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STRONG LAWS FOR QUANTILES CORRESPONDING TO MOVING BLOCKS OF RANDOM VARIABLES

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Let $U_1, U_2, \dots$ be a sequence of independent uniform $(0,1)$ random variables, and for $1 \le k \le n$ let $\xi_p(n, k)$ denote the pth quantile, $0 < p < 1$, corresponding to the block $U_{n-k+1}, \dots, U_n$. In this paper we investigate the a.s. limiting behavior of $\xi_p(n, a_n)$ when a_n is an integer sequence, $1 \le a_n \le n$, and $\lim_{n\to\infty} a_n/\log n = \beta \in [0, \infty]$. In addition, we investigate the a.s. limiting behavior of $\max_{a_n \le k \le n} \xi_p(n, k)$ and other maxima involving the $\xi_p(n, k)$'s.

1. Introduction. Let $U_1, U_2, \dots$ be a sequence of independent random variables, uniformly distributed on the unit interval, and for $1 \le i \le k \le n$ let $U_{nk}(i)$ denote the ith order statistic among $U_{n-k+1}, U_{n-k+2}, \dots, U_n$ (the block with right endpoint n and length k). For fixed $0 < p < 1$ and $0 \le \pi \le 1$ define the corresponding pth quantile

$$(1.1)\quad \xi_p(n, k) = \begin{cases} \pi U_{nk}(pk) + (1-\pi) U_{nk}(pk+1), & \text{if } pk = [pk], \\ U_{nk}([pk]+1), & \text{if } pk \ne [pk], \end{cases}$$

where $[y]$ denotes the integer part of y. If a_n is an integer sequence satisfying $1 \le a_n \le n$ for $n \ge 1$, then

$$(1.2)\qquad \lim_{n\to\infty} \xi_p(n, a_n) = p \quad \text{a.s.}$$

when $a_n \equiv n$, and $\xi_p(n, a_n) = U_n$ diverges ("oscillates" over $[0,1]$) almost surely when $a_n \equiv 1$. The condition "$a_n \to \infty$" is necessary for (1.2) to hold, but it is not sufficient. How fast must a_n grow for (1.2) to hold? When (1.2) holds, what is the rate of convergence? When (1.2) fails, how does $\xi_p(n, a_n)$ behave? In this paper we answer these questions for the case where $\lim_{n\to\infty} a_n/\log n = \beta$, $0 \le \beta \le \infty$. In addition, we investigate the almost sure limiting behavior of $\max_{a_n \le k \le n} \xi_p(n, k)$ and other maxima involving the $\xi_p(n, k)$'s.

2. Results. In Theorem 2.1 we assume $\beta = \infty$. We associate with each difference, $\xi_p(n, k) - p$, the normalizing constant

$$\phi(n, k) = k^{1/2}\left\{2p(1-p)\left(\log\frac{n}{k} + \log\log k\right)\right\}^{-1/2}, \qquad 3 \le k \le n,$$

and consider $\alpha_{n,0} = (\xi_p(n, a_n) - p)\phi(n, a_n)$, the maxima

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$$\alpha_{n,1} = \max_{a_n \le k \le n} \big(\xi_p(n,k) - p\big)\phi(n,k),$$

$$\alpha_{n,2} = \max_{0 \le k \le n-a_n} \big(\xi_p(k+a_n, a_n) - p\big)\phi(k+a_n, a_n),$$

$$\alpha_{n,3} = \max_{\substack{0 \le j < k \le n \\ a_n \le k-j}} \big(\xi_p(k, k-j) - p\big)\phi(k, k-j),$$

and $\alpha_{n,ia}$, $i = 1,2,3$, defined as $\alpha_{n,i}$ with $|\xi_p - p|$ replacing $(\xi_p - p)$. In Theorem 2.2 we assume $0 \le \beta < \infty$. We consider $\gamma_{n,0} = \xi_p(n, a_n)$ and the maxima

$$\gamma_{n,1} = \max_{a_n \le k \le n} \xi_p(n,k),$$

$$\gamma_{n,2} = \max_{0 \le k \le n-a_n} \xi_p(k+a_n, a_n)$$

and

$$\gamma_{n,3} = \max_{\substack{0 \le j < k \le n \\ a_n \le k-j}} \xi_p(k, k-j).$$

Figure 1 indicates the relationship among the various maxima we have in mind.

THEOREM 2.1. *Suppose* $1 \le a_n \le n$, $a_n/\log n \to \infty$ *as* $n \to \infty$, $r = \liminf_{n\to\infty}(\log(n/a_n)/\log\log n)$ *and that* $r/(r+1) = 1$ *for* $r = \infty$. *Then, with probability one*, $\liminf_{n\to\infty}\alpha_n$ *and* $\limsup_{n\to\infty}\alpha_n$ *are as indicated in the following table*:

	$\liminf_{n\to\infty}$		$\limsup_{n\to\infty}$	
$\alpha_{n,0}$	-1		1	(a)
$\alpha_{n,1}$	$= F(a)$, $\in [-1, F(a)]$,	*if* $a_n/n \to a$ *if* $\liminf_{n\to\infty} a_n/n = a$	1	(b)
$\alpha_{n,1a}$	0		1	(c)
$\alpha_{n,2}$ *and* $\alpha_{n,3}$	$= \left(\dfrac{r}{r+1}\right)^{1/2}$, $\in [-1, 0]$, $= G(a)$,	*if* $0 < r \le \infty$ *if* $r = 0$ *if* $r = 0$ *and* $a_n/n \to a$	1	(d)
$\alpha_{n,2a}$ *and* $\alpha_{n,3a}$	$\left(\dfrac{r}{r+1}\right)^{1/2}$		1	(e)
	(1)		(2)	

where

$$F(a) = \begin{cases} 0, & a = 0, \\ -\left(1 - \dfrac{\log a}{4}\right)^{-1/2}, & 0 < a \le 1, \end{cases}$$

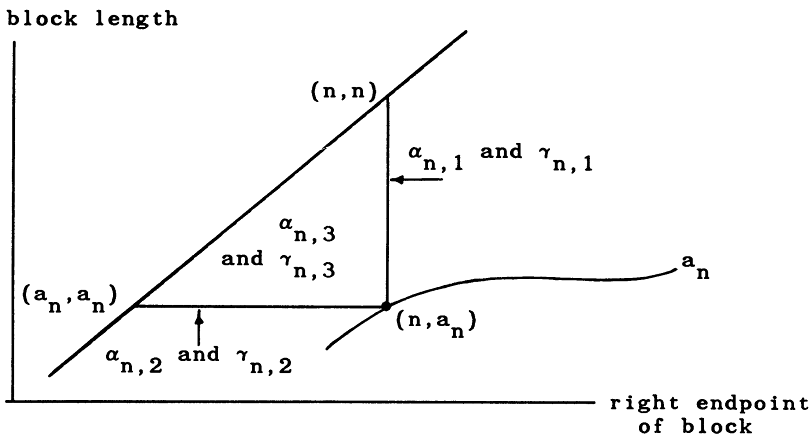

FIG. 1.

and

$$G(a)=\begin{cases}0, & a=0,\\ -\left(\dfrac{(2n+1)a-1}{n(n+1)a}\right)^{1/2}, & \dfrac{1}{n+1}<a\le\dfrac{1}{n},\ n=1,2,\dots.\end{cases}$$

THEOREM 2.2. *Suppose* $1 \le a_n \le n$ *and* $a_n/\log n \to \beta \in [0,\infty)$, *as* $n \to \infty$. *Then, with probability one,* $\liminf_{n\to\infty}\gamma_n$ *and* $\limsup_{n\to\infty}\gamma_n$ *are as indicated in the following table*:

	$\beta=0$		$0<\beta<\infty$		
	$\liminf_{n\to\infty}$	$\limsup_{n\to\infty}$	$\liminf_{n\to\infty}$	$\limsup_{n\to\infty}$	
$\gamma_{n,0}$	0	1	$1-\psi_{1-p}^{-1}(\beta)$	$\psi_p^{-1}(\beta)$	(f)
$\gamma_{n,1}$	p	1	p	$\psi_p^{-1}(\beta)$	(g)
$\gamma_{n,2}$ *and* $\gamma_{n,3}$	1	1	$\psi_p^{-1}(\beta)$	$\psi_p^{-1}(\beta)$	(h)
	(1)	(2)	(3)	(4)	

where ψ_p^{-1} *denotes the inverse of the function*

$$\psi_p(\lambda)=-\left(\log\left\{\left(\frac{\lambda}{p}\right)^{p}\left(\frac{1-\lambda}{1-p}\right)^{1-p}\right\}\right)^{-1},\qquad p<\lambda<1.$$

REMARKS. When $\lim_{n\to\infty}a_n/\log n=\beta$, statements (a) and (f) say that "$\beta=\infty$" is necessary and sufficient for (1.2) to hold. Convergence rates in this case are given by (a). The oscillatory behavior of $\xi_p(n,a_n)$ (when $0\le\beta<\infty$) is indicated by (f). Note that $\psi_p^{-1}(\beta)\to p$ as $\beta\to\infty$ and that $\psi_p^{-1}(\beta)\to 1$ as $\beta\to 0$. When $a_n=[\beta\log n]$, $0<\beta<\infty$, (3h) and (4h) applied to $\gamma_{n,2}$ give Book and Truax's

"Erdös–Rényi law for sample quantiles" in [3]. In (1b) and (1d), lim inf is not given when a_n/n diverges. However, it can be shown by example that "lim inf" is not determined by $\liminf a_n/n = b_1$ and $\limsup a_n/n = b_2$ when $b_1 \neq b_2$, without additional assumptions on a_n.

3. Proofs. Throughout this section $\#[A]$ denotes the number of elements in the set A, and C denotes various positive constants whose exact values do not matter so that, for example, $1 + C = C$ might appear in this notation. For notational convenience we define for $3 \le k \le n$, $\phi^{-1}(n, k) = (\phi(n, k))^{-1}$ and $d(n, k) = \{2k(\log(n/k) + \log\log k)\}^{1/2}$, and for $0 \le u < v \le 1$, $B_{nk}(u, v) = \#[i\colon n - k + 1 \le i \le n,\ u \le U_i < v]$.

LEMMA 3.1. *If* $1 \le a_n \le n$ *and* $a_n/\log n \to \infty$ *as* $n \to \infty$, *then*

$$P\Big\{\limsup_{n\to\infty} \max_{a_n \le k \le n} |B_{nk}(0, p) - kp| k^{-1}\phi(n, k) = 1\Big\} = 1.$$

PROOF. For $n \ge 1$ define

$$Y_n = \frac{p - B_{n1}(0, p)}{(p(1-p))^{1/2}} \quad \text{and} \quad S_n = Y_1 + \cdots + Y_n. \tag{3.1}$$

Since $|B_{nk}(0, p) - kp|k^{-1}\phi(n, k) = |S_n - S_{n-k}|/d(n, k)$ for $3 \le k \le n$, the above lemma follows from Theorem 5.1 of [5]. □

For $\lambda > 0$ and $3 \le k \le n$ define the events

$$\Lambda_{nk\lambda} = \Big\{\big|B_{nk}\big(p, p + \lambda\phi^{-1}(n, k)\big) - k\lambda\phi^{-1}(n, k)\big| > 4\big(k\lambda\phi^{-1}(n, k)\log n\big)^{1/2}\Big\},$$

$$\Lambda'_{nk\lambda} = \Big\{\big|B_{nk}\big(p - \lambda\phi^{-1}(n, k), p\big) - k\lambda\phi^{-1}(n, k)\big| > 4\big(k\lambda\phi^{-1}(n, k)\log n\big)^{1/2}\Big\}$$

and

$$E_\lambda = \Bigg(\limsup_{n\to\infty} \bigcup_{k=a_n}^{n} \Lambda_{nk\lambda}\Bigg) \cup \Bigg(\limsup_{n\to\infty} \bigcup_{k=a_n}^{n} \Lambda'_{nk\lambda}\Bigg) = E_{\lambda 1} \cup E_{\lambda 2}.$$

LEMMA 3.2. *Suppose that* $\lambda > 0$. *If* $1 \le a_n \le n$ *and* $a_n/\log n \to \infty$ *as* $n \to \infty$, *then* $P(E_\lambda) = 0$.

PROOF. We will prove that $P(E_{\lambda 1}) = 0$. The proof that $P(E_{\lambda 2}) = 0$ is similar and is omitted. If $3 \le k \le n$, then by Bernstein's inequality (see [9])

$$\begin{aligned} P(\Lambda_{nk\lambda}) &\le 2\exp\left\{\frac{-16k\lambda\phi^{-1}(n, k)\log n}{2k\lambda\phi^{-1}(n, k) + \frac{8}{3}\big(k\lambda\phi^{-1}(n, k)\log n\big)^{1/2}}\right\} \\ &\le 2\exp\left\{-\frac{1}{2}\min\Big(8\log n, 6\big(k\lambda\phi^{-1}(n, k)\log n\big)^{1/2}\Big)\right\}. \end{aligned} \tag{3.2}$$

Since $a_n/\log n \to \infty$, we have $(k\lambda\phi^{-1}(n, k)\log n)^{1/2}(\log n)^{-1} \to \infty$ uniformly in

k for $a_n \le k \le n$. Thus from (3.2), $P(\Lambda_{nk\lambda}) \le 2n^{-4}$ for $a_n \le k \le n$ and n sufficiently large so that

$$\sum_{n=3}^{\infty} P\left(\bigcup_{k=a_n}^{n} \Lambda_{nk\lambda}\right) \le C + \sum_{n=3}^{\infty} (n - a_n)n^{-4} < \infty. \qquad \square$$

LEMMA 3.3. *If* $1 \le a_n \le n$ *and* $a_n/\log n \to \infty$ *as* $n \to \infty$, *then*

$$\xi_p(n, k) = 2p - k^{-1}B_{nk}(0, p) + e_{nk}, \qquad 1 \le k \le n, \tag{3.3}$$

where as $n \to \infty$, $\max_{a_n \le k \le n} |e_{nk}| \phi(n, k) = o(1)$ *a.s.*

PROOF. Fix $0 < \delta < 1$, an integer $M > 2/\delta$, and $\omega \in D = \bigcap_{j=1}^{M+1}(E_{j\delta}^c \cap A)$, where A is the event in Lemma 3.1. By Lemmas 3.1 and 3.2, $P(D) = 1$. Since $k\phi^{-1}(n, k)/(k\phi^{-1}(n, k)\log n)^{1/2} \to \infty$ as $n \to \infty$, uniformly in k for $a_n \le k \le n$, we have for all large n, $n > n_0$, and $a_n \le k \le n$,

$$|B_{nk}(0, p) - kp| \le 2k\phi^{-1}(n, k), \tag{3.4}$$

$$\omega \notin \bigcup_{j=1}^{M+1} \left(\Lambda_{n,k,j\delta} \cup \Lambda'_{n,k,j\delta}\right) \tag{3.5}$$

and

$$\delta k\phi^{-1}(n, k) > 4\big((M+1)\delta k\phi^{-1}(n, k)\log n\big)^{1/2}. \tag{3.6}$$

Fix $n > n_0$ and suppose that $a_n \le k \le n$. By (3.4) there exists $1 \le j_{nk} = j \le M$ such that either

$$kp - j\delta k\phi^{-1}(n, k) \le B_{nk}(0, p) < kp - (j-1)\delta k\phi^{-1}(n, k) \tag{3.7}$$

or

$$kp + (j-1)\delta k\phi^{-1}(n, k) \le B_{nk}(0, p) < kp + j\delta k\phi^{-1}(n, k). \tag{3.7a}$$

Suppose that (3.7) holds [the proof is the same if (3.7a) holds]. Then, by (3.5) and (3.6)

$$\begin{aligned}
&B_{nk}(0, p + (j+1)\delta\phi^{-1}(n, k)) \\
&\quad = B_{nk}(0, p) + B_{nk}(p, p + (j+1)\delta\phi^{-1}(n, k)) \\
&\quad \ge kp - j\delta k\phi^{-1}(n, k) + (j+1)\delta k\phi^{-1}(n, k) \\
&\qquad - 4\big((j+1)\delta k\phi^{-1}(n, k)\log n\big)^{1/2} \\
&\quad \ge kp + \delta k\phi^{-1}(n, k) - 4\big((M+1)\delta k\phi^{-1}(n, k)\log n\big)^{1/2} > kp,
\end{aligned}$$

so that $\xi_p(n, k) \le p + (j+1)\delta\phi^{-1}(n, k)$. By (3.3) and (3.7)

$$\begin{aligned}
e_{nk} &= \xi_p(n, k) - 2p + k^{-1}B_{nk}(0, p) \\
&\le (j+1)\delta\phi^{-1}(n, k) - p + k^{-1}\big(kp - (j-1)\delta k\phi^{-1}(n, k)\big) \\
&= 2\delta\phi^{-1}(n, k).
\end{aligned}$$

From a similar calculation, $e_{nk} \geq -2\delta\phi^{-1}(n, k)$. Since δ, k and ω are arbitrary, the lemma follows. □

PROOF OF (1a) AND (2a). With S_n defined as in (3.1), Lemma 3.3 says that the collection of limit points, L, of $\alpha_{n,0}$ is almost surely that of $(S_n - S_{n-a_n})/d(n, a_n)$. By Theorem 1 of [8] we can define a standard Wiener process $\{W(T), T \geq 0\}$ and the sequence $Y_1, Y_2, \ldots$ on a new probability space (say Ω') so as to have $S_n = W(n) + O(\log n)$ a.s. Since $a_n/\log n \to \infty$ we get $(\log n)/d(n, a_n) \to 0$, so that the collection of limit points of $\alpha_{n,0}$ is almost surely that of $(W(n) - W(n - a_n))/d(n, a_n)$. By Corollary 2.3 of [6], $P\{L = [-1, 1]\} = 1$. □

PROOF OF (2b)–(2e). With S_n defined as in (3.1) it follows by Lemma 3.3 and Theorem 5.3 of [5] that

$$\limsup_{n\to\infty} \alpha_{n,3a} = \limsup_{n\to\infty} \max_{\substack{0\leq j<k\leq n\\ a_n\leq k-j}} \frac{|S_k - S_j|}{d(k, k-j)} = 1 \quad \text{a.s.}$$

Thus, statements (2b)–(2e) follow from (2a) since $\alpha_{n,0} \leq \alpha_{n,i} \leq \alpha_{n,3a}$ for $i = 1, 1a, 2, 2a, 3$. □

PROOF OF (1b)–(1e). Again, we go to Ω'. By Lemma 3.3 and Theorem 1 of [8] we have, as in the proof of (1a) and (2a),

$$\liminf_{n\to\infty} \alpha_{n,2} = \liminf_{n\to\infty} \max_{0\leq k\leq n-a_n} \frac{W(k + a_n) - W(k)}{d(k + a_n, a_n)} \quad \text{a.s.}$$

By Theorem 2 of [7], to prove our result for $\liminf_{n\to\infty}\alpha_{n,2}$ it suffices to prove that

$$\begin{aligned}(3.8)\qquad &\liminf_{n\to\infty} \max_{0\leq k\leq n-a_n} \frac{W(k + a_n) - W(k)}{d(k + a_n, a_n)}\\ &\quad= \liminf_{T\to\infty} \max_{0\leq t\leq T-a_T} \frac{W(t + a_T) - W(t)}{d(t + a_T, a_T)} \quad \text{a.s.,}\end{aligned}$$

where $a_T = a_{[T]}$ for $T \geq 1$ and $a_T = T$ for $0 < T < 1$. For $T > 1$ and $0 \leq t \leq T - a_T$,

$$\begin{aligned}&\frac{W([t] + a_{[T]}) - W([t])}{d([t] + a_{[T]}, a_{[T]})}\\ &\quad= \frac{W(t + a_T) - W(t)}{d(t + a_T, a_T)} \frac{d(t + a_T, a_T)}{d([t] + a_{[T]}, a_{[T]})}\\ &\qquad - \frac{W(t + a_T) - W([t] + a_{[T]})}{d([t] + a_{[T]}, a_{[T]})} + \frac{W(t) - W([t])}{d([t] + a_{[T]}, a_{[T]})}\\ &\quad= \left(\frac{W(t + a_T) - W(t)}{d(t + a_T, a_T)}\right) A(t, T) + B(t, T) + C(t, T).\end{aligned}$$

By Theorem of 3.3B of [5], if $M_T \to \infty$, then

$$\lim_{T\to\infty} \max_{\substack{0\le s<t\le T\\ |t-s|\le 1}} \frac{|W(t)-W(s)|}{(M_T \log T)^{1/2}} = 0 \quad \text{a.s.} \tag{3.9}$$

Now, $A(t, T) \to 1$ uniformly in t, $0 \le t \le T - a_T$, as $T \to \infty$. By (3.9), with probability one, $B(t, T) \to 0$ and $C(t, T) \to 0$ uniformly in t, $0 \le t \le T - a_T$, as $T \to \infty$. This proves (3.8). A similar proof (which also uses Theorems 1 and 3 of [7]) yields the remaining lim inf statements of Theorem 2.1. □

PROOF OF (4g) AND (4h). The following relationship is evident from Figure 1 or Lemma 2.1 of [5]:

$$\limsup_{n\to\infty} \gamma_{n,2} \le \limsup_{n\to\infty} \gamma_{n,3} = \limsup_{n\to\infty} \gamma_{n,1}. \tag{3.10}$$

(Note: To use Lemma 2.1, one needs to replace the original sequence, a_n, by a nondecreasing sequence.) Define for $p < \lambda < 1$,

$$h(\lambda) = (\lambda/p)^p((1-\lambda)/(1-p))^{1-p}.$$

By Theorem 1 of [1] (see also Lemma 1 of [2]) there exist constants $0 < C_{1\lambda} < C_{2\lambda} < \infty$ such that for all large k and $n \ge k$,

$$C_{1\lambda} k^{-1/2} h(\lambda)^k \le P\{\xi_p(n,k) > \lambda\} \le C_{2\lambda} k^{-1/2} h(\lambda)^k. \tag{3.11}$$

Fix $\psi_p^{-1}(\beta) < \lambda < 1$. By (3.11), for n sufficiently large

$$\begin{aligned} P(G_n) &= P\left(\bigcup_{k=a_n}^{n} \{\xi_p(n,k) > \lambda\}\right) \le C \sum_{k=a_n}^{n} h(\lambda)^k \\ &= C \sum_{k=a_n}^{n} \exp(-k/\psi_p(\lambda)) \le C \int_{a_N-1}^{n} \exp(-x/\psi_p(\lambda))\,dx \\ &\le C \exp(-a_n/\psi_p(\lambda)) \le C n^{-((a_n/\psi_p(\lambda))/\log n)}. \end{aligned} \tag{3.12}$$

Since $0 < \psi_p(\lambda) < \beta$ and $a_n/\log n \to \beta$, it follows from (3.12) that $\sum_{n=1}^{\infty} P(G_n) < \infty$. Since λ is arbitrary we get $\limsup_{n\to\infty} \gamma_{n,1} \le \psi_p^{-1}(\beta)$ almost surely. Thus, (4g) and (4h) follow from (3.10), (3.16) in the proof of (4f), and the observation that $\gamma_{n,i} \ge \gamma_{n,0}$ for $1 \le i \le 3$. □

PROOF OF (3h). Fix $p < \lambda < \psi_p^{-1}(\beta)$. By (3.11), for n sufficiently large

$$\begin{aligned} P(J_n) &= P\left(\bigcap_{j=1}^{[n/a_n]} \{\xi_p(ja_n, a_n) < \lambda\}\right) \\ &\le \prod_{j=1}^{[n/a_n]} \left(1 - C a_n^{-1/2} h(\lambda)^{a_n}\right) \\ &\le \exp\left(-C a_n^{-1/2} h(\lambda)^{a_n} [n/a_n]\right) \\ &\le \exp\left(-C a_n^{-3/2} n^{1-((a_n/\psi_p(\lambda))/\log n)}\right). \end{aligned}$$

Since $\psi_p(\lambda) > \beta$ and $a_n/\log n \to \beta$, $\sum_{n=1}^{\infty} P(J_n) < \infty$. Since λ is arbitrary, (3h) follows from (4h). □

PROOF OF (1g) AND (3g). We will prove that

$$\text{if} \quad 1 \le a_n \le n, \quad \text{then} \quad \liminf_{n\to\infty} \gamma_{n,1} = p \quad \text{a.s.} \tag{3.13}$$

(Note: The assumption that $\lim a_n/\log n$ exists is not needed in this result.) Fix $0 < \delta < 1 < B$ so that $p^{2[\delta \log n]} \ge n^{-1}$, all $n \ge 1$, and $\psi_p(p+\delta) < B$. By (4g), $\limsup_{n\to\infty} \max_{[B \log n] \le k \le n} \xi_p(n,k) \le p + \delta$ almost surely, so that

$$\liminf_{n\to\infty} \gamma_{n,1} \le \max\left\{ \liminf_{n\to\infty} \max_{1 \le k \le [B \log n]} \xi_p(n,k),\, p+\delta \right\} \quad \text{a.s.} \tag{3.14}$$

Let $n_0 \ge 1$ be such that $(n_0 - 1)^2 < n_0^2 - [B \log n_0^2]$, and for $n \ge n_0$ define $A_n = \{\max_{1 \le k \le [B \log n^2]} \xi_p(n^2, k) \le p + \delta\}$. We will show that

$$P\left(\limsup_{n\to\infty} A_n\right) = 1.$$

Let $k_n' = [B \log n^2]/[\delta \log n]$ and suppose $n \ge n_0$. The set

$$\{n^2 - [B \log n^2] + 1, \dots, n^2\}$$

can be partitioned into $k_n = [k_n']$ groups,

$$g_j = \{h_n + (k_n - j)[\delta \log n] + 1, \dots, h_n + (k_n - j + 1)[\delta \log n]\},$$

for $2 \le j \le k_n$ and $g_1 = \{h_n + (k_n - 1)[\delta \log n] + 1, \dots, n^2\}$, where $h_n = n^2 - [B \log n^2]$. Note that $\#[g_j] = [\delta \log n]$ for $2 \le j \le k_n$ and that $[\delta \log n] \le \#[g_1] < 2[\delta \log n]$. Define $\nabla_j = \{\#[i\colon i \in g_j,\, U_i \le p + \delta] \ge p[\delta \log n]\}$ for $2 \le j \le k_n$. Note that $P(\nabla_j) \ge 1/2$ and $k_n \le 3B/\delta$ for n sufficiently large, so by choice of δ and B,

$$\begin{aligned} P(A_n^*) &= P\left(\{U_i \le p + \delta \text{ all } i \in g_1\} \cap \left(\bigcap_{j=2}^{k_n} \nabla_j \right) \right) \\ &\ge p^{2[\delta \log n]} (1/2)^{k_n} \ge n^{-1} (1/2)^{3B/\delta}, \end{aligned} \tag{3.15}$$

for n sufficiently large. For $n \ge n_0$, $A_n \supset A_n^*$ so by (3.15), $\sum_{n=n_0}^{\infty} P(A_n) = \infty$. Since $A_{n_0}, A_{n_0+1}, \dots$ are independent, $P(\limsup_{n\to\infty} A_n) = 1$ as asserted. Now, since δ was arbitrary, (3.13) follows from (3.14), and the observation that $\gamma_{n,1} \ge \xi_p(n,n) \to p$ almost surely. □

PROOF OF (3f) AND (4f). Let K denote the collection of limit points of the sequence $\gamma_{n,0}$. We first prove that

$$P\{K \supset [p, \psi_p^{-1}(\beta)]\} = 1. \tag{3.16}$$

Suppose that $p < \lambda_1 < \lambda_2 < \psi_p^{-1}(\beta)$. Fix $\varepsilon > 0$, and for $n \ge 1$ let $y(n) = [n^{1+\varepsilon}]$. For some $n_0 \ge 1$, $\xi_p(y(n_0), a_{y(n_0)}), \xi_p(y(n_0+1), a_{y(n_0+1)}), \dots$ are independent random variables. The function h defined in the proof of (4g) and (4h) is decreasing, so by (3.11), for n sufficiently large,

$$P\left\{\lambda_1 < \xi_p\left(y(n), a_{y(n)}\right) \le \lambda_2\right\} \ge C(\log n)^{-1/2}\left(h(\lambda_1)\right)^{a_{y(n)}}$$
$$= C(\log n)^{-1/2} n^{-(a_{y(n)}/a_n)((a_n/\psi_p(\lambda_1))/\log n)} \ge Cn^{-1},$$

for ε sufficiently small since $\psi_p(\lambda_1) > \beta$, $a_n/\log n \to \beta$ and $a_{y(n)}/a_n \to 1 + \varepsilon$. Thus, (3.16) holds. Now, consider the sequence $U_1' = 1 - U_1$, $U_2' = 1 - U_2, \dots$. Let $\xi_{1-p}'(n, a_n)$ denote a $(1-p)$th quantile [use (1.1) with "π" $= 1 - \pi$] of the subsequence $U_{n-a_n+1}', U_{n-a_n+2}', \dots, U_n'$ and let K' denote the collection of limit points of $\xi_{1-p}'(n, a_n)$. By applying (3.16) to the U_n' sequence we get $P\{K' \supset [1-p, \psi_{1-p}^{-1}(\beta)]\} = 1$, or equivalently

$$P\left\{K \supset \left[1 - \psi_{1-p}^{-1}(\beta), p\right]\right\} = 1. \tag{3.17}$$

By (3.16), (3.17) and an application of (4g) to the U_n and U_n' sequences we get $P\{K = [1 - \psi_{1-p}^{-1}(\beta), \psi_p^{-1}(\beta)]\} = 1$. □

Proof of (1f) and (2f). Define K as in the preceding proof. Suppose $0 \le a < b \le 1$. Define for $n \ge 1$, $M_n = \{a < U_i < b$, all $n^2 - a_{n^2} + 1 \le i \le n^2\}$. For n sufficiently large

$$P(M_n) = (b-a)^{a_{n^2}} = (b-a)^{(a_{n^2}/\log n^2)\log n^2} \ge n^{-1},$$

since $a_{n^2}/\log n^2 \to 0$. The events $M_{n_0}, M_{n_0+1}, \dots$ are independent for some $n_0 \ge 1$, so that $P\{K = [0,1]\} = 1$. □

Proof of (1h), (2g) and (2h). Statements (2g) and (2h) follow from (2f). To prove (1h) fix $0 < \delta < 1$ and define $H_n = \bigcup_{j=1}^{k_n}\{1 - \delta < U_i < 1$ all $(j-1)a_n < i \le ja_n\}$ for $n \ge 1$ where $k_n = [n/a_n]$. Since $a_n/\log n \to 0$, we have for n sufficiently large

$$P(H_n^c) = (1 - \delta^{a_n})^{k_n} \le \exp(-\delta^{a_n}k_n) \le \exp(-n^{1/2}),$$

so that $\sum_{n=1}^{\infty} P(H_n^c) < \infty$. Thus, $P(\liminf_{n\to\infty} H_n) = 1$ (note that this fact follows also from well-known results on the longest head run in Bernoulli trials). Now note that $\{\gamma_{n,3} \ge 1 - \delta\} \supset \{\gamma_{n,2} \ge 1 - \delta\} \supset H_n$ for $n \ge 1$, and that δ is arbitrary. □

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