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COMPARISON OF THRESHOLD STOP RULES AND MAXIMUM FOR INDEPENDENT NONNEGATIVE RANDOM VARIABLES

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Let $X_i \geq 0$ be independent, $i = 1, \cdots, n$, and $X_n^* = \max(X_1, \cdots, X_n)$. Let $t(c)$ $(s(c))$ be the threshold stopping rule for $X_1, \cdots, X_n$, defined by $t(c)$ = smallest i for which $X_i \geq c$ ($s(c)$ = smallest i for which $X_i > c$), $= n$ otherwise. Let m be a median of the distribution of X_n^*. It is shown that for every n and $\underline{X}$ either $EX_n^* \leq 2EX_{t(m)}$ or $EX_n^* \leq 2EX_{s(m)}$. This improves previously known results, [1], [4]. Some results for i.i.d. X_i are also included.

1. Introduction and theorem. Let $X_i \geq 0$, $i = 1, \cdots, n$ be independent random variables, and denote by T_n the set of stopping rules for the variables $X_1, \cdots, X_n$. Let $X_n^* = \max(X_1, \cdots, X_n)$ and $V_n(\underline{X}) = \sup\{EX_t; t \in T_n\}$. Krengel and Sucheston (1978) show that $EX_n^* \leq 2V_n(\underline{X})$ and Hill and Kertz (1981) show that, in fact, strict inequality holds in all but trivial cases. The interpretation of this result is that the expected return of an optimal gambler is at least half the expected return of a prophet, with complete foresight. The constant "2" in the above statement cannot be improved upon, for any $n \geq 2$, as is easily seen by taking $X_{n-1} = \mu$ and $X_n = 1$ and 0 with probability μ and $1 - \mu$, respectively, and taking $X_1, \cdots, X_{n-2}$ smaller than μ. Letting $\mu \to 0$ yields the result.

For i.i.d. X_i, 2 is no longer the best constant. Hill and Kertz (1982) show that the best constant, a_n, depends on n, and is bounded by 1.6 for all n, but 1.6 is not the best bound. This result has recently been improved on (see Kertz, 1983). He conjectures that the best bound is $1 + \alpha^* = 1.341\cdots$, where α^* is the unique solution to $\int_0^1 [y - y \ln y + \alpha]^{-1} dy = 1$. He proves that $\lim a_n = 1 + \alpha^*$.

Optimal rules are nice in theory, but they are often difficult to compute, and sometimes difficult to implement, even if computed. In the present note we therefore consider the class of "threshold rules" which are simple in practical implementation and are defined as follows. Let $c \geq 0$ be a constant. Let $t(c)$ = smallest $i < n$ such that $X_i \geq c$, $t(c) = n$ otherwise; and let $s(c)$ = smallest $i < n$ such that $X_i > c$, $s(c) = n$ otherwise.

The purpose of the present note is to show that the constant "2" can be achieved as a bound when one uses a good threshold rule, rather than an optimal stopping rule. To abbreviate, set $E^+X_{t(c)} = E[X_{t(c)}I(X_{t(c)} \geq c)]$ and $E^+X_{s(c)} = E[X_{s(c)}I(X_{s(c)} > c)]$. Let m be a median of the distribution of X_n^*, i.e.

$$(1.1) \qquad P(X_n^* < m) = q \leq 1/2, \quad P(X_n^* > m) = p \leq 1/2.$$

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Let

$$\beta = \textstyle\sum_{i=1}^{n} E(X_i - m)^+. \tag{1.2}$$

THEOREM 1. *Let $X_1, \cdots, X_n$ be independent nonnegative random variables, $X_n^* = \max(X_1, \cdots, X_n)$ and m and β satisfy* (1.1) *and* (1.2).

(I) *If $\beta \ge m$ then $EX_n^* \le 2E^+X_{s(m)} \le 2EX_{s(m)}$.*
(II) *If $\beta \le m$ then $EX_n^* \le 2E^+X_{t(m)} \le 2EX_{t(m)}$.*

PROOF. First note

$$EX_n^* \le m + E(X_n^* - m)^+ \le m + \beta. \tag{1.3}$$

Suppose $\beta \ge m$. By (1.1)

$$\begin{aligned} E^+X_{s(m)} &= mp + E(X_{s(m)} - m)^+ = mp + E \textstyle\sum_{i=1}^{n} (X_i - m)^+ I(s(m) = i) \\ &= mp + \textstyle\sum_{i=1}^{n} E[(X_i - m)^+ I(s(m) > i - 1)] \\ &= mp + \textstyle\sum_{i=1}^{n} E(X_i - m)^+ P(s(m) > i - 1) \\ &\ge mp + \beta(1 - p) \ge (m + \beta)/2 \ge EX_n^*/2. \end{aligned} \tag{1.4}$$

The fourth equality in (1.4) uses independence. Similarly, when $\beta \le m$,

$$\begin{aligned} E^+X_{t(m)} &= m(1 - q) + \textstyle\sum_{i=1}^{n} E(X_i - m)^+ P(t(m) > i - 1) \\ &\ge m(1 - q) + \beta q \ge (m + \beta)/2 \ge EX_n^*/2. \end{aligned}$$

Note that in nontrivial cases strict inequality holds in (1.3), and hence also in the results of (I) and (II).

NOTE. The "median rule" of Theorem 1 is not (necessarily) the only threshold rule for which the constant 2 is achieved.

Let a^* be the unique solution to $a = E(X_n^* - a)^+$ and b^* be the unique solution to $b = E \sum_{i=1}^{n} (X_i - b)^+$. Clearly $a^* \le b^*$, and since for any a $X_n^* \le a + (X_n^* - a)^+$ it follows that $EX_n^* \le 2a^*$. We have

ASSERTION. *Let $a^* \le c \le b^*$. Then $EX_n^* \le 2E^+X_{t(c)} \le 2EX_{t(c)}$.*

PROOF. Similarly to (1.4)

$$\begin{aligned} E^+X_{t(c)} &= cP\{X_n^* \ge c\} + E \textstyle\sum_{i=1}^{n} (X_i - c)^+ I(t(c) > i - 1) \\ &\ge cP\{X_n^* \ge c\} + E \textstyle\sum_{i=1}^{n} (X_i - b^*)^+ P\{X_n^* < c\} \\ &\ge a^*P\{X_n^* \ge c\} + b^*P\{X_n^* < c\} \ge a^*. \end{aligned}$$

2. Identically distributed X_i. Let T_n^* be the set of all threshold rules $s(c)$ and $t(c)$, $c \ge 0$, for $X_1, \cdots, X_n$. We shall show that the constant 2 cannot be improved upon (for large n), when considering threshold rules, even when the X_i

are i.i.d. We have

THEOREM 2. *Let $X_i \geq 0$ be i.i.d.*

$$\sup_n \sup_X \left[\frac{EX_n^*}{\sup_{t \in T_n^*} E^+ X_t}\right] = \sup_n \sup_X \left[\frac{EX_n^*}{\sup_{t \in T_n^*} EX_t}\right] = 2.$$

PROOF. Because of Theorem 1, it suffices to exhibit i.i.d. $X_1^{(n)}, \cdots, X_n^{(n)}$ such that $\lim_{n \to \infty} EX_n^{(n)*}/(\sup_{t \in T_n^*} EX_t^{(n)})$ is arbitrarily close to 2. This can be achieved by $X_i^{(n)} = 0$, a and 1 with probabilities $1 - (b + c)/n$, c/n and b/n respectively, where $0 < a < 1$, $0 < b$, $0 < c$, $b + c < n$ will be chosen later. It is easily seen that for fixed a, b, c

$$E = \lim_{n \to \infty} EX_n^{(n)*} = 1 - e^{-b} + a\{e^{-b} - e^{-b-c}\}.$$

There are essentially only two competing rules in T_n^*, viz. $t(a)$ and $t(1)$. Easy computations yield

$$W(a) = \lim_{n \to \infty} EX_{t(a)}^{(n)} = \frac{(1 - e^{-b-c})(b + ac)}{b + c}$$

$$W(1) = \lim_{n \to \infty} EX_{t(1)}^{(n)} = 1 - e^{-b}.$$

If we let $a = a^*$ where

$$a^* = \frac{c(1 - e^{-b}) - be^{-b}(1 - e^{-c})}{c(1 - e^{-b-c})}$$

then $W(1) = W(a^*)$, and $0 < a^* < 1$. Thus

$$Q(b, c) = \lim_{n \to \infty} \frac{E(X_n^{(n)*})}{\sup_{t \in T_n^*} EX_t^{(n)}} = 1 + \frac{e^{-b} - e^{-b-c}}{1 - e^{-b-c}} - \frac{b(e^{-b} - e^{-b-c})^2}{c(1 - e^{-b-c})(1 - e^{-b})}.$$

Now $Q(b, c)$ can be arbitrarily close to 2, since as $c \to \infty$ and $b \to 0$, $Q(b, c) \to 2$ (e.g. $Q(10^{-2}, 10^3) = 1.99$).

REMARK 1. If the X_i can achieve only two values (whether identically distributed or not), then $EX_n^* \equiv \sup_{t \in T^*n} EX_t$. Thus the r.v. X_i must take on at least three values to achieve the bound 2 for $EX_n^*/\sup_{t \in T^*n} EX_t$.

REMARK 2. Let $\underline{X} = (X_1, \cdots, X_n)$ where n is fixed and the X_i are i.i.d. Let

$$\alpha_n = \sup_X \left\{\frac{EX_n^*}{\sup_{t \in T_n^*} EX_t}\right\}, \quad \beta_n = \sup_X \left\{\frac{EX_n^*}{\sup_{t \in T_n^*} E^+ X_t}\right\}.$$

Then $\alpha_n \leq \beta_n$, and the proof of Theorem 2 shows that $\lim \alpha_n = \lim \beta_n = 2$. By considering X_i taking on two values only, it is easy to show that $\beta_n \geq 2 - 1/n$, (and presumably equality holds). The values α_n are harder to compute, e.g. $\alpha_2 = 4 - 2^{3/2} = 1.171 \cdots$, and coincides with the extremal value for the prophet problem comparison with optimal stopping rules. See [2].

REMARK 3. Consider the goal of stopping *at* the maximal observation with high probability. Professor Aryeh Dvoretzky has recently shown (oral communication) that for continuous, i.i.d. random variables there exists a threshold rule $t(c)$ for which $P\{X_{t(c)} = X_n^*\} = g_n$ where $g_n \downarrow g = \max_{v \geq 0} e^{-v} \int_0^v (e^u - 1)u^{-1}\, du = .517\cdots$. This interesting fact stands in no contradiction to the results in the present paper. It indicates that when $t(c)$ stops at the maximal observation, that observation has a (comparatively) low value.

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