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A COMPARISON THEOREM FOR STOCHASTIC EQUATIONS WITH VOLTERRA DRIFTS

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A comparison theorem is proved for one-dimensional stochastic equations driven by continuous semimartingales and having Volterra-type drifts. A counterexample which shows that the coefficient of the continuous martingale term cannot be Volterra-type is given. Then the comparison result is used in order to obtain the existence of strong solutions when the Lipschitz condition is replaced by a Hölder-type one.

Assume $(\Omega, \mathscr{F}, P, (\mathscr{F}_t)_{t\geq 0})$ is a filtered probability space satisfying the usual assumptions, that is, $\mathscr{F}$ is complete with respect to P, the null sets in $\mathscr{F}$ belong to $\mathscr{F}_0$ and the filtration $(\mathscr{F}_t)_{t\geq 0}$ is right-continuous. Let M be a continuous real-valued local martingale, N be a continuous real-valued increasing process and define the increasing process C by $C(t) = t + N(t) + \langle M\rangle(t)$. We introduce the class of functions

$$LS = \left\{\rho\colon \rho \text{ is strictly increasing, concave and } \int_{0+}^{1} du/\rho(u) = \infty\right\}.$$

For example functions of the form $x|\log x|^{1-\varepsilon}$ in the neighborhood of 0 which are strictly increasing and concave are in LS. By using Picard iteration and the stochastic Gronwall lemma [5], we obtain as in Protter [4] the following result.

Theorem 1. *Let Z be a real semimartingale with the control process Q in the Metivier–Pellaumail sense* ([3]) *and denote $L(t) = t + Q(t)$. Let $F(t, s, \omega, x)\colon R_+^2 \times \Omega \times R \to R$ be a measurable function in all variables. Assume that*:

1. *For every t, s, x, $F(t, s, \cdot, x)$ is $\mathscr{F}_s$-measurable and for every t, ω, x, $F(t, \cdot, \omega, x)$ is left-continuous with right-hand limits.*
2. *There exist $\rho \in LS$ and the increasing predictable processes γ, $\gamma^r (r > 0)$ locally L-integrable such that*

$$|F(s, s, \omega, x) - F(s, s, \omega, y)|^2 \leq \gamma^r(s)\rho(|x-y|^2)$$

for any $r > 0$, $s \geq 0$, $\omega \in \Omega$ and $|x| \leq r$, $|y| \leq r$, and

$$|F(s, s, \omega, x)|^2 \leq \gamma(s)(1 + |x|^2)$$

for any $s \geq 0$, $\omega \in \Omega$, $x \in R$.

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3. *$\partial F(t, s, \omega, x)/\partial t$ exists and there are $\rho_1 \in LS$, γ_1: $R_+ \to R_+$ locally integrable with respect to Lebesgue measure and the increasing processes γ_2, γ_2^r, $r > 0$, which are predictable, locally L-integrable and such that*

$$|\partial F(t, s, \omega, x)/\partial t - \partial F(t, s, \omega, y)/\partial t|^2 \le \gamma_1(t)\gamma_2^r(s)\rho_1(|x - y|^2)$$

for every t, s, ω and $|x| \le r$, $|y| \le r$, and

$$|\partial F(t, s, \omega, x)/\partial t|^2 \le \gamma(t)\gamma_2(s)(1 + |x|^2)$$

for every t, s, ω, x.

Then for every adapted process H which is continuous on the right with limits on the left there exists a pathwise unique strong solution of

$$X(t) = H(t) + \int_0^t F(t, s, X(s-))\, dZ(s). \tag{1}$$

The following comparison result holds.

THEOREM 2. *Let A_1, A_2: $R_+^2 \times \Omega \times R \to R$, B: $R_+ \times \Omega \times R \to R$ be measurable functions such that:*

(a) *A_i, $\partial A_i(t, s, \omega, x)/\partial t$ are continuous, bounded and*

$$A_1(\cdot, x) \ge A_2(\cdot, y), \qquad \partial A_1(\cdot, x)/\partial t \ge \partial A_2(\cdot, y)/\partial t \quad \text{if } x \ge y.$$

(b) *There exist $\rho_1, \rho_2 \in LS$, γ_1: $R_+ \to R_+$, γ_1 locally integrable with respect to Lebesgue measure and γ_2, γ_3: $R_+ \to R_+$ which are locally C-integrable and such that for every t, s, ω, x, y and $i = 1$ or $i = 2$,*

$$|A_i(t, s, \omega, x) - A_i(t, s, \omega, y)|^2 + |\partial A_i(t, s, \omega, x)/\partial t - \partial A_i(t, s, \omega, y)/\partial t|^2$$
$$\le \gamma_1(t)\gamma_2(s)\rho_2(|x - y|^2),$$
$$|B(s, x) - B(s, y)|^2 \le \gamma_3(s)\rho_2(|x - y|^2).$$

Let X_1, X_2 be solutions of

$$X_i(t) = X_0 + \int_0^t A_i(t, s, X_i(s))\, dN(s) + \int_0^t B(s, X_i(s))\, dM(s), \tag{2}$$

where X_0 is a $\mathscr{F}_0$-measurable random variable. Then $X_1(t) \ge X_2(t)$ a.s. for every $t \ge 0$.

PROOF. Assume $i = 1$ in (b). Then by Theorem 1, X_1 is the unique strong solution of (2). For $\varepsilon > 0$ let X^ε be the strong solution of

$$X^\varepsilon(t) = X_0 + \int_0^t B(s, X^\varepsilon(s))\, dM(s) + \int_0^t [A_1(s, s, X^\varepsilon(s)) + \varepsilon]\, dN(s) + \int_0^t \left(\int_0^s [\partial A_1(s, u, X^\varepsilon(u))/\partial s + \varepsilon]\, dN(u) \right) ds \tag{3}$$

(see [5], Theorem 1).

By the functional comparison theorem of Mel'nikov [2], Theorem 2, we have $X^\varepsilon(t) \geq X^\eta(t)$ for $\varepsilon > \eta$ and $X^\varepsilon(t) \geq X_2(t)$ a.s. Of course $\lim_{\varepsilon \to 0} X^\varepsilon(t) = X_1(t)$ a.s. from which the conclusion is clear. □

REMARK 1. Theorem 2 holds if we take B a Volterra coefficient which is independent of space variable.

REMARK 2. Theorem 2 fails if B is of Volterra-type and depends on the space variable.

Indeed we consider $X_0 = -1$, $N(t) = t$ for all t, M a Brownian motion,

$$A_1(t, s, \omega, x) = 0, \qquad A_2(t, s, \omega, x) = 1,$$

$$B(t, s, \omega, x) = \frac{t-1}{s-1} x \lambda_{R \setminus (1-\varepsilon, 1+\varepsilon)}(s)$$

for all t, s, ω, x and for some $\varepsilon > 0$.

Let, for $a \geq 0$, $\{M^a(t)\}_t$ be the Brownian motion after a and let ξ^a be the exponential martingale defined by $\xi_t^a = \exp\{M^a(t) - t/2\}$. Then for $t \geq 0$ we have

$$X_1(t+1+\varepsilon) = (t+\varepsilon)\xi_t^{1+\varepsilon}\left[-\frac{1}{\varepsilon} + \int_0^{1-\varepsilon} \frac{X_1(s)}{s-1}\, dM(s) + \int_0^t \left(\xi_s^{1+\varepsilon}\right)^{-1} \frac{ds}{(s+\varepsilon)^2} \right],$$

$$X_2(t+1+\varepsilon) = (t+\varepsilon)\xi_t^{1+\varepsilon}\left[1 + \int_0^{1-\varepsilon} \frac{X_2(s)}{s-1}\, dM(s) \right].$$

Utilizing the law of the iterated logarithm, one obtains that

$$\lim_{t \to \infty} \int_0^t \left(\xi_s^{1+\varepsilon}\right)^{-1} \frac{ds}{(s+\varepsilon)^2} = \infty \quad \text{a.s.}$$

and this implies that $\lim_{t \to \infty} P(X_1(t) > X_2(t)) = 1$.

Theorem 3. *Let* $A(t, s, \omega, x)$: $R_+^2 \times \Omega \times R \to R$, $B(s, \omega, x)$: $R_+ \times \Omega \times R \to R$ *be such that*:

(a) A, $\partial A(t, s, \omega, x)/\partial t$ *are bounded and continuous.*
(b) *There exist* $\rho \in LS$ *and* γ: $R_+ \to R_+$ *locally C-integrable such that*

$$|B(t, x) - B(t, y)|^2 \leq \gamma(t)\rho(|x - y|^2).$$

Let X_0 *be a* $\mathscr{F}_0$*-measurable random variable. Then there exists a strong solution of*

$$X(t) = X_0 + \int_0^t A(t, s, X(s))\, dN(s) + \int_0^t B(s, X(s))\, dM(s). \tag{4}$$

Proof. Choose a sequence of functions A^n such that:

(i) $A^n(t, s, \omega, x) \searrow A(t, s, \omega, x)$, $\partial A^n(t, s, \omega, x)/\partial t \searrow \partial A(t, s, \omega, x)/\partial t$ for all s, t, ω, x.
(ii) A^n, $\partial A^n/\partial t$ are bounded and satisfy the Lipschitz condition. According to Theorem 1, there is a pathwise unique strong solution X^n of

$$X^n(t) = X_0 + \int_0^t A^n(t, s, X^n(s))\, dN(s) + \int_0^t B(s, X^n(s))\, dM(s). \tag{5}$$

By Theorem 2 we have

$$X^1(t) \geq X^2(t) \geq \cdots \geq X^n(t) \geq \cdots \quad P\text{-a.s. for all } t \geq 0.$$

Next we show that $X(t) = \lim_n X^n(t)$ is a solution of (4). Let $\{\sigma(n)\}_n$ be a sequence of stopping times reducing M. Then

$$E\left[\sup_{t \leq \sigma(n)} \left| \int_0^t (B(s, X^n(s)) - B(s, X(s)))\, dM(s) \right|^2\right]$$
$$\leq 4E\left[\int_0^{\sigma(n)} |B(s, X^n(s)) - B(s, X(s))|^2\, d\langle M\rangle(s)\right] \to 0$$

by the dominated convergence theorem. Therefore

$$\int_0^t B(s, X^n(s))\, dM(s) \to \int_0^t B(s, X(s))\, dM(s)$$

in probability, uniformly on every compact interval. Since

$$|A^n(t, s, X^n(s)) - A(t, s, X(s))|$$
$$\leq \sup_{x \in [X(s), X^1(s)]} |A^n(t, s, x) - A(t, s, x)|$$
$$+ |A(t, s, X^n(s)) - A(t, s, X(s))| \to 0,$$

it follows by the dominated convergence theorem that

$$\int_0^t [A^n(t, s, X^n(s)) - A(t, s, X(s))] \, dN(s) \to 0 \quad P\text{-a.s. for all } t.$$

The above computation justifies the passing to the limit in (5). □

REMARK 3. Theorem 3 includes as a special case the result of Mel'nikov [2], Theorem 1, and generalizes to a certain extent some results of Barlow and Perkins [1] in the continuous case.

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