le b_{n_k})$. This condition forces $\{n_k\}$ to grow sufficiently fast to capitalize on the asymptotic independence of distant events among $\{M_n \le b_n\}_{n\ge1}$. These two opposing constraints on n_k are not mutually exclusive, as we now indicate.

Fix $0 < \lambda < 1$. Suppose $\{b_n\}$ is non-decreasing and, to avoid trivialities, suppose $P(M_n \le b_n) \to 0$ (otherwise $P(M_n \le b_n \text{ i.o.}) = 1$). Let $n_1 = 1$. Having defined $n_1, \cdots, n_k$, let

$$(3) \qquad n_{k+1} = \text{1st } j > n_k : P(M_j \le b_j \mid M_{n_k} \le b_{n_k}) \le \lambda.$$

The events $\{M_{n_k} \le b_{n_k}\}$ are sufficiently uncorrelated to yield $P(M_{n_k} \le b_{n_k} \text{ i.o.}) = 1$ iff $\sum_k P(M_{n_k} \le b_{n_k}) = \infty$. Moreover, from (3) it follows that $P(M_{n_{k+1}} \le b_{n_{k+1}} \mid M_j \le b_j) > \lambda$ for $n_k < j \le n_{n+1}$; in consequence, $P(M_n \le b_n \text{ i.o.}) = P(M_{n_k} \le b_{n_k} \text{ i.o.})$. Thus indeed (2) holds with this choice of $\{n_k\}$ (see Theorem 3).

Besides the above construction of n_k, two additional equally viable alternatives are presented. Specifically, if n_{k+1} satisfies either

$$(4) \qquad n_{k+1} = \text{1st } j > n_k : P(M_j \le b_{n_k} \mid M_{n_k} \le b_{n_k}) \le \lambda$$

or

$$n_{k+1} = \begin{cases} \text{1st } j > n_k : P(M_{n_k} \le b_{j_k} \mid M_{n_k} \le b_j) \le \lambda \\ n_k + 1 \quad \text{if no such } j \text{ exists.} \end{cases} \tag{5}$$

Then (2) holds, provided $\{b_n\}$ is non-decreasing and $P(M_n \le b_n) \to 0$. Actually, slightly more general constructions of feasible n_k's are given.

We have asserted that no loss of generality ensues from the assumption that b_n be non-decreasing. To see this, note first that we may assume $P(M_n \le b_n) \to 0$, as usual. Hence there exists n^* such that $P(X > b_n) > 0$ for $n \ge n^*$. Let

$$b_n^* = \begin{cases} b_{n^*} & \text{if } 1 \le n \le n^* \\ \max_{n^* \le j \le n} b_j & \text{if } n \ge n^*. \end{cases} \tag{6}$$

Clearly, $P(M_n \le b_n \text{ i.o.}) \le P(M_n \le b_n^* \text{ i.o.})$. Now if $\lim_{n\to\infty} P(X \le b_n^*) \equiv p < 1$, then $\sum_n P(M_n \le b_n^*) \le \sum_n p^n < \infty$ and so $P(M_n \le b_n^* \text{ i.o.}) = 0$; a fortiori, $P(M_n \le b_n \text{ i.o.}) = 0$. If $p = 1$, let $t_1 = n^*$ and, having defined $t_1, \cdots, t_k$, let $t_{k+1} = \text{1st } n > t_k : b_n = \max_{n^* \le j \le n} b_j$. ($t_{k+1}$ is defined for every $k \ge 1$ since $p = 1$.) Since $b_n^* = b_{t_k}$ for every $t_k \le n < t_{k+1}$,

$$\begin{aligned} P(M_n \le b_n^* \text{ i.o.}) &= P(\cup_{t_k \le n \le t_{k+1}} \{M_n \le b_n^*\} \text{ i.o.}) \\ &= P(\cup_{t_k \le n < t_{k+1}} \{M_n \le b_{t_k}\} \text{ i.o.}) \\ &= P(M_{t_k} \le b_{t_k} \text{ i.o.}) \qquad (\text{since } M_n \nearrow) \\ &\le P(M_n \le b_n \text{ i.o.}). \end{aligned}$$

Therefore $P(M_n \le b_n \text{ i.o.}) = P(M_n \le b_n^* \text{ i.o.})$, regardless of the value of p, and so $\{b_n\}$ may always be replaced by an equivalent, non-decreasing sequence.

We proceed with the results.

2. The theorems. Theorems 1, 2 and 3 of this section verify that the various constructions of n_k's satisfying either (5), (4) or (3), respectively, each yield the criterion given in (2) for determining the value of $P(M_n \le b_n \text{ i.o.})$.

THEOREM 1. *Let $X, X_1, X_2, \cdots$ be* i.i.d. *random variables and put $M_n = \max_{1\le j\le n} X_j$. Let $\{b_n\}$ be a sequence of non-decreasing constants. To avoid trivialities, assume X and $\{b_n\}$ satisfy*

$$P(X > b_n) \to 0 \quad \textit{and} \tag{7}$$

$$P(M_n \le b_n) \to 0 \quad (\textit{equivalently, } nP(X > b_n) \to \infty). \tag{8}$$

Take any reals $0 < \lambda_* \le \lambda^* < 1$. Construct $1 \le n_1 < n_2 < \cdots$ such that

$$P(M_{n_k} \le b_{n_k} \mid M_{n_k} \le b_j) \begin{cases} \le \lambda^* & \textit{for } j \ge n_{k+1} \\ \ge \lambda_* & \textit{for } j < n_{k+1}. \end{cases} \tag{9}$$

Then

$$P(M_n \le b_n \text{ i.o.}) = \begin{cases} 0 & \textit{if } \sum_k P(M_{n_k} \le b_{n_k}) < \infty \\ 1 & \textit{if } \sum_k P(M_{n_k} \le b_{n_k}) = \infty. \end{cases} \tag{10}$$

PROOF. If $\sum_k P(M_{n_k} \le b_{n_k}) < \infty$, then

$$P(M_n \le b_n \text{ i.o.})$$

$$= \lim_{N\to\infty} P(\cup_{j\ge n_N} \{M_j \le b_j\}) \le \lim_{N\to\infty} \sum_{k=N}^{\infty} P(\cup_{n_k \le j < n_{k+1}} \{M_j \le b_j\})$$

$$\le \lim_{N\to\infty} \sum_{k=N}^{\infty} P(M_{n_k} \le b_{n_{k+1}-1})$$

$$\le \lim_{N\to\infty} \sum_{k=N}^{\infty} P(M_{n_k} \le b_{n_{k+1}-1}) \frac{P(M_{n_k} \le b_{n_k} \mid M_{n_k} \le b_{n_{k+1}-1})}{\lambda_*}$$

$$= (\lambda_*)^{-1} \lim_{N\to\infty} \sum_{k=N}^{\infty} P(M_{n_k} \le b_{n_k}) = 0.$$

Henceforth assume $\sum_k P(M_{n_k} \le b_{n_k}) = \infty$.

We group the events $\{M_{n_k} \le b_{n_k}\}$ into blocks. Fix $0 < \gamma < 1$. Let $m_1 = n_1$. Having defined $m_1, \cdots, m_i$, let

$$m_{i+1} = 1\text{st } n_k > m_i : P(M_{m_i} \le b_{n_k}) \ge \gamma. \tag{11}$$

(m_{i+1} is always defined and finite since $P(X > b_n) \to 0$. Moreover, since $P(M_n \le b_n) \to 0$, $P(M_{m_i} \le b_{n_k}) < \gamma$ for $n_k < m_{i+1}$ for all i large.) Let $A_i = \cup_{m_i \le n_k < m_{i+1}} \{M_{n_k} \le b_{n_k}\}$ and $A_i' = \cup_{m_i \le n_k < m_{i+1}} \{\max_{m_{i-1} < j \le n_k} X_j \le b_{n_k}\}$, where $m_0 \equiv 0$. For $j = 0$ and $j = 1$, the events $\{A'_{2i+j} : i \ge 1\}$ are independent. Applying the Borel-Cantelli lemma separately to the even indices and odd indices, note that if

$$\sum_i P(A_i') = \infty, \tag{12}$$

then $P(A_i' \text{ i.o.}) = 1$. We claim that, in fact, (12) entails $P(M_{n_k} \le b_{n_k} \text{ i.o.}) = 1$. To see this, suppose (12) holds and fix $\varepsilon > 0$. For each i there exists $c_i < \infty$ such that $P(\cup_{j=i}^{c_i} A_j') > 1 - \varepsilon$. Let

$$\tau_i = \begin{cases} \text{last} \quad i \le j \le c_i \quad \text{such that } A_j' \text{ occurs} \\ \infty \quad \text{if no such } j \text{ exists.} \end{cases}$$

Note that $P(\tau_i \ne \infty) = P(\cup_{j=i}^{c_i} A_j') > 1 - \varepsilon$. Then

$$\begin{aligned} P(\cup_{j=i}^{c_i} (\cup_{m_j \le n_k < m_{j+1}} \{M_{n_k} \le b_{n_k}\})) &= P(\cup_{j=i}^{c_i} A_j) \ge P(\tau_i \ne \infty, A_{\tau_i}) \\ &\ge \sum_{j=i}^{c_i} P(\tau_i = j, M_{m_{j-1}} \le b_{m_j}) \\ &= \sum_{j=i}^{c_i} P(\tau_i = j) P(M_{m_{j-1}} \le b_{m_j}) \\ &\ge \gamma \sum_{j=i}^{c_i} P(\tau_i = j) = \gamma P(\tau_i \ne \infty) > \gamma(1 - \varepsilon). \end{aligned}$$

By the Hewitt-Savage Zero-One Law, we may conclude that $P(M_{n_k} \le b_{n_k} \text{ i.o.}) = 1$. Therefore, it is sufficient to prove that divergence of $\sum_k P(M_{n_k} \le b_{n_k})$ implies that

$$\sum_i P(A_i) = \infty \tag{13}$$

(since (13) implies (12)). We must lower-bound $P(A_i)$. To do so, we partition A_i

into sub-blocks of events. Let $m_{i,1} = m_i$, and having defined $m_{i,1}, \cdots, m_{i,j}$, let

$$m_{i,j+1} = \begin{cases} \text{1st } n_k \geq m_{i,j} + m_i & \text{if such } \quad n_k \leq m_{i+1} \quad \text{exists} \\ m_{i+1} & \text{otherwise.} \end{cases} \tag{14}$$

Then set $l(i) = \text{last } j : m_{i,j} \leq m_{i+1}$. For $1 \leq j < l(i)$, let

$$A_{i,j} = \cup_{m_{i,j} \leq n_k < m_{i,j+1}} \{M_{n_k} \leq b_{n_k}\}.$$

Thus $A_i = \cup_{j=1}^{l(i)} A_{i,j}$. To obtain disjoint sub-blocks, let

$$B_{i,j} = \cup \, \{\{X_k > b_i^*\} : m_{i,j+1} < k \leq m_{i,j+1} + m_i\}, \tag{15}$$

$$\text{where} \quad i^* = \max\{n_k : n_k < m_{i+1}\}.$$

Note that $A_{i,j} \cap B_{i,j}$ is disjoint from $A_{i,j'}$ for $j' \geq j + 2$. Moreover, since $P(M_n \leq b_n) \to 0$, the construction of $\{m_i\}$ yields $P(B_{i,j}) = 1 - P(M_{m_i} \leq b_{i^*}) > 1 - \gamma$ for i large. Thus for i large,

$$\begin{aligned} P(A_i) &\geq P(\cup_{j=1}^{l(i)} (A_{i,j} \cap B_{i,j})) \\ &\geq 2^{-1}(P(\cup_{j \text{ even}} (A_{i,j} \cap B_{i,j})) + P(\cup_{j \text{ odd}} (A_{i,j} \cap B_{i,j}))) \\ &= 2^{-1} \textstyle\sum_{j=1}^{l(i)} P(A_{i,j} \cap B_{i,j}) \quad \text{(by disjointness)} \\ &= 2^{-1} \textstyle\sum_{j=1}^{l(i)} P(A_{i,j})P(B_{i,j}) \quad \text{(by independence)} \\ &\geq (1 - \gamma)2^{-1} \textstyle\sum_{j=1}^{l(i)} P(A_{i,j}). \end{aligned}$$

Finally, we extract the individual events which comprise $A_{i,j}$.

$$\begin{aligned} P(A_{i,j}) &= P(\cup_{m_{i,j} \leq n_k < m_{i,j+1}} \{M_{n_k} \leq b_{n_k}\}) \\ &\geq P(M_{m_{i,j}} \leq b_{m_{i,j}}) + \textstyle\sum_{m_{i,j} < n_k < m_{i,j+1}} P(M_{n_k} \leq b_{n_k}, M_{m_{i,j}} > b_{n_{k-1}}) \\ &= P(M_{m_{i,j}} \leq b_{m_{i,j}}) + \textstyle\sum_{m_{i,j} < n_k < m_{i,j+1}} P(M_{n_k} \leq b_{n_k}) \\ &\quad \times (1 - P(M_{m_{i,j}} \leq b_{n_{k-1}} \mid M_{n_k} \leq b_{n_k})) \\ &= P(M_{m_{i,j}} \leq b_{m_{i,j}}) + \textstyle\sum_{m_{i,j} < n_k < m_{i,j+1}} P(M_{n_k} \leq b_{n_k}) \\ &\quad \times (1 - \{P(M_{n_{k-1}} \leq b_{n_{k-1}} \mid M_{n_{k-1}} \leq b_{n_k})\}^{m_{i,j}/n_{k-1}}) \\ &\geq \textstyle\sum_{m_{i,j} \leq n_k < m_{i,j+1}} P(M_{n_k} \leq b_{n_k})(1 - (\lambda^*)^{m_{i,j}/n_{k-1}}) \quad \text{(by (9))} \\ &\geq (1 - \sqrt{\lambda^*}) \textstyle\sum_{m_{m_{i,j}} \leq n_k < m_{i,j+1}} P(M_{n_k} \leq b_{n_k}). \end{aligned}$$

Consequently, there exists $\delta > 0$ such that

$$\textstyle\sum_{i=1}^{\infty} P(A_i) \geq \delta \sum_{k=1}^{\infty} P(M_{n_k} \leq b_{n_k}) = \infty. \quad \square$$

REMARK 1. Theorem 1 can be restated directly in terms of the X-distribution as follows: assume $b_n \nearrow$, $P(X > b_n) \to 0$, and $nP(X > b_n) \to \infty$. Take reals $0 < \lambda_{**} \leq \lambda^{**} < 1$ and construct $1 \leq n_1 \leq n_2 < \cdots$ such that

$$n_k P(b_{n_k} < X \leq b_j) \begin{cases} \leq \lambda^{**} & \text{for} \quad j < n_{k+1} \\ \geq \lambda_{**} & \text{for} \quad j \geq n_{k+1}. \end{cases} \tag{16}$$

Then

$$P(M_n \le b_n \text{ i.o.}) = \begin{cases} 1 & \text{if } \sum_k e^{-n_k P(X > b_{n_k})} = \infty \\ 0 & \text{if } \sum_k e^{-n_k P(X > b_{n_k})} < \infty. \end{cases} \tag{17}$$

To verify this fact, note first that

$$\lim_{k\to\infty}\sup_{j\ge n_k} | P(M_{n_k} \le b_{n_k} | M_{n_k} \le b_j) - e^{-n_k P(b_{n_k} < X \le b_j)} | = 0.$$

Hence Theorem 1 applies. Secondly, note that the convergence or divergence of both series $\sum_k P(M_{n_k} \le b_{n_k})$ and $\sum_k e^{-n_k P(X > b_{n_k})}$ depend only on those terms for which $n_k P(X > b_{n_k}) < 2 \log k$. For such terms k,

$$\frac{P(M_{n_k} \le b_{n_k})}{\exp(-n_k P(X > b_{n_k}))} \to 1.$$

Hence both series converge or diverge together.

THEOREM 2. *Let* $X, X_1, X_2, \cdots$ *be* i.i.d. *random variables and put* $M_n = \max_{1\le j\le n} X_j$. *Let* $\{b_n\}$ *be any non-decreasing sequence of constants. Suppose*

$$P(X > b_n) \to 0 \quad \textit{and} \tag{18}$$

$$P(M_n \le b_n) \to 0 \quad (\textit{equivalently, } nP(X > b_n) \to \infty). \tag{19}$$

Take any $0 < \lambda_* \le \lambda^* < 1$. *Choose any* $1 \le n_1 < n_2 < \cdots$ *such that*

$$P(M_j \le b_{n_k}) \begin{cases} \le \lambda^* & \textit{for } j \ge n_{k+1} - n_k \\ \ge \lambda_* & \textit{for } j < n_{k+1} - n_k. \end{cases} \tag{20}$$

Then

$$P(M_n \le b_n \text{ i.o.}) \begin{cases} = 1 & \textit{if } \sum_k P(M_{n_k} \le b_{n_k}) = \infty \\ = 0 & \textit{if } \sum_k P(M_{n_k} \le b_{n_k}) < \infty. \end{cases} \tag{21}$$

PROOF. Define m_i, $m_{i,j}$, $l(i)$, A_i, and $A_{i,j}$ as the proof of Theorem 1, using the present sequence of n_k's. If $\sum_k P(M_{n_k} \le b_{n_k}) < \infty$, then

$$\begin{aligned} P(M_n \le b_n \text{ i.o.}) &= \lim_{N\to\infty} P(\cup_{k=N}^{\infty} \cup_{n_k < j \le n_{k+1}} \{M_j \le b_j\}) \\ &\le \lim_{N\to\infty} \textstyle\sum_{k=N}^{\infty} P(\cup_{n_k < j \le n_{k+1}} \{M_j \le b_j\}) \\ &\le \lim_{N\to\infty} \textstyle\sum_{k=N}^{\infty} P(M_{n_k+1} \le b_{n_{k+1}}) \\ &\le \lim_{N\to\infty} \textstyle\sum_{k=N}^{\infty} (P(M_{n_{k+1}-n_k-1} \le b_{n_{k+1}}))^{-1} P(M_{n_{k+1}} \le b_{n_{k+1}}) \\ &\le \lim_{N\to\infty} \textstyle\sum_{k=N}^{\infty} (\lambda_*)^{-1} P(M_{n_{k+1}} \le b_{n_{k+1}}) = 0. \end{aligned}$$

Next, assume $\sum_k P(M_{n_k} \le b_{n_k}) = \infty$. Arguing as in the proof of Theorem 1, there exists $c > 0$ such that $P(A_i) \geq c \sum_{j=1}^{l(i)} P(A_{i,j})$. To lower-bound

$$P(A_{i,j}) = P(\cup_{m_{i,j} \leq n_k < m_{i,j+1}} \{M_{n_k} \leq b_{n_k}\})$$

in terms of $\sum_{m_{i,j} \leq n_k < m_{i,j+1}} P(M_{n_k} \leq b_{n_k})$ it suffices to discard any n_k's for which $P(M_{n_k} \leq b_{n_k})$ is abnormally small relative to the other terms. With this in mind, fix (i, j) and let

$$k^* = \text{last } k : n_k < m_{i,j+1} \tag{22}$$

and

$$k_* = \text{first } k : n_k \geq m_{i,j} \quad \text{and} \quad P^{1/4}(X \leq b_{n_k}) \geq P(X \leq b_{n_{k^*}}). \tag{23}$$

For $m_{i,j} \leq n_k < m_{i,j+1}$, let

$$B_{n_k} = \{\max_{n_k < l \leq n_{k+1}} X_l > b_{n_{k^*}}\}. \tag{24}$$

Note that for $k_* \leq k \leq k^*$,

$$\begin{aligned} P(B_{n_k}) &= 1 - P(M_{n_{k+1}-n_k} \leq b_{n_{k^*}}) = 1 - \{P(X \leq b_{n_{k^*}})\}^{n_{k+1}-n_k} \\ &\geq 1 - \{P(X \leq b_{n_k})\}^{(n_{k+1}-n_k)/4} = 1 - \{P(M_{n_{k+1}-n_k} \leq b_{n_k})\}^{1/4} \\ &\geq 1 - (\lambda^*)^{1/4}. \end{aligned}$$

Lower-bounding,

$$\begin{aligned} P(A_{i,j}) &\geq P(\cup_{k_* \leq k \leq k^*} \{M_{n_k} \leq b_{n_k}\} \cap B_{n_k}) \\ &= \textstyle\sum_{k_* \leq k \leq k^*} P(\{M_{n_k} \leq b_{n_k}\} \cap B_{n_k}) \quad \text{(by disjointness)} \\ &= \textstyle\sum_{k_* \leq k \leq k^*} P(M_{n_k} \leq b_{n_k}) P(B_{n_k}) \quad \text{(by independence)} \\ &\geq (1 - (\lambda^*)^{1/4}) \textstyle\sum_{k_* \leq k \leq k^*} P(M_{n_k} \leq b_{n_k}) \\ &\geq (1 - (\lambda^*)^{1/4})\, C \textstyle\sum_{m_{i,j} \leq n_k < m_{i,j+1}} P(M_{n_k} \leq b_{n_k}), \quad \text{(by Lemma 1 to follow),} \end{aligned}$$

$$\text{where} \quad C = (2 + 3\,(\log\,(\lambda^*)^{-1})^{-1})^{-1}.$$

Consequently, $\sum_i P(A_i) = \infty$.

Arguing again as in the proof of Theorem 1, we may conclude that $P(M_n \leq b_n \text{ i.o.}) = 1$. □

REMARK 2. As noted in Remark 1, the conditions satisfied by n_k's of Theorem 2 may be expressed directly in terms of tail probabilities. Thus, if there are $0 < \lambda_{**} \leq \lambda^{**} < \infty$ such that

$$jP(X > b_{n_k}) \begin{cases} \geq \lambda_{**} & \text{for } j \geq n_{k+1} - n_k \\ \leq \lambda^{**} & \text{for } j < n_{k+1} - n_k \end{cases} \tag{25}$$

then

$$P(M_n \leq b_n \text{ i.o.}) \begin{cases} = 1 & \text{if } \sum_k P(M_{n_k} \leq b_{n_k}) = \infty \\ = 0 & \text{if } \sum_k P(M_{n_k} \leq b_{n_k}) < \infty. \end{cases} \tag{26}$$

Moreover, the series $\sum_k P(M_{n_k} \le b_{n_k})$ and $\sum_k e^{-n_k P(X > b_{n_k})}$ converge or diverge together.

LEMMA 1. *Let* $0 < x_1 \le x_2 \le \cdots \le x_L \le 2x_1$, *and* $0 < P_1 \le P_2 \le \cdots \le P_L$. *Suppose* $P_i^{x_i} \le e^{-1}$ *for* $1 \le i \le L$ *and that* $P_i^{x_{i+1}-x_i} \le \lambda$ *for* $1 \le i \le L$ *and some* $0 < \lambda < 1$. *Then*

$$\sum_{i=1}^L P_i^{x_i} \le (2 + 3(\log \lambda^{-1})^{-1}) \sum_{i=L_*}^L P_i^{x_i}, \tag{27}$$

where

$$L_* = \min\{1 \le i \le L : P_i > P_L^4\}.$$

PROOF. Let $\mathscr{C}_j = \{1 \le k \le L : (P_L)^{4^j} < P_k \le (P_L)^{4^{j-1}}\}$. For simplicity write P_i as e^{-y_i}. Then

$$\begin{aligned} |\mathscr{C}_j| &\le 1 + \sum_{\{i<L: i \in \mathscr{C}_j\}} y_i(x_{i+1} - x_i)/\log \lambda^{-1} \\ &\le 1 + 4^j y_L \sum_{\{i<L: i \in \mathscr{C}_j\}} (x_{i+1} - x_i)/\log \lambda^{-1} \\ &\le 1 + 4^j y_L(x_L - x_1)/\log \lambda^{-1} \le 1 + 4^j x_L y_L/2 \log \lambda^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{i=1}^L P_i^{x_i} &= \sum_{j=1}^L \sum_{i \in \mathscr{C}_j} P_i^{x_i} \le \sum_{i \in \mathscr{C}_1} P_i^{x_i} + \sum_{j=2}^L \sum_{i \in \mathscr{C}_j} P_L^{4^{j-1} x_i} \\ &\le \sum_{i \in \mathscr{C}_1} P_i^{x_i} + \sum_{j=2}^L |\mathscr{C}_j| \, P_L^{4^{j-1} x_L/2} \\ &\le \sum_{i \in \mathscr{C}_1} P_i^{x_i} + \sum_{j=2}^L (1 + 2^{2j} x_L y_L/2 \log \lambda^{-1})(e^{-x_L y_L})^{2^{2j-3}} \\ &\le \sum_{i=L_*}^L P_i^{x_i} + \sum_{j=2}^\infty (e^{-x_L y_L})^{2^{2j-3}} \\ &\quad + (8 x_L y_L/\log \lambda^{-1}) e^{-2x_L y_L} \sum_{j=2}^\infty (4e^{-8x_L y_L})^{j-2} \\ &\le \sum_{i=L_*}^L P_i^{x_i} + e^{-2x_L y_L}(1 - e^{-8x_L y_L})^{-1} \\ &\quad + (8 x_L y_L/\log \lambda^{-1}) e^{-2x_L y_L}(1 - 4e^{-8x_L y_L})^{-1} \\ &\le \sum_{i=L_*}^L P_i^{x_i} + (1 + 3(\log \lambda^{-1})^{-1}) e^{-x_L y_L} \quad \text{since} \quad x_L y_L \ge 1 \\ &\le (2 + 3(\log \lambda^{-1})^{-1}) \sum_{i=L_*}^L P_i^{x_i}. \quad \square \end{aligned}$$

We now verify that the intuitive construction of $\{n_k\}$ given in (3) satisfies (2).

THEOREM 3. *Let* $X, X_1, X_2, \cdots$ *be* i.i.d. *random variables and put* $M_n = \max_{1 \le j \le n} X_j$. *Let* $\{b_n\}$ *be any non-decreasing sequence of constants. Suppose*

$$P(X > b_n) \to 0 \tag{28}$$

and

$$P(M_n \le b_n) \to 0. \tag{29}$$

Take any $0 < \lambda_* \le \lambda^* < 1$. *Choose any* $1 \le n_1 < n_2 < \cdots$ *such that*

$$P(M_j \le b_{n_k+j}) \begin{cases} \le \lambda^* & \text{for } j = n_{k+1} - n_k \\ \ge \lambda_* & \text{for } j < n_{k+1} - n_k. \end{cases} \tag{30}$$

Then

$$P(M_n \le b_n \text{ i.o.}) \begin{cases} = 1 & \text{if } \sum_k P(M_{n_k} \le b_{n_k}) = \infty \\ = 0 & \text{if } \sum_k P(M_{n_k} \le b_{n_k}) < \infty. \end{cases} \tag{31}$$

PROOF. Suppose $\sum_k P(M_{n_k} \le b_{n_k}) < \infty$. Notice that

$$\liminf_{k\to\infty} P(M_{n_{k+1}-n_k} \le b_{n_{k+1}}) = \liminf_{k\to\infty} P(M_{n_{k+1}-n_k-1} \le b_{n_{k+1}}) \ge \lambda_*.$$

Hence

$$\begin{aligned} P(M_n \le b_n \text{ i.o.}) &\le \lim_{N\to\infty} \sum_{k=N}^{\infty} P(\cup_{n_k \le j \le n_{k+1}} \{M_j \le b_j\}) \\ &\le \lim_{N\to\infty} \sum_{k=N}^{\infty} P(M_{n_k} \le b_{n_{k+1}}) \\ &\le \lim_{N\to\infty} \sum_{k=N}^{\infty} P(M_{n_{k+1}} \le b_{n_{k+1}})/\lambda_* = 0. \end{aligned}$$

Finally, assume $\sum_k P(M_{n_k} \le b_{n_k}) = \infty$. Since

$$P(M_{n_{k+1}-n_k} \le b_{n_k}) \le P(M_{n_{k+1}-n_k} \le b_{n_{k+1}}) \le \lambda^*,$$

the proof of Theorem 2 applies verbatim, yielding $P(M_{n_k} \le b_{n_k} \text{ i.o.}) = 1$.

By interchanging inequalities and strict inequalities, companions to the preceding theorems can be derived. To illustrate the point we state such a companion result, giving the full strength of what has actually been proved.

THEOREM 4. *Let* $X, X_1, X_2, \cdots$ *be* i.i.d. *random variables and put* $M_n = \max\{X_1, \cdots, X_n\}$. *Let* $\{b_n\}$ *be any non-decreasing sequence of constants. Suppose*

$$P(X \ge b_n) \to 0 \tag{32}$$

and

$$P(M_n < b_n) \to 0. \tag{33}$$

Take any $0 < \lambda_* \le \lambda^* < 1$, *and choose any* $1 \le n_1 < n_2 \cdots$.

If

$$(n_{k+1} - n_k) P(X \ge b_{n_k}) \ge \lambda_* \tag{34}$$

then

$$P(M_{n_k} < b_{n_k} \text{ i.o.}) = 1 \quad \text{if} \quad \sum_k e^{-n_k P(X \ge b_{n_k})} = \infty. \tag{35}$$

If

$$(n_{k+1} - n_k) P(X \ge b_{n_k}) \le \lambda^* \tag{36}$$

then

$$P(M_n < b_n \text{ i.o.}) = 0 \quad \textit{if} \quad \sum_k e^{-n_k P(X \geq b_{n_k})} < \infty. \tag{37}$$

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