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THE CUBE OF A NORMAL DISTRIBUTION IS INDETERMINATE

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It is established that if X is a stochastic variable with a normal distribution, then X^{2n+1} has an indeterminate distribution for $n \geq 1$. Furthermore, the distribution of $|X|^\alpha$ is determinate for $0 < \alpha \leq 4$ while indeterminate for $\alpha > 4$.

1. Introduction. Let $\mathscr{M}^*$ denote the set of probability measures on the real axis having moments of all orders. The kth moment of $\mu \in \mathscr{M}^*$ is the number

$$s_k(\mu) = \int x^k \, d\mu(x), \qquad k = 0, 1, \dots .$$

Two distributions $\mu, \nu \in \mathscr{M}^*$ are called equivalent if $s_k(\mu) = s_k(\nu)$ for $k = 0, 1, 2, \dots$, and μ is called *determinate* (in the Hamburger sense), if the equivalence class $[\mu]$ containing μ is equal to $\{\mu\}$, and *indeterminate* otherwise.

It is well known that there exist indeterminate distributions. This observation goes back to Stieltjes (1894/1895), who proved that the distributions on $(0, \infty)$ with the densities

$$a \exp(-t^{1/4}) \quad \text{and} \quad b_k t^{k - \log t}$$

are indeterminate [see Stieltjes (1894), Sections 55 and 56]. Here $a, b_k > 0$ are normalization constants and $k \in \mathbb{Z}$. For $k = -1$ we get the density of a log-normal distribution. Heyde (1963) pointed out that distributions commonly used in statistics need not be determinate, and as an example he gave the log-normal distribution.

If X is a stochastic variable with a normal distribution, then $\exp(X)$ has a log-normal distribution. The purpose of this note is to point out that even simpler transformations of X may lead to indeterminate distributions, viz. X^3 has an indeterminate distribution. Murad Taqqu has observed that the Carleman condition fails for X^3, and this suggests that X^3 is not determinate, although it is well known that the Carleman condition is not necessary for determinacy.

More generally, we prove that X^{2n+1} has an indeterminate distribution for $n \geq 1$, and the distribution of $|X|^\alpha$ is determinate for $0 < \alpha \leq 4$ while indeterminate for $\alpha > 4$. We are thus in the strange situation that X^3 and X^5 are indeterminate, whereas $|X^3|$ and X^4 are determinate.

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2. Statements and proofs.

PROPOSITION 1. *If X has a normal distribution, then X^{2n+1} is indeterminate for $n \geq 1$.*

PROOF. We may assume that X has the density $(1/\sqrt{\pi})\exp(-x^2)$, and then X^3 has the density

$$d(x) = \frac{1}{3\sqrt{\pi}}|x|^{-2/3}\exp\left(-|x|^{2/3}\right).$$

The function

$$d(x)\left\{1 + r\left(\cos\left(\sqrt{3}|x|^{2/3}\right) - \sqrt{3}\sin\left(\sqrt{3}|x|^{2/3}\right)\right)\right\}, \qquad x \in \mathbb{R},$$

is easily seen to be ≥ 0 for $|r| \leq \frac{1}{2}$, and it is a probability density with the same moments as d for these r. To see this, it suffices to prove that

$$s_k = \int_{-\infty}^{\infty} x^k d(x)\left(\cos\left(\sqrt{3}|x|^{2/3}\right) - \sqrt{3}\sin\left(\sqrt{3}|x|^{2/3}\right)\right) dx = 0, \quad \text{for} \quad k = 0, 1, \ldots.$$

This is clear for k odd, and we find

$$\begin{aligned} s_{2k} &= \frac{2}{3\sqrt{\pi}}\int_0^{\infty} x^{2k-2/3}\exp\left(-x^{2/3}\right)\left(\cos\left(\sqrt{3}x^{2/3}\right) - \sqrt{3}\sin\left(\sqrt{3}x^{2/3}\right)\right) dx \\ &= \frac{1}{\sqrt{\pi}}\int_0^{\infty} x^{3k-1/2}e^{-x}\left(\cos\left(\sqrt{3}x\right) - \sqrt{3}\sin\left(\sqrt{3}x\right)\right) dx. \end{aligned}$$

Using

$$\int_0^{\infty} x^{c-1}e^{-xz}\, dx = z^{-c}\Gamma(c), \quad \text{for} \quad c > 0, \operatorname{Re} z > 0,$$

we get for $z = 1 + i\beta$,

$$(*) \qquad \int_0^{\infty} x^{c-1}e^{-x}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(\beta x)\, dx = \left(1 + \beta^2\right)^{-c/2}\Gamma(c)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}(c\arctan\beta).$$

Putting $c = 3k + \frac{1}{2}$, $\beta = \sqrt{3}$ we see that

$$s_{2k} = \left(1/\sqrt{\pi}\right)2^{-c}\Gamma(c)\left(\cos\left(c\frac{\pi}{3}\right) - \sqrt{3}\sin\left(c\frac{\pi}{3}\right)\right) = 0. \qquad \square$$

EXTENSION. The density of X^{2n+1} is

$$d_n(x) = \frac{1}{(2n+1)\sqrt{\pi}}|x|^{-2n/(2n+1)}\exp\left(-|x|^{2/(2n+1)}\right),$$

and in this case we consider

$$d_n(x)\{1 + r(\cos(\beta_n|x|^{2/(2n+1)}) - \gamma_n\sin(\beta_n|x|^{2/(2n+1)}))\},$$

with

$$\beta_n = \tan\frac{\pi}{2n+1}, \quad \gamma_n = \cot\frac{\pi}{2(2n+1)}.$$

PROPOSITION 2. *If X has a normal distribution and $\alpha > 0$, then $|X|^\alpha$ is determinate for $\alpha \le 4$ and indeterminate for $\alpha > 4$.*

PROOF. If X has the density $(1/\sqrt{\pi})\exp(-x^2)$, then $|X|^\alpha$ has the density

$$d_\alpha(x) = \frac{2}{\alpha\sqrt{\pi}} x^{1/\alpha - 1}\exp(-x^{2/\alpha}),$$

with respect to Lebesgue measure on $]0, \infty[$. For $\alpha > 4$ we consider

$$d_\alpha(x)\{1 + r(\cos(\beta_\alpha x^{2/\alpha}) - \gamma_\alpha\sin(\beta_\alpha x^{2/\alpha}))\},$$

where

$$\beta_\alpha = \tan\frac{2\pi}{\alpha}, \qquad \gamma_\alpha = \cot\frac{\pi}{\alpha},$$

and this is a nonnegative function on $]0, \infty[$ for $|r| \le \sin \pi/\alpha$ with the same moments as d_α for these r, since

$$s_k = \int_0^\infty x^k d_\alpha(x)(\cos(\beta_\alpha x^{2/\alpha}) - \gamma_\alpha\sin(\beta_\alpha x^{2/\alpha}))\,dx = 0, \quad \text{for} \quad k = 0, 1, \ldots.$$

This shows that $|X|^\alpha$ is indeterminate for $\alpha > 4$. Notice that $|X|^\alpha$ is indeterminate even in the Stieltjes sense, i.e., there exist different probabilities on $[0, \infty[$ with the same moments.

To see that d_α is determinate for $0 < \alpha \le 4$ we use

$$s_k(d_\alpha) = \int_0^\infty x^k d_\alpha(x)\,dx = \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{\alpha k + 1}{2}\right), \qquad k = 0, 1, \ldots,$$

and by Stirling's formula $s_k(d_\alpha)^{1/k} \sim ck^{\alpha/2}$ for $k \to \infty$, where c is a suitable constant. This shows that

$$\sum_1^\infty s_k(d_\alpha)^{-1/2k} = \infty,$$

for $\alpha \le 4$, so by a theorem of Carleman [see Shohat and Tamarkin (1943), page 20] d_α is determinate in the Stieltjes sense. That d_α is the only measure on the whole real line with the same moments as d_α is then a consequence of a result of Chihara (1968), page 481, stating: If μ is determinate in the Stieltjes sense and indeterminate in the Hamburger sense, then μ is a Nevanlinna extremal measure and in particular discrete. □

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