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ESTIMATES OF THE RATE OF CONVERGENCE FOR MAX-STABLE PROCESSES

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Several estimates of the rate of convergence of normalized maxima of random processes are exhibited by using the theory of probability metrics. In contrast to the summation scheme, uniform estimates for normalized maxima of random sequences will be derived.

1. Introduction. Let $\mathbf{B} = (\mathbf{l}_r[T], \|\cdot\|_r)$, $1 \le r \le \infty$, be the separable Banach space of all measurable functions $x: T \to \mathbb{R}$ (T is a Borel subset of $\mathbb{R}$) with finite norm $\|x\|_r$, where

$$(1.1) \qquad \|x\|_r = \left\{ \int_T |x(t)|^r \, dt \right\}^{1/r}, \qquad 1 \le r < \infty,$$

and if $r = \infty$, $\mathbf{l}_\infty(T)$ is assumed to be the space of all continuous functions on a compact subset T with the norm

$$(1.2) \qquad \|x\|_\infty = \sup_{t \in T} |x(t)|.$$

Suppose $\mathbf{X} = \{X_n, n \ge 1\}$ is a sequence of (dependent) random variables taking values in $\mathbf{B}$. Let $\mathscr{C}$ be the class of all sequences $\mathbf{C} = \{c_j(n);\ j, n = 1, 2, \dots\}$ satisfying the conditions

$$(1.3) \qquad c_1(n) > 0, \qquad c_j(n) \ge 0,\ j = 1, 2, \dots, \qquad \sum_{j=1}^{\infty} c_j(n) = 1.$$

For any $\mathbf{X}$ and $\mathbf{C}$ define the normalized maxima $\overline{X}_n := \bigvee_{j=1}^{\infty} c_j(n) X_j$, where $\bigvee :=$ max and $\overline{X}_n(t) := \bigvee_{j=1}^{\infty} c_j(n) X_j(t)$, $t \in T$.

We will be interested in the limit behavior of $\overline{X}_n$. To this end, we explore an approximation $(\overline{Y}_n)$ of $\overline{X}_n$ with a known limit behavior. More precisely, let $\mathbf{Y} = \{Y_n,\ n \ge 1\}$ be a sequence of i.i.d. r.v.'s and define $\overline{Y}_n = \bigvee_{j=1}^{\infty} c_j(n) Y_j$. Assuming that

$$(1.4) \qquad \overline{Y}_n =_d Y_1 \quad \text{for any } \mathbf{C} \in \mathscr{C}$$

($\overline{Y}_n =_d Y_1$ means $\Pr_{\overline{Y}_n} = \Pr_{Y_1}$), we will be interested in estimates of the deviation between $\overline{X}_n$ and $\overline{Y}_n$. The class of r.v.'s Y_1 satisfying (1.4) was introduced by de Haan (1984) and will be called the class of *simple max-stable processes*.

EXAMPLE [de Haan (1984)]. Consider a Poisson point process on $\mathbb{R}_+ \times [0, 1]$ with intensity measure $(dx/x^2)\, dy$. With probability 1 there are denumerably

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many points in the point process. Let $\{\xi_k, \eta_k\}$, $k = 1, 2, \dots$, be an enumeration of the points in the process. Consider a family of nonnegative functions $\{f_t(\cdot), t \in T\}$ defined on $[0,1]$. Suppose for fixed $t \in T$ the function $f_t(\cdot)$ is measurable and $\int_0^1 f_t(v)\,dv < \infty$. We claim that the family of random variables $Y(t) := \sup_{k\ge 1} f_t(\eta_k)\xi_k$ form a simple max-stable process. Clearly, it is sufficient to show that for any $\mathbf{C} \in \mathscr{C}$ and any $0 < t_1 < \cdots < t_k \in T$ the joint distribution of $(Y(t_1), \dots, Y(t_k))$ satisfies the equality

$$\prod_{j=1}^{\infty} \Pr\{c_j Y(t_1) \le y_1, \dots, c_j Y(t_k) \le y_k\}$$
$$= \Pr\{Y(t_1) \le y_1, \dots, Y(t_k) < y_k\}, \quad \text{where } c_j = c_j(n).$$

Now

$$\prod_{j=1}^{\infty} \Pr\{c_j Y(t_1) \le y_1, \dots, c_j Y(t_k) \le y_k\}$$
$$= \prod_{j=1}^{\infty} \Pr\{f_{t_i}(\eta_m)\xi_m \le y_i/c_j,\ i = 1, \dots, k;\ m = 1, 2, \dots\}$$
$$= \prod_{j=1}^{\infty} \Pr\Big\{\text{there are no points of the point process above the graph of the function } g(v) = (1/c_j) \min_{i\le k} y_i/f_{t_i}(v),\ v \in [0,1]\Big\}$$
$$= \prod_{j=1}^{\infty} \exp\left(-\int_0^1 \left[\int_{\{x>g(v)\}} x^{-2}\,dx\right] dv\right)$$
$$= \prod_{j=1}^{\infty} \exp\left(-\int_0^1 \left(c_j \max_{i\le k} f_{t_i}(v)/y_i\right) dv\right)$$
$$= \exp \sum_{j=1}^{\infty} c_j\left(-\int_0^1 \max_{i\le k} f_{t_i}(v)/y_i\right) dv$$
$$= \exp\left(-\int_0^1 \left(\max_{i\le k} f_{t_i}(v)/y_i\right) dv\right)$$
$$= \Pr\{Y(t_1) \le y_1, \dots, Y(t_k) \le y_k\}.$$

In this article we seek the weakest conditions providing an estimate of the deviation $\mu(\overline{X}_n, \overline{Y}_n)$ with respect to a given metric μ. Such a metric will be defined on the space $\mathscr{X}(\mathbf{B})$ of all r.v.'s X: $(\Omega, \mathscr{A}, \Pr) \to (\mathbf{B}, \mathscr{X}_{\mathbf{B}})$, where the probability space $(\Omega, \mathscr{A}, \Pr)$ is assumed to be nonatomic. In particular, if the sequence $\mathbf{X}$ consists of i.i.d. r.v.'s we will derive estimates of the rate of convergence of $\overline{X}_n$ to Y_1 in terms of *the minimal metric* $\hat{\mu}$ defined by

$$\hat{\mu}(X, Y) := \hat{\mu}(\Pr_X, \Pr_Y) := \inf\{\mu(X', Y'): X', Y' \in \mathscr{X}(\mathbf{B}),\ X' =_d X,\ Y' =_d Y\} \tag{1.5}$$

[see Zolotarev (1976, 1983a), Dudley (1976), Rachev (1985) and Rachev and Shortt (1988)].

In addition, all considered metrics μ will be such that $\hat{\mu}$-convergence implies the weak convergence in the space $\mathscr{P}(\mathbf{B})$ of distributions $\Pr_X$. Further, let $Z_1, Z_2, \dots$ be a sequence of i.i.d. r.v.'s taking values in the Hilbert space $\mathbf{H} = (\mathbb{R}^\infty, \|\cdot\|_2)$ with $\mathbf{E}Z_1 = 0$ and covariance operator $\mathbf{V}$. The central limit theorem in $\mathbf{H}$ states that the distribution of the normalized sums $\tilde{Z}_n = n^{-1/2}\sum_{i=1}^n Z_i$ weakly tends to the normal distribution of a r.v. $Z \in \mathscr{X}(\mathbf{H})$ with mean 0 and covariance operator $\mathbf{V}$. However, the uniform convergence

$$\rho(F_{\tilde{Z}_n}, F_Z) := \sup_{X \in \mathbb{R}^\infty} |F_{\tilde{Z}_n}(x) - F_Z(x)| \to 0, \qquad n \to \infty,$$

where

$$F_X(x) := \Pr\left\{ \bigcap_{i=1}^{\infty} [X^{(i)} < x^{(i)}] \right\}, \qquad X = (X^{(1)}, X^{(2)}, \dots), x = (x^{(1)}, x^{(2)}, \dots),$$

may fail [see, for example, Sazonov (1981), pages 69–70]. In contrast to the summation scheme we will show that under some tail conditions the distribution function (d.f.) of the normalized maxima $\overline{X}_n$ of i.i.d. r.v.'s $X_i \in \mathscr{X}(\mathbb{R}^\infty)$ converges uniformly to the d.f. of a simple max-stable sequence Y. Moreover, the rate of uniform convergence is nearly the same as in the univariate case $(X_i \in \mathscr{X}(\mathbb{R}^1))$ [see Omey and Rachev (1988)]. Moreover, in our investigations we will not assume that $\mathbb{R}^\infty$ has the structure of Hilbert or even normed space.

Our method is based on exploring compound metrics for which

$$\mu_r(c(X_1 \vee Y), c(X_2 \vee Y)) \le c^r \mu_r(X_1, X_2), \quad X_1, X_2, Y \in \mathscr{X}(\mathbf{B}),\ c > 0, \tag{1.6}$$

where r is an arbitrary positive number [see Rachev (1987)]. This is actually the main difference between our approach to the maxima scheme and Zolotarev's approach to the rate of convergence of normalized sums [see Zolotarev (1976, 1983a)], because metrics υ_r with the property

$$\upsilon_r(c(X_1 + Y), c(X_2 + Y)) \le c^r \upsilon_r(X_1, X_2), \quad X_1, X_2, Y \in \mathscr{X}(\mathbb{R}),\ c > 0, \tag{1.7}$$

for $r > 1$, take values 0 and ∞.

The outline of the article is as follows. In Section 2 we will list the main structural and topological properties of the metrics we will use further on. Section 3 (Section 4) deals with the convergence rate problem for normalized maxima in terms of minimal (resp. uniform) metrics.

2. Preliminaries: probability metrics. In this section we summarize the facts on probability metrics which we need in the sequel. For general acquaintance with the theory of probability metrics we refer to Zolotarev (1976, 1983a), Dudley (1976), Rachev (1984b, 1985) and Rachev and Shortt (1988).

Let $(\Omega, \mathscr{A}, \Pr)$ be a nonatomic probability space, $U = (U, d)$ be universally measurable separable metric space with metric d and $\mathscr{X} = \mathscr{X}(U)$ be the space of all r.v.'s X: $(\Omega, \mathscr{A}, \Pr) \to (U, \mathscr{B}(U))$ [$\mathscr{B}(U)$ is the Borel σ-algebra generated by d]. Note that any complete separable metric space is universally measurable.

Using the convention $X = Y$, $X, Y \in \mathscr{X}(U)$, iff $\Pr(X = Y) = 1$, we say that μ: $\mathscr{X}^2 \to [0, \infty]$ is a compound (probability) metric if μ is a metric in $\mathscr{X}$ possibly taking infinite values. Since the set $\mathscr{P}_2(U)$ of all probability measures on the Cartesian product $(U \times U, \mathscr{B}(U \times U))$ coincides with the space $\mathscr{L}\mathscr{X}_2(U)$ of all laws $\Pr_{X,Y}$, $X, Y \in \mathscr{X}$ [see Rachev and Shortt (1988)], any compound metric μ is well defined on the set $\mathscr{P}_2(U)$ and we will use the notation $\mu(P)$, $P \in \mathscr{P}_2(U)$, and $\mu(X, Y)$, $X, Y \in \mathscr{X}$. Further, let $\mathscr{P}(U)$ be the set of all probability measures on $(U, \mathscr{B}(U))$. Then $\mathscr{P}(U)$ coincides with the space $\mathscr{L}\mathscr{X}_1(U)$ of all laws P_X, $x \in \mathscr{X}$. So, each metric ν on $\mathscr{P}(U) \equiv \mathscr{L}\mathscr{X}_1(U)$, possibly taking infinite values, will be called a simple (probability) metric and we will use the notation $\nu(P_1, P_2)$ $P_1, P_2 \in \mathscr{P}(U)$, and $\nu(X, Y) := \nu(\Pr_X, \Pr_Y)$, $X, Y \in \mathscr{X}(U)$, interchangeably.

The main relationship between compound and simple metrics is realized by the notion of minimal metric. Namely, the simple metric $\hat{\mu}$ is said to be a minimal metric with respect to the compound metric μ if

$$\hat{\mu}(P_1, P_2) = \inf\{\mu(P): P \in \mathscr{P}_2(U),\ P(A \times U) = P_1(A),\ P(U \times A) = P_2(A) \text{ for any } A \in \mathscr{B}(U)\} \tag{2.1}$$

[see also (1.5)].

One of the main parts of the theory of probability metrics concerns the dual and explicit representations of the minimal metrics $\hat{\mu}$ as well as the topological structure of the spaces $(\mathscr{P}(U), \hat{\mu})$. In the next sections we will use the following probability metrics and relations between them [see (i), (ii) and (iii)].

(i) *Compound probability metrics.*

(i.1) *Ky Fan metric* (distance in probability):

$$K(X, Y) := \inf\{\varepsilon > 0: \Pr(d(X, Y) > \varepsilon) \le \varepsilon\}. \tag{2.2}$$

(i.2) $\mathscr{L}_p$*-metric* $(1 \le p \le \infty)$:

$$\mathscr{L}_p(X, Y) := \{\mathbf{E} d^p(X, Y)\}^{1/p}, \qquad 1 \le p < \infty, \tag{2.3}$$

$$\mathscr{L}_\infty(X, Y) := \operatorname{ess\,sup} d(X, Y) := \inf\{\varepsilon > 0: \Pr\{d(X, Y) > \varepsilon\} = 0\}. \tag{2.4}$$

(i.3) τ_Q*-metric:*

$$\tau_Q(X, Y) := \mathbf{E} d(QX, QY), \tag{2.5}$$

where Q: $U \to U$ is a homeomorphism of U.

(i.4) χ_p*-metric* $(p > 0)$:

$$\chi_p(X, Y) := \left[\sup_{t > 0} t^p \Pr\{d(X, Y) > t\}\right]^{1/(1+p)}. \tag{2.6}$$

LEMMA 2.1. *For any $p > 0$, χ_p is a compound probability metric.*

PROOF. Let us check the triangle inequality. For any $\alpha \in [0,1]$ and any $t > 0$,

$$\Pr\{d(X,Y) > t\} \leq \Pr\{d(X,Z) > \alpha t\} + \Pr\{d(Z,Y) > (1-\alpha)t\}$$

and hence

$$\chi_p^{p+1}(X,Y) \leq \alpha^{-p}\chi_p^{p+1}(X,Z) + (1-\alpha)^{-p}\chi_p^{p+1}(Z,Y).$$

Minimizing the right-hand side of the last inequality over all $\alpha \in (0,1)$, we obtain

$$\chi_p(X,Y) \leq \chi_p(X,Z) + \chi_p(Z,Y). \qquad \square$$

(ii) *Simple probability metrics.*

(ii.1) *Prokhorov metric*:

$$\pi(X,Y) = \inf\{\varepsilon > 0 \colon \Pi_\varepsilon(X,Y) < \varepsilon\}, \tag{2.7}$$

where $\Pi_\varepsilon(X,Y) = \sup\{\Pr\{X \in A\} - \Pr\{Y \in A^\varepsilon\} \colon A \in \mathscr{B}(U)\}$ and A^ε is the ε-neighborhood of A.

Lemmas 2.2–2.5 will deal with the dual and explicit representations of the simple metrics under consideration as well as the convergences in $\mathscr{P}(U)$ which these metrics metrize.

LEMMA 2.2. *$\hat{K} = \pi$ metrizes the weak convergence in $\mathscr{P}(U)$.*

PROOF. See Prohorov (1956), Strassen (1965) and Dudley (1976). $\square$

(ii.2) *l_p-metric* $(1 \leq p \leq \infty)$ *for* $p \in (1,\infty)$:

$$\begin{aligned} l_p^p(X,Y) = \sup\{&\mathbf{E}f(X) + \mathbf{E}g(Y) \colon f \colon U \to \mathbb{R},\ g \colon U \to \mathbb{R}, \\ &\|f\|_\infty := \sup\{|f(x)| \colon x \in U\} < \infty,\ \|g\|_\infty < \infty, \\ &\operatorname{Lip}(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)} < \infty,\ \operatorname{Lip}(g) < \infty, \\ &f(x) + g(y) \leq d^p(x,y) \text{ for any } x, y \in U\}, \end{aligned}$$

and for $p = 1$, $p = \infty$,

$$\begin{aligned} l_1(X,Y) &= \sup\{|\mathbf{E}f(X) - \mathbf{E}f(Y)| \colon f \colon U \to \mathbb{R}, \\ &\qquad \|f\|_\infty < \infty, \operatorname{Lip}(f) \leq 1\}, \\ l_\infty(X,Y) &= \inf\{\varepsilon > 0 \colon \Pi_\varepsilon(X,Y) = 0\}. \end{aligned} \tag{2.8}$$

LEMMA 2.3. (a) *The functional l_p has the following dual representation*:

$$l_p = \hat{\mathscr{L}}_p \quad \textit{for any } p \in [1,\infty] \tag{2.9}$$

and, in particular, l_p *is a simple probability metric.*

(b) *Let* $p \in [1, \infty)$, *a be some fixed element of* U *and*

$$\omega_X(N) = \{\mathbf{E}d^p(X, a)I\{d(X, a) > N\}\}^{1/p}, \qquad N > 0,\ X \in \mathscr{X}(U).$$

Then for any $N > 0$, $X, Y \in \chi(U)$,

$$l_p(X, Y) \le \pi(X, Y) + 2N\pi^{1/p}(X, Y) + \omega_X(N) + \omega_Y(N), \tag{2.10}$$

$$l_p(X, Y) \ge \pi^{(p+1)/p}(X, Y), \qquad l_\infty(X, Y) \ge \pi(X, Y) \tag{2.11}$$

and

$$\omega_X(N) \le 3\big(l_p(X, Y) + \omega_Y(N)\big). \tag{2.12}$$

In particular, if $L_p(X_n, a) + L_p(X, a) < \infty$, $n = 1, 2, \dots,$ *then*

$$l_p(X_n, X) \to 0 \quad \textit{iff } \pi(X_n, X) \to 0 \textit{ and } \lim_{N \to \infty} \sup_n \omega_{X_n}(N) = 0. \tag{2.13}$$

(c) *If* $U = \mathbb{R}$, $d(x, y) = |x - y|$, *then* l_p *has the explicit representation*

$$l_p(X, Y) = \left[\int_0^1 \left|F_X^{\text{inv}}(x) - F_Y^{\text{inv}}(x)\right|^p dx\right]^{1/p}, \qquad 1 \le p < \infty, \tag{2.14}$$

$$l_\infty(X, Y) = \sup\left\{\left|F_X^{\text{inv}}(x) - F_Y^{\text{inv}}(x)\right| : x \in [0, 1]\right\}, \tag{2.15}$$

where F_X^{inv} *is the function inverse to the d.f.* F_X *of* X.

PROOF. (a) See Rachev (1984b) and Kellerer (1984a, b).

(b) See Zolotarev (1975, 1976) and Rachev (1982, 1984a, 1985).

(c) See Cambanis, Simons and Stout (1976) and Rachev (1981). □

(ii.3) *Q-difference pseudomoment* [see Zolotarev (1976)]:

$$\begin{aligned}\kappa_Q(X, Y) := \sup\{&|\mathbf{E}f(X) - \mathbf{E}f(Y)| : f : U \to \mathbb{R}, \|f\|_\infty < \infty, \\ &|f(x) - f(y)| \le d(Qx, Qy) \text{ for all } x, y \in U\},\end{aligned} \tag{2.16}$$

where $Q: U \to U$ is a homeomorphism of U.

As a corollary of Lemma 2.3 one can easily obtain the following metric and topological characterizations of κ_Q.

LEMMA 2.4. (a) *For any* $X, Y \in \mathscr{X}(U)$,

$$\kappa_Q(X, Y) = l_1(QX, QY) = \hat{\tau}_Q(X, Y). \tag{2.17}$$

(b) *If* $\mathbf{E}QX_n + \mathbf{E}QX < \infty$, $n = 1, 2, \dots,$ *then*

$$\kappa_Q(X_n, X) \to 0 \quad \textit{iff } \pi(X_n, X) \to 0 \textit{ and } \mathbf{E}QX_n \to \mathbf{E}QX. \tag{2.18}$$

(c) *If* $U = \mathbb{R}$, $d(x, y) = |x - y|$, *then*

$$\kappa_Q(X, Y) = \int_{-\infty}^{\infty} |F_{QX}(x) - F_{QY}(x)|\, dx. \tag{2.19}$$

(ii.4) *The minimal metric w.r.t.* χ_p [see (2.6)]:

$$\xi_p(X, Y) := \hat{\chi}_p(X, Y), \qquad p > 0. \tag{2.20}$$

LEMMA 2.5. (a) *Let* $a \in U$,

$$\tilde{\omega}_X(N) := \left[\sup_{t>N} t^p \Pr\{d(X, a) > t\}\right]^{1/(p+1)}, \qquad N > 0, \tag{2.21}$$

and

$$\eta_p(X, Y) := \left[\sup_{t>0} t^p \Pi_t(X, Y)\right]^{1/(1+p)}. \tag{2.22}$$

Then for any $N > 0$ *and* $p > 0$,

$$\pi \le \eta_p \le \xi_p \le \begin{cases} l_p^{p/(1+p)} & \text{if } p \ge 1, \\ l_p^{1/(1+p)} & \text{if } p \le 1, \end{cases} \tag{2.23}$$

where l_p, $p < 1$, *in the space* $\mathscr{X}(U, d)$ *is defined as* l_1 [*see* (2.8)] *in the space* $\mathscr{X}((U, d^p))$. *Moreover,*

$$\tilde{\omega}_X(N) \le 2^{p/(1+p)}\left[\eta_p(X, Y) + \tilde{\omega}_Y(N/2)\right] \tag{2.24}$$

and

$$\xi_p^{p+1}(X, Y) \le \max\left[\pi^p(X, Y), (2N)^p \pi(X, Y), 2^p\left(\tilde{\omega}_X^p(N) + \tilde{\omega}_Y^p(N)\right)\right]. \tag{2.25}$$

(b) *In particular, if* $\lim_{N\to\infty}(\tilde{\omega}_{X_n}(N) + \tilde{\omega}_X(N)) = 0$, $n \ge 1$, *then the following statements are equivalent*:

$$\xi_p(X_n, X) \to 0, \tag{2.26}$$

$$\eta_p(X_n, X) \to 0, \tag{2.27}$$

$$\pi(X_n, X) \to 0 \quad \text{and} \quad \lim_{N\to\infty} \sup_{n\ge1} \tilde{\omega}_{X_n}(N) = 0. \tag{2.28}$$

REMARK 2.1. Note that the equality $\eta_p = \xi_p$ may fail in general. The problem of getting explicit representation for ξ_p is still open.

(iii) Further, let $U = \mathbb{R}^\infty$ and consider the space of $\mathscr{X}^\infty = \mathscr{X}(\mathbb{R}^\infty)$ of all random sequences $X = (X^{(1)}, X^{(2)}, \dots)$. We will use the following simple metrics in $\mathscr{X}^\infty$ and relations between them.

(iii.1) *Lévy metric*:

$$L(X, Y) := \inf\{\varepsilon > 0\colon F_X(x - \varepsilon\iota) - \varepsilon \le F_Y(x) \le F_X(x + \varepsilon\iota) + \varepsilon\} \tag{2.29}$$

for all $x \in \mathbb{R}^\infty$, where $F_X(x) := \Pr\{\bigcap_{i=1}^\infty [X^{(i)} \le x^{(i)}]\}$, $x = (x^{(1)}, x^{(2)}, \dots)$, is the d.f. of X and $\iota := (1, 1, \dots)$.

(iii.2) *Kolmogorov* (*uniform*) *metric*:

$$\rho(X, Y) := \sup\{|F_X(x) - F_Y(x)|\colon x \in \mathbb{R}^\infty\}. \tag{2.30}$$

(iii.2) *Weighted Kolmogorov (semi-)metric:*

$$(2.31)\quad \rho_p(X,Y) := \sup\{M^p(x)|F_X(x) - F_Y(x)|: x \in \mathbb{R}^\infty\}, \qquad p > 0,$$

where $M(x) := \inf_{i\geq 1}|x^{(i)}|$, $x \in \mathbb{R}^\infty$.

Lemma 2.6. (a) *For any* $\beta > 0$, $X, Y \in \mathscr{X}^\infty$,

$$(2.32)\qquad L^{\beta+1}(X,Y) \leq \mathbf{E}|X - Y|_\infty^\beta,$$

where $\|x\|_\infty := \sup_{i\geq 1}|x^{(i)}|$,

$$(2.33)\qquad L(X,Y) \leq \rho(X,Y)$$

and

$$(2.34)\qquad L^{p+1}(X,Y) \leq 2^p\rho_p(X,Y).$$

(b) *If* $Y = (Y^{(1)}, Y^{(2)}, \dots)$ *has bounded marginal densities* $p_{Y^{(i)}}$, $i = 1, 2, \dots,$ *with* $A_i := \sup_{x\in\mathbb{R}} p_{Y^{(i)}}(x) < \infty$ *and* $A := \sum_{i=1}^\infty A_i$, *then*

$$(2.35)\qquad \rho(X,Y) \leq (1 + A)L(X,Y).$$

Moreover, if $X, Y \in \mathscr{X}_+^\infty = \mathscr{X}(\mathbb{R}_+^\infty)$ *(i.e.* X, Y *have nonnegative components), then*

$$(2.36)\qquad L^{p+1}(X,Y) \leq \rho_p(X,Y)$$

and

$$(2.37)\qquad \rho(X,Y) \leq \Lambda(p)A^{p/(1+p)}\rho_p^{1/(1+p)}(X,Y), \qquad p > 0,$$

where

$$(2.38)\qquad \Lambda(p) := (1 + p)p^{-p/(1+p)}.$$

Proof. (a) The inequalities (2.32) and (2.33) are obvious. One can obtain (2.34) in the same manner as (2.36) which we are going to prove completely.

(b) Let $L(X,Y) < \varepsilon$. Further, for each $x \in \mathbb{R}^\infty$ and $n = 1, 2, \dots,$ let $x_n := (x^{(1)}, \dots, x^{(n)}, \infty, \infty, \dots)$. Then

$$\begin{aligned} F_X(x_n) - F_Y(x_n) &\\ &\leq \varepsilon + F_Y(x_n + \varepsilon\iota) - F_Y(x_n)\\ &\leq \varepsilon + \left[F_Y(x_n + \varepsilon\iota) - F_Y\left((x^{(1)}, x^{(2)} + \varepsilon, \dots, x^{(n)} + \varepsilon, \infty, \infty, \dots)\right)\right]\\ &\quad + \cdots + \left[F_Y\left((x^{(1)}, \dots, x^{(n-1)}, x^{(n)} + \varepsilon, \infty, \infty, \dots)\right) - F_Y(x_n)\right]\\ &\leq \varepsilon + [A_1 + \cdots + A_n]\varepsilon. \end{aligned}$$

Analogously,

$$F_Y(x_n) - F_X(x_n) \leq F_Y(x_n) - F_Y(x_n - \varepsilon\iota) + \varepsilon \leq \varepsilon + [A_1 + \cdots + A_n]\varepsilon.$$

Letting $n \to \infty$, we obtain $\rho(X,Y) \leq (1 + A)\varepsilon$ which proves (2.35).

Further, let $L(X,Y) > \varepsilon > 0$. Then there exists $x_0 \in \mathbb{R}_+^\infty$ such that $|F_X(x) - F_Y(x)| > \varepsilon$ for all $x \in [x_0, x_0 + \varepsilon\iota]$ (i.e. $x^{(i)} \in [x_0^{(i)}, x_0^{(i)} + \varepsilon]$ for all $i \geq 1$). Hence,

$$\begin{aligned}\rho_p(X,Y) &\geq \sup\{M^p(x)\varepsilon: x \in [x_0, x_0 + \varepsilon\iota]\}\\ &\geq \varepsilon \inf_{z \in \mathbb{R}_+^\infty} \sup_{x \in [z, z+\varepsilon\iota]} M^p(x) = \varepsilon \sup_{x \in [0\iota, \varepsilon\iota]} M^p(x) = \varepsilon^{1+p}.\end{aligned}$$

Letting $\varepsilon \to L(X,Y)$, we obtain (2.36).

By (2.35) and (2.36), we obtain

$$\rho(X,Y) \leq (1+A)\rho_p^{1/(1+p)}(X,Y). \tag{2.39}$$

Next, we shall use the homogeneity of ρ and ρ_p in order to improve (2.39). Namely, using the equalities

$$\rho(cX, cY) = \rho(X,Y), \qquad \rho_p(cX, cY) = c^p\rho_p(X,Y), \qquad c > 0, \tag{2.40}$$

we have, by (2.39),

$$\begin{aligned}\rho(X,Y) &\leq \left(1 + \frac{1}{c}A\right)\rho_p^{1/(1+p)}(cX, cY)\\ &= \left(c^{p/(1+p)} + c^{-1/(1+p)}A\right)\rho_p^{1/(1+p)}(X,Y).\end{aligned} \tag{2.41}$$

Minimizing the right-hand side of (2.41) w.r.t. $c > 0$ we obtain (2.37). □

REMARK 2.2. The inequality (2.39) was proved by Zolotarev (1983b) in the case of real-valued X and Y. The inequality (2.37) is an improvement of (2.39). For example, if $p \to 0$, then the right-hand sides of (2.39) and (2.37) tend to $(1+A)\rho(X,Y)$ and $\rho(X,Y)$, respectively.

3. Convergence rate for maxima of processes. Let $U = \mathbf{B} := \mathbf{l}_r[T]$, $d(x,y) = \|x - y\|_r$ [see (1.1) and (1.2)] and, as in (2.3), (2.4) and (2.5), define the following compound metrics in $\mathscr{X}(\mathbf{B})$: For any $r \in [1, \infty]$,

$$\mathscr{L}_{p,r}(X,Y) = \left[\mathbf{E}\|X - Y\|_r^p\right]^{1/p}, \qquad p \geq 1, \tag{3.1}$$

$$\mathscr{L}_{\infty,r}(X,Y) = \operatorname{ess\,sup}\|X - Y\|_r, \tag{3.2}$$

$$\tau_{s,r}(X,Y) = \mathbf{E}\|Q_sX - Q_sY\|_r, \qquad s > 0, \tag{3.3}$$

where the homeomorphism Q_s on U is defined by

$$(Q_sx)(t) = |x(t)|^s \operatorname{sign} x(t). \tag{3.4}$$

Let $l_{p,r} = \hat{\mathscr{L}}_{p,r}$ and $\kappa_{s,r} = \hat{\tau}_{s,r}$ (see Lemmas 2.3 and 2.4). Further, let $\mathbf{X} = \{X_i, i \geq 1\}$, $\mathbf{Y} = \{Y_i, i \geq 1\}$ be two sequences of *dependent* r.v.'s on $\mathscr{X}(\mathbf{B})$ and let us fix a sequence $\mathbf{C} \in \mathscr{C}$ [see (1.3)].

Define the sample maxima with normalizing constants $c_j(n)$ by

$$\overline{\mathbf{X}}_n = \bigvee_{j=1}^{\infty} c_j(n) X_j, \qquad \overline{\mathbf{Y}}_n = \bigvee_{j=1}^{\infty} c_j(n) Y_j \tag{3.5}$$

(see Section 1). In the next two theorems we will obtain estimates of the closeness between $\overline{\mathbf{X}}_n$ and $\overline{\mathbf{Y}}_n$ in terms of the metrics $\mathscr{L}_{p,r}$ and $\tau_{p,r}$. In particular, if $\mathbf{X}$ and $\mathbf{Y}$ have i.i.d. components and Y_1 is a simple max-stable process [see (1.4)], we will obtain the rate of convergence of $\overline{X}_n$ to Y_1 in terms of $\ell_{p,r}$ and $\kappa_{s,r}$. With this aim in mind we need some conditions on the sequences $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{C}$.

(C.1) Let

$$a_p(n) = \left[\sum_{j=1}^{\infty} c_j^p(n)\right]^{\bar{p}} \quad \text{for } p \in (0,\infty),\ \bar{p} = \begin{cases} 1, & p \le 1, \\ 1/p, & p > 1, \end{cases} \tag{3.6}$$

and

$$a_\infty(n) = \sup_{j \ge 1} c_j(n). \tag{3.7}$$

Assume that

$$\begin{aligned} &a_\alpha(n) < \infty \quad \text{for some fixed } \alpha \in (0,1) \text{ and all } n \ge 1, \\ &a_1(n) = 1 \quad \text{for all } n \ge 1, \\ &a_p(n) \to 0 \quad \text{as } n \to \infty \text{ for all } p > 1. \end{aligned} \tag{3.8}$$

The main examples of $\mathbf{C}$ satisfying the condition (C.1) are the Cesàro and Abel summation schemes.

Cesàro sum:

$$c_j(n) = \begin{cases} 1/n, & j = 1,2,\dots,n, \\ 0, & j = n+1, n+2, \dots, \end{cases} \tag{3.9}$$

$$a_p(n) = \begin{cases} n^{1-p} & \text{for } p \in (0,1], \\ n^{1/p-1} & \text{for } p \in [1,\infty]. \end{cases} \tag{3.10}$$

Abel sum:

$$c_j(n) = (e^{1/n} - 1)e^{-j/n}, \qquad j = 1,2,\dots, \ n = 1,2,\dots, \tag{3.11}$$

$$a_p(n) = \begin{cases} (1 - e^{-1/n})^p/(1 - e^{-p/n}) \sim (1/p)n^{1-p} \\ \qquad\qquad \text{as } n \to \infty \text{ for any } p \in (0,1), \\ (1 - e^{-1/n})(1 - e^{-p/n})^{-1/p} \sim p^{-1/p} n^{1/p-1} \\ \qquad\qquad \text{as } n \to \infty \text{ for any } p \in [1,\infty), \\ 1 - e^{-1/n} \sim 1/n \quad \text{as } n \to \infty \text{ for } p = \infty. \end{cases} \tag{3.12}$$

The following condition concerns the sequences **X** and **Y**.

(C.2) Let $\alpha \in (0,1)$ be such that $a_\alpha(n) < \infty$ [see (3.8)] and assume that

$$\sup_{j\geq 1} \mathbf{E}\left|X_j(t)\right|^\alpha < \infty \quad \text{for any } t \in T, \tag{3.13}$$

$$\sup_{j\geq 1} \mathbf{E}\left|Y_j(t)\right|^\alpha < \infty \quad \text{for any } t \in T. \tag{3.14}$$

Condition (C.2) is quite natural. For example, if Y_j, $j \geq 1$, are independent copies of a max-stable process [see de Haan (1984)], then all one-dimensional marginal d.f.'s are of the form $\exp\{-\beta(t)/x\}$, $x \geq 0$ [for some $\beta(t) \geq 0$] and hence (3.14) holds. In the simplest one-dimensional case $T = \{t_0\}$ [$\{X_j = X_j(t_0), j \geq 1\}$ are i.i.d. r.v.'s as well as $\{Y_j = Y_j(t_0),\ j \geq 1\}$ are i.i.d. r.v.'s with $F_{Y_1}(x) := \exp\{-1/x\}$, $x \geq 0$], one can easily check that the condition (3.13) is necessary to have a polynomial rate of the uniform convergence of the d.f. of $(1/n)\bigvee_{j=1}^n X_j$ to the extreme-value distribution F_{Y_1} [see Omey and Rachev (1988)].

THEOREM 3.1. (a) *Let* **X**, **Y** *and* **C** *satisfy* (C.1) *and* (C.2). *Let* $1 < p \leq r \leq \infty$ *and*

$$\mathscr{L}_{p,r}(X_j, Y_j) \leq \mathscr{L}_{p,r}(X_1, Y_1) < \infty \quad \textit{for all } j = 1, 2, \dots . \tag{3.15}$$

Then

$$\mathscr{L}_{p,r}(\overline{X}_n, \overline{Y}_n) \leq a_p(n)\mathscr{L}_{p,r}(X_1, Y_1) \to 0 \quad \textit{as } n \to \infty. \tag{3.16}$$

(b) *If* **X** *and* **Y** *have i.i.d. components,* $1 < p \leq r \leq \infty$ *and* $l_{p,r}(X_1, Y_1) < \infty$, *then*

$$l_{p,r}(\overline{X}_n, \overline{Y}_n) < a_p(n) l_{p,r}(X_1, Y_1) \to 0 \quad \textit{as } n \to \infty. \tag{3.17}$$

In particular, if **Y** *satisfies the "max-stable property"*

$$\overline{Y}_n =_d Y_1, \tag{3.18}$$

then

$$l_{p,r}(\overline{X}_n, Y_1) \leq a_p(n) l_{p,r}(X_1, Y_1) \to 0 \quad \textit{as } n \to \infty. \tag{3.19}$$

PROOF. (a) Let $1 < p < r < \infty$. By (C.1), (C.2) and the Tchebycheff inequality, we have

$$\Pr\{\overline{X}_n(t) > \lambda\} \leq \lambda^{-\alpha} a_\alpha(n) \sup_{j\geq 1} \mathbf{E} X_j(t)^\alpha \to 0 \quad \text{as } \lambda \to \infty$$

and hence

$$\Pr\{\overline{X}_n(t) + \overline{Y}_n(t) < \infty\} = 1 \quad \text{for any } t \in T.$$

For any $\omega \in \Omega$ such that $\overline{X}_n(t)(\omega) + \overline{Y}_n(t)(\omega) < \infty$, we have

$$\overline{X}_n(t)(\omega) = \bigvee_{j=1}^{m} c_j(n) X_j(t)(\omega) + \varepsilon_\omega(m), \qquad \lim_{m\to\infty} \varepsilon_\omega(m) = 0,$$

$$\overline{Y}_n(t)(\omega) = \bigvee_{j=1}^{m} c_j(n) Y_j(t)(\omega) + \delta_\omega(m), \qquad \lim_{m\to\infty} \delta_\omega(m) = 0$$

and hence

$$\left|\overline{X}_n(t)(\omega) - \overline{Y}_n(t)(\omega)\right| \le \bigvee_{j=1}^{m} \left|c_j(n) X_j(t)(\omega) - c_j(n) Y_j(t)(\omega)\right| + |\varepsilon_\omega(m)| + |\delta_\omega(m)|.$$

So, with probability 1,

$$\left|\overline{X}_n(t) - \overline{Y}_n(t)\right| \le \bigvee_{j=1}^{\infty} c_j(n) \left|X_j(t) - Y_j(t)\right|. \tag{3.20}$$

Using the Minkowski inequality and the fact that $p/r \le 1$, we obtain

$$\begin{aligned}
\mathscr{L}_{p,r}(\overline{X}_n, \overline{Y}_n) &= \left\{ \mathbf{E}\left[\int_T \left|\overline{X}_n(t) - \overline{Y}_n(t)\right|^r dt\right]^{p/r} \right\}^{1/p} \\
&\le \left\{ \mathbf{E}\left[\int_T \left[\bigvee_{j=1}^{\infty} c_j(n) \left|X_j(t) - Y_j(t)\right|\right]^r dt\right]^{p/r} \right\}^{1/p} \\
&= \left\{ \mathbf{E}\left[\int_T \bigvee_{j=1}^{\infty} c_j^r(n) \left|X_j(t) - Y_j(t)\right|^r dt\right]^{p/r} \right\}^{1/p} \\
&\le \left\{ \mathbf{E}\left[\int_T \sum_{j=1}^{\infty} c_j^r(n) \left|X_j(t) - Y_j(t)\right|^r dt\right]^{p/r} \right\}^{1/p} \\
&\le \left\{ \mathbf{E} \sum_{j=1}^{\infty} c_j^p(n) \left[\int_T \left|X_j(t) - Y_j(t)\right|^r dt\right]^{p/r} \right\}^{1/p} \\
&= \left\{ \sum_{j=1}^{\infty} c_j^p(n) \mathscr{L}_{p,r}^p(X_j, Y_j) \right\}^{1/p} \\
&\le a_p(n) \mathscr{L}_{p,r}(X_1, Y_1).
\end{aligned}$$

The statement for $p < r = \infty$, $p = r = \infty$ can be proved in an analogous way.

(b) By the definition of the minimal metric [see (1.5) and (2.1)], we have

$$\begin{aligned}
\hat{\mathscr{L}}_{p,r}(\overline{X}_n, \overline{Y}_n) &\leq \inf\Bigg\{\mathscr{L}_{p,r}(\tilde{X}, \tilde{Y})\colon \tilde{X} = \bigvee_{j=1}^{\infty} c_j(n)\tilde{X}_j,\ \tilde{Y} = \bigvee_{j=1}^{\infty} c_j(n)\tilde{Y}_j, \\
&\qquad \text{where } \{\tilde{X}_j, j \geq 1\} \text{ and } \{\tilde{Y}_j, j \geq 1\} \text{ have i.i.d. components} \\
&\qquad \text{and } (\tilde{X}_j, \tilde{Y}_j) =_d (\tilde{X}_1, \tilde{Y}_1),\ \tilde{X}_1 =_d X_1,\ \tilde{Y}_1 =_d Y_1\Bigg\} \\
&\leq \inf\Bigg\{\left[\sum_{j=1}^{\infty} c_j^p(n)\mathscr{L}_{p,r}^p(\tilde{X}_j, \tilde{Y}_j)\right]^{1/p}\colon \\
&\qquad \{\tilde{X}_j,\ j \geq 1\} \text{ are i.i.d.}, \{\tilde{Y}_j,\ j \geq 1\} \text{ are i.i.d.}, \\
&\qquad (\tilde{X}_j, \tilde{Y}_j) =_d (\tilde{X}_1, \tilde{Y}_1),\ \tilde{X}_1 =_d X_1,\ \tilde{Y}_1 =_d Y_1\Bigg\} \\
&= \inf\Bigg\{\left[\sum_{j=1}^{\infty} c_j^p(n)\mathscr{L}_{p,r}^p(\tilde{X}_1, \tilde{Y}_1)\right]^{1/p}\colon \\
&\qquad \tilde{X}_1 =_d X_1,\ \tilde{Y}_1 =_d Y_1\Bigg\} \\
&= a_p(n)\hat{\mathscr{L}}_{p,r}(X_1, Y_1).
\end{aligned}$$

By Lemma 2.3 [see (2.9)] we obtain (3.17).

Finally, (3.19) follows immediately from (3.17) and (3.18). □

Even in the univariate case the estimate (3.19) is new and seems to be very hard to obtain using traditional techniques.

COROLLARY 3.1. *Let* $\{X_j,\ j \geq 1\}$ *and* $\{Y_j,\ j \geq 1\}$ *be random sequences with i.i.d. real-valued components and* $F_{Y_1}(x) = \exp\{-1/x\}$, $x \geq 0$. *Then*

$$l_p\left(\bigvee_{j=1}^{\infty} c_j(n)X_j, Y_1\right) \leq a_p(n)l_p(X_1, Y_1), \qquad p \in [1, \infty], \tag{3.21}$$

where the metric l_p *is given by* (2.14) *and* (2.15).

Let π_r be the Prohorov metric [see (2.7)] in the space $\mathscr{X}(\mathbf{B}, \|\cdot\|_r)$). Using the relationship between π_r and $\mathscr{L}_{p,r}$ [see (2.11)], we get the following rate of convergence of $\overline{X}(n)$ to Y_1 under the assumptions of Theorem 3.1(b).

COROLLARY 3.2. *Let the assumption of Theorem* 3.1(b) *be valid and* (3.18) *hold. Then*

$$\pi_r(\overline{X}(n), Y_1) \leq a_p(n)^{p/(1+p)} l_{p,r}(X_1, Y_1)^{p/(1+p)}. \tag{3.22}$$

The next theorem is devoted to a similar estimate of the closeness between $\overline{X}_n$ and $\overline{Y}_n$ but now in terms of the metric $\tau_{p,r}$ [see (3.3)], $p > 0$, and its corresponding minimal metric $\kappa_{p,r}$. Moreover, we shall relax the restriction $1 < p \le r \le \infty$ imposed in Theorem 3.1.

THEOREM 3.2. (a) *Let* (C.1) *and* (C.2) *hold,* $p > 0$ *and* $1/p < r \le \infty$. *Assume that*

$$\tau_{p,r}(X_j, Y_j) \le \tau_{p,r}(X_1, Y_1) < \infty, \qquad j = 1, 2, \dots. \tag{3.23}$$

Then

$$\tau_{p,r}(\overline{X}_n, \overline{Y}_n) \le \alpha_{\bar{p}}(n)\tau_{p,r}(X_1, Y_1) \to 0 \quad \text{as } n \to \infty, \tag{3.24}$$

where $\alpha_{\bar{p}}(n) := \sum_{j=1}^{\infty} c_j^{\bar{p}}(n)$, $\bar{p} := p\min(1, r)$.

(b) *If* $\mathbf{X}$ *and* $\mathbf{Y}$ *consist of i.i.d. r.v.'s, then* $\kappa_{p,r}(X_1, Y_1) < \infty$ *implies*

$$\kappa_{p,r}(\overline{X}_n, \overline{Y}_n) \le \alpha_{\bar{p}}(n)\kappa_{p,r}(X_1, Y_1) \to 0 \quad \text{as } n \to \infty. \tag{3.25}$$

Moreover, assuming that (3.18) *holds, we have*

$$\kappa_{p,r}(\overline{X}_n, Y_1) \le \alpha_{\bar{p}}(n)\kappa_{p,r}(X_1, Y_1) \to 0 \quad \text{as } n \to \infty. \tag{3.26}$$

PROOF. (a) By (C.1) and (C.2),

$$\Pr\left(\bigvee_{j=1}^{\infty} c_j^p(n)(Q_pX_j)(t) + \bigvee_{j=1}^{\infty} c_j^p(n)(Q_pY_j)(t) < \infty\right) = 1.$$

Hence, as in Theorem 3.1, we have

$$\begin{aligned}
&\left|Q_p\left(\bigvee_{j=1}^{\infty} c_j(n)X_j\right)(t) - Q_p\left(\bigvee_{j=1}^{\infty} c_j(n)Y_j\right)(t)\right| \\
&\qquad = \left|\bigvee_{j=1}^{\infty} c_j^p(n)(Q_pX_j)(t) - \bigvee_{j=1}^{\infty} c_j^p(n)(Q_pY_j)(t)\right| \\
&\qquad \le \bigvee_{j=1}^{\infty} c_j^p(n)\left|(Q_pX_j)(t) - (Q_pY_j)(t)\right|.
\end{aligned}$$

Next, denote $\tilde{r} = \min(1, 1/r)$ and then

$$\begin{aligned}
&\tau_{p,r}(\overline{X}_n, \overline{Y}_n) \\
&\quad = \mathbf{E}\left[\int_T\left|Q_p\left(\bigvee_{j=1}^{\infty} c_j(n)X_j\right)(t) - Q_p\left(\bigvee_{j=1}^{\infty} c_j(n)Y_j\right)(t)\right|^r dt\right]^{\tilde{r}} \\
&\quad \le \mathbf{E}\left[\int_T \bigvee_{j=1}^{\infty} c_j^{pr}(n)\left|(Q_pX_j)(t) - (Q_pY_j)(t)\right|^r dt\right]^{\tilde{r}} \\
&\quad \le \mathbf{E}\left[\sum_{j=1}^{\infty} \int_T c_j^{pr}(n)\left|(Q_pX_j)(t) - (Q_pY_j)(t)\right|^r dt\right]^{\tilde{r}} \\
&\quad \le \sum_{j=1}^{\infty} c_j^{pr\tilde{r}}(n)\tau_{p,r}(X_j, Y_j) \le \alpha_{\bar{p}}(n)\tau_{p,r}(X_1, Y_1).
\end{aligned}$$

(b) Passing to the minimal metrics, as in Theorem 3.1(b), we obtain (3.25) and (3.26). □

Note that (3.26) may be viewed as infinite analog of the following well-known estimate of the convergence rate for the univariate maxima.

COROLLARY 3.3. *Let* **X** *and* **Y** *consist of i.i.d. real-valued r.v.'s and* $F_{Y_1}(x) = \exp\{-1/x\}$, $x \geq 0$. *Then*

$$\kappa_p(X_n, Y_1) \leq \alpha_p(n)\kappa_p(X_1, Y_1), \qquad p > 1, \tag{3.27}$$

where $\alpha_p(n) = \sum_{j=1}^{\infty} c_j^p(n)$ *and*

$$\kappa_p(X, Y) := p\int_{-\infty}^{\infty} |x|^{p-1}|F_X(x) - F_Y(x)|\,dx. \tag{3.28}$$

The proof of (3.27) follows immediately from (3.26) and (2.19) with $Q\colon \mathbb{R} \to \mathbb{R}$ given by $Qx = |x|^p \operatorname{sign} x$.

The main purpose of the next theorem is to refine the estimate (3.22). By Lemmas 2.3 and 2.5, we know that $l_{p,\infty}$ is topologically stronger than

$$\xi_{p,\infty} = \hat{\chi}_{p,\infty}, \tag{3.29}$$

where $\chi_{p,\infty}(X, Y) := [\sup_{t>0} t^p \Pr\{\|X - Y\|_\infty > t\}]^{1/(1+p)}$, $X, Y \in \mathscr{X}(\mathbf{1}_\infty)$. So, in the next lemma we shall show that it is possible to replace $l_{p,\infty}$ with $\xi_{p,\infty}$ in the right-hand side of the inequality (3.22) ($r = \infty$).

THEOREM 3.3. (a) *Let* (C.1) *and* (C.2) *hold and* **X** *and* **Y** *be sequences of r.v.'s taking values in* $\mathscr{X}(\mathbf{1}_\infty)$ *such that*

$$\chi_{p,\infty}(X_j, Y_j) \leq \chi_{p,\infty}(X_1, Y_1) < \infty \quad \textit{for all } j \geq 1. \tag{3.30}$$

Then

$$\chi_{p,\infty}(\overline{X}_n, \overline{Y}_n) \leq \alpha_p^{1/(1+p)}(n)\chi_{p,\infty}(X_1, Y_1) \to 0 \quad \textit{as } n \to \infty, \tag{3.31}$$

where $\alpha_p := a_p^p$, $p > 1$.

(b) *If* **X** *and* **Y** *have i.i.d. components and* (3.18) *holds, then*

$$\xi_{p,\infty}(\overline{X}_n, Y_1) \leq \alpha_p^{1/(1+p)}(n)\xi_{p,\infty}(X_1, Y_1). \tag{3.32}$$

In particular,

$$\begin{aligned}\pi_\infty(\overline{X}_n, Y_1) &\leq \alpha_p^{1/(1+p)}(n)\xi_{p,\infty}(X_1, Y_1)\\ &\leq \alpha_p^{1/(1+p)}(n) l_{p,\infty}(X_1, Y_1)^{p/(1+p)}.\end{aligned} \tag{3.33}$$

PROOF. (a) By (1.2) and (3.20),

$$\begin{aligned}
&\chi_{p,\infty}^{1+p}(\overline{X}_n, \overline{Y}_n) \\
&\le \sup_{u>0} u^p \Pr\left\{ \sup_{t\in T} \bigvee_{j=1}^{\infty} |c_j(n)X_j(t) - c_j(n)Y_j(t)| > u \right\} \\
&\le \sup_{u>0} u^p \Pr\left\{ \sup_{t\in T} |X_j(t) - Y_j(t)| > u/c_j(n) \text{ for some } j \ge 1 \right\} \\
&\le \sum_{j=1}^{\infty} \sup_{u>0} u^p \Pr\left\{ \sup_{t\in T} |X_j(t) - Y_j(t)| > u/c_j(n) \right\} \\
&= \sum_{j=1}^{\infty} c_j^p(n) \chi_{p,\infty}^{1+p}(X_j, Y_j) \le \alpha_p(n) \chi_{p,\infty}^{1+p}(X_1, Y_1).
\end{aligned}$$

(b) Passing to the minimal metrics in (3.31) we get (3.32). Finally, using the inequality (2.23), we obtain (3.33). □

4. Uniform rate of convergence of the distributions of maxima of random sequences. In this section we will always assume $\mathbf{X} := \{X, X_j,\ j \ge 1\}$, $\mathbf{Y} := \{Y, Y_j,\ j \ge 1\}$ are sequences of i.i.d. r.v.'s taking values in $\mathbb{R}_+^\infty$ and

$$\overline{X}_n := \bigvee_{j=1}^{\infty} c_j(n) X_j, \tag{4.1}$$

$$\overline{Y}_n := \bigvee_{j=1}^{\infty} c_j(n) Y_j, \tag{4.2}$$

where the components $Y^{(i)}$, $i \ge 1$, of $\mathbf{Y}$ have extreme-value distribution $F_{Y^{(i)}}(x) = G(x) = e^{-1/x}$, $x \ge 0$.

Further, we shall consider $\mathbf{C} \in \mathscr{C}$ [see (1.3)] subject to the condition

$$\alpha_p(n) := \sum_{j=1}^{\infty} c_j^p(n) \to 0 \quad \text{as } n \to \infty \text{ for any } p > 1. \tag{4.3}$$

Denote $a \circ x := (a^{(1)}x^{(1)}, a^{(2)}x^{(2)}, \dots)$, $bx := (bx^{(1)}, bx^{(2)}, \dots)$ for any $a = (a^{(1)}, a^{(2)}, \dots) \in \mathbb{R}^\infty$, $x = (x^{(1)}, x^{(2)}, \dots) \in \mathbb{R}^\infty$, $b \in \mathbb{R}$.

This section is devoted to the rate of convergence problem $\rho(\overline{X}_n, Y) \to 0$ (as $n \to \infty$) where ρ is the Kolmogorov metric (2.30).

First, note that the assumption, the components $X_j^{(k)}$ of X_j are nonnegative, is not a restriction since

$$\rho(X_n, Y) = \rho\left(\bigvee_{j=1}^{\infty} c_j(n) \tilde{X}_j, Y \right),$$

where $\tilde{X}_j^{(k)} := \max(X_j^{(k)}, 0)$, $k \ge 1$. First, we shall obtain an estimate, unimprovable as a general estimate, of the rate of convergence of $\overline{X}_n$ to Y in terms of weighted Kolmogorov metric ρ_p, $p > 1$ [see (2.31)].

LEMMA 4.1. *Let* $p > 1$. *Then*

$$\rho_p(\overline{X}_n, Y) \le \alpha_p(n)\rho_p(X, Y). \tag{4.4}$$

PROOF. For any $x \in \mathbb{R}^\infty$,

$$\begin{aligned} M^p(x)\left|F_{\overline{X}_n}(x) - F_Y(x)\right| &= M^p(x)\left|F_{\overline{X}_n}(x) - F_{\overline{Y}_n}(x)\right| \\ &\le \lim_{m\to\infty} M^p(x)\prod_{j=1}^{m}\left|F_{X_j}(x/c_j(n)) - F_{Y_j}(x/c_j(n))\right| \\ &\le \sum_{j=1}^{\infty} M^p(x)\left|F_{X_j}(x/c_j(n)) - F_{Y_j}(x/c_j(n))\right| \\ &\le \alpha_p(n)\rho_p(X, Y). \end{aligned}$$

□

REMARK 4.1. The estimate (4.4) was shown by Zolotarev (1983b) in the case of real-valued r.v.'s X_j and Y_j. The fact that such an estimate cannot be improved, without additional assumptions, was shown by Omey and Rachev (1988).

THEOREM 4.1. *Let* $\gamma > 0$ *and* $a = (a^{(1)}, a^{(2)}, \dots) \in \mathbb{R}_+^\infty$ *be such that* $A(a, \gamma) := \sum_{k=1}^{\infty}(a^{(k)})^{-1/\gamma} < \infty$. *Then for any* $p > 1$ *there exists a constant* $c = c(a, p, \gamma)$ *such that*

$$\rho(\overline{X}_n, Y) \le c\alpha_p(n)^{1/(1+p\gamma)}\rho_p(a \circ X, a \circ Y)^{1/(1+p\gamma)}. \tag{4.5}$$

REMARK 4.2. In the estimate (4.5) the "convergence index" $\alpha_p(n)^{1/(1+p\gamma)}$ tends to the right one $\alpha_p(n)$ as $\gamma \to 0$. The constant c has the form

$$c := (1 + \tilde{p})\tilde{p}^{-\tilde{p}/(1+\tilde{p})}[A(a, \gamma)\lambda(\gamma)]^{\tilde{p}/(1+\tilde{p})}, \tag{4.6}$$

where $\tilde{p} := p\gamma$ and

$$\lambda(\gamma) := \gamma \exp\left[\left(1 + \frac{1}{\gamma}\right)\left(\ln\left(1 + \frac{1}{\gamma}\right) - 1\right)\right]. \tag{4.7}$$

Choosing $a = a(\gamma) \in \mathbb{R}^\infty$ such that $(a^{(k)})^{-1/\gamma}\lambda(\gamma) = k^{-\theta}$ for any $k \ge 1$ and some $\theta > 1$, one can obtain that

$$c(a(\gamma), p, \gamma) \to 1 \quad \text{as } \gamma \to 0.$$

However, in this case, $a^{(k)} = a^{(k)}(\gamma) \to \infty$ as $\gamma \to 0$ for any $k \ge 1$ and hence $\rho_p(a \circ X, a \circ Y) \to \infty$ as $\gamma \to 0$.

Proof of Theorem 4.1. Denote

$$\tilde{X}_j := a \circ X_j,\ \tilde{Y}_j =: a \circ Y_j,\ p_k(\gamma) := \sup_{x \geq 0} p_{(\tilde{Y}^{(i)})^{1/\gamma}}(x), \tag{4.8}$$

where $p_X(\cdot)$ means the density of a real-valued r.v. X. Using the inequality (2.37), we have that for any $\tilde{p} > \gamma$,

$$\begin{aligned} \rho(\overline{X}_n, Y) &= \rho\left(\left(a \circ \bigvee_{j=1}^{\infty} c_j(n) X_j\right)^{1/\gamma}, \tilde{Y}^{1/\gamma}\right) \\ &= \rho\left(\bigvee_{j=1}^{\infty} c_j(n)^{1/\gamma} \tilde{X}_j^{1/\gamma}, \tilde{Y}^{1/\gamma}\right) \\ &\leq \Lambda(\tilde{p})\left(\sum_{k=1}^{\infty} p_k(\gamma)\right)^{\tilde{p}/(1+\tilde{p})} \rho_{\tilde{p}}^{1/(1+\tilde{p})}\left(\bigvee_{j=1}^{\infty} c_j(n)^{1/\gamma} \tilde{X}_j^{1/\gamma}, \tilde{Y}^{1/\gamma}\right), \end{aligned} \tag{4.9}$$

where $\Lambda(\tilde{p})$ is given by (2.38). Next we exploit Lemma 4.1 and obtain

$$\begin{aligned} \rho_{\tilde{p}}\left(\bigvee_{j=1}^{\infty} c_j(n)^{1/\gamma} \tilde{X}^{1/\gamma}, \tilde{Y}^{1/\gamma}\right) &\leq \alpha_{\tilde{p}/\gamma}(n) \rho_{\tilde{p}}\left(\tilde{X}_j^{1/\gamma}, \tilde{Y}_j^{1/\gamma}\right) \\ &= \alpha_{\tilde{p}/\gamma}(n) \rho_{\tilde{p}/\gamma}\left(\tilde{X}_j, \tilde{Y}_j\right). \end{aligned} \tag{4.10}$$

Now we can choose $\tilde{p} := p\gamma$. Then, by (4.9) and (4.10),

$$\rho(\overline{X}_n, Y) \leq \Lambda(\tilde{p})\left(\sum_{k=1}^{\infty} p_k(\gamma)\right)^{\tilde{p}/(1+\tilde{p})} \alpha_p(n)^{1/(1+\tilde{p})} \rho_p\left(\tilde{X}_j, \tilde{Y}_j\right)^{1/(1+\tilde{p})}. \tag{4.11}$$

Finally, note that since the components of Y have common d.f. G, then $p_k(\gamma) = (a^{(k)})^{-1/\gamma} \lambda(\gamma)$, where $\lambda(\gamma)$ is given by (4.7). □

In Theorem 4.1 we have no restrictions on the sequence of **C** of normalizing constants $c_j(n)$ [see (1.3) and (4.3)]. However, the rate of convergence $\alpha_p(n)^{1/(1+p\gamma)}$ is close but not equal to the exact rate of convergence $\alpha_p(n)$.

In the next theorem we impose the following conditions on **C** which allow us to reach the exact rate of convergence.

(A.1) There exist an absolute constant $K_1 > 0$ and a sequence of integers $m(n)$, $n = 2, 3, \dots$, such that

$$\sum_{j=1}^{m(n)} c_j(n) \geq K_1 \leq \sum_{j=m(n)+1}^{\infty} c_j(n) \tag{4.12}$$

and $m(n) < n$.

(A.2) There exist constants $\beta \in (0, 1)$, $\theta \geq 0$, $\varepsilon_m(n)$ and $\delta_{im}(n)$, $i = 1, 2, \dots, n = 2, 3, \dots$, such that

$$c_{i+m}(n) = \varepsilon_m(n) c_i(n - m) + \delta_{im}(n) \tag{4.13}$$

and

$$\left\{\sum_{i=1}^{\infty}|\delta_{im}(n)|^{\beta}\right\}^{1/(1+\beta)} \le \theta\alpha_p(n) \tag{4.14}$$

for all $i = 1,2,\ldots, n = 2,3,\ldots$ and $m = m(n)$ defined by (A.1).

(A.3) There exists a constant K_2 such that

$$\alpha_p(n - m(n)) \le K_2\alpha_p(n). \tag{4.15}$$

We shall check now that the Cesàro sum (for any $p > 1$) and the Abel sum (for $1 < p < 1 + 3[\beta/(1+\beta)]$ satisfy (A.1)–(A.3).

EXAMPLE 4.1. Cesàro sum [see (3.9)]. For any $p \ge 1$ we have $\alpha_p(n) = n^{1-p}$.

(A.1) Take $m(n) = [n/2]$, where $[a]$ means the integer part of $a \ge 0$. Then (4.12) holds with $K_1 \le \frac{1}{2}$ and, obviously, $m(n) < n$.

(A.2) The equality (4.13) is valid with $\varepsilon_m(n) = (n - m)/n$ and $\delta_{im} = 0$. Hence, $\theta = 0$ in (4.14).

(A.3) $K_2 := 2^{p-1}$.

EXAMPLE 4.2. Abel sum [see (3.11)]. For any $p \ge 1$ we have

$$\alpha_p(n) = (1 - e^{-1/n})^p/(1 - e^{-p/n}) \sim (1/p)n^{1-p} \quad \text{as } n \to \infty.$$

(A.1) Since $\sum_{j=1}^{k} c_j(n) = 1 - e^{-k/n}$ one can take $m = m(n) = [n \ln 2]$.

(A.2) Let $\varepsilon_m(n) = [Q(n)/Q(n-m)]A_m(n)$, where $Q_n = e^{1/n} - 1$ and $A_m = (n - m)/n$. Then

$$\begin{aligned}\delta_{im}(n) &= (e^{1/n} - 1)e^{-(1+m)/n} - \frac{Q(n)}{Q(n-m)}A_m(n)(e^{-1/(n-m)} - 1)e^{-i/(n-m)}\\ &= Q(n)\left[e^{-(i+m)/n} - A_m(n)e^{-i/(n-m)}\right].\end{aligned}$$

Hence, as $n \to \infty$,

$$\begin{aligned}\sum_{i=1}^{\infty}|\delta_{im}(n)|^{\beta} &= Q^{\beta}(n)\sum_{i=1}^{\infty}\left|e^{-(i+m)/n} - A_m(n)e^{-i/(n-m)}\right|^{\beta}\\ &= Q^{\beta}(n)\sum_{i=1}^{\infty}\left|\left(1 - \frac{i+m}{n}\right) - A_m(n)\left(1 - \frac{i}{n-m}\right)\right|^{\beta}\\ &\quad + Q^{\beta}(n)O(n^{-2\beta})\\ &= Q^{\beta}(n)\sum_{i=1}^{\infty}|n - i - m|^{\beta}\left|\frac{1}{n} - A_m(n)\frac{1}{n-m}\right|^{\beta} + O(n^{-3\beta}).\end{aligned}$$

Since $1/n - A_m(n)[1/(n-m)] = 0$ for any $n \geq 1$ we have

$$\left\{ \sum_{i=1}^{\infty} |\delta_{im}(n)|^{\beta} \right\}^{1/(1+\beta)} = O(n^{-3\beta/(1+\beta)}) \quad \text{as } n \to \infty.$$

Thus (4.14) holds if $3\beta/(1+\beta) \geq p-1$, that is, $p \leq 1 + 3[\beta/(1+\beta)]$.

(A.3) Since $\alpha_p(n-m)/\alpha_p(n) \sim 2^{p-1}$ as $n \to \infty$, there exists K_2 such that (4.15) holds.

THEOREM 4.2. *Let* Y *be max-stable sequences* [*see* (1.4)] *and* $\mathbf{C}$ *satisfy* (A.1)–(A.3). *Let* $a \in \mathbb{R}_+^{\infty}$ *be such that*

$$\mathscr{A}(a) := \sum_{i=1}^{\infty} 1/a^{(i)} < \infty. \tag{4.16}$$

Let $p > 1$, $\tilde{X} = a \circ X$, $\tilde{Y} = a \circ Y$ *and*

$$\lambda_p := \lambda_p(\tilde{X}, \tilde{Y}) := \max\left(\rho_p^{1/(p+1)}(\tilde{X}, \tilde{Y}), \rho_p(\tilde{X}, \tilde{Y}), \Gamma_\beta\right),$$

where

$$\Gamma_\beta := \theta\left\{\left[\mathbf{E}\|\tilde{X}\|_\infty^\beta\right]^{1/(1+\beta)} + \left[\mathbf{E}\|\tilde{Y}\|_\infty^\beta\right]^{1/(1+\beta)}\right\}$$

and β, θ *are given by* (A.2). *Then there exist absolute constants* A *and* B *such that*

$$\lambda_p \leq \underline{A} \Rightarrow \rho(\overline{X}_n, Y) \leq \underline{B}\lambda_p \alpha_p(n). \tag{4.17}$$

REMARK 4.3. As appropriate pairs $(\underline{A}, \underline{B})$ satisfying (4.17) one can take any $\underline{A}$ and $\underline{B}$ such that

$$\underline{A} \leq \underline{C}_8(p, a), \qquad \underline{B} \geq \underline{C}_9(p, a),$$

where the constants $\underline{C}_8$ and $\underline{C}_9$ are defined in the following way. Denote

$$\underline{C}_1(a) := \left(1 + (2/e)^2\right)\mathscr{A}(a)/K_1, \qquad \underline{C}_2(a) := \underline{c}_1(a)(1 + K_2), \tag{4.18}$$

$$\underline{C}_3(a) := (2/e)^2 \mathscr{A}(a), \qquad \underline{C}_4(p, a) := (p/e)^p \mathscr{B}(a)^{-p} \tag{4.19}$$

[where $\mathscr{B}(a) := \min_{i \geq 1} a^{(i)} > 0$ by (4.16)],

$$\underline{C}_5(p, a) := 4\underline{C}_4(p, a)K_1^{-p}, \qquad \underline{C}_6(p, a) := \Lambda(p)\left(\frac{\underline{C}_3(a)}{K_1}\right)^{p/(1+p)} \tag{4.20}$$

[where $\Lambda(p)$ is given by (2.38)],

$$\begin{aligned} &\underline{C}_7(p, a) := \Lambda(p)\underline{C}_3(a)^{p/(1+p)}, \\ &\underline{C}_8(p, a) := \left(2\underline{C}_6(p, a)\underline{C}_2(a)\right)^{-1-p} \end{aligned} \tag{4.21}$$

and

$$\underline{C}_9(p, a) := \max\left\{1, \underline{C}_5(p, a), \underline{C}_7(p, a)\left(1 \vee \alpha_p(2)\right)^{-p/(1+p)}\right\}.$$

The proof of Theorem 4.2 is essentially based on the next lemma. [In the sequel $X' \vee X''$, $X', X'' \in \mathscr{X}(\mathbb{R}_+^\infty)$, always means a random sequence with d.f. $F_{X'}(x)F_{X''}(x)$, $x \in \mathbb{R}_+^\infty$, as well as $\tilde{X}$ means $a \circ X$ where $a \in \mathbb{R}_+^\infty$ satisfies (4.16).]

LEMMA 4.2. (a) *For any* $X', X'', Z \in \mathscr{X}(\mathbb{R}_+^\infty)$ *and* $c > 0$,

$$\rho_p(cX', cX'') = c^p \rho_p(X', X''), \qquad p > 0,$$

$$\rho_p(X' \vee Z, X'' \vee Z) \le \rho_p(X', X'').$$

(b) *If* Y *is a simple max-stable sequence, then for any* $X', X'' \in \mathscr{X}(\mathbb{R}_+^\infty)$ *and* $\delta > 0$,

$$\rho(X' \vee \delta\tilde{Y}, X'' \vee \delta\tilde{Y}) \le \underline{C}_4(p, a)\delta^{-p}\rho_p(X', X''), \tag{4.22}$$

$$\rho(X', \tilde{Y}) \le \underline{C}_7(p, a)\rho_p^{1/(1+p)}(X', \tilde{Y}), \tag{4.23}$$

where $\underline{C}_4, \underline{C}_7$ *are given by* (4.19) *and* (4.21), *respectively.*

(c) *For any* $X', X'', U, V \in \mathscr{X}(\mathbb{R}_+^\infty)$,

$$\rho(X' \vee U, X'' \vee U) \le \rho(X', X'')\rho(U, V) + \rho(X' \vee V, X'' \vee V). \tag{4.24}$$

PROOF. (a) and (c) are obvious.

(b) Let

$$\underline{C}(p) := (p/e)^p = \sup_{x>0} x^{-p} G(x). \tag{4.25}$$

Then

$$\begin{aligned} F_{\tilde{Y}}(x/\delta) &\le \min_{i\ge 1} F_{a^{(i)}Y^{(i)}}(x^{(i)}/\delta) \\ &= \min_{i\ge 1} G(x^{(i)}/a^{(i)}\delta) \le \underline{C}(p)\delta^{-p} \min_{i\ge 1}\left(x^{(i)}/a^{(i)}\right)^p \\ &\le \underline{C}(p)\mathscr{B}(a)^{-p} M(x)^p \delta^{-p}. \end{aligned}$$

Hence, by (4.19) and (4.25),

$$\begin{aligned} \rho(X' \vee \delta\tilde{Y}, X'' \vee \delta\tilde{Y}) &= \sup_{x\in\mathbb{R}_+^\infty} F_{\tilde{Y}}(x/\delta)\left|F_{X'}(x) - F_{X''}(x)\right| \\ &\le \underline{C}_4(p, a)\delta^{-p}\rho_p(X', X''), \end{aligned}$$

which proves (4.22).

Further, by Lemma 2.6 [see (2.37)], we have

$$\begin{aligned}\rho(X', \tilde{Y}) &\le \Lambda(p)\left(\sum_{i=1}^{\infty} \sup_{x>0} p_{a^{(i)}Y^{(i)}}(x)\right)^{p/(1+p)} \rho_p(X', \tilde{Y})^{1/(1+p)} \\ &\le \Lambda(p)\left(\underline{C}(2) \sum_{i=1}^{\infty} 1/a^{(i)}\right)^{p/(1+p)} \rho_p(X', \tilde{Y})^{1/(1+p)} \\ &= \underline{C}_7(p, a)\rho_p(X', \tilde{Y})^{1/(1+p)}. \qquad \square\end{aligned}$$

PROOF OF THEOREM 4.2. If $n = 1, 2$, then by (4.23) and Lemma 4.1 we have

$$\begin{aligned}\rho(\overline{X}_n, Y) &= \rho\left(\bigvee_{i=1}^{\infty} c_i(n)\tilde{X}_i, \tilde{Y}\right) \\ &\le \underline{C}_7(p, a)\rho_p^{1/(1+p)}\left(\bigvee_{i=1}^{\infty} c_i(n)\tilde{X}_i, \tilde{Y}_i\right) \\ &\le \underline{C}_7(p, a)\alpha_p(n)^{1/(1+p)}\rho_p^{1/(1+p)}(\tilde{X}, \tilde{Y}).\end{aligned}$$

Since $\lambda_p \ge \rho_p^{1/(1+p)}(\tilde{X}, \tilde{Y})$ and $\underline{C}_7(p, a)\alpha_p(n)^{1/(1+p)} \le \underline{\mathscr{B}}\alpha_p(n)$ for $n = 1, 2$, we have proved (4.17) for any $\underline{A}$ and $n = 1, 2$.

We now proceed by induction.

Suppose that

$$\rho\left(\bigvee_{j=1}^{\infty} c_j(l)\tilde{X}_j, Y\right) \le \underline{\mathscr{B}}\lambda_p\alpha_p(l) \quad \text{for all } l = 1, \dots, n-1. \tag{4.26}$$

Let $m = m(n)$, $n \ge 3$, be given by (A.1). Then using the triangle inequality we obtain

$$\rho\left(\bigvee_{j=1}^{\infty} c_j(n)\tilde{X}_j, \tilde{Y}\right) \le J_1 + J_2, \tag{4.27}$$

where

$$J_1 := \rho\left(\bigvee_{j=1}^{m} c_j(n)\tilde{X}_j \vee \bigvee_{j=m+1}^{\infty} c_j(n)\tilde{X}_j, \bigvee_{j=1}^{m} c_j(n)\tilde{Y}_j \vee \bigvee_{j=m+1}^{\infty} c_j(n)\tilde{X}_j\right)$$

and

$$J_2 := \rho\left(\bigvee_{j=1}^{m} c_j(n)\tilde{Y}_j \vee \bigvee_{j=m+1}^{\infty} c_j(n)\tilde{X}_j, \tilde{Y}\right).$$

Now we will use the inequality (4.24) in order to estimate J_1:

$$J_1 \le J_1' + J_1'', \tag{4.28}$$

where

$$J_1' := \rho\left(\bigvee_{j=1}^{m} c_j(n)\tilde{X}_j,\ \bigvee_{j=1}^{m} c_j(n)\tilde{Y}_j\right)\rho\left(\bigvee_{j=m+1}^{\infty} c_j(n)\tilde{X}_j,\ \bigvee_{j=m+1}^{\infty} c_j(n)\tilde{Y}_j\right)$$

and

$$J_1'' := \rho\left(\bigvee_{j=1}^{m} c_j(n)\tilde{X}_j \vee \bigvee_{j=m+1}^{\infty} c_j(n)\tilde{Y}_j,\ \bigvee_{j=1}^{m} c_j(n) \vee \bigvee_{j=m+1}^{\infty} c_j(n)\tilde{Y}_j\right).$$

Let us estimate J_1'. Since Y is a simple max-stable sequence

$$\bigvee_{j=m+1}^{\infty} c_j(n)\tilde{Y}_j =_d a \circ \left(\sum_{j=m+1}^{\infty} c_j(n)\right) Y \tag{4.29}$$

[see (1.3) and (1.4)]. Hence, by (4.29), (2.35), (A.1) and (A.2), we have

$$\begin{aligned}
&\rho\left(\bigvee_{j=m+1}^{\infty} c_j(n)\tilde{X}_j,\ \bigvee_{j=m+1}^{\infty} c_j(n)\tilde{Y}_j\right)\\
&\quad\le \left(1 + \left(\frac{2}{l}\right)^2 \sum_{i=1}^{\infty}\left(a^{(i)} \sum_{j=m+1}^{\infty} c_j(n)\right)\right)\\
&\qquad \times L\left(\bigvee_{j=1}^{\infty} c_{j+m}(n)\tilde{X}_j,\ \bigvee_{j=1}^{\infty} c_{j+m}(n)\tilde{Y}_j\right)\\
&\quad\le \underline{C}_1(a) L\left(\bigvee_{j=1}^{\infty} (\varepsilon_m(n)c_j(n-m) + \delta_{jm}(n))\tilde{X}_j,\right.\\
&\qquad\qquad \left.\bigvee_{j=1}^{\infty} (\varepsilon_m(n)c_j(n-m) + \delta_{jm}(n))\tilde{Y}_j\right)\\
&\quad\le \underline{C}_1(a)\left[L\left(\bigvee_{j=1}^{\infty} (\varepsilon_m(n)c_j(n-m) + \delta_{jm}(n))\tilde{X}_j,\ \bigvee_{j=1}^{\infty} \varepsilon_m(n)c_j(n-m)\tilde{X}_j\right)\right.\\
&\qquad + L\left(\bigvee_{j=1}^{\infty} \varepsilon_m(n)c_j(n-m)\tilde{X}_j,\ \bigvee_{j=1}^{\infty} \varepsilon_m(n)c_j(n-m)\tilde{Y}_j\right)\\
&\qquad \left. + L\left(\bigvee_{j=1}^{\infty} \varepsilon_m(n)c_j(n-m)\tilde{Y}_j,\ \bigvee_{j=1}^{\infty} (\varepsilon_m(n)c_j(n-m) + \delta_{jm}(n))\tilde{Y}_j\right)\right]\\
&\quad =: \underline{C}_1(a)(I_1 + I_2 + I_3),
\end{aligned} \tag{4.30}$$

where $\underline{C}_1(a)$ is given by (4.18). Let us estimate I_1 by using (A.2) and the inequality (2.32):

$$I_1 \le \left\{ \mathbf{E} \left\| \bigvee_{j=1}^{\infty} \left(\varepsilon_m(n) c_j(n-m) + \delta_{jm}(n) \right) \tilde{X}_j - \bigvee_{j=1}^{\infty} \varepsilon_m(n) c_j(n-m) \tilde{X}_j \right\|_{\infty}^{\beta} \right\}^{1/(1+\beta)}$$

$$\le \left\{ \mathbf{E} \bigvee_{j=1}^{\infty} \left\| \left(\varepsilon_m(n) c_j(n-m) + \delta_{jm}(n) \right) \tilde{X}_j - \varepsilon_m(n) c_j(n-m) \tilde{X}_j \right\|_{\infty}^{\beta} \right\}^{1/(1+\beta)}$$

$$(4.31) \qquad = \left\{ \mathbf{E} \bigvee_{j=1}^{\infty} \left\| \delta_{jm}(n) \tilde{X}_j \right\|_{\infty}^{\beta} \right\}^{1/(1+\beta)}$$

$$\le \left\{ \mathbf{E} \sum_{j=1}^{\infty} |\delta_{jm}|^{\beta} \|\tilde{X}_j\|_{\infty}^{\beta} \right\}^{1/(1+\beta)}$$

$$\le \left\{ \sum_{j=1}^{\infty} |\delta_{jm}|^{\beta} \right\}^{1/(1+\beta)} \left\{ \mathbf{E} \|\tilde{X}\|_{\infty}^{\beta} \right\}^{1/(1+\beta)}$$

$$\le \theta \alpha_p(n) \left\{ \mathbf{E} \|\tilde{X}\|_{\infty}^{\beta} \right\}^{1/(1+\beta)}.$$

Analogously,

$$(4.32) \qquad I_3 \le \theta \alpha_p(n) \left\{ \mathbf{E} \|\tilde{Y}\|_{\infty}^{\beta} \right\}^{1/(1+\beta)}.$$

In order to estimate I_2 we use the inductive assumption (4.26), condition (A.3) and (2.33):

$$I_2 \le \rho\left(\bigvee_{j=1}^{\infty} \varepsilon_m(n) c_j(n-m) \tilde{X}_j, \bigvee_{j=1}^{\infty} \varepsilon_m(n) c_j(n-m) \tilde{Y}_j \right)$$

$$(4.33) \qquad = \rho\left(\bigvee_{j=1}^{\infty} c_j(n-m) \tilde{X}_j, \bigvee_{j=1}^{\infty} c_j(n-m) \tilde{Y}_j \right)$$

$$\le \underline{\mathscr{B}} \lambda_p \alpha_p(n-m) \le K_2 \underline{\mathscr{B}} \lambda_p \alpha_p(n).$$

Hence, by (4.30)–(4.33) and (4.18), we have

$$(4.34) \qquad \rho\left(\bigvee_{j=m+1}^{\infty} c_j(n) \tilde{X}_j, \bigvee_{j=m+1}^{\infty} c_j(n) \tilde{Y}_j \right) \le \underline{C}_1(a) \left[\Gamma_\beta + K_2 \underline{\mathscr{B}} \lambda_p \right] \alpha_p(n)$$

$$\le \underline{C}_1(a)(1 + K_2 \underline{\mathscr{B}}) \lambda_p \alpha_p(n) \le \underline{C}_2(a) \underline{\mathscr{B}} \lambda_p \alpha_p(n).$$

Next, let us estimate $\rho(\bigvee_{j=1}^{m} c_j(n)\tilde{X}_j, \bigvee_{j=1}^{m} c_j(n)\tilde{Y}_j)$ in J_1'. Since Y is a simple max-stable sequence [see (1.3) and (1.4)] we have

$$\bigvee_{j=1}^{m} c_j(n)\tilde{Y}_j =_d \sum_{j=1}^{m} c_j(n)\tilde{Y}. \tag{4.35}$$

Thus, by (2.37), (4.20) and (A.1),

$$\begin{aligned} &\rho\left(\bigvee_{j=1}^{m} c_j(n)\tilde{X}_j, \bigvee_{j=1}^{m} c_j(n)\tilde{Y}_j\right) \\ &\quad = \rho\left(\bigvee_{j=1}^{m} c_j(n)\tilde{X}_j, \sum_{j=1}^{m} c_j(n)\tilde{Y}\right) \\ &\quad \le \Lambda(p)\left[(2/e)^2 \sum_{i=1}^{\infty}\left(a^{(i)}\sum_{j=1}^{m} c_j(n)\right)^{-1}\right]^{p/(1+p)} \rho_p(\tilde{X}, Y)^{1/(1+p)} \\ &\quad \le \underline{C}_6(p,a)\rho_p(\tilde{X}, Y)^{1/(1+p)} \\ &\quad \le \underline{C}_6(p,a)\lambda_p^{1/(1+p)} \le \underline{C}_6(p,a)\underline{A}^{1/(1+p)}. \end{aligned} \tag{4.36}$$

Using the estimates in (4.34) and (4.36), we obtain the following bound for J_1':

$$J_1' \le \underline{C}_6(p,a)\underline{A}^{1/(1+p)}\underline{C}_2(a)\mathscr{B}\lambda_p\alpha_p(n) \le \tfrac{1}{2}\mathscr{B}\lambda_p\alpha_p(n). \tag{4.37}$$

Now, let us estimate J_1''. By (4.22), (4.29), (A.1) and (4.4), we have

$$\begin{aligned} J_1'' &\le \underline{C}_4(p,a)\rho_p\left(\bigvee_{j=1}^{m} c_j(n)\tilde{X}_j, \bigvee_{j=1}^{m} c_j(n)\tilde{Y}_j\right)\left(\sum_{j=m+1}^{\infty} c_j(n)\right)^{-p} \\ &\le \underline{C}_4(p,a)K_1^{-p}\sum_{j=1}^{m} c_j^p(n)\rho_p(\tilde{X}, \tilde{Y}) \\ &\le \underline{C}_4(p,a)K_1^{-p}\lambda_p\alpha_p(n). \end{aligned} \tag{4.38}$$

Analogously, we estimate J_2 [see (4.27)]:

$$\begin{aligned} J_2 &\le \underline{C}_4(p,a)\rho_p\left(\bigvee_{j=m+1}^{\infty} c_j(n)\tilde{X}_j, \bigvee_{j=m+1}^{\infty} c_j(n)\tilde{Y}_j\right)\left(\sum_{j=1}^{m} c_j(n)\right)^{-p} \\ &\le \underline{C}_4(p,a)K_1^{-p}\sum_{j=m+1}^{\infty} c_j^p(n)\rho_p(\tilde{X}, \tilde{Y}) \\ &\le \underline{C}_4(p,a)K_1^{-p}\lambda_p\alpha_p(n). \end{aligned} \tag{4.39}$$

Since $2\underline{C}_4(p,a)K_1^{-p} \le \mathscr{B}/2$ (see Remark 4.3),

$$J_1'' + J_2 \le \tfrac{1}{2}\mathscr{B}\lambda_p\alpha_p(n) \tag{4.40}$$

by (4.38) and (4.39). Finally, using (4.27), (4.28), (4.37) and (4.40), we obtain (4.26) for $l = n$. □

In the case of the Cesàro sum [see (3.9)] one can refine Theorem 4.2 following the proof of the theorem and using some simplifications (see Example 4.1). Namely, the following assertion holds.

COROLLARY 4.1. *Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. r.v.'s taking values in $\mathbb{R}_+^\infty$. Let $Y = (Y^{(1)}, Y^{(2)}, \dots)$ be a max-stable sequence [see de Haan (1984)] with $F_{Y^{(i)}}(x) = e^{-1/x}$, $x > 0$. Let $a \in \mathbb{R}^\infty$ satisfy* (4.16). *Denote $\bar{\lambda}_p := \lambda_p(\tilde{X}, \tilde{Y}) := \max\{\rho(\tilde{X}, \tilde{Y}), \rho_p(\tilde{X}, \tilde{Y})\}$, $\tilde{X} := a \circ X$, $\tilde{Y} := a \circ \tilde{Y}$. Then there exist constants $\underline{C}$ and $\underline{D}$ such that*

$$\bar{\lambda}_p \le \underline{C} \Rightarrow \rho\left((1/n) \bigvee_{k=1}^{n} X_k, Y\right) \le \underline{D} \bar{\lambda}_p n^{1-p}. \tag{4.41}$$

REMARK 4.4. As an example of a pair $(\underline{C}, \underline{D})$ that fulfills (4.41) one can choose any $(\underline{C}, \underline{D})$ satisfying the inequalities

$$CD\left(\tfrac{2}{3}\right)^{p-1} \le \tfrac{1}{2}, \qquad \underline{D} \ge \max\left(2^p, 4\underline{C}_4(p, a)(2^{p-1} + 6^p)\right),$$

where $\underline{C}_4(p, a)$ is defined by (4.19).

REMARK 4.5. Smith (1982), Cohen (1982), Resnick (1987) and Balkema and de Haan (1988) consider the univariate case ($X, X_1, X_2, \dots \in \mathscr{X}(\mathbb{R})$) of general normalized maxima

$$\rho\left(a_n \bigvee_{i=1}^{n} X_i - b_n, Y\right) \le c(X_1, Y)\phi_{X_1}(n), \qquad n = 1, 2, \dots.$$

In order to extend these type results to the multivariate case ($X, X_1, X_2, \dots \in \mathscr{X}(\mathbf{B})$) using the method developed here in this article, one needs to generalize (1.6) by determining metrics $\underline{\mu}_\phi$ and $\bar{\mu}_\phi$ in $\mathscr{X}(\mathbf{B})$ such that for any $X_1, X_2, Y \in \mathscr{X}(\mathbf{B})$ and $c > 0$,

$$\underline{\mu}_\phi(c(X_1 \vee Y), c(X_2 \vee Y)) \le \phi(c)\bar{\mu}_\phi(X_1, X_2),$$

where ϕ: $[0, \infty) \to [0, \infty)$ is suitably chosen strictly increasing continuous function, $\phi(0) = 0$.

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REFERENCES

BALKEMA, A. A. and DE HAAN, L. (1988). A convergence rate in extreme-value theory. Preprint.

CAMBANIS, S., SIMONS, G. and STOUT, W. (1976). Inequalities for $Ek(X, Y)$ when the marginals are fixed. *Z. Wahrsch. verw. Gebiete* **36** 285–294.

COHEN, J. P. (1982). Convergence rate for the ultimate and penultimate approximation in extreme-value theory. *Adv. in Appl. Probab.* **14** 833–854.

DE HAAN, L. (1984). A spectral representation for max-stable processes. *Ann. Probab.* **12** 1194–1204.

DUDLEY, R. M. (1976). *Probability and Metrics.* Aarhus Univ., Aarhus.

KELLERER, H. G. (1984a). Duality theorems for marginal problems. *Z. Wahrsch. verw. Gebiete* **67** 399–432.

KELLERER, H. G. (1984b). Duality theorems and probability metrics. In *Proc. Seventh Brasov Conf. 1982* 211–220.

OMEY, E. and RACHEV, S. T. (1988). On the rate of convergence in extreme value theory. *Theory Probab. Appl.* To appear.

PROHOROV, YU. V. (1956). Convergence of random processes and limit theorems in probability theory. *Theory Probab. Appl.* **1** 157–214.

RACHEV, S. T. (1981). On minimal metrics in a space of real-valued random variables. *Soviet Math. Dokl.* **23** 425–428.

RACHEV, S. T. (1982). Minimal metrics in the random variables space. In *Probability and Statistical Inference. Proc. Second Pannonian Symp.* (W. Grossman, G. Ch. Pflug and W. Wertz, eds.) 319–327. Reidel, Dordrecht.

RACHEV, S. T. (1984a). On a class of minimal functionals on a space of probability measures. *Theory Probab. Appl.* **29** 41–49.

RACHEV, S. T. (1984b). The Monge–Kantorovich mass transference problem and its stochastic applications. *Theory Probab. Appl.* **29** 647–676.

RACHEV, S. T. (1985). Extreme functionals in the space of probability measures. *Stability Problems for Stochastic Models. Lecture Notes in Math.* **1155** 320–348. Springer, New York.

RACHEV, S. T. (1987). Extreme functionals in the space of probability measures. In *Probability Theory and Mathematical Statistics* (Yu. V. Prohorov et al., eds.) **2** 471–476. VNU Science Press, Utrecht.

RACHEV, S. T. and IGNATOV, TS. G. (1986). Minimality of ideal probabilities metrics. *J. Soviet Math.* **32** 595–608.

RACHEV, S. T. and SHORTT, R. M. (1988). Classification problem for probability metrics. In *Measure and Measurable Dynamics. Contemp. Math.* Amer. Math. Soc., Providence, R.I.

RESNICK, S. I. (1987). Uniform rates of convergence to extreme-value distributions. In *Probability and Statistics: Essays in Honor of Franklin A. Graybill* (J. Srivastava, ed.). North-Holland, Amsterdam.

SAZONOV, V. V. (1981). *Normal Approximation—Some Recent Advances. Lecture Notes in Math.* **879**. Springer, New York.

SMITH, R. L. (1982). Uniform rates of convergence in extreme-value theory. *Adv. in Appl. Probab.* **14** 600–622.

STRASSEN, V. (1965). The existence of probability measures with given marginals. *Ann. Math. Statist.* **36** 423–439.

ZOLOTAREV, V. M. (1975). On the continuity of stochastic sequences generated by recurrence procedures. *Theory Probab. Appl.* **20** 819–832.

ZOLOTAREV, V. M. (1976). Metric distances in spaces of random variables and their distributions. *Math. USSR-Sb.* **3** 373–401.

ZOLOTAREV, V. M. (1983a). Probability metrics. *Theory Probab. Appl.* **28** 278–302.

ZOLOTAREV, V. M. (1983b). Foreword. *Stability Problems for Stochastic Models. Lecture Notes in Math.* **982** vii–xvii. Springer, New York.

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