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MATRIX NORMALIZED SUMS OF INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VECTORS[1]

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Let $X_1, X_2, \dots$ be a sequence of independent identically distributed random vectors and $S_n = X_1 + \cdots + X_n$. Necessary and sufficient conditions are given for there to exist matrices B_n and vectors γ_n such that $\{B_n(S_n - \gamma_n)\}$ is stochastically compact, i.e., $\{B_n(S_n - \gamma_n)\}$ is tight and no subsequential limit is degenerate. When this condition holds we are able to obtain precise estimates on the distribution of S_n. These results are then specialized to the case where X_1 is in the generalized domain of attraction of an operator stable law and a local limit theorem is proved which generalizes the classical local limit theorem where the normalization is done by scalars.

1. Introduction. The motivation for this paper came originally from two different sources. The first was to try to obtain a suitable analogue of the one-dimensional results in Griffin, Jain, and Pruitt 1984 (GJP), for random variables taking values in $\mathbb{R}^d$. We will now take this opportunity to briefly describe one of the main results in GJP.

Let $X, X_1, X_2, \dots$ be a sequence of independent identically distributed random variables taking values in $\mathbb{R}^d$. We will always assume that X is full, i.e., the distribution of X is not supported on a $d-1$-dimensional hyperplane. For $r > 0$ define

$$G(r) = P\{|X| > r\}, \qquad K(r) = r^{-2}\int_{|X| \le r} |X|^2\, dP, \tag{1.1}$$

$$Q(r) = G(r) + K(r) = E\left(r^{-1}|X| \wedge 1\right)^2. \tag{1.2}$$

One easily checks that Q is continuous, strictly decreasing for large r, and $Q(r) \to 0$ as $r \to \infty$. Thus for large n we can define an increasing sequence a_n by

$$Q(a_n) = \frac{1}{n}. \tag{1.3}$$

To describe the main probability estimate in GJP we will assume for simplicity that X is lattice valued and that the correct lattice is the integers $\mathbb{Z}$. Furthermore we will assume that S_n is strongly aperiodic (Spitzer, p. 42). (None of these assumptions are needed.)

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THEOREM (GJP). *Assume that*

$$(A_1) \qquad \liminf_{r\to\infty} K(r)/G(r) > 0.$$

Then for all $\varepsilon > 0$ *there exist positive constants* $c_1 = c_1(\varepsilon)$, $M = M(\varepsilon)$, c_2 *independent of* ε, *and centering terms* $\delta_n = \delta_n(\varepsilon)$ *such that for sufficiently large* n,

$$(1.4) \qquad P\{|S_n - \delta_n| \le Ma_n\} \ge 1 - \varepsilon,$$

$$(1.5) \qquad P\{S_n = x\} \le c_2/a_n \quad \text{for all } x \in \mathbb{Z},$$

$$(1.6) \qquad P\{S_n = x\} \ge c_1/a_n \quad \text{if } x \in \mathbb{Z} \text{ and } |x - \delta_n| \le Ma_n.$$

Thus one obtains a very good description of the distribution of S_n under the assumption (A_1). Several other interesting probabilistic equivalences of (A_1) can be found in GJP.

In earlier work (Griffin (1983)) some d-dimensional results had been obtained. In particular under (A_1), radial symmetry of X and a geometric condition on the distribution of X, it was shown that there exist positive constants, c_1, c_2, and λ_0 such that for all $\lambda \ge \lambda_0$ and all n

$$(1.7) \qquad c_1(\lambda/a_n \wedge 1)^d \le P\{S_n \in C(0,\lambda)\} \le c_2(\lambda/a_n \wedge 1)^d,$$

where $C(x, \lambda)$ is the cube of side length 2λ centered at $x \in \mathbb{R}^d$.

The geometric condition essentially ensured that the distribution of S_n spread out at the same rate in all directions. If this is not the case then the scalar sequence a_n does not contain enough information about the distribution of S_n. For example if X has independent, symmetric stable components of indices α and β, respectively, where $\alpha < \beta$, then for each $\lambda > 0$

$$P\{S_n \in C(0,\lambda)\} \sim \frac{c\lambda^2}{n^{1/\alpha}n^{1/\beta}},$$

while $a_n \sim cn^{1/\alpha}$ (Example 3.7 in Griffin, 1983). In this example it is clear that to avoid losing information, one should normalize S_n by

$$A_n = \begin{pmatrix} n^{-1/\alpha} & 0 \\ 0 & n^{-1/\beta} \end{pmatrix}.$$

Then since A_nS_n has the same distribution as X,

$$P\{S_n \in C(0,\lambda)\} = P\{X \in A_nC(0,\lambda)\} \sim c\lambda^2|\det A_n|.$$

This idea leads to our second motivation; if X is in the generalized domain of attraction of a full operator stable law Y, i.e., there exist matrices B_n and vectors γ_n such that $B_n(S_n - \gamma_n) \to Y$, does a local limit theorem hold? For example is it true that

$$P\{S_n - \gamma_n \in C(0,\lambda)\} \sim c\lambda^d|\det B_n|?$$

This would generalize the classical local limit theorem for random variables in the domain of attraction of a stable law, i.e., where the normalization is done by scalars (Stone, 1965). For further information on operator stable laws see Sharpe

(1969) and for a complete characterization of their generalized domain of attraction (GDOA) see Hahn and Klass (1985).

To describe our results we must first introduce some further notation. Let S^{d-1} be the unit sphere in $\mathbb{R}^d$ and $\langle\ ,\ \rangle$ be the usual inner product on $\mathbb{R}^d$. For $r > 0$ and $\theta \in S^{d-1}$ define

$$(1.8)\quad G(\theta, r) = P\{\,|\langle X, \theta\rangle| > r\}, \qquad K(\theta, r) = r^{-2}\int_{|\langle X, \theta\rangle| \le r} \langle X, \theta\rangle^2\, dP,$$

$$(1.9)\quad Q(\theta, r) = G(\theta, r) + K(\theta, r) = E\big(r^{-1}|\langle X, \theta\rangle| \wedge 1\big)^2.$$

As before for n sufficiently large we can define for each $\theta \in S^{d-1}$, an increasing sequence $a_n(\theta)$ by

$$(1.10)\qquad Q(\theta, a_n(\theta)) = \frac{1}{n}.$$

For each n we will show how to construct a particular orthonormal basis $\{\theta_{n1}, \dots, \theta_{nd}\}$ for $\mathbb{R}^d$ which we will call the minimal orthonormal basis (MONB). We then define A_n: $\mathbb{R}^d \to \mathbb{R}^d$ by

$$(1.11)\qquad A_n\theta_{ni} = a_n^{-1}(\theta_{ni})\theta_{ni}, \qquad 1 \le i \le d.$$

The matrix sequence $\{A_n\}$ will be our replacement for the scalar sequence $\{a_n\}$. (To be precise, it is actually A_n^{-1} that will play the role of a_n.) In particular we can prove the following analogue of Theorem 1 in GJP. Again for convenience we will assume that X is lattice-valued, the correct lattice for X being the integer lattice $\mathbb{Z}^d$ in $\mathbb{R}^d$, and that S_n is strongly aperiodic. We then have the following case of Theorem 5.3.

THEOREM. *Assume that*

$$(\mathrm{A})\qquad \liminf_{r\to\infty}\ \inf_{\theta\in S^{d-1}} \frac{K(\theta, r)}{G(\theta, r)} > 0.$$

Then for all $\varepsilon > 0$, *there exist positive constants* $c_1 = c_1(\varepsilon)$ *and* c_2, *and regions* $R_n = R_n(\varepsilon)$ *such that for all n sufficiently large*

$$(1.12)\qquad P\{S_n \in R_n\} \ge 1 - \varepsilon,$$

$$(1.13)\qquad P\{S_n = x\} \le c_2|\det A_n| \quad \text{for all } x \in \mathbb{Z}^d,$$

$$(1.14)\qquad P\{S_n = x\} \ge c_1|\det A_n| \quad \text{if } x \in \mathbb{Z}^d \cap R_n.$$

In general, unlike the one-dimensional case, we have been unable to give a direct construction of the regions R_n from knowing just the distribution of X. In some special cases, however, this can be done; for example if X is radially symmetric then one can take $R_n = A_n^{-1}\, C(0, M)$ where M is chosen sufficiently large depending only on ε.

The method of proof of the probability estimates involves us in proving some results which are of independent interest. In particular we show that (A) is equivalent to the following probabilistic statement:

(C) There exist B_n and γ_n such that $\{B_n(S_n - \gamma_n)\}$ is stochastically compact, i.e. every subsequence contains a further subsequence which converges weakly to a full limit.

In the case that (C) holds we also show that the normalizing matrix may be taken to be A_n.

The techniques developed in proving these results enable us to prove rather easily the local limit theorem for random variables in the GDOA of a full operator stable law; see Theorem 6.4 for a precise statement of the result.

In concluding the introduction, we should point out that the equivalence of (A) and a condition similar to (C) has been established independently by Hahn and Klass (1985). In the introduction of their paper, they point out that their interest (and ours) in matrix normalization arises from trying to approximate the distribution of S_n. Theorems 5.3 and 6.4 mentioned above are our attempts at doing this.

2. Properties of *G*, *K*, and *Q*. In this section we describe some of the properties of the functions G, K, and Q that will be needed. In addition the MONB will be defined and a crucial inequality between $a_n(\theta)$ and $|A_n^{-1}\theta|$ will be proved.

We begin by recalling the definition of Q as given by (1.2). By Lemma 2.1 of Pruitt (1981)

$$Q(r) = r^{-2}\int_0^r 2uG(u)\,du. \tag{2.1}$$

Set $r_0 = \sup\{r\colon P\{0 < |X| \le r\} = 0\}$. From (2.1) or (1.2) it follows that Q is positive, continuous, $Q(r) = Q(r_0) \le 1$ for $0 < r \le r_0$, Q is strictly decreasing for $r \ge r_0$, and $Q(r)\downarrow 0$ as $r\uparrow\infty$. Observe that analogous statements also hold for $Q(\theta, r)$ defined by (1.9).

LEMMA 2.1.

(i) $\lim_{r\to\infty}\sup_{\theta\in S^{d-1}} Q(\theta, r) = 0.$

(ii) $Q(\theta, r)$ *is jointly continuous on* $S^{d-1}\times(0,\infty)$.

(iii) If $\theta_n \to \theta$ *and* $r_n \to 0$, then

$$\liminf_{n\to\infty} Q(\theta_n, r_n) \ge \lim_{n\to\infty} Q(\theta, r_n) = G(\theta, 0).$$

(iv) *There exists* r_0 *such that for all* $r \ge r_0$ *and all* $\theta\in S^{d-1}$, $Q(\theta, r)$ *is strictly decreasing.*

PROOF. (i) By Lemma 2.1 of Pruitt (1981), for any $\theta \in S^{d-1}$

$$\begin{aligned} Q(\theta, r) &= r^{-2}\int_0^r 2uG(\theta, u)\,du \\ &\le r^{-2}\int_0^r 2uG(u)\,du \\ &= Q(r) \to 0 \quad \text{as } r\to\infty. \end{aligned} \tag{2.2}$$

(ii) Assume that $\theta_n \to \theta$ and $r_n \to r > 0$. Then

$$1\{0 \le u \le r_n\}G(\theta_n, u) \to 1\{0 \le u \le r\}G(\theta, u)$$

for all but countably many values of u. Thus by (2.2) and bounded convergence,

$$Q(\theta_n, r_n) \to Q(\theta, r).$$

(iii) Assume that $\theta_n \to \theta$ and $r_n \to 0$. Then

$$\liminf_{n\to\infty} G(\theta_n, r_n) \ge G(\theta, 0).$$

Hence, by (2.2)

$$\begin{aligned}\liminf_{n\to\infty} Q(\theta_n, r_n) &= \liminf_{n\to\infty} r_n^{-2}\int_0^{r_n} 2uG(\theta_n, u)\,du \\ &\ge \liminf_{n\to\infty} r_n^{-2}\int_0^{r_n} 2uG(\theta_n, r_n)\,du \\ &\ge G(\theta, 0).\end{aligned}$$

Also by (2.2), since $G(\theta, \cdot)$ is right continuous,

$$\lim_{n\to\infty} Q(\theta, r_n) = G(\theta, 0).$$

(iv) Let $r_0(\theta) = \text{suP}\{r\colon\ P\{0 < |\langle X, \theta\rangle| \le r\} = 0\}$. Then $Q(\theta, r)$ is strictly decreasing for $r \ge r_0(\theta)$ and $Q(\theta, r) = Q(\theta, r_0)$ for $0 < r \le r_0(\theta)$. If $r_0(\theta)$ is not a bounded function of θ, then there exists a sequence $\theta_n \to \sigma \in S^{d-1}$ such that $r_0(\theta_n) \to \infty$. Now by (ii)

$$Q(\theta_n, r_0(\sigma) + 1) \to Q(\sigma, r_0(\sigma) + 1).$$

Also, if n is large enough that $r_0(\theta_n) \ge r_0(\sigma) + 1$, then

$$Q(\theta_n, r_0(\sigma) + 1) = Q(\theta_n, r_0(\sigma)) \to Q(\sigma, r_0(\sigma)).$$

Hence $Q(\sigma, r_0(\sigma)) = Q(\sigma, r_0(\sigma) + 1)$, which is a contradiction. Thus $r_0(\theta)$ is a bounded function of θ and so we may let $r_0 = \sup\{r_0(\theta)\colon \theta \in S^{d-1}\}$.

Since $Q(\cdot, r_0)$ is a positive, continuous function, $q_0 = \min\{Q(\theta, r_0)\colon \theta \in S^{d-1}\} > 0$. Thus letting $[x]$ = greatest integer $\le x$, we see by Lemma 2.1(iv) that for all $n \ge n_0 = [q_0^{-1}] + 1$ and all $\theta \in S^{d-1}$ there is an increasing sequence $a_n(\theta)$ defined by

$$Q(\theta, a_n(\theta)) = \frac{1}{n}. \tag{2.3}$$

LEMMA 2.2. (i) $\lim_{n\to\infty} \inf_{\theta\in S^{d-1}} a_n(\theta) = \infty$.
(ii) *There exists* n_1 *such that for all* $n \ge n_1$, $a_n(\theta)$ *is a continuous function of* θ.

PROOF. If (i) fails then there exist $\theta_k \in S^{d-1}$ and a subsequence n_k such that $\theta_k \to \theta$ and $a_{n_k}(\theta_k) \to a$ for some $\theta \in S^{d-1}$ and some $a \in [0, \infty)$. By Lemma 2.1(ii), (iii)

$$\liminf_{k\to\infty} Q(\theta_k, a_{n_k}(\theta_k)) \ge Q(\theta, a) > 0,$$

which is a contradiction since $Q(\theta_k, a_{n_k}(\theta_k)) = n_k^{-1} \to 0$.

Now choose $n_1 \geq n_0$ such that $\inf\{a_{n_1}(\theta)\colon \theta \in S^{d-1}\} > 0$. Fix $n \geq n_1$ and assume that $\theta_k \to \theta$. Observe that $a_n(\theta_k)$ cannot be unbounded, for if it were then

$$\frac{1}{n} = Q(\theta_k, a_n(\theta_k)) \to 0$$

as $k \to \infty$ by Lemma 2.1(i). Assume that $a_n(\theta_k) \to a$ along some subsequence as $k \to \infty$. Then, since $a > 0$,

$$\frac{1}{n} = Q(\theta_k, a_n(\theta_k)) \to Q(\theta, a)$$

by Lemma 2.1(ii). However, $a_n(\theta)$ is the unique solution of $Q(\theta, a) = n^{-1}$ and so $a_n(\theta_k) \to a_n(\theta)$.

Definition. For $n \geq n_1$ define the sequence $\{\theta_{n1}, \dots, \theta_{nd}\}$ by

$$a_n(\theta_{n1}) = \min\{a_n(\theta)\colon \theta \in S^{d-1}\},$$
$$a_n(\theta_{nk}) = \min\{a_n(\theta)\colon \theta \in S^{d-1} \text{ and } \langle\theta, \theta_{ni}\rangle = 0 \text{ for } 1 \leq i < k\}.$$

The choice of $\theta_{n1}, \dots, \theta_{nd}$, whilst it may not be unique, is possible by Lemma 2.2(ii). We will refer to $\theta_{n1}, \dots, \theta_{nd}$ as the minimal orthonormal basis (MONB) at time n. This orthonormal basis has appeared earlier in the work of Hahn and Klass in their study of the GDOA of operator stable laws.

Definition. For $n \geq n_1$ the linear transformation $A_n\colon \mathbb{R}^d \to \mathbb{R}^d$ is defined by

$$A_n\theta_{ni} = a_n^{-1}(\theta_{ni})\theta_{ni}, \qquad 1 \leq i \leq d. \tag{2.4}$$

The basic assumption that we will be making about the underlying distribution is

$$\liminf_{r\to\infty} \inf_{\theta\in S^{d-1}} \frac{K(\theta, r)}{G(\theta, r)} > 0. \tag{A}$$

By Lemma 2.4 of Pruitt (1981) it follows that there exist $p > 0$ and $r_0 > 0$ such that

$$r^p Q(\theta, r) \quad \text{decreases} \tag{2.5}$$

for all $\theta \in S^{d-1}$ and all $r \geq r_0$. From now on r_0 will always refer to this fixed constant and n_0 will always be $\max\{n_1, [(\min\{Q(\theta, r_0)\colon \theta \in S^{d-1}\})^{-1}] + 1\}$. In particular this choice of r_0 works in Lemma 2.1(iv) and with this choice of n_0, (2.3) and Lemma 2.2(ii) hold for all $n \geq n_0$. One consequence of (2.5) is that there exists a positive constant c such that for all $\theta \in S^{d-1}$ and all $n \geq n_0$

$$a_n(\theta) \geq c a_{2n}(\theta). \tag{2.6}$$

A second consequence is that for $n \geq n_0$

$$\max_{\theta\in S^{d-1}} a_n(\theta) \leq r_0 n^{1/p}. \tag{2.7}$$

Definition. For $r > 0$ and $\alpha, \alpha_1, \dots, \alpha_k \in S^{d-1}$, set

$$R(\alpha, r) = \{x \in \mathbb{R}^d: |\langle x, \alpha \rangle| > r\}$$

$$V(\alpha_1, \dots, \alpha_k) = \left\{ \sum_{i=1}^{k} \lambda_i \alpha_i: \lambda_i \in \mathbb{R}, i = 1, \dots, k \right\} \cap S^{d-1}.$$

Lemma 2.3. *Assume that $\alpha, \beta \in S^{d-1}$ are such that $\langle \alpha, \beta \rangle = 0$ and*

$$a_n(\alpha) = \min\{a_n(\theta): \theta \in V(\alpha, \beta)\}, \tag{2.8}$$

where $n \geq n_0$. If (A) holds, then there exists a constant $c \geq 1$, independent of n, such that for all $\gamma \in V(\alpha, \beta)$

$$c a_n^2(\gamma) \geq a_n^2(\alpha)\langle \alpha, \gamma \rangle^2 + a_n^2(\beta)\langle \beta, \gamma \rangle^2. \tag{2.9}$$

Proof. We first observe that (2.9) is trivial if $\gamma = \alpha$ or $\gamma = \beta$. Thus we assume that $\gamma \neq \alpha$ and $\gamma \neq \beta$. By elementary geometry one can check that for all $r > 0$,

$$R(\beta, r) \subseteq R\left(\gamma, \frac{r}{2}\langle \beta, \gamma \rangle\right) \cup R\left(\alpha, \frac{r}{2}\langle \beta, \gamma \rangle\right).$$

Thus $G(\beta, r) \leq G(\gamma, (r/2)\langle \beta, \gamma \rangle) + G(\alpha, (r/2)\langle \beta, \gamma \rangle)$. It then follows from (2.2) by a change of variables that

$$Q(\beta, r) \leq Q\left(\gamma, \frac{r}{2}\langle \beta, \gamma \rangle\right) + Q\left(\alpha, \frac{r}{2}\langle \beta, \gamma \rangle\right). \tag{2.10}$$

Setting $r = 2a_n(\gamma)/\langle \beta, \gamma \rangle$ and using Lemma 2.1(iv) and (2.8) we have

$$Q\left(\beta, \frac{2a_n(\gamma)}{\langle \beta, \gamma \rangle}\right) \leq \frac{2}{n}.$$

Thus

$$a_{n/2}(\beta) \leq \frac{2a_n(\gamma)}{\langle \beta, \gamma \rangle}.$$

Hence by (2.6) there is a positive constant c, independent of α, β, γ, and n, such that $a_n(\beta)\langle \beta, \gamma \rangle \leq c a_n(\gamma)$. Finally by (2.8) we trivially have $a_n(\alpha)\langle \alpha, \gamma \rangle \leq a_n(\gamma)$ and so (2.9) holds.

Lemma 2.4. *Assume that (A) holds and for $1 \leq k \leq d$ let $1 \leq m(1) < \cdots < m(k) \leq d$. Then for all $n \geq n_0$, all $1 \leq k \leq d$, and all $\theta \in V(\theta_{nm(1)}, \dots, \theta_{nm(k)})$,*

$$c^{k-1} a_n^2(\theta) \geq \sum_{i=1}^{k} a_n^2(\theta_{nm(i)})\langle \theta, \theta_{nm(i)} \rangle^2, \tag{2.11}$$

where c is the constant appearing in (2.9).

Proof. The proof is by induction on k. If $k = 1$ then it is immediate. Now assume (2.11) is true for $k < d$ and let $1 \leq m(1) < \cdots < m(k+1) \leq d$. For

$\theta \in V(\theta_{nm(1)}, \ldots, \theta_{nm(k+1)})$ set

$$\sigma = \left(\sum_{i=2}^{k+1} \langle \theta, \theta_{nm(i)} \rangle \theta_{nm(i)}\right)\left(\sum_{i=2}^{k+1} \langle \theta, \theta_{nm(i)} \rangle^2\right)^{-1/2}.$$

Then by the induction hypothesis

$$c^{k-1}a_n^2(\sigma) \geq \sum_{i=2}^{k+1} a_n^2(\theta_{nm(i)})\langle \sigma, \theta_{nm(i)}\rangle^2. \tag{2.12}$$

By Lemma 2.3 and the definition of $\{\theta_{n1}, \ldots, \theta_{nd}\}$

$$ca_n^2(\theta) \geq a_n^2(\theta_{nm(1)})\langle \theta, \theta_{nm(1)}\rangle^2 + a_n^2(\sigma)\langle \theta, \sigma\rangle^2. \tag{2.13}$$

A simple computation shows that for $2 \leq i \leq k+1$,

$$\langle \theta, \sigma\rangle\langle \sigma, \theta_{nm(i)}\rangle = \langle \theta, \theta_{nm(i)}\rangle; \tag{2.14}$$

(2.11) then follows from (2.12), (2.13), and (2.14).

COROLLARY 2.5. *Assume that* (A) *holds; then there exists a positive constant* c_0 *such that for all* $n \geq n_0$ *and all* $\theta \in S^{d-1}$

$$c_0 a_n(\theta) \geq |A_n^{-1}\theta|. \tag{2.15}$$

PROOF. This follows immediately from (2.4) and (2.11).

3. Bounds on the characteristic function of X. The characteristic function of X will be denoted by φ, i.e., for $t \in \mathbb{R}^d$,

$$\varphi(t) = E\exp(i\langle t, X\rangle).$$

For $u \in \mathbb{R}$ and $\theta \in S^{d-1}$, the characteristic function of $\langle X, \theta\rangle$ is given by

$$\varphi(\theta, u) = E\exp(iu\langle X, \theta\rangle).$$

Observe that if $t = |t|\theta$ then

$$\varphi(t) = \varphi(\theta, |t|). \tag{3.1}$$

In order to get the necessary bound on φ, we need to consider the symmetrized random variable $\langle X^s, \theta\rangle = \langle X_1 - X_2, \theta\rangle$. This gives rise to the functions $G^s(\theta, r)$, $K^s(\theta, r)$, and $Q^s(\theta, r)$ where for example $G^s(\theta, r) = P\{|\langle X^s, \theta\rangle| < r\}$.

The following result follows from the proof of Lemma 2.7 in Griffin (1983); one only needs to observe that the proof can be made uniform in $\theta \in S^{d-1}$.

LEMMA 3.1. *Assume that X is full, then*

(i) *there exists* r_1 *such that for all* $r \geq r_1$ *and all* $\theta \in S^{d-1}$

$$\tfrac{1}{2}G^s(\theta, 2r) \leq G(\theta, r) \leq 2G^s(\theta, r/2); \tag{3.2}$$

(ii) *there exist positive constants* c_1 *and* c_2 *such that for all* $\theta \in S^{d-1}$ *and all* $r > 0$

$$c_1 Q^s(\theta, r) \leq Q(\theta, r) \leq c_2 Q^s(\theta, r). \tag{3.3}$$

One further result, which is a uniform version of Lemma 2.5 in Griffin (1983), will also prove useful.

LEMMA 3.2. *Assume that X is full and* (A) *holds, then*

$$\liminf_{r\to\infty}\ \inf_{\theta\in S^{d-1}} \frac{K^s(\theta,r)}{G^s(\theta,r)} > 0. \tag{3.4}$$

REMARK. By increasing r_0 if necessary, we may assume that

$$\inf_{r\geq r_0}\ \inf_{\theta\in S^{d-1}} \frac{K^s(\theta,r)}{G^s(\theta,r)} > 0, \tag{3.5}$$

where r_0 is defined as in (2.5).

LEMMA 3.3. *Assume that X is full and* (A) *holds. Then there exists a positive constant c such that for all $n \geq n_0$, all $\theta \in S^{d-1}$, and all $1 \leq |u| \leq r_0^{-1}a_n(\theta)$,*

$$|\varphi^n(\theta, ua_n^{-1}(\theta))| \leq \exp(-c|u|^p), \tag{3.6}$$

where p is given by (2.5).

PROOF.

$$\begin{aligned} 1-|\varphi(\theta,v)|^2 &\geq \int_{|\langle X^s,\theta\rangle|\leq|v|^{-1}} [1-\cos(v\langle X^s,\theta\rangle)]\,dP \\ &\geq c_1|v|^2\int_{|\langle X^s,\theta\rangle|\leq|v|^{-1}} \langle X^s,\theta\rangle^2\,dP \\ &= c_1K^s(\theta,|v|^{-1}) \\ &\geq c_2Q^s(\theta,|v|^{-1}) \end{aligned}$$

if $|v|^{-1} \geq r_0$ by (3.5). Thus by (3.3) for $|v|^{-1} \geq r_0$,

$$\begin{aligned} 1-|\varphi(\theta,v)| &\geq \tfrac{1}{2}(1-|\varphi(\theta,v)|^2) \\ &\geq cQ(\theta,|v|^{-1}). \end{aligned}$$

Next using the inequality $1 - x \leq e^{-x}$ we obtain for $|v|^{-1} \geq r_0$,

$$|\varphi(\theta,v)|^n \leq \exp(-cnQ(\theta,|v|^{-1})). \tag{3.7}$$

For $1 \leq |u| \leq r_0^{-1}a_n(\theta)$ let $v = ua_n^{-1}(\theta)$. Then by (2.5) we have that $Q(\theta, a_n(\theta)|u|^{-1}) \geq |u|^p n^{-1}$ and so by (3.7)

$$|\varphi(\theta, ua_n^{-1}(\theta))|^n \leq \exp(-c|u|^p).$$

In order to obtain our final estimate on the characteristic function of X we need the following simple lemma which we state without proof.

LEMMA 3.4. *Assume that the matrix* $A\colon \mathbb{R}^d \to \mathbb{R}^d$ *is invertible. Define* $S, T\colon S^{d-1} \to S^{d-1}$ *by* $S\sigma = A\sigma|A\sigma|^{-1}$ *and* $T\theta = A^{-1}\theta|A^{-1}\theta|^{-1}$. *Then* $S \circ T = T \circ S = \mathrm{id}$ *and furthermore if* $S\sigma = \theta$ *then* $|A\sigma| = |A^{-1}\theta|^{-1}$.

DEFINITION. $E_n = \{x \in \mathbb{R}^d\colon x \in A_n^{-1}B(r_0^{-1})\}$ where $B(\lambda)$ is the ball of radius λ centered at the origin in $\mathbb{R}^d$.

Recalling the definition of c_0 from (2.15) we now prove

LEMMA 3.5. *Assume that X is full and* (A) *holds; then there exists a positive constant c such that for all* $n \geq n_0$ *and all* $s \in E_n \setminus B(c_0)$

$$|\varphi^n(A_n s)| \leq \exp(-c|s|^p). \tag{3.8}$$

PROOF. Fix $s \in E_n \setminus B(c_0)$ and let $\theta, \sigma \in S^{d-1}$ be such that $s = |s|\sigma$ and $A_n s = |A_n s|\theta$. Then by (3.6)

$$\begin{aligned} |\varphi^n(A_n s)| &= |\varphi^n(\,|A_n s|\theta)| \\ &= |\varphi^n(\theta, |A_n s|\,)| \\ &\leq \exp(-c_1|A_n s|^p a_n^p(\theta)), \end{aligned}$$

provided $a_n^{-1}(\theta) \leq |A_n s| \leq r_0^{-1}$.

First observe that by definition of E_n we have that $|A_n s| \leq r_0^{-1}$. Further by (2.15) and Lemma 3.4

$$\begin{aligned} |A_n s| a_n(\theta) &= |s|\,|A_n\sigma| a_n(\theta) \\ &= |s|\,|A_n^{-1}\theta|^{-1} a_n(\theta) \\ &\geq c_0^{-1}|s|. \end{aligned}$$

Thus for $s \notin B(c_0)$, $|A_n s| \geq a_n^{-1}(\theta)$ and further, for $s \in E_n \setminus B(c_0)$,

$$|\varphi^n(A_n s)| \leq \exp(-c|s|^p).$$

4. Equivalence of (A) and (C). In this section we will prove the equivalence of the statements (A) and (C) as defined in the introduction. The proofs are based on the one-dimensional proofs given in GJP and a result of Hahn and Klass (1979) which characterizes the feasibility of matrix normalizing a sequence of random vectors to obtain a full limit distribution.

THEOREM 4.1. *Assume that X is full and* (A) *holds; then there exists a centering sequence* δ_n *such that* $\{A_n(S_n - \delta_n)\}$ *is stochastically compact.*

PROOF. For $\theta \in S^{d-1}$ and $\eta > 0$ define

$$U_n(\theta, \eta) = \sum_{k=1}^{n} \langle X_k, \theta\rangle 1\{\,|\langle X_k, \theta\rangle| \leq \eta a_n(\theta)\}.$$

Set $\delta_n(\theta) = EU_n(\theta, 2^{1/p})$. Then for all $\theta \in S^{d-1}$ and all $\eta \geq 2^{1/p}$,

$$\begin{aligned}|EU_n(\theta,\eta) - \delta_n(\theta)| &\leq n\int_{\{2^{1/p}a_n(\theta) < |\langle X,\theta\rangle| \leq \eta a_n(\theta)\}} |\langle X,\theta\rangle|\, dP \\ &\leq n\eta a_n(\theta) G\big(\theta, 2^{1/p}a_n(\theta)\big) \\ &\leq n\eta a_n(\theta) Q\big(\theta, 2^{1/p}a_n(\theta)\big) \\ &\leq \frac{\eta}{2} a_n(\theta)\end{aligned}$$

for all n sufficiently large independent of θ by Lemma 2.2(i) and (2.5). Thus

$$\begin{aligned}&P\{\,|\langle S_n,\theta\rangle - \delta_n(\theta)| \geq \eta a_n(\theta)\} \\ &\quad\leq P\{\langle S_n,\theta\rangle \neq U_n(\theta,\eta)\} + P\{\,|U_n(\theta,\eta) - \delta_n(\theta)| \geq \eta a_n(\theta)\} \\ &\quad\leq nG(\theta,\eta a_n(\theta)) + P\{\,|U_n(\theta,\eta) - EU_n(\theta,\eta)| \geq \eta a_n(\theta)/2\} \\ &\quad\leq nG(\theta,\eta a_n(\theta)) + 4nK(\theta,\eta a_n(\theta)) \\ &\quad\leq 4nQ(\theta,\eta a_n(\theta)) \\ &\quad\leq \frac{4}{\eta^p}\end{aligned}$$

for n sufficiently large as before. Now set

$$\delta_n = \sum_{i=1}^{d} \delta_n(\theta_{ni})\theta_{ni}. \tag{4.1}$$

Then since A_n is self-adjoint, i.e., $A_n = A_n^*$,

$$\begin{aligned}P\{\,|A_n(S_n - \delta_n)| \geq M\} &\leq \sum_{i=1}^{d} P\{\,|\langle A_n(S_n - \delta_n), \theta_{ni}\rangle| \geq Md^{-1/2}\} \\ &= \sum_{i=1}^{d} P\{\,|\langle S_n, A_n\theta_{ni}\rangle - \langle \delta_n, A_n\theta_{ni}\rangle| \geq Md^{-1/2}\} \\ &= \sum_{i=1}^{d} P\{\,|\langle S_n, \theta_{ni}\rangle - \delta_n(\theta_{ni})| \geq Md^{-1/2}a_n(\theta_{ni})\} \\ &\leq 4d\left(\frac{d^{1/2}}{M}\right)^p,\end{aligned}$$

which proves that $\{A_n(S_n - \delta_n)\}$ is tight. Thus given any subsequence there is a further subsequence n_k such that $A_{n_k}(S_{n_k} - \delta_{n_k})$ converges weakly to some limit, say V. We must show that V is full. Let ψ be the characteristic function of V; then by (3.8) for $|s| \geq c_0$

$$|\psi(s)| \leq \exp(-c|s|^p). \tag{4.2}$$

Thus not only is V full, it has a C^∞ density.

In order to prove that (C) implies (A) we will need some preliminary results. First we state for convenience the polar decomposition of an invertible linear transformation on $\mathbb{R}^d$ (Halmos, 1958, p. 169).

Polar decomposition. Let B be an invertible linear transformation of $\mathbb{R}^d$. Then there exists an orthonormal basis $\{\theta_1, \dots, \theta_d\}$, a transformation D which is diagonal with respect to this basis, and a unitary transformation U such that $B = U \circ D$. If

$$D\theta_i = b_i^{-1}\theta_i,$$

then we set

$$\bar{b}^2(\theta) = \sum_{i=1}^{d} \langle \theta, \theta_i \rangle^2 b_i^2 = |D^{-1}\theta|^2. \tag{4.3}$$

Observe that if $\{B_n(S_n - \delta_n)\}$ is stochastically compact then for large n, B_n must be invertible and hence has such a polar decomposition.

Let ρ be the Prohorov metric defined on the space of all d-dimensional random variables by

$$\begin{aligned}\rho(X, Y) &= \inf\{\varepsilon > 0\colon P(X \in A) \le P(Y \in A^\varepsilon) + \varepsilon \text{ for all } A \in \mathscr{B}\} \\ &= \inf\{\varepsilon > 0\colon P(Y \in A) \le P(X \in A^\varepsilon) + \varepsilon \text{ for all } A \in \mathscr{B}\},\end{aligned}$$

where $A^\varepsilon = \{x \in \mathbb{R}^d\colon |x - y| < \varepsilon \text{ for some } y \in A\}$ and $\mathscr{B}$ = Borel sets in $\mathbb{R}^d$.

LEMMA 4.2. *Assume that $\{B_n S_n\}$ is stochastically compact; then for any sequence $\theta_n \in S^{d-1}$, $\{\langle S_n, \theta_n \rangle / \bar{b}_n(\theta_n)\}$ is stochastically compact.*

PROOF. Let $B_n = U_n \circ D_n$ be the polar decomposition of B_n. Given any subsequence, choose a further subsequence along which $B_n S_n$ converges weakly to some limit V. By Theorem 2 of Hahn and Klass (1979), along this subsequence

$$\lim_{n \to \infty} \sup_{\theta \in S^{d-1}} \rho\left[\langle S_n, \theta \rangle / \bar{b}_n(\theta), \langle V, U_n D_n^{-1}\theta / |D_n^{-1}\theta| \rangle\right] = 0. \tag{4.4}$$

Now choose a further subsequence along which

$$\frac{U_n D_n^{-1}\theta_n}{|D_n^{-1}\theta_n|} \to \sigma \in S^{d-1}. \tag{4.5}$$

Then by (4.4) and (4.5) along this subsequence

$$\begin{aligned}\rho\left[\langle S_n, \theta_n \rangle / \bar{b}_n(\theta_n), \langle V, \sigma \rangle\right] &\le \rho\left[\langle S_n, \theta_n \rangle / \bar{b}_n(\theta_n), \langle V, U_n D_n^{-1}\theta_n / |D_n^{-1}\theta_n| \rangle\right] \\ &\quad + \rho\left[\langle V, U_n D_n^{-1}\theta_n / |D_n^{-1}\theta_n| \rangle, \langle V, \sigma \rangle\right] \\ &\to 0.\end{aligned}$$

Further since V is full, $\langle V, \sigma \rangle$ is nondegenerate and so $\{\langle S_n, \theta_n \rangle / \bar{b}_n(\theta_n)\}$ is stochastically compact.

The next Lemma is not needed in this section, but since it is an easy consequence of Lemma 4.2, we shall prove it now.

LEMMA 4.3. *Let $\{B_n\}$ and $\{C_n\}$ be sequences of matrices and $\{\zeta_n\}$ and $\{\xi_n\}$ be centering terms such that both $\{B_n(S_n - \zeta_n)\}$ and $\{C_n(S_n - \xi_n)\}$ are stochastically compact. Then there exist positive constants c_1, c_2, and n_1 such that for all*

$n \geq n_1$ and all $\theta \in S^{d-1}$

$$c_1 \leq \bar{b}_n(\theta)/\bar{c}_n(\theta) \leq c_2. \tag{4.6}$$

PROOF. We begin by observing that under the hypotheses, both $\{B_n S_n^s\}$ and $\{C_n S_n^s\}$ are stochastically compact. Now assume that (4.6) is false, so we may assume without loss of generality that there exist $\theta_k \in S^{d-1}$ and $n_k \to \infty$ such that

$$\bar{b}_{n_k}(\theta_k)/\bar{c}_{n_k}(\theta_k) \to 0. \tag{4.7}$$

By Lemma 4.2, there exists a further subsequence and nondegenerate random variables Z_1 and Z_2 such that

$$\langle S_{n_k}^s, \theta_k \rangle / \bar{b}_{n_k}(\theta_k) \to Z_1,$$
$$\langle S_{n_k}^s, \theta_k \rangle / \bar{c}_{n_k}(\theta_k) \to Z_2.$$

However, by the convergence of types theorem this contradicts (4.7).

One final result that we need is the following version of Lemma 1 in GJP. Since the proof is similar it will not be given here.

LEMMA 4.4. *Assume that* $\liminf_{r\to\infty} \inf_{\theta \in S^{d-1}} K(\theta, r)/G(\theta, r) = 0$; *then there exist $\theta_j \in S^{d-1}$ and integers $m_j, n_j \to \infty$ with $m_j < n_j$ such that*

$$\frac{m_j}{n_j} \to 1, \qquad \frac{a_{m_j}(\theta_j)}{a_{n_j}(\theta_j)} \to 0.$$

Furthermore if $x_j \in [a_{m_j}(\theta_j), a_{n_j}(\theta_j)]$ then

$$\frac{K(\theta_j, x_j)}{G(\theta_j, x_j)} \to 0.$$

THEOREM 4.5. *Assume that X is full; then the statements* (A) *and* (C) *are equivalent.*

PROOF. We have already seen in Theorem 4.1 that (A) $\Rightarrow$ (C).

(C) $\Rightarrow$ (A). First observe that since $\{B_n(S_n - \gamma_n)\}$ is stochastically compact, so is $B_n S_n^s$. Now assume that (A) fails and apply Lemma 4.4. Define $\nu_j = \min\{k: \bar{b}_k(\theta_j) \geq a_{m_j}(\theta_j)\}$ and suppose that along some subsequence $\nu_j/m_j \to \xi \in [0, \infty]$.

CASE 1. $\xi > 0$. Set $k_j = \nu_j - 1$ and observe that by Lemmas 3.1 and 4.4 for any $M > 1/2$ and all j sufficiently large

$$\begin{aligned} 2k_j G^s\big(\theta_j, M\bar{b}_{k_j}(\theta_j)\big) &\geq 2k_j G^s\big(\theta_j, M a_{m_j}(\theta_j)\big) \\ &\geq k_j G\big(\theta_j, 2M a_{m_j}(\theta_j)\big) \\ &\geq k_j G\big(\theta_j, a_{n_j}(\theta_j)\big) \\ &\sim k_j Q\big(\theta_j, a_{n_j}(\theta_j)\big) \\ &= k_j/n_j \to \xi. \end{aligned} \tag{4.8}$$

Now

$$2P\left\{ |\langle S^s_{k_j}, \theta_j\rangle| \geq M\bar{b}_{k_j}(\theta_j)\right\} \geq P\left\{ \max_{1\leq i\leq k_j} |\langle S^s_i, \theta_j\rangle| \geq M\bar{b}_{k_j}(\theta_j)\right\}$$
$$\geq 1 - P\left\{ \max_{1\leq i\leq k_j} |\langle X^s_i, \theta_j\rangle| \leq 2M\bar{b}_{k_j}(\theta_j)\right\}$$
$$= 1 - \left[1 - G^s\left(\theta_j, 2M\bar{b}_{k_j}(\theta_j)\right)\right]^{k_j}$$
$$\geq 1 - \exp\left\{-k_j G^s\left(\theta_j, 2M\bar{b}_{k_j}(\theta_j)\right)\right\}.$$

Thus by (4.8), $\{\langle S^s_{k_j}, \theta_j\rangle / \bar{b}_{k_j}(\theta_j)\}$ is not tight which contradicts Lemma 4.2.

CASE 2. $\xi = 0$. Set $k_j = \nu_j$ and observe that for any $\varepsilon > 0$, by truncating at $\pm a_{m_j}(\theta_j)$, we obtain

$$P\left\{ |\langle S^s_{k_j}, \theta_j\rangle| \geq \varepsilon\bar{b}_{k_j}(\theta_j)\right\} \leq P\left\{ |\langle S^s_{k_j}, \theta_j\rangle| \geq \varepsilon a_{m_j}(\theta_j)\right\}$$
$$\leq \frac{k_j a^2_{m_j}(\theta_j) K^s\left(\theta_j, a_{m_j}(\theta_j)\right)}{\varepsilon^2 a^2_{m_j}(\theta_j)} + k_j G^s\left(\theta_j, a_{m_j}(\theta_j)\right)$$
$$\leq k_j(1 + \varepsilon^{-2}) Q^s\left(\theta_j, a_{m_j}(\theta_j)\right)$$
$$\leq c(1 + \varepsilon^{-2}) k_j / m_j \to 0,$$

where we have used (3.3) in obtaining the last inequality. Thus $\{\langle S^s_{k_j}, \theta_j\rangle / \bar{b}_{k_j}(\theta_j)\}$ is not stochastically compact, which again contradicts Lemma 4.2.

5. Probability estimates under (A). We will begin this section by constructing the regions R_n alluded to in the introduction. From (A), it follows by Theorem 4.1 that $A_n(S_n - \delta_n)$ is stochastically compact. Set

$$\mathscr{G} = \left\{ V\colon V \text{ is a subsequential limit of } \{A_n(S_n - \delta_n)\}\right\}.$$

Since $\{A_n(S_n - \delta_n)\}$ is stochastically compact, it immediately follows that

$$\inf_{V\in\mathscr{G}} \rho\left[A_n(S_n - \delta_n), V\right] \to 0 \tag{5.1}$$

as $n \to \infty$. Further one can easily check that $(\mathscr{G}, \rho)$ is a compact metric space and so there exists $V_n \in \mathscr{G}$ such that

$$\rho\left[A_n(S_n - \delta_n), V_n\right] = \inf_{V\in\mathscr{G}} \rho\left[A_n(S_n - \delta_n), V\right]. \tag{5.2}$$

Let ψ_n be the characteristic function of V_n. By (3.8) (see also (4.2)) for any n and all $|s| > c_0$

$$|\psi_n(s)| \leq \exp(-c|s|^p). \tag{5.3}$$

Thus by the inversion formula, V_n has a density f_n given by

$$f_n(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle t, x\rangle} \psi_n(t)\, dt. \tag{5.4}$$

From (5.3) and (5.4), one can easily check that

(5.5) there exists a constant c_1 such that for all $x, y \in \mathbb{R}^d$ and all n

$$|f_n(x) - f_n(y)| \le c_1|x - y|,$$

(5.6) there exists a constant β such that for all $x \in \mathbb{R}^d$ and all n

$$|f_n(x)| \le \beta,$$

(5.7) if $V_{n_k} \to V$ then $f_{n_k} \to f$ uniformly in $x \in \mathbb{R}^d$.

From (5.1) and (5.2) we see that if $A_{n_k}(S_{n_k} - \delta_{n_k}) \to V$, then $V_{n_k} \to V$. Thus it follows that $\{V_n\}$ is stochastically compact and in particular, $\{V_n\}$ is tight.

Given $\varepsilon \in (0,1)$ choose M large enough that for all n

$$P\{\,|V_n| \ge M - 1\} < \varepsilon/6. \tag{5.8}$$

Next choose n_1 so that for all $n \ge n_1$

$$\rho[A_n(S_n - \delta_n), V_n] < \frac{\varepsilon}{6(1 + 2c_1)V(M)} \equiv \alpha, \tag{5.9}$$

where $V(M)$ is the volume of the ball of radius M in $\mathbb{R}^d$ and c_1 is given by (5.5). Set

$$P_n = \{x \in \mathbb{R}^d: |x| \le M \text{ and } f_n(x) > \alpha\}.$$

Finally let

$$R_n = \delta_n + A_n^{-1}P_n = \{x \in \mathbb{R}^d: A_n(x - \delta_n) \in P_n\}. \tag{5.10}$$

Lemma 5.1. *With ε, α, R_n, and V_n as above,*

$$P\{S_n \in R_n\} \ge 1 - \varepsilon, \tag{5.11}$$

$$\textit{if } x_n \in R_n \textit{ then } \liminf_{n\to\infty} f_n(A_n(x_n - \delta_n)) \ge \alpha. \tag{5.12}$$

Proof. (5.12) is immediate from the construction of R_n. To prove (5.11) observe that $P\{S_n \in R_n\} = P\{A_n(S_n - \delta_n) \in P_n\}$ and by (5.9)

$$\begin{aligned} P\{V_n \in P_n\} - P\{A_n(S_n - \delta_n) \in P_n\} &\le P\{A_n(S_n - \delta_n) \in P_n^\alpha \setminus P_n\} + \alpha \\ &\le P\{V_n \in (P_n^\alpha \setminus P_n)^\alpha\} + 2\alpha. \end{aligned} \tag{5.13}$$

Let ∂P_n be the boundary of P_n. Then $(P_n^\alpha \setminus P_n)^\alpha \subseteq \{x \in \mathbb{R}^d: |x - y| < 2\alpha$ for some $y \in \partial P_n\}$. By continuity $f_n(y) = \alpha$ if $y \in \partial P_n$ and $|y| \ne M$. Since $\alpha < \frac{1}{6}$, if $x \in (P_n^\alpha \setminus P_n)^\alpha \cap B(M - 1)$ it follows from (5.5) that

$$f_n(x) \le \alpha + c_1 2\alpha. \tag{5.14}$$

Hence by (5.8) and (5.14)

$$\begin{aligned} P\{V_n \in (P_n^\alpha \setminus P_n)^\alpha\} &\le \int_{\substack{(P_n^\alpha\setminus P_n)^\alpha \\ |x| \le M-1}} f_n(x)\,dx + \int_{|x| \ge M-1} f_n(x)\,dx \\ &\le \alpha(1 + 2c_1)V(M - 1) + \frac{\varepsilon}{6} \\ &\le \frac{\varepsilon}{3}. \end{aligned}$$

Thus by (5.13)

$$P\{A_n(S_n - \delta_n) \in P_n\} \geq P\{V_n \in P_n\} - \frac{2\varepsilon}{3}.$$

Finally

$$P\{V_n \in P_n\} \geq 1 - \int_{B(M)\setminus P_n} f_n(x)\,dx - \int_{|x|>M} f_n(x)\,dx$$
$$\geq 1 - \alpha V(M) - \frac{\varepsilon}{6} \geq 1 - \frac{\varepsilon}{3}.$$

We will now assume that the distribution of X is normalized in the following sense (Stone, 1965): There exists an integer k, $0 \leq k \leq d$ and real numbers $\alpha_1, \dots, \alpha_k$ such that

$$\varphi(2\pi n_1, \dots, 2\pi n_k, 0, \dots, 0) = \exp\{2\pi i(n_1\alpha_1 + \cdots + n_k\alpha_k)\} \tag{5.15}$$

for integral values of $n_1, \dots, n_k$ and $|\varphi(t)| < 1$ for all other values of $t \in \mathbb{R}^d$. If (5.15) failed then we could find an invertible transformation U of $\mathbb{R}^d$ such that the characteristic function of $Y_i = UX_i$ satisfied (5.15), and we would then work with the random variables $\{Y_i\}$. The condition $|\varphi(t)| < 1$ means that the random walk is strongly aperiodic (Spitzer, p. 42) in the lattice directions. This merely avoids further technical details concerned with periodicity. This aspect is discussed in the one-dimensional case in GJP Section 2.

Observe that the distribution of S_n is supported by

$$D_n = \{x \in \mathbb{R}^d\colon x_i - n\alpha_i \text{ is an integer } 1 \leq i \leq k\}, \tag{5.16}$$

where $x = (x_1, \dots, x_d)$. It will be convenient to write $x = (\tilde{x}, \bar{x})$ where $\tilde{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$ and $\bar{x} = (x_{k+1}, \dots, x_d) \in \mathbb{R}^{d-k}$. Define

$$\bar{C}(x, \lambda) = \{y \in \mathbb{R}^d\colon \tilde{x} = \tilde{y} \text{ and } |x_i - y_i| \leq \lambda \text{ for } k < i \leq d\}$$
$$R(\pi, a) = \{t \in \mathbb{R}^d\colon |t_i| \leq \pi \text{ for } 1 \leq i \leq k,\ |t_i| \leq a \text{ for } k < i \leq d\}.$$

We will need the following inversion formula (Stone, 1965): For $x \in D_n$, $\lambda > 0$ and $a > 0$,

$$\begin{aligned} &(2\lambda)^{k-d}\int_{\mathbb{R}^{d-k}} P\{S_n \in \bar{C}((\tilde{x}, \bar{x} - \bar{y}), \lambda)\} a^{d-k} H(a\bar{y})\,d\bar{y} \\ &\quad = (2\pi)^{-d}\int_{R(\pi, a)} \exp(-i\langle t, x\rangle)\varphi^n(t)k(\lambda t)h(a^{-1}t)\,dt, \end{aligned} \tag{5.17}$$

where

$$k(t) = \prod_{i=k+1}^{d} (\sin t_i)/t_i$$
$$h(t) = \prod_{i=k+1}^{d} (1 - |t_i|)^+$$
$$H(\bar{y}) = \prod_{i=k+1}^{d} (1 - \cos y_i)/\pi y_i^2.$$

LEMMA 5.2. *Assume* (A) *and set*

$$I_n(x, \lambda, a) = (2\pi)^{-d} \int_{R(\pi, a)} \exp(-i\langle t, x\rangle) \varphi^n(t) k(\lambda t) h(a^{-1}t)\, dt.$$

Then for each $\varepsilon > 0$, $\lambda > 0$, *and* $a > 0$,

$$\alpha \le \liminf_{n\to\infty} \inf_{x \in R_n(\varepsilon)} \frac{I_n(x, \lambda, a)}{|\det A_n|} \le \limsup_{n\to\infty} \sup_{x \in \mathbb{R}^d} \frac{I_n(x, \lambda, a)}{|\det A_n|} \le \beta, \tag{5.18}$$

where α *and* β *are defined by* (5.9) *and* (5.6), *respectively.*

PROOF. Fix $\lambda > 0$ and $a > 0$ and set

$$q = \max\{|\varphi(t)| : t \in R(\pi, a) \setminus B(r_0^{-1})\},$$

where r_0 is defined in (2.5). Then $q < 1$ since X is assumed to be normalized. Thus

$$\begin{aligned} &|(2\pi)^{-d} \int_{R(\pi, a)\setminus B(r_0^{-1})} \exp(-i\langle t, x\rangle) \varphi^n(t) k(\lambda t) h(a^{-1}t)\, dt|\, |\det A_n|^{-1} \\ &\quad \le (2\pi)^{k-d} (2a)^{d-k} q^n |\det A_n|^{-1} \to 0 \end{aligned} \tag{5.19}$$

by (2.4) and (2.7).

By a change of variable and observing that A_n is self-adjoint, we see that

$$\begin{aligned} &(2\pi)^{-d} \int_{B(r_0^{-1})} \exp(-i\langle t, x\rangle) \varphi^n(t) k(\lambda t) h(a^{-1}t)\, dt \\ &\quad = |\det A_n| (2\pi)^{-d} \int_{A_n^{-1}B(r_0^{-1})} \exp(-i\langle s, A_n(x - \delta_n)\rangle) \varphi^n(A_n s) \\ &\qquad\qquad \cdot \exp(-i\langle s, A_n\delta_n\rangle) k(\lambda A_n s) h(a^{-1}A_n s)\, ds \\ &\quad = |\det A_n| J_n(x, \lambda, a). \end{aligned}$$

Now given any subsequence, choose a further subsequence along which $A_n(S_n - \delta_n)$ converges weakly to some limit, say V. As mentioned earlier $V_n \to V$ along this same subsequence. Hence along this subsequence for any $x_n \in \mathbb{R}^d$,

$$\begin{aligned} &| J_n(x_n, \lambda, a) - f_n(A_n(x_n - \delta_n)) | \\ &\quad \le \Big| (2\pi)^{-d} \int_{A_n^{-1}B(r_0^{-1})} \exp(-i\langle s, A_n(x_n - \delta_n)\rangle) \\ &\qquad \cdot [\varphi^n(A_n s) \exp(-i\langle s, A_n\delta_n\rangle) k(\lambda A_n s) h(a^{-1}A_n s) - \psi_n(s)]\, ds \Big| \\ &\qquad + \Big| (2\pi)^{-d} \int_{\mathbb{R}^d \setminus A_n^{-1}B(r_0^{-1})} \exp(-i\langle s, A_n(x_n - \delta_n)\rangle) \psi_n(s)\, ds \Big|. \end{aligned}$$

By Lemma (2.2)(i) and (2.4), $A_n^{-1}B(r_0^{-1}) \to \mathbb{R}^d$ as $n \to \infty$; thus by (3.8), (5.3), and dominated convergence

$$|J_n(x_n, \lambda, a) - f_n(A_n(x_n - \delta_n))| \to 0. \tag{5.20}$$

The upper bound in (5.18) now follows immediately from (5.6), (5.19), and (5.20) while the lower bound follows from (5.12), (5.19), and (5.20).

THEOREM 5.3. *Assume that* X *is full, normalized, and* (A) *holds. Then for all* $\varepsilon > 0$ *there exist regions* $R_n = R_n(\varepsilon)$ *and positive constants* $c_1 = c_1(\varepsilon)$ *and* c_2, c_2 *independent of* ε, *such that for all* $\lambda > 0$

(i) $P\{S_n \in R_n\} \geq 1 - \varepsilon$;
(ii) $\limsup_{n \to \infty} \sup_{x \in D_n} (\lambda^{d-k}|\det A_n|)^{-1} P\{S_n \in \bar{C}(x, \lambda)\} \leq c_2$;
(iii) $\liminf_{n \to \infty} \inf_{x \in R_n \cap D_n} (\lambda^{d-k}|\det A_n|)^{-1} P\{S_n \in \bar{C}(x, \lambda)\} \geq c_1$,

where D_n *is defined by* (5.16) *and* k *by* (5.15).

PROOF. Given $\varepsilon > 0$, let $R_n = R_n(\varepsilon)$ be the region described in (5.10); thus (i) follows immediately from (5.11). With the bounds from Lemma 5.2 the proofs of (ii) and (iii) are somewhat standard so we will only outline the arguments; essentially the same arguments are given in Feller (1965), Stone (1965), and Griffin (1983).

Given $\lambda > 0$ choose a large enough that

$$\int_{\mathbb{R}^{d-k} \setminus C^{d-k}(0, \lambda/2)} a^{d-k} H(a\bar{y})\, d\bar{y} = \frac{2^{d-k}}{16 \cdot 3^{d-k}} \frac{\alpha}{\beta}, \tag{5.21}$$

where $C^{d-k}(0, \lambda)$ is the cube of side length 2λ centered at the origin in $\mathbb{R}^{d-k}$.

Observe that if $\varepsilon \in (0, \lambda)$ and $\bar{y} \in C^{d-k}(0, \varepsilon)$ then

$$\bar{C}((\tilde{x}, \bar{x} - \bar{y}), \lambda + \varepsilon) \supseteq \bar{C}((\tilde{x}, \bar{x}), \lambda), \tag{5.22}$$

$$\bar{C}((\tilde{x}, \bar{x} - \bar{y}), \lambda - \varepsilon) \subseteq \bar{C}((\tilde{x}, \bar{x}), \lambda). \tag{5.23}$$

Thus by (5.17), (5.18), (5.21), and (5.22), for $x \in D_n$ and n sufficiently large

$$\begin{aligned} 2\beta|\det A_n| &\geq (3\lambda)^{k-d} \int_{C^{d-k}(0, \lambda/2)} P\{S_n \in \bar{C}((\tilde{x}, \bar{x} - \bar{y}), 3\lambda/2)\} a^{d-k} H(a\bar{y})\, d\bar{y} \\ &\geq (3\lambda)^{k-d} P\{S_n \in \bar{C}(x, \lambda)\} \int_{C^{d-k}(0, \lambda/2)} a^{d-k} H(a\bar{y})\, d\bar{y} \\ &\geq (1/2)(3\lambda)^{k-d} P\{S_n \in \bar{C}(x, \lambda)\}, \end{aligned}$$

which proves the upper bound with $c_2 = 4 \cdot 3^{d-k}\beta$. For the lower bound, we have by (5.17), (5.18), (5.21), (5.23), and the upper bound just derived, for $x \in D_n \cap R_n$ and n sufficiently large

$$\begin{aligned} \frac{\alpha}{2}|\det A_n| &\leq \lambda^{k-d} \int_{C^{d-k}(0, \lambda/2)} P\{S_n \in \bar{C}((\tilde{x}, \bar{x} - \bar{y}), \lambda/2)\} a^{d-k} H(a\bar{y})\, d\bar{y} \\ &\quad + \lambda^{k-d} c_2 (\lambda/2)^{d-k} |\det A_n| \int_{\mathbb{R}^{d-k} \setminus C^{d-k}(0, \lambda/2)} a^{d-k} H(a\bar{y})\, d\bar{y} \\ &\leq \lambda^{k-d} P\{S_n \in \bar{C}(x, \lambda)\} + (\alpha/4)|\det A_n|, \end{aligned}$$

which proves the lower bound with $c_1 = \alpha/4$.

REMARK. If there exist sequences $\{B_n\}$ and $\{\gamma_n\}$ such that $\{B_n(S_n - \gamma_n)\}$ is stochastically compact, then by Theorem 4.5, (A) holds and so we can apply Theorem 5.3. One might expect that in this case, $|\det A_n|$ could be replaced with $|\det B_n|$ in (ii) and (iii). This is indeed the case. A proof can be based on Lemma 4.3 or alternatively one can modify the proof of Theorem 5.3 ((6.4) would be needed in this case).

As mentioned earlier if X is not normalized, then there exists an invertible linear transformation U of $\mathbb{R}^d$ such that the characteristic function of $Y = UX$ satisfies (5.15). Let $T_n = UX_1 + \cdots + UX_n = US_n$. Again assume for convenience that T_n is strongly aperiodic. Observe that if (A) holds for X, by Theorem 4.1, $\{A_n(S_n - \delta_n)\}$ is stochastically compact. Thus $\{B_n(T_n - \gamma_n)\}$ is stochastically compact where $B_n = A_nU^{-1}$ and $\gamma_n = U\delta_n$. Hence by the above remark, since $|\det B_n| = |\det U^{-1}|\,|\det A_n|$ we can prove

THEOREM 5.4. *Assume that X is full and* (A) *holds. Then for all $\varepsilon > 0$ there exist regions $R_n = R_n(\varepsilon)$ and positive constants $c_1 = c_1(\varepsilon)$ and c_2, c_2 independent of ε, such that for all $\lambda > 0$*

(i) $P\{S_n \in R_n\} \geq 1 - \varepsilon$;
(ii) $\limsup_{n\to\infty}\sup_{x\in D_n}(\lambda^{d-k}|\det A_n|)^{-1}P\{S_n \in U^{-1}\overline{C}(x, \lambda)\} \leq c_2$;
(iii) $\liminf_{n\to\infty}\inf_{x\in R_n\cap D_n}(\lambda^{d-k}|\det A_n|)^{-1}P\{S_n \in U^{-1}\overline{C}(x, \lambda)\} \geq c_1$,

where $R_n = U^{-1}R'_n$, $D_n = U^{-1}D'_n$, and R'_n, D'_n are defined by (5.10) *and* (5.16) *for Y.*

REMARKS. As mentioned in the introduction, one would ideally like to construct the regions R_n knowing just the distribution of X, as was done in the one-dimensional case in GJP. However, we have been unable to do this, although in the special case where X is radially symmetric, we can show that $R_n(\varepsilon)$ may be taken to be the region $A_n^{-1}C(0, M)$ where M is chosen sufficiently large depending only on ε. In general we suspect that $R_n(\varepsilon)$ may be taken to be a cone with vertex at some point $\gamma_n \in \mathbb{R}^d$, intersected with $A_n^{-1}C(\gamma_n, M)$ where M is chosen large enough depending only on ε.

Hall (1983) considered the behavior of the concentration function under (A) in $d = 1$ dimension. In particular if we let

$$Q(S_n, \lambda) = \sup_{x\in\mathbb{R}} P\{|S_n - x| \leq \lambda\},$$

then he showed that for any $\lambda > 0$

$$0 < \liminf_{n\to\infty} a_n Q(S_n, \lambda) \leq \limsup_{n\to\infty} a_n Q(S_n, \lambda) < \infty,$$

where a_n is defined by (1.3). The generalization of this result to $\mathbb{R}^d$ is an immediate consequence of Theorem 5.4.

In GJP Theorem 1 it was shown that in $d = 1$ dimension, (A) is equivalent to several different probabilistic statements. It would be interesting to determine whether the appropriate analogues hold in higher dimensions.

As one final remark we note that if there exist B_n and γ_n such that $\{B_n(S_n - \gamma_n)\}$ is tight and the upper bound holds with $|\det B_n|$ replacing $|\det A_n|$ then (A) holds. This is because these two conditions imply (C), which in turn implies (A) by Theorem 4.5.

6. Generalized domains of attraction. Recall that X is in the GDOA of a full operator stable law Y if there exist normalizing matrices B_n and centering terms γ_n such that

$$B_n(S_n - \gamma_n) \to Y. \tag{6.1}$$

As mentioned in the introduction Marjorie Hahn and Michael Klass have now obtained a complete characterization of the GDOA of a full operator stable law. It is interesting to note how little information we need to know about the distribution of X in order to prove the local limit theorem.

Observe that by (6.1) and Theorem 4.5, (A) holds, and thus all of the results in the previous sections, which were proved under assumption (A), are valid for X.

We will assume as before that X is normalized (see Section 5). The proof of the local limit theorem is very similar to the proof of Theorem 5.3. We will state separately the following two lemmas which enable us to apply dominated convergence as before.

LEMMA 6.1. *Let B be an invertible linear transformation of $\mathbb{R}^d$ with polar decomposition $B = U \circ D$. If $B^*\sigma = |B^*\sigma|\theta$ where $\theta, \sigma \in S^{d-1}$, then*

$$|B^*\sigma|\,|D^{-1}\theta| = 1. \tag{6.2}$$

PROOF. If $B^*\sigma = |B^*\sigma|\theta$, then

$$|B^*\sigma|\,|B^{*-1}\theta| = 1. \tag{6.3}$$

Now since U is unitary and D is self-adjoint

$$\begin{aligned}|B^{*-1}\theta| &= |U^{*-1}D^{*-1}\theta|\\ &= |D^{-1}\theta|,\end{aligned}$$

which proves (6.2).

LEMMA 6.2. *There exist positive constants c_1 and c_2 such that if $s \in (B_n^*)^{-1}B(r_0^{-1}) \setminus B(c_1)$ and n is sufficiently large,*

$$|\varphi^n(B_n^* s)| \le \exp(-c_2|s|^p), \tag{6.4}$$

where p is given by (2.5).

PROOF. Since (A) holds, by Lemma 4.3 there exists a constant c such that for all $\theta \in S^{d-1}$ and all n sufficiently large

$$\bar{b}_n(\theta) \le c\bar{a}_n(\theta) = c|A_n^{-1}\theta|. \tag{6.5}$$

Set $c_1 = cc_0$ where c_0 is given by (2.15). Fix $s \in (B_n^*)^{-1}B(r_0^{-1})\setminus B(c_1)$ and let θ, $\sigma \in S^{d-1}$ be such that $s = |s|\sigma$ and $B_n^*s = |B_n^*s|\theta$. Then by (3.6)

$$(6.6)\qquad \begin{aligned} |\varphi^n(B_n^*s)| &= |\varphi^n(\theta, |B_n^*s|)| \\ &\le \exp\left(-c|B_n^*s|^p a_n(\theta)^p\right) \end{aligned}$$

provided $a_n^{-1}(\theta) \le |B_n^*s| \le r_0^{-1}$. The upper bound is immediate since $B_n^*s \in B(r_0^{-1})$. By Lemma 6.1

$$|B_n^*s| = |s|\,|B_n^*\sigma| = |s|\,|D_n^{-1}\theta|^{-1} = |s|/\bar{b}_n(\theta)$$

and by (6.5) and (2.15)

$$\bar{b}_n(\theta) \le c|A_n^{-1}\theta| \le cc_0 a_n(\theta).$$

Thus

$$(6.7)\qquad |B_n^*s|a_n(\theta) \ge c_1^{-1}|s|,$$

which proves the lower bound for $s \notin B(c_1)$. Finally (6.4) follows from (6.6) and (6.7).

Let ψ be the characteristic function of Y. Then by (6.4), for $|s| \ge c_1$

$$(6.8)\qquad |\psi(s)| \le \exp(-c_2|s|^p).$$

Thus in particular Y has a C^∞ density, call it g.

LEMMA 6.3. *Set*

$$I_n(x, \lambda, a) = (2\pi)^{-d}\int_{R(\pi, a)} \exp(-i\langle t, x\rangle)\varphi^n(t)k(\lambda t)h(a^{-1}t)\,dt.$$

Then for each $a > 0$, $\lambda > 0$

$$(6.9)\qquad \lim_{n\to\infty}\ \sup_{x\in\mathbb{R}^d} |I_n(x, \lambda, a)/|\det B_n| - g(B_n(x - \gamma_n))| = 0.$$

PROOF. First observe that by (2.7), (2.15), (4.3), and (4.6), for any $q \in (0,1)$, $q^n|\det B_n|^{-1} \to 0$ as $n \to \infty$. Thus proceeding as in Lemma 5.2 we see that

$$\left|\int_{R(\pi, a)\setminus B(r_0^{-1})} \exp(-i\langle t, x\rangle)\varphi^n(t)k(\lambda t)h(a^{-1}t)\,dt\right| = o(|\det B_n|)$$

uniformly in $x \in \mathbb{R}^d$. Hence by a change of variable

$$\begin{aligned} &|I_n(x, \lambda, a)/|\det B_n| - g(B_n(x - \gamma_n))| \\ &\quad\le \left|(2\pi)^{-d}\int_{(B_n^*)^{-1}B(r_0^{-1})} \exp(-i\langle s, B_n(x - \gamma_n)\rangle)\right. \\ &\qquad\quad \left.\cdot\left[\varphi^n(B_n^*s)\exp(-i\langle s, B_n\gamma_n\rangle)k(\lambda B_n^*s)h\left(a^{-1}B_n^*s\right) - \psi(s)\right] ds\right| \\ &\qquad + \left|(2\pi)^{-d}\int_{\mathbb{R}^d\setminus(B_n^*)^{-1}B(r_0^{-1})} \exp(-i\langle s, B_n(x - \gamma_n)\rangle)\psi(s)\,ds\right| + o(1). \end{aligned}$$

By Lemma 2.2(i), (4.3), and (4.6), $(B_n^*)^{-1}B(r_0^{-1}) \to \mathbb{R}^d$ as $n \to \infty$; thus by (6.4), (6.8), and dominated convergence (6.9) holds.

Theorem 6.4. *Assume that X is normalized and $B_n(S_n - \gamma_n) \to Y$ where Y is a full operator stable law with density g. Then for each $\lambda > 0$*

$$(6.10)\quad P\{S_n \in \bar{C}(x, \lambda)\} = (2\lambda)^{d-k}|\det B_n| g(B_n(x - \gamma_n)) + o(|\det B_n|)$$

uniformly in $x \in D_n$, i.e.,

$$(6.11)\quad \lim_{n\to\infty} \sup_{x\in D_n} \Big| P\{S_n \in \bar{C}(x, \lambda)\} - (2\lambda)^{d-k}|\det B_n| g(B_n(x - \gamma_n)) \Big| |\det B_n|^{-1} = 0.$$

Proof. Fix $\lambda > 0$. Given $\varepsilon > 0$, choose a large enough that

$$\int_{C^{d-k}(0,\varepsilon)} a^{d-k} H(a\bar{y})\, d\bar{y} = 1 - \varepsilon.$$

Then using (5.22) and (6.9) and proceeding as in the proof of Theorem 5.3 we obtain uniformly in $x \in D_n$

$$(2(\lambda + \varepsilon))^{k-d} P\{S_n \in \bar{C}(x, \lambda)\}(1 - \varepsilon) \le g(B_n(x - \gamma_n))|\det B_n| + o(|\det B_n|).$$

Since g is a bounded function, we can rewrite this as

$$(6.12)\quad \begin{aligned} &P\{S_n \in \bar{C}(x, \lambda)\} \\ &\quad \le (2\lambda)^{d-k}|\det B_n| g(B_n(x - \gamma_n)) + \delta_1(\varepsilon)|\det B_n| + o(|\det B_n|) \end{aligned}$$

uniformly in $x \in D_n$, where $\delta_1(\varepsilon)$ is independent of x and n and $|\delta_1(\epsilon)| \to 0$, $\varepsilon \to 0$. Using (6.12) together with (5.23) and (6.9) we obtain as before, uniformly in $x \in D_n$

$$\begin{aligned} &g(B_n(x - \gamma_n))|\det B_n| + o(|\det B_n|) \\ &\quad \le (2(\lambda - \varepsilon))^{k-d} P\{S_n \in \bar{C}(x, \lambda)\} + (2(\lambda - \varepsilon))^{k-d}\varepsilon \\ &\qquad \cdot \left[(2\lambda)^{d-k}|\det B_n|\beta + \delta_1(\varepsilon)|\det B_n| + o(|\det B_n|)\right], \end{aligned}$$

where β is an upper bound for g. We can rewrite this as

$$(6.13)\quad \begin{aligned} &P\{S_n \in \bar{C}(x, \lambda)\} \\ &\quad \ge (2\lambda)^{d-k}|\det B_n| g(B_n(x - \gamma_n)) + \delta_2(\varepsilon)|\det B_n| + o(|\det B_n|) \end{aligned}$$

uniformly in $x \in D_n$, where $\delta_2(\varepsilon)$ is independent of x and n and $|\delta_2(\varepsilon)| \to 0$ as $\varepsilon \to 0$. (6.11) now follows from (6.12) and (6.13).

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