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TRAVELING WAVES IN INHOMOGENEOUS BRANCHING BROWNIAN MOTIONS. I

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Consider a branching Brownian motion for which the instantaneous branching rate of a particle at position x is given by $\beta(x)$. We assume that β is an integrable continuous function converging to 0 as $x \to \pm\infty$. Let $R(t)$ be the position of the rightmost descendant at the time t of a simple particle starting from position 0 at time 0. We show that there exists a constant $\lambda_0 > 0$ such that $R(t) - \sqrt{\lambda_0/2}\,t$ converges in distribution as $t \to \infty$ to a location mixture of the extreme value distribution $\exp(e^{-\sqrt{2\lambda_0 x}})$.

1. Introduction. A (homogeneous) branching Brownian motion is defined as follows. At time $t = 0$ a single particle begins a standard Brownian motion $X_1(t)$ starting at $X_1(0) = 0$. At a random time T, independent of the motion $X_1(t)$ and with $P\{T > t\} = e^{-t}$, $t \geq 0$, the particle produces a replicate particle, also located at $X_1(T)$. The two particles continue along independent Brownian paths, each subject to the same law of reproduction.

Let $R(t)$ denote the position of the rightmost particle at time t, and let $u(t, x) = P\{R(t) \leq x\}$. It is by now well known ([1] and [6]) that, as $t \to \infty$, $u(t, x)$ approaches a traveling wave with velocity $\sqrt{2}$, i.e., there exists a function $w(x)$ such that for every $x \in \mathbb{R}$,

$$u(t, m_t + x) \to w(x), \tag{1.1}$$

where m_t is the median of the distribution of $R(t)$, and

$$m_t/t \to \sqrt{2}.$$

The purpose of this paper is to prove a similar result for a related process, which we call the inhomogeneous branching Brownian motion (IBBM). This process evolves in the same manner as the homogeneous branching Brownian motion, except that the instantaneous rate of reproduction is no longer identically 1, but depends on the spatial position of the particle. The initial particle follows a Brownian path $X_1(t)$, starting at $X_1(0) = 0$, producing its first offspring at time T, where

$$P(T > t | X_1(s), s \geq 0) = \exp\left\{-\int_0^t \beta(X_1(s))\, ds\right\}$$

and $\beta(x) \geq 0$ is continuous and bounded. New particles follow Brownian paths independent of the old particles, and obey the same reproduction law.

Let $R(t)$ be the position of the rightmost particle at time t.

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THEOREM 1. *Assume that*

$$\beta(x) \to 0, \quad as\ |x| \to \infty \tag{1.2}$$

and

$$\int_{-\infty}^{\infty} \beta(x)\,dx < \infty. \tag{1.3}$$

There exist constants $\lambda_0 > 0$, $\gamma > 0$ *and a random variable* $Z > 0$ *such that for every* $x \in \mathbb{R}$,

$$\lim_{t\to\infty} P\left\{R(t) \le \sqrt{\lambda_0/2}\,t + x\right\} = E\exp\left\{-Z\gamma e^{-\sqrt{2\lambda_0}x}\right\}. \tag{1.4}$$

The constant λ_0 is the solution of an eigenvalue problem, and Z is the limit of a positive martingale. See Section 2. An explicit formula for γ is given in Section 4.

This result complements the main result of [3], which states that, under (1.2), $R(t)/t \to \sqrt{\lambda_0/2}$, a.s. It differs from the corresponding results (1.1) for homogeneous branching Brownian motion in that the behavior of the median m_t of the distribution of $R(t)$ is different: For homogeneous branching Brownian motion [1], as $t \to \infty$,

$$m_t = \sqrt{2}\,t - (3/2\sqrt{2})\log t + \text{constant} + o(1),$$

whereas, for IBBM satisfying (1.2) and (1.3),

$$m_t = \sqrt{\lambda_0/2}\,t + \text{constant} + o(1).$$

The proof of Theorem 1 is completely unlike that of (1.1). The proof of (1.1) follows from the fact that $u(t,x)$ solves the K-P-P/Fisher equation $u_t = \frac{1}{2}u_{xx} + u^2 - u$ [6]. No such proof seems possible for IBBM, because there seems to be no parabolic PDE for $P\{R(t) \le x\}$. Our methods involve stochastic comparisons rather than analytic comparisons.

The proof of Theorem 1 is carried out in Sections 4–5 under the additional hypothesis that the branching rate function $\beta(x)$ has compact support; the general case is discussed in Sections 6–7. An auxiliary process, the Poisson tidal wave, is introduced in Section 3. Section 2 gives some preliminary information about the growth of the IBBM.

We have also established that the traveling wave phenomenon holds for the IBBM whose branching rate function $\beta(x)$ satisfies

$$\beta(x) \ge b > 0, \qquad x \in \mathbb{R},$$

and

$$\beta(x) - b \text{ has compact support.}$$

This is discussed in [5].

2. Watanabe's theorem. For $J \subset \mathbb{R}$ and $t \ge 0$, let $N(t; J)$ denote the number of IBBM particles in J at time t. For bounded intervals J the asymptotic growth of $N(t; J)$ was described by Watanabe [7].

Consider the differential operator $g \mapsto \frac{1}{2}g'' + \beta g$. Since $\beta \geq 0$ and β satisfies (1.2) and (1.3), the differential operator has a largest positive eigenvalue λ_0 and a unique corresponding eigenfunction $\varphi_0(x)$ satisfying (cf. [4], Chapter XI)

$$\varphi_0(x) > 0, \qquad x \in \mathbb{R},$$

$$\varphi_0(0) = 1, \qquad \int \varphi_0(x)^2\, dx < \infty$$

and

$$\varphi_0(x) \sim C_{\pm} e^{-\sqrt{2\lambda_0}|x|}, \text{ as } x \to \pm\infty.$$

Define $Z_t = e^{-\lambda_0 t}\int_{\mathbb{R}}\varphi_0(x)N(t;\, dx)$; then $Z_t > 0$ and Z_t is a martingale. Define

$$Z = \lim_{t \to \infty} Z_t.$$

It is easily established that $Z > 0$ with probability 1. For Borel measurable $J \subset \mathbb{R}$, define

$$\nu(J) = \int_J \varphi_0(x)\, dx.$$

WATANABE'S THEOREM. *For every bounded interval* $J \subset \mathbb{R}$,

$$\lim_{t \to \infty} N(t;\, J)/e^{\lambda_0 t} = Z\nu(J), \quad a.s.$$

For every nonnegative, continuous function $f(x)$ *with compact support*

$$\lim_{t \to \infty} \int f(x) N(t;\, dx)/e^{\lambda_0 t} = Z \int f\, d\nu, \quad a.s.$$

3. Poisson tidal waves. A Poisson tidal wave is a particle system defined as follows. Particles are born at times $t \in (-\infty, \infty)$ and locations $x \in (-\infty, \infty)$; the collection of all birth points (t, x) constitutes a Poisson point process in $\mathbb{R}^2$ with intensity measure $Ce^{\lambda t}\mu(dx)\, dt$, where $C > 0$, $\lambda > 0$ and μ is a probability measure. Individual particles execute independent Brownian motions forward in time, starting at their birth points. (A referee has asked whether this is a new process. To our knowledge it is.)

Observe that at each time $t \in (-\infty, \infty)$ the number of particles in existence at time t is finite, almost surely, since the expected number is

$$\int_{-\infty}^{t} \int_{\mathbb{R}} Ce^{\lambda s}\mu(dx)\, ds < \infty.$$

Moreover, at each time $t > -\infty$ the positions of the existing particles constitute a Poisson point process on $\mathbb{R}^1$ with intensity measure $i_t(x)\, dx$, where

$$\begin{aligned} i_t(x) &= \int_{-\infty}^{t} \int_{-\infty}^{\infty} \frac{e^{-(x-y)^2/2(t-s)}Ce^{\lambda s}}{\sqrt{2\pi(t-s)}} \mu(dy)\, ds \\ &= \frac{Ce^{\lambda t}}{\sqrt{2\lambda}} \int_{-\infty}^{\infty} e^{-\sqrt{2\lambda}|y-x|}\mu(dy). \end{aligned} \tag{3.1}$$

Assume now that the measure μ has support entirely contained in $(-\infty, x_*]$ for some $x_* < \infty$. Then for $x \geq x_*$,

$$(3.2)\qquad \begin{aligned} i_t(x) &= \frac{Ce^{\lambda t - \sqrt{2\lambda}x}}{\sqrt{2\lambda}} \int e^{\sqrt{2\lambda}y}\mu(dy) \\ &= \frac{C}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}(x - \sqrt{\lambda/2}\,t)} \int e^{\sqrt{2\lambda}y}\mu(dy), \end{aligned}$$

from which it is apparent that $i_t(x)$ is a traveling wave with velocity $\sqrt{\lambda/2}$ in the region $x \geq x_*$. Hence the name "Poisson tidal wave."

Let $R^*(t)$ be the position of the rightmost particle in the tidal wave at time t [if no particles have been born by time t, then $R^*(t) = -\infty$]. Let $v(t,x) = P\{R^*(t) \leq x\}$. Then for $x \geq x_*$,

$$(3.3)\qquad \begin{aligned} v(t,x) &= \exp\left\{-\int_x^\infty i_t(y)\,dy\right\} \\ &= \exp\{-Ke^{\lambda t - \sqrt{2\lambda}x}\}, \end{aligned}$$

where

$$(3.4)\qquad K = (C/2\lambda)\int e^{\sqrt{2\lambda}y}\mu(dy).$$

Thus $v(t,x)$ is a traveling wave with velocity $\sqrt{\lambda/2}$. Observe that (3.3) immediately implies that $R^*(t)/t \to_P \sqrt{\lambda/2}$. (In fact, it may be shown that the convergence holds almost surely.)

The analysis of the two preceding paragraphs breaks down if μ is not supported by $(-\infty, x_*]$ for some $x_* < \infty$. However, (3.1) implies that

$$(3.5)\qquad \begin{aligned} i_t(x) = &\frac{Ce^{\lambda t - \sqrt{2\lambda}x}}{\sqrt{2\lambda}} \int_{-\infty}^x e^{\sqrt{2\lambda}y}\mu(dy) \\ &+ \frac{Ce^{\lambda t + \sqrt{2\lambda}x}}{\sqrt{2\lambda}} \int_x^\infty e^{-\sqrt{2\lambda}y}\mu(dy), \end{aligned}$$

so if $\int_{-\infty}^\infty e^{\sqrt{2\lambda}y}\mu(dy) < \infty$, then as $x \to \infty$,

$$(3.6)\qquad i_t(x) \sim \frac{Ce^{\lambda t - \sqrt{2\lambda}x}}{\sqrt{2\lambda}} \int_{-\infty}^\infty e^{\sqrt{2\lambda}y}\mu(dy),$$

uniformly in t. Thus the intensity $i_t(x)$ *approaches* a traveling wave as $t \to \infty$. If $R^*(t)$ is the position of the rightmost particle at time t and $v(t,x) = P\{R^*(t) \leq x\}$, then

$$v(t,x) = \exp\left\{-\int_x^\infty i_t(y)\,dy\right\},$$

so

$$(3.7)\qquad \log v(t,x) \sim -K\exp\{\lambda t - \sqrt{2\lambda}x\}, \quad \text{as } x \to \infty,$$

where K is defined by (3.4), uniformly in t. Consequently, $v(t, x)$ approaches a traveling wave with velocity $\sqrt{\lambda/2}$ for large t.

Recall that, with probability 1, only finitely many births occur before time $s < \infty$. Therefore, at time s there exist only finitely many particles, which are located at various points of R. Regardless of their histories up to time s, these particles move along independent Brownian trajectories after s. Since $R^*(t)/t \to_P \sqrt{\lambda/2} > 0$, the law of large numbers for zero-drift Brownian motion implies that for each $s < \infty$,

$$\lim_{t \to \infty} P\{\text{rightmost particle at time } t \text{ was born before time } s\} = 0. \tag{3.8}$$

Thus the history of the Poisson tidal wave up to time s has little effect on the traveling wave phenomenon for the distribution of $R^*(t)$ for large t.

4. A heuristic argument. In this section we discuss the IBBM whose branching rate function $\beta(x)$ has compact support. We shall argue that, in the vicinity of the "frontier," the IBBM looks like a Poisson tidal wave, at least for large times t.

For $J \subset \mathbb{R}$ let $N(t; J)$ denote the number of IBBM particles in J at time t. Recall Watanabe's theorem (Section 2): For some positive constant λ_0, finite positive measure $\nu(dx)$ and positive random variable Z,

$$N(t, J)/e^{\lambda_0 t} \to Z\nu(J), \quad \text{a.s.}, \tag{4.1}$$

as $t \to \infty$, for *every* bounded interval J. Now the births of IBBM particles in J constitute a point process whose intensity is

$$\int_J \beta(x) N(t; dx); \tag{4.2}$$

if J is a very short interval, so that β is nearly constant on J, then this intensity is approximately

$$N(t; J) \int_J \beta(x)\, dx/|J|. \tag{4.3}$$

Thus for short intervals J and large times t the intensity of the point process of births in J is asymptotic to

$$Ze^{\lambda_0 t}\nu(J) \int_J \beta(x)\, dx/|J|. \tag{4.4}$$

Suppose we could condition on the value of Z, say $Z = C$. Suppose also that instead of merely being asymptotic to (4.4) the intensities of the birth processes in all short intervals J were *equal* to (4.4) for all $t > -\infty$. Then the point process of births in space–time would be a Poisson process with intensity measure

$$Ce^{\lambda_0 t}\beta(x)\nu(dx)\, dt, \tag{4.5}$$

just as for the Poisson tidal wave.

The only difference between the IBBM and a Poisson tidal wave is the birth process: In both processes, particles move according to independent Brownian motions subsequent to their births. The argument just completed suggests that for large time the birth process in the IBBM looks very much like the birth process for a Poisson tidal wave (conditional on $Z = C$) with intensity (4.5). Conditional on $Z = C$, how much is the distribution of the position of the rightmost particle affected by the early part of the birth process? Not much, because by (3.8) the chance that the rightmost particle was born early is negligible.

The upshot of all this is that, for large time t, the distribution of the position $R(t)$ of the rightmost particle in an IBBM should be approximately a mixture of the distributions of the positions $R^*(t)$ in Poisson tidal waves with birth intensity measures (4.5). The mixing is done by setting $C = Z$. Hence, by (3.3)

$$\lim_{t\to\infty} P\left\{R(t) \le x + \sqrt{\lambda_0/2}\,t\right\} = E\exp\left\{-Z\gamma e^{-\sqrt{2\lambda_0}x}\right\}, \tag{4.6}$$

where

$$\gamma = (2\lambda_0)^{-1}\int e^{\sqrt{2\lambda_0}y}\beta(y)\nu(dy). \tag{4.7}$$

This argument not only establishes the traveling wave phenomenon for the distribution of $R(t)$, but also gives a fairly complete picture of the IBBM in the vicinity of the frontier for large t. In particular, near the frontier ($x = \sqrt{\lambda_0/2}\,t$) the point process consisting of the positions of IBBM particles at time t looks like a doubly stochastic Poisson process ([2], Chapter II) with (random) intensity

$$I_t(x) = \frac{Z}{\sqrt{2\lambda_0}}\exp\left\{-\sqrt{2\lambda_0}\left(x - \sqrt{\lambda_0/2}\,t\right)\right\}\int \exp\left(\sqrt{2\lambda_0}\,y\right)\beta(y)\nu(dy). \tag{4.8}$$

5. A coupling construction. We shall make rigorous the heuristic argument of the preceding section by a coupling construction. Once again we consider the IBBM whose branching rate function $\beta(x)$ has compact support.

PROPOSITION 1. *Let $0 < C_0 < C_1 < \infty$ be arbitrary constants, and let $3\delta = C_1 - C_0$. On some probability space may be constructed a copy of the IBBM and Poisson tidal waves W_0, W_1 with birth intensity measures $C_i e^{\lambda_0 t}\beta(x)\nu(dx)\,dt$, $i = 0, 1$, in such a way that*

(i) *if $R(t)$, $R_0^*(t)$, $R_1^*(t)$ are the positions of the rightmost particles in the IBBM and the Poisson tidal waves W_0, W_1, respectively, then on $\{C_0 + \delta < Z < C_1 - \delta\}$,*

$$R_0^*(t) - \delta \le R(t) \le R_1^*(t) + \delta, \tag{5.1}$$

eventually with probability 1; *and*

(ii) *for all t, the histories of particles in W_0 and W_1 born after t are independent of the histories of all particles in the IBBM, W_0 and W_1 up to time t.*

Before giving the proof of Proposition 1, we shall indicate how it implies (1.4). Recall that $Z = \lim_{s\to\infty} Z_s$, where Z_s is a function of the positions of the IBBM particles at time s (Section 2). It follows that, for large s, the symmetric difference of the events $\{C_0 + \delta < Z < C_1 - \delta\}$ and $\{C_0 + \delta < Z_s < C_1 - \delta\}$ has vanishingly small probability. Hence, for each $\varepsilon > 0$ there exists s_* sufficiently large that for all $s \geq s_*$, $t \geq 0$, $x \in \mathbb{R}$ and $i = 0, 1$,

$$\begin{aligned} &\Big| P\Big\{ R_i^*(t) \leq \sqrt{\lambda_0/2}\, t + x;\, C_0 + \delta < Z_s < C_1 - \delta \Big\} \\ &\quad - P\Big\{ R_i^*(t) \leq \sqrt{\lambda_0/2}\, t + x;\, C_0 + \delta < Z < C_1 - \delta \Big\} \Big| \\ &\quad < \varepsilon. \end{aligned} \tag{5.2}$$

According to statement (ii) of Proposition 1, the histories of particles in W_0, W_1 born after time s are independent of the positions of the IBBM particles at time s, and consequently are independent of Z_s. By (3.8) the distributions of $R_0^*(t)$ and $R_1^*(t)$ are not much affected by the histories of particles in W_0 and W_1 born after time s, provided $t \gg s$. Therefore, (3.3) implies

$$\begin{aligned} &\lim_{t \to \infty} P\Big\{ R_i^*(t) \leq \sqrt{\lambda_0/2}\, t + x;\, C_0 + \delta < Z_s < C_1 - \delta \Big\} \\ &\quad = \exp\Big\{ -C_i \gamma e^{-\sqrt{2\lambda_0}\, x} \Big\} P\{ C_0 + \delta < Z_s < C_1 - \delta \}, \end{aligned}$$

where $\gamma = (2\lambda_0)^{-1} \int e^{-\sqrt{2\lambda_0}\, y} \beta(y) \nu(dy)$, for $i = 0, 1$. Letting $s \to \infty$ and using (5.2), we obtain

$$\begin{aligned} &\lim_{t \to \infty} P\Big\{ R_i^*(t) \leq \sqrt{\lambda_0/2}\, t + x;\, C_0 + \delta < Z < C_1 - \delta \Big\} \\ &\quad = \exp\Big\{ -C_i \gamma e^{-\sqrt{2\lambda_0}\, x} \Big\} P\{ C_0 + \delta < Z < C_1 - \delta \}. \end{aligned} \tag{5.3}$$

By statement (i) of Proposition 1 the distribution of $R(t)$ is nearly bracketed between the distributions of $R_0^*(t) - \delta$ and $R_1^*(t) + \delta$ for large t. Consequently, (5.3) implies

$$\begin{aligned} &\exp\Big\{ -C_0 \gamma e^{-\sqrt{2\lambda_0}(x + \delta)} \Big\} \\ &\quad \geq \limsup_{t \to \infty} P\Big\{ R(t) \leq \sqrt{\lambda_0/2}\, t + x \,|\, C_0 + \delta < Z < C_1 - \delta \Big\} \\ &\quad \geq \liminf_{t \to \infty} P\Big\{ R(t) \leq \sqrt{\lambda_0/2}\, t + x \,|\, C_0 + \delta < Z < C_1 - \delta \Big\} \\ &\quad \geq \exp\Big\{ -C_1 \gamma e^{-\sqrt{2\lambda_0}(x - \delta)} \Big\}. \end{aligned}$$

Letting $\delta \to 0$, we obtain (1.4).

In proving Proposition 1, we will make use of a strong law of large numbers for point processes. Let $N(t)$, $t \geq 0$, be the counting process associated with a point process on $(0, \infty)$; let $\mathscr{F}_t$ be a filtration to which $N(t)$ is adapted; and let $\lambda(t)$ be an $\mathscr{F}_t$-measurable intensity for the point process. [A counting process $N(t)$ is an increasing, integer-valued right-continuous, adapted stochastic process that satisfies $N(0) = 0$ and whose jumps are all of size $+1$. Its intensity $\lambda(t)$ is its dual predictable projection, so that $N(t) - \lambda(t)$ is a local martingale. See [2], Chapter II.]

SLLN. *If there is a continuous function* $f\colon [0,\infty) \to [0,\infty)$ *such that* $\int_0^\infty f(t)\,dt = \infty$ *and* $\lambda(t)/f(t) \to Z$, *a.s. for some positive r.v.* Z, *then*

$$\frac{N(t)}{\int_0^t f(s)\,ds} \to Z, \quad a.s.$$

This follows immediately from the representation of $N(t)$ as a Poisson process with a random time change ([2], Section II.6).

PROOF OF PROPOSITION 1. Start with a copy of the IBBM and independent Poisson processes on $\mathbb{R}^2$ with intensities $C_i e^{\lambda_0 t}\beta(x)\nu(dx)\,dt$, $i = 0, 1$. The Poisson processes are to be the point processes of births for the Poisson tidal waves W_0, W_1, respectively. An individual particle in the tidal wave W_i executes a Brownian motion (starting at its birth point) independent of the IBBM, the birth processes and the motions of all other particles in W_0 and W_1 until the instant it is "paired" with an IBBM particle. (The pairing scheme is explained in the following discussion; the pairing times are stopping times.) Hereafter, the W_i-particle "shadows" the IBBM particle with which it is paired, i.e., it follows the trajectory that keeps its distance from the IBBM particle constant over time.

The pairing laws are such that each IBBM particle is paired with at most one W_0-particle and one W_1-particle. Since the pairing times are stopping times, the movements of individual particles in W_0 and W_1 are Brownian motions. Since IBBM particles follow independent trajectories, particles of W_i follow independent trajectories, $i = 0, 1$. Therefore, the processes W_0 and W_1 are Poisson tidal waves. Moreover, it is evident from the construction that the histories of particles in W_0 and W_1 born after time t are independent of the histories of W_0, W_1 and the IBBM up to time t, for any t.

The pairing laws for W_1-particles are as follows. Let J be an interval containing the support of β: Note that no particles are born outside J. Let $J = \bigcup_{i=1}^k J_i$, where $J_1, J_2, \ldots, J_k$ are disjoint intervals such that for each $i = 1, \ldots, k$, J_i has length $\leq \varepsilon$ ($\varepsilon > 0$ will be specified later). Let $\alpha > 0$. Any IBBM particle born in J_i during $[n\alpha, (n+1)\alpha)$ is immediately paired with the oldest unpaired W_1-particle born in J_i during $[(n-1)\alpha, n\alpha)$, which has not traveled farther than $\delta/2$ from its birthplace. If no such W_1-particle exists, the IBBM particle remains unpaired until it meets an unpaired W_1-particle older than 2α, at which time these two particles are paired.

The pairing laws for W_0-particles are similar, but the roles are partially reversed. Any W_0-particle born in J_i during $[n\alpha, (n+1)\alpha)$ is immediately paired with the oldest unpaired IBBM particle born in J_i during $[(n-1)\alpha, n\alpha)$, which has not traveled farther than $\delta/2$ from its birthplace. If no such IBBM particle exists, the W_0-particle remains unpaired until it meets an unpaired IBBM particle older than 2α, at which time these two particles are paired.

(*Note*: The two pairing schemes work separately. In particular, a W_0-particle considers an IBBM particle unpaired if the IBBM particle is not yet paired with another W_0-particle; whether the IBBM particle is paired with a W_1-particle is irrelevant.)

It remains to show that $\varepsilon > 0$ and $\alpha > 0$ may be chosen so that (5.1) holds on $\{C_0 + \delta < Z < C_1 - \delta\}$ for all large t. Observe that if $\varepsilon < \delta/2$, then paired particles are always within δ of their partners. Consequently, to prove (5.1), it suffices to show that on $\{C_0 + \delta < Z < C_1 - \delta\}$, almost surely, (a) all IBBM particles born after a certain time are immediately paired with W_1-particles, and all W_0-particles born after a certain time are immediately paired with IBBM particles, and (b) the finite number of W_0-particles and IBBM particles not paired immediately at birth are eventually paired with IBBM particles and W_1-particles, respectively.

For $\alpha > 0$, let $p(\alpha)$ be the probability that a standard Wiener process in $\mathbb{R}^1$ started at 0 does not exit the interval $[-\delta/2, \delta/2]$ before time 2α. Since Wiener paths are continuous, $p(\alpha) \uparrow 1$ as $\alpha \to 0$. Consequently, there exists $\alpha > 0$ sufficiently small that

$$C_1 p(\alpha) e^{-\lambda_0 \alpha} > C_1 - \delta \tag{5.4}$$

and

$$(C_0 + \delta) p(\alpha) e^{-\lambda_0 \alpha} > C_0. \tag{5.5}$$

Associated with each W_i-particle is a standard Wiener process, independent of the IBBM, the birth processes for W_0 and W_1 and the Wiener processes associated with all other W_0- and W_1-particles. The motion of a W_i-particle is determined by its associated Wiener process up to the time the particle is paired, after which the associated Wiener process ceases to play any role in the evolution of the particle system W_i. Label a W_1-particle "good" if its associated Wiener process does not exit the interval of radius $\delta/2$ centered at the initial point of the process before 2α time units elapse. Otherwise, label the W_1-particle "bad."

The point process consisting of the birth locations (in space–time) of all *good* W_1-particles born in J_i is a Poisson point process with intensity measure

$$p(\alpha) C_1 e^{\lambda_0 t} \beta(x) 1_{J_i}(x) \nu(dx)\, dt.$$

If $N_t^*(J_i)$ is the total number of good W_1-particles born in J_i up to time t, then, by the SLLN for point processes,

$$\frac{N_t^*(J_i)}{e^{\lambda_0 t}} \to p(\alpha) C_1 \lambda_0^{-1} \int_{J_i} \beta(x) \nu(dx), \quad \text{a.s.}, \tag{5.6}$$

as $t \to \infty$.

Consider now the point process of births of IBBM particles in J_i. This has intensity $\int_{J_i} \beta(x) N(t; dx)$, where $N(t, A)$ is the number of IBBM particles in A at time t. It follows from Watanabe's theorem that

$$\frac{\int_{J_i} \beta(x) N(t; dx)}{e^{\lambda_0 t}} \to Z \int_{J_i} \beta(x) \nu(dx), \quad \text{a.s.}, \tag{5.7}$$

as $t \to \infty$. Therefore, if $N_t^{**}(J_i)$ is the number of IBBM births in J_i up to time t, then

$$\frac{N_t^{**}(J_i)}{e^{\lambda_0 t}} \to Z \lambda_0^{-1} \int_{J_i} \beta(x) \nu(dx), \quad \text{a.s.}, \tag{5.8}$$

by the SLLN for point processes.

Relations (5.4), (5.6) and (5.8) imply that, on $\{C_0 + \delta < Z < C_1 - \delta\}$, the number of good W_1-births in J_i during $[(n-1)\alpha, n\alpha)$ exceeds the number of IBBM births in J_i during $[n\alpha, (n+2)\alpha)$ for all large n, almost surely. Therefore, on this event all IBBM particles born after a certain time are immediately paired with W_1-particles. Furthermore, there is an infinite "surplus" of W_1-particles, so the finitely many IBBM particles that are not paired at birth eventually meet and pair with W_1-particles.

A similar argument shows that, on $\{C_0 + \delta < Z < C_1 - \delta\}$, all W_0-particles are eventually paired with IBBM particles, and all but finitely many are paired at birth. □

6. The general case. Consider now the IBBM with branching rate function $\beta(x)$ satisfying (1.2) and (1.3). If $\beta(x)$ does not have compact support, then particles may be born at arbitrarily large distances from 0.

PROPOSITION 2. *For each $\varepsilon > 0$ there exists $A = A(\varepsilon)$ sufficiently large that for all large t,*

$$(6.1) \qquad P\{\text{rightmost particle at time } t \text{ was born outside } [-A, A]\} \le \varepsilon.$$

Proposition 2 will be proved in Section 7.

Let $R(t)$ denote the position of the rightmost particle at time t, and let $R_A(t)$ denote the position of the rightmost particle at time t born in $[-A, A]$. [Note that the truncation here refers only to individual particles and not to their parents. A particle born outside $[-A, A]$ is neglected in determining $R_A(t)$, but its offspring born inside $[-A, A]$ are not.] By (6.1), for t sufficiently large and all $x \in \mathbb{R}$,

$$(6.2) \qquad P\{R_A(t) \le x\} - \varepsilon \le P\{R(t) \le x\} \le P\{R_A(t) \le x\}.$$

Consequently, to prove (1.4), it suffices to demonstrate that, for each $A < \infty$, the distribution of $R_A(t)$ approaches a traveling wave as $t \to \infty$, and that these waves coalesce as $A \to \infty$.

The distribution of $R_A(t)$ may be studied by the methods of Sections 4–5, as only particles born in $[-A, A]$ affect the value of $R_A(t)$. Let Z, φ_0, λ_0 be as in Watanabe's theorem. Then for all $x \in \mathbb{R}$, $0 < A < \infty$, by the arguments of Section 5,

$$(6.3) \qquad \lim_{t\to\infty} P\left\{R_A(t) \le x + \sqrt{\lambda_0/2}\, t\right\} = E\exp\left\{-Z\gamma_A e^{-\sqrt{2\lambda_0}x}\right\},$$

where

$$\gamma_A = (2\lambda_0)^{-1}\int_{-A}^{A} e^{\sqrt{2\lambda_0}y}\beta(y)\varphi_0(y)\,dy.$$

Recall that $\varphi_0(x) \sim C_\pm \exp\{-\sqrt{2\lambda_0}\,|x|\}$ as $x \to \pm\infty$, and $\int\beta(y)\,dy < \infty$. Therefore,

$$(6.4) \qquad \gamma = \lim_{A\to\infty}\gamma_A = (2\lambda_0)^{-1}\int_{-\infty}^{\infty} e^{\sqrt{2\lambda_0}y}\beta(y)\varphi_0(y)\,dy < \infty.$$

Now (6.2)–(6.4) imply (1.4).

7. Proof of Proposition 2. We assume, as in the preceding section, that the branching rate function $\beta(x)$ satisfies (1.2) and (1.3). Let Z, λ_0, φ_0, and $N(t; J)$ be as in Watanabe's theorem.

LEMMA. *For each $\delta > 0$ there exists $K = K(\delta) < \infty$ such that for every continuous f: $\mathbb{R} \to [0, \infty)$ with compact support contained in $\mathbb{R} \setminus [-\delta, \delta]$, and every $t \geq 0$,*

$$E\left(\int_{\mathbb{R}} f(x) N(t; dx)\right) \leq K e^{\lambda_0 t} \int_{\mathbb{R}} f(x) e^{-\sqrt{2\lambda_0}|x|}\, dx. \tag{7.1}$$

PROOF (adapted from [3], Section 3, step 1). Let E^x denote the expectation operator for an IBBM that starts with a single particle located at position x at time 0 (thus $E^0 = E$). Consider the semigroup T_t: $C_b(\mathbb{R}) \to C_b(\mathbb{R})$ defined by

$$T_t g(x) = \frac{e^{-\lambda_0 t}}{\varphi_0(x)} E^x\left(\int_{\mathbb{R}} g(y) \varphi_0(y) N(t; dy)\right).$$

This is the transition semigroup of a diffusion process on $\mathbb{R}$ with generator

$$\frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{\varphi_0'(x)}{\varphi_0(x)}\right)\frac{d}{dx}$$

and invariant probability measure $\pi(dx) = \varphi_0(x)^2\, dx / \int \varphi_0(y)^2\, dy$. A coupling argument for this diffusion process shows that, for each $\delta > 0$, there exists $K' = K'(\delta) < \infty$ such that

$$T_t g(0) \leq K' \int_{\mathbb{R}} T_t g(x) \pi(dx), \tag{7.2}$$

for every nonnegative $g \in C_b(\mathbb{R})$ vanishing on $[-\delta, \delta]$, and every $t \geq 0$. [The invariance of $\pi(dx)$ is not used in obtaining (7.2).] Since $f \geq 0$,

$$\begin{aligned}
&E\left(\int_{\mathbb{R}} f(x) N(t; dx)\right)\\
&\quad = e^{\lambda_0 t} \varphi_0(0) T_t(f/\varphi_0)(0)\\
&\quad \leq K' \varphi_0(0) e^{\lambda_0 t} \int_{\mathbb{R}} T_t(f/\varphi_0)(x) \pi(dx)\\
&\quad = K' \varphi_0(0) e^{\lambda_0 t} \int_{\mathbb{R}} (f/\varphi_0)(x) \pi(dx)\\
&\quad = K'' e^{\lambda_0 t} \int f(x) \varphi_0(x)\, dx.
\end{aligned}$$

The result (7.1) follows, since $\varphi_0(x) \sim C_{\pm} e^{-\sqrt{2\lambda_0}|x|}$ as $x \to \pm\infty$. □

Fix $x \in \mathbb{R}$, $t > 0$. Any particle of the IBBM whose position at time t is to the right of $x + \sqrt{\lambda_0/2}\, t$ was born at some (s, y), where $0 \leq s \leq t$ and $y \in \mathbb{R}$. Since the intensity of the birth process is $\beta(y)N(s;dy)\,ds$,

$$EN_A\big(t;[x+\sqrt{\lambda_0/2}\,t,\infty)\big)$$

$$(7.3)\qquad = E\left\{\int_{s=0}^{t}\int_{y\in\mathbb{R}\setminus[-A,A]}\int_{z=x+\sqrt{\lambda_0/2}\,t}^{\infty}\beta(y)\frac{e^{-(z-y)^2/2(t-s)}}{\sqrt{2\pi(t-s)}}\,dz\,N(s;dy)\,ds\right\},$$

where $N_A(t;J)$ denotes the number of particles in J at time t that were born outside $[-A,A]$. It follows from (7.1) that

$$EN_A\big(t;[x+\sqrt{\lambda_0/2}\,t,\infty)\big)$$

$$\leq \int_{s=0}^{t}\int_{|y|>A}\int_{z=x+\sqrt{\lambda_0/2}\,t}^{\infty}\frac{Ke^{\lambda_0 s}e^{-\sqrt{2\lambda_0}|y|}}{\sqrt{2\pi(t-s)}}\beta(y)e^{-(z-y)^2/2(t-s)}\,dz\,dy\,ds$$

$$\leq \int_{|y|>A}\int_{z=x+\sqrt{\lambda_0/2}\,t}^{\infty}\frac{Ke^{\lambda_0 t}}{\sqrt{2\lambda_0}}e^{-\sqrt{2\lambda_0}|y|-\sqrt{2\lambda_0}|z-y|}\beta(y)\,dz\,dy.$$

Since $\int\beta(y)\,dy<\infty$, it is apparent that, for any $x\in\mathbb{R}$, $\varepsilon>0$, there is an A so large that

$$EN_A\big(t;[x+\sqrt{\lambda_0/2}\,t,\infty)\big)\leq\varepsilon/2,$$

for all $t\geq 0$.

To prove Proposition 2, it now suffices to show that there exists $x\in\mathbb{R}$ such that

$$P\big\{R(t)\leq x+\sqrt{\lambda_0/2}\,t\big\}<\varepsilon/2,$$

for all $t\geq 0$. This follows from the arguments of Section 6. If $R_A(t)$ denotes the position of the rightmost particle at time t born in $[-A,A]$, then obviously $R_A(t)\leq R(t)$ (regardless of what A is). By (6.3)

$$P\big\{R_A\leq x+\sqrt{\lambda_0/2}\,t\big\}\to E\exp\big\{-Z\gamma_A e^{-\sqrt{2\lambda_0}x}\big\},$$

as $t\to\infty$; for x sufficiently small this limit is $<\varepsilon/4$. □

REFERENCES

[1] BRAMSON, M. (1983). Convergence of solutions of the Kolmogorov equation to travelling waves. *Mem. Amer. Math. Soc.* **44**.

[2] BREMAUD, P. (1981). *Point Processes and Queues*. Springer, New York.

[3] ERICKSON, K. B. (1984). Rate of expansion of an inhomogeneous branching process of Brownian particles. *Z. Wahrsch. verw. Gebiete* **66** 129–140.

[4] HARTMAN, P. (1964). *Ordinary Differential Equations*. Wiley, New York.

[5] LALLEY, S. and SELLKE, T. (1988). Traveling waves in inhomogeneous branching Brownian motions. II. *Ann. Probab.* To appear.

[6] MCKEAN, H. (1975). Application of Brownian motion to the equation of Kolmogorov–Petrovski–Piskunov. *Comm. Pure Appl. Math.* **28** 323–331.

[7] WATANABE, S. (1967). Limit theorem for a class of branching processes. In *Markov Processes and Potential Theory* (J. Chover, ed.). Wiley, New York.

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