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THE BRANCHING ANNIHILATING PROCESS: AN INTERACTING PARTICLE SYSTEM

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The branching annihilating process (BAP) is a special case of the branching annihilating walk of Bramson and Gray. In the BAP each particle places offspring on neighbouring sites at unit rate, but when two particles occupy the same site they annihilate each other. We show that the product measure with density 1/2 is the limit starting from any $A \neq \varnothing$.

Also considered will be the DBAP (double BAP) in one dimension. In this model a particle always places offspring on both neighbouring sites. The limiting measure here depends critically on whether the initial number of particles is odd or even.

1. Introduction. Bramson and Gray (1985) introduced a model they called the branching annihilating random walk. Particles are placed on the one-dimensional lattice $\mathbb{Z}$. Each can do two different things: (1) give birth at exponential rate 1 to a new particle at one of the neighbouring sites; (2) jump to a neighbouring site at rate ρ. When a particle attempts to occupy a site already occupied both particles are annihilated. Bramson and Gray show that for some $\rho < 1/100$ the system can survive indefinitely starting from a finite number of particles, but that if ρ is large enough extinction is certain.

In this paper we consider the special case $\rho = 0$, which we shall call the branching annihilating process (BAP). We shall, however, chiefly consider the process in $\mathbb{Z}^d$.

We define $\beta_t^A \subset \mathbb{Z}^d$ to be the set of occupied sites at time t in a BAP with initial occupied set A, and $\beta_t^A(x) = 1$ or 0 according as site x is occupied or not at time t. The BAP is what is known as a *spin system* since changes in configuration only occur at one site at a time. It may thus be defined by its *flip rates*, $c(x, \beta)$, the rates at which the coordinate $\beta(x)$ flips from 0 to 1 or from 1 to 0 when the system is in state β. For the BAP on $\mathbb{Z}^d$,

$$c(x,\beta) = \sum_{|y-x|=1} \frac{1}{2d}\beta(y), \tag{1}$$

and the infinitesimal generator of the process is

$$Lf(\beta) = \sum c(x,\beta)(f(\beta^x) - f(\beta)),$$

where β^x is the configuration β with the value at x flipped. If we allow branching from x to $x+y$ with rate $p(y)$, $\sum p(y) < \infty$, then we have a generalised BAP with flip rates

$$c(x,\beta) = \sum \beta(x-y)p(y). \tag{2}$$

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For the BAP survival is not a problem since the last particle has no means of annihilating itself, rather interest turns to the limiting distribution. Now the BAP is a process in which the flip rate at a site does not depend on whether there is a particle at the site or not, and it is known [see Durrett (1981), page 111] that if (a) (unlike the BAP) the flip rate is strictly positive, and (b) only determined by sites to the right of the given site, then the limiting distribution is $\nu_{1/2}$, the product measure density $1/2$. In this paper we shall show that for the BAP that is the only nonempty limiting measure.

Further, in Section 9 we shall introduce the double branching annihilating process (DBAP) on $\mathbb{Z}$. In this model a particle always places annihilating offspring on both neighbouring sites. We shall show:

(a) if the initial number of particles is odd, then the limiting distribution is $\nu_{1/2}$;
(b) if the initial number of particles is even, then the number of particles remains bounded;
(c) if the initial measure is ν_p, $0 < p \leq 1$, then the limiting measure is $\nu_{1/2}$;
(d) $\nu_{1/2}$ is not the only limiting measure with density greater than 0;
(e) the stationary measures form a one-parameter family, the borders of product measures ν_p, $0 \leq p \leq 1/2$.

An annihilating random walk with rate ρ is added to the DBAP in Section 10. We show that for this model $\varnothing$ is the only limiting measure.

2. Finite graphs. Consider a finite graph G for which there is a path from each vertex to any other. Initially at least one of the vertices is occupied by a particle. The particles interact as a generalised BAP in which the rate at which a vertex V_i receives offspring is a nonnegative function $f(N_i)$, where N_i is the set of occupied neighbouring sites of V_i, and $f = 0$ only when $N_i = \varnothing$. The state space is $\{0,1\}^G/\varnothing$.

THEOREM 1. *Consider a generalised BAP on a connected finite graph G with initial state nonempty. In the limiting distribution each occupied state has probability* $1/(2^{|G|} - 1)$.

PROOF. Let β and β^* be states of the system such that it is possible for β to jump to β^*. For this to be possible β and β^* can only differ at one vertex V_i with $N_i \neq \varnothing$. The rates at which β jumps to β^* and β^* to β are both $f(N_i)$.

This means the Markov process is doubly stochastic. It is also irreducible since, starting from any occupied state, it is clear the state "all occupied" can be reached, and any state can then be reached by reversing the process. Thus, in the limit, each state has equal probability, in this case $(2^{|G|} - 1)^{-1}$. □

THEOREM 2. *Consider a BAP on $\mathbb{Z}$ where branching only takes place to the right, that is, from n to $n + 1$. Suppose in the initial state the leftmost particle is at* 0. *Then the limiting measure on* $\{0,1\}^{\mathbb{Z}^+}$ *is* $\nu_{1/2}$.

PROOF. Consider the set of sites $[1, n]$. Even though branching only takes place to the right, the Markov process on $\{0, 1\}^{[1, n]}$ is still irreducible, since, starting from any occupied state, it is clear the state "all occupied" can be reached, and any state can then be reached by reversing the process. Further, the rate at which β jumps to β^* is the same as that for which β^* jumps to β. Thus, in the limiting distribution all states have equal probability 2^{-n}, since the all-empty state is included. □

3. The rate of spread of the nearest-neighbour BAP on $\mathbb{Z}$. Given β_t^B, B finite, we may define a stochastic process $G(t)$ on $\mathbb{Z}^+ \cup \{0\} \cup \{\infty\}$ as the number of unoccupied sites between the rightmost particle of β_t^B and the nearest particle to it. $G = \infty$ if there is only one particle. Then the following transitions in $G(t)$ may occur: $0 \to \{1, \dots, \infty\}$ at rate 1, $1 \to 0$ at rate $3/2$ and $\{2, \dots, \infty\} \to 0$ at rate 1. Thus, if we define $H(t) = 0$ when $G(t) = 0$ and $H(t) = 1$ when $G(t) \geq 1$, we have the following bounds on the transition rates for $H(t)$: $0 \to 1$ at rate $\alpha_0 = 1$ and $1 \to 0$ at rate $1 \leq \alpha_1 \leq 3/2$. We shall show the following theorem.

THEOREM 3. *If $R(t)$ and $L(t)$ are the positions of the rightmost and leftmost particles of a BAP, β_t^B, B finite, then*

$$\frac{2}{5} \leq \liminf \frac{R(t) - L(t)}{t} \leq \limsup \frac{R(t) - L(t)}{t} \leq \frac{1}{2} \quad a.s.$$

To prove Theorem 3, we shall need the following lemma.

LEMMA 1. *If there is a sequence of random variables $\{\xi_i\}$ s.t. for some positive integer n_0,*

$$(3) \qquad E\{\xi_{n_0+m} | \xi_m, \dots, \xi_1\} \geq \mu, \qquad m = 1, \dots, \text{ for all } \omega,$$

and $\operatorname{var}(\xi_i)$ *is bounded, then*

$$P\left\{\liminf \frac{1}{n} \sum_{m=1}^{n} \xi_m \geq \mu\right\} = 1.$$

PROOF. From Loève (1955), page 387, we know that for any sequence of random variables $\{\xi_i\}$ with bounded variances

$$\frac{1}{n}\left[\xi_1 - E\{\xi_1\} + \sum_{m=2}^{n} \left(\xi_m - E\{\xi_m | \xi_{m-1}, \dots, \xi_1\}\right)\right] \to 0 \quad \text{a.s.}$$

For each $0 < r \leq n_0$ we apply this result to the subsequence $\xi_r, \xi_{r+n_0}, \dots, \xi_{r+(n-1)n_0}, \dots$, to obtain

$$\frac{1}{n}\left[\xi_r - E\{\xi_r\} + \sum_{m=1}^{n-1} \left(\xi_{r+mn_0} - E\{\xi_{r+mn_0} | \xi_{r+(m-1)n_0}, \dots, \xi_r\}\right)\right] \to 0 \quad \text{a.s.},$$
$$0 < r \leq n_0.$$

Combined with (3), this gives

$$P\left\{\liminf \frac{1}{n} \sum_{m=0}^{n-1} \xi_{r+mn_0} \geq \mu\right\} = 1, \qquad 0 < r \leq n_0.$$

Since this is true for all the subsequences $0 < r \leq n_0$ the lemma follows. □

PROOF OF THEOREM 3. Let $h(t) = P\{H(t) = 0\}$. Then, since $\alpha_0 = 1$ and $1 \leq \alpha_1 \leq 3/2$,

$$\tfrac{3}{2}(1 - h(t)) - h(t) \geq h'(t) \geq 1 - h(t) - h(t),$$

from which follows

$$\begin{aligned}(4)\qquad \tfrac{3}{5} + \left(\tfrac{2}{5} - h_s\right)e^{-5(t-s)/2} &\geq P\{H(t) = 0 | H(s) = h_s\} \\ &\geq \tfrac{1}{2} + \left(\tfrac{1}{2} - h_s\right)e^{-2(t-s)}.\end{aligned}$$

These mixing inequalities show the rate at which $H(t)$ "forgets" its value at time $t - s$ earlier. Let

$N^{\pm}(t)$ = number of jumps of ± 1 made by the rightmost particle up to time t.

Then $N^{+}(t)$ is a Poisson process with rate $1/2$, so that $N^{+}(t)/t \to 1/2$ a.s. $N^{-}(t)$ is a process with rate $1/2I\{H(t) = 0\}$. Let

ξ_n^- = the number of increments of $N^-(t)$ in $[n\delta, (n+1)\delta)$, $\quad n = 0, \dots$.

Then

$$E\{\xi_n^- | H(n\delta) = 0\} = \frac{\delta}{2} + \varepsilon_1(\delta),$$

$$E\{\xi_n^- | H(n\delta) = 1\} = \varepsilon_2(\delta),$$

where $\varepsilon_1(\delta), \varepsilon_2(\delta) = O(\delta^2)$. Using (4), it follows that

$$(5)\qquad E\{\xi_{n_0+m}^- | \xi_m^-, \dots, \xi_1^-\} \leq \left(\frac{3}{5} + \frac{2}{5}e^{-5n_0\delta/2}\right)\left(\frac{\delta}{2} + \varepsilon_1(\delta)\right) + \varepsilon_2(\delta).$$

From (5) and the lemma it follows that for all n_0, δ,

$$(6)\quad P\left\{\liminf \frac{1}{n} \sum_{m=0}^{n-1} \xi_m^- \leq \left(\frac{3}{5} + \frac{2}{5}e^{-5n_0\delta/2}\right)\left(\frac{1}{2}\delta + \varepsilon_1(\delta)\right) + \varepsilon_2(\delta)\right\} = 1.$$

Now $\sum_{m=0}^{n-1} \xi_m^- = N^-(n\delta)$, so putting $n_0 = [\delta^{-2} - 1]$ in (6), we obtain

$$P\left\{\liminf \frac{1}{n\delta} N^-(n\delta) \leq \left(\tfrac{3}{5} + \tfrac{2}{5}e^{-5/2\delta}\right)\left(\frac{1}{2} + \frac{\varepsilon_1(\delta)}{\delta}\right) + \frac{\varepsilon_2(\delta)}{\delta}\right\} = 1.$$

Letting $\delta \to 0$, it follows that

$$P\left\{\limsup \frac{N^-(t)}{t} \leq \frac{3}{10}\right\} = 1.$$

Similar arguments give

$$P\left\{\liminf \frac{N^-(t)}{t} \geq \frac{1}{4}\right\} = 1.$$

Since $R(t) = R(0) + N^+(t) - N^-(t)$, and similar arguments apply for $L(t)$, the theorem follows. □

4. The BAP on $\mathbb{Z}^d$. As remarked, the BAP is a special case of processes for which the flip rate at a site does not depend on whether the site is occupied or not. It is not hard to see that $\nu_{1/2}$ is an equilibrium measure for such processes. Let S be the set of $\eta \subset \{0,1\}^{\mathbb{Z}^d}$ for which $\eta(x_1) = i_1, \dots, \eta(x_n) = i_n$, $i_r = 0$ or 1, $r = 1, \dots, n$. Given η, define η_{x_r} so that $\eta_{x_r}(y) = \eta(y)$, $y \neq x_r$ and $\eta_{x_r}(x_r) = 1 - \eta(x_r)$. The balance equation for $\mu(\eta_t\colon \eta_t \in S)$ is

$$\int_S c(\eta, x_r)\, d\mu(\eta) = \int_S c(\eta_{x_r}, x_r)\, d\mu(\eta_{x_r}),$$

where $c(\eta, x_r)$ is the flip rate at x_r when the process is in state η. We have assumed $c(\eta, x_r) = c(\eta_{x_r}, x_r)$ so the balance equation is satisfied when $\mu(\eta)$ is the uniform measure, $\nu_{1/2}$.

It should be noted that the flip rates for the BAP are those of a stochastic Ising model with $J_R \equiv 0$, using the notation of Liggett (1985), Chapter 4. However, because some of the flip rates are 0, the results of that chapter cannot be used.

When the flip rate c has the property that $c(\phi, x) = 0$ for all x, then $\delta_\varnothing$ is also an equilibrium. We shall show that mixtures of $\nu_{1/2}$ and $\delta_\varnothing$ are the only limiting measures for the nearest-neighbour BAP on $\mathbb{Z}^d$. This will be done in the following stages: (1) in Section 5 we shall show that with initial occupied set A the probability a set B contains an even number of particles at time $t =$ the probability that independent BAPs starting from A and B, respectively, have an even number of occupied sites in common at $t/2$; (2) in Sections 6, 7 and 8 that the number in common approaches ∞; (3) that this implies the probability that the number of sites in common is even approaches $1/2$; and (4) that using (1) the probability that B has an even number of occupied sites at time $t \to 1/2$ as $t \to \infty$, which implies the limiting measure, is $\nu_{1/2}$.

5. The dual process for the BAP. To construct the dual of the generalised BAP in $\mathbb{Z}^d$, we shall first define a percolation substructure of the kind invented by Harris (1978) and developed by Griffeath (1979). Let $\{U_z(t)\colon t \geq 0\}$, $z \in \mathbb{Z}^d$, be independent Poisson processes with rate 1. Let $T_{z,n}$ be the time of the nth event in U_z. Let $\{Y_z(n)\colon n \geq 1\}$, $z \in \mathbb{Z}^d$, be independent i.i.d. sequences with the property that $P\{Y_z(n) = y\} = p(y)$ for all y, z, n. We define a process β with values the subsets of $\mathbb{Z}^d$ so that if $\beta(z) = 1$ at $T_{z,n}$ the value at $z + Y_{z,n}$ is flipped. β has the correct flip rates for a generalised BAP [equation (2)].

The percolation substructure, $P_{[s,t]}$ is obtained by drawing the family of line segments $\{z\} \times [s,t]$, $z \in \mathbb{Z}^d$, and drawing arrows from $(z, T_{z,n})$ to $(z + Y_{z,n}, T_{z,n})$, $s \le T_{z,n} \le t$. We shall abbreviate $P_{[0,t]}$ by P_t. A path from (A, s) to (B, t) is any continuous path consisting of line segments and arrows in which the direction of time is always nondecreasing and which starts at (x, s), $x \in A$, and ends at (y, t), $y \in B$. [There is said to be a path from (x, s) to (x, s).] We define

$$N^A_{[s,t]} = \text{the number of paths from } (A, s) \text{ to } (B, t),$$

and once again we shall abbreviate the subscript $[0, t]$ by t.

LEMMA 2. *The process*

$$\beta^A_t = \{x\colon N^A_t(x) \text{ is odd}\}$$

is a BAP.

PROOF. First, $\beta^A_0 = \{x\colon N^A_0(x) = 1\} = A$. Further, if there is an arrow from $(z, T_{z,n})$ to $(z + Y_{z,n}, T_{z,n})$, then

$$N^A_{T_{z,n}}(z + Y_{z,n}) = N^A_{T^-_{z,n}}(z + Y_{z,n}) + N^A_{T^-_{z,n}}(z),$$

so that the parity at $z + Y_{z,n}$ is flipped iff $N^A_{T^-_{z,n}}(z)$ is odd. Thus β^A_t is a BAP. □

If there is a path from $(x, 0)$ up to (y, t) (time is normally represented as going up), then there is a path from (y, t) down to $(x, 0)$, obtained by reversing time and reversing the direction of all arrows. This defines a dual substructure $\hat{P}_t$. We define

$$\hat{N}^{(B,t)}_s(A) = \text{the number of paths down from } (B, t) \text{ to } (A, t - s).$$

Then the process

$$\hat{\beta}^B_s = \{x\colon \hat{N}^{(B,t)}_s(x) \text{ is odd}\}$$

is a generalised BAP with flip rates

$$\hat{c}(x, \hat{\beta}) = \sum \hat{\beta}(x - y) p(-y).$$

The definition of β^A_t given in the above lemma means that BAPs are cancellative systems. They satisfy standard duality equations [see Griffeath (1979), Chapter 3, Proposition (1.5)], of which the first identity of the next theorem is an example.

THEOREM 4. *For B finite*

$$P\{|\beta^A_t \cap B| \text{ is even}\} = P\{|\hat{\beta}^B_t \cap A| \text{ is even}\}$$
$$= P\{|\beta^A_s \cap \hat{\beta}^B_{t-s}| \text{ is even}\},$$

where the β and $\hat{\beta}$ are independent BAPs.

PROOF. The dual percolation structure $\hat{P}_t$ can be constructed from P_t by reversing time and all arrows. The event "$|\beta_t^A \cap B|$ is even" is equivalent to "$N_t^A(B)$ is even," and since $N_t^A(B) = \hat{N}_t^{(B,t)}(A)$ the first identity follows.

Because of the "forgetfulness" property of Poisson processes, $P_{[0,s]}$ and $P_{[s,t]}$ are independent. Thus, any independent processes β_s^A and $\hat{\beta}_{t-s}^B$ may be constructed from a single realisation P_t with $\hat{P}_{t-s}$ equivalent to $P_{[s,t]}$. Now

$$N_t^A(B) = \sum_x N_s^A(x) N_{[s,t]}^x(B) = \sum_x N_s^A(x) \hat{N}_{t-s}^{(B,t)}(x)$$
$$= \sum_x \beta_s^A(x) \hat{\beta}_{t-s}^B(x) \pmod 2 = \left|\beta_s^A \cap \hat{\beta}_{t-s}^B\right| \pmod 2,$$

and the second identity follows. □

The sort of duality identity which involves the meeting of a process and its dual was used by Griffeath (1978).

We wish to prove $P\{|\beta_t^A \cap B| \text{ is even}\} \to 1/2$ and, putting $s = t/2$ in Theorem 4, this is equivalent to showing $P\{|\beta_t^A \cap \hat{\beta}_t^B \text{ is even}\} \to 1/2$. Now Arratia (1981), pages 921–922, has shown the following.

ARRATIA'S LEMMA. *If a stochastic process on $\mathbb{Z}$ has the properties*

(a) *$X(t)$ increments by ± 1 at a rate which $\to_p \infty$,*
(b) *the change in rate across an increment is bounded,*

then $P\{X(t)$ is even$\} \to 1/2$.

The rate of change of $|\beta_t^A \cap \hat{\beta}_t^B|$ is equal to $2^{-d+1}\sum_{|x-y|=1}\beta_t^A(x)\hat{\beta}_t^B(y)$. Without loss of generality it will thus be sufficient to demonstrate that $|\beta_t^A \cap \hat{\beta}_t^B| \to_p \infty$ for any sets A and B, B finite.

6. The distance to the nearest occupied site. In the next three sections it will be shown that $|\beta_t^A \cap \hat{\beta}_t^B| \to_p \infty$ for any sets A, B. The method will be roughly as follows: A particular site is considered and the "distance" from it to the nearest occupied site. It will be shown that this "distance" may be bounded above by something akin to a negative drift random walk, from which it follows that the probability that the particular site is occupied is bounded away from 0. Further, it will be shown that distant sites do not affect the particular site sufficiently to prevent the conditional probability of occupation being bounded away from 0. The required result will then quickly follow.

In what follows we shall abbreviate β_t^A by β_t. Points in $\mathbb{Z}^d$ will be represented by Roman letters x, y, etc. Where necessary these same letters may also represent the position vector of the points, so we may write $\|x - y\|$ as the Euclidean distance between the points x and y. Further $x = (x_1, \dots, x_d)$ so that x_i is always the ith coordinate of x.

The process β_t defines another process $Q(\mathscr{B}_t) \in \beta_t$, where $\mathscr{B}_t = \{\beta_s : s \le t\}$. Roughly, $Q(\mathscr{B}_t)$ represents a sequence of occupied sites that tends to get closer to the origin, 0. Sometimes we shall abbreviate the process $Q(t)$. $Q_i(t)$ is the component in the direction of the ith coordinate axis.

First, we define $Q(0)$. Consider the set of $y \in \beta_0$ with $\|y\| = \min_{x \in \beta_0}\|x\|$. In this set select those y with minimum y_1. Within this set choose those y with minimum y_2. Continue until a unique point is reached. This is $Q(0)$.

$Q(t)$ is to evolve in such a manner that it has no occupied neighbours closer to the origin than itself. If the particle at $Q(t-)$ branches to such a site, then that site will become $Q(t)$, unless it has occupied neighbours closer to the origin, in which case one of those neighbours is to be $Q(t)$, unless it has an occupied neighbour closer to the origin, etc. An example in two dimensions will help clarify this; $Q(t-)$ is in the position occupied by $*$, the labels $Q(t), y^1, y^0$ appear above the sites they represent:

$$\begin{array}{cccccc}
 & Q(t) & y^1 & y^0 & & \\
0 & 1 & 1 & 0 & \leftarrow & * \\
 & 0 & 1 & 0 & & 0 \\
 & & 1 & & & \\
(0,0) & & & & &
\end{array}$$

At time t, $\beta_t(y^0)$ becomes 1, but $Q(t) \neq y^0$ because at time t, y^0 has an occupied neighbour y^1. y^1 has two occupied neighbours closer to the origin; we choose the neighbour in the direction of the ith axis with i a minimum (i.e., preferring the x_1 direction over the x_2 direction).

If $Q(t-)$ is annihilated by a neighbour at time t, then $Q(t)$ jumps to that neighbour, except for the provisos given in the above paragraph to ensure $Q(t)$ never has occupied neighbours closer to the origin than itself. Another example will help clarify this, with once again $Q(t-)$ represented by *:

$$\begin{array}{ccccc}
 & & y^2 & y^1 & y^0 \\
 & 0 & 1 & 1 & 1 \\
 & & y^3 & & \downarrow \\
 & 0 & 1 & 0 & * \\
 & & y^4 = Q(t) & & \\
 & 0 & 1 & 0 & 0 \\
 & & 0 & & \\
(0,0) & & & &
\end{array}$$

Formally, if either y^0 annihilates the particle at $Q(t-)$ at time t, or the particle at $Q(t-)$ branches to a y^0 which is closer to the origin, then $Q(t) = y^0$, unless there is a set of sites $y^1, \dots, y^n$ such that

(a) $y^1, \dots, y^n \in \beta_t$,
(b) y^i, y^{i+1} are neighbouring sites, $i = 0, \dots, n-1$,
(c) y^i is closer to 0 than y^{i-1}, $i = 1, \dots, n$,
(d) if there are several neighbours of y^i, $y^i - \hat{y}^i_{j_k} \in \beta_t$, then y^{i+1} is the neighbour with least j_k,

in which case $Q(t) = y^n$. (Note: $\hat{y}^i_j$ means the unit vector component of y^i.) That is, $Q(t)$ passes instantaneously through a series of occupied sites each of which is closer to the origin than the last. Provision (d) ensures that the path is uniquely defined. (a), (b) and (c) ensure there is no occupied site neighbouring $Q(t)$ which is closer to 0.

We now define the displacement of $Q(t)$ from the origin in the x_i direction

$$D_i(\mathscr{B}_t) = D_i(t) = |Q_i(t)|. \tag{7}$$

$D_i(t)$ is a jump process with increments less than or equal to 1 which is reflected at the origin. We now define a coupled jump process $D_i^*(t)$ and will show that its increments have negative mean.

1 $D_i^*(0) = D_i(0)$;
2. when $D_i(t)$ jumps, $D_i^*(t)$ jumps by the same increment;
3. in addition to (2), when $D_i(t) = 0$, $D_i^*(t)$ jumps by -1 at Poisson rate $1/2d$.

Let the increments in $D_i^*(t)$ be $X_1, X_2, \dots, X_n, \dots$ at times $\tau_1, \tau_2, \dots, \tau_n, \dots$. Put $\tau_0 = 0$. From (1), (2) and (3) we may deduce that

$$D_i(\tau_n) = \max(0, D_i(\tau_n -) + X_n). \tag{8}$$

We aim to show the following lemma.

LEMMA 3.

$$P\{X_n = 1 | \mathscr{B}_s, 0 \le s \le \tau_{n-1}\} < \frac{1 + 2d^2}{3 + 4d^2} < \frac{1}{2}.$$

PROOF. Define

$$\delta_i(t) = \beta_t(Q(t) + \hat{Q}_i(t)).$$

When $\delta_i(t) = 1$, $Q(t)$ has a neighbour in the x_i direction which could branch to annihilate the particle at $Q(t)$, and thus increase $D_i(t)$ [and $D_i^*(t)$], except that a chain of occupied sites might end in them actually decreasing! In any case, the instantaneous rate of increase of $D_i^*(t)$ is $\le \delta_i(t)/2d$ and its rate of decrease is $\ge 1/2d$, where once again the $>$ sign reflects the possibility of chains of occupied neighbours.

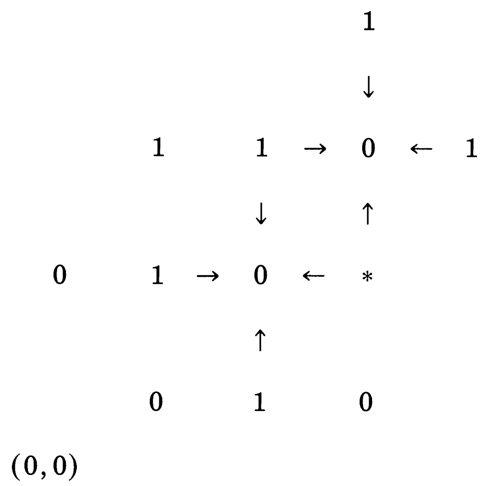
Let $Y_i(\mathscr{B}_t)$ be the first increment in $D_i^*(\mathscr{B}_s)$, $s > t$. Define

$$S_1 = \{\mathscr{B}_t \colon \delta_i(\mathscr{B}_t) = 1\}, \qquad S_0 = \{\mathscr{B}_t \colon \delta_i(\mathscr{B}_t) = 0\}.$$

Suppose $\mathscr{B}_t \in S_1$. We condition on the first of three competing events to occur, $D_i^*(\cdot)$ increments by 1, $D_i^*(\cdot)$ increments by ≤ -1, or neither occurs but $\delta_i(\cdot)$ goes from 1 to 0. The respective instantaneous rates are $\le 1/2d$, $\ge 1/2d$ and $\ge 1/2d$. Since $P\{Y_i(\mathscr{B}_s) = 1\} \le 1/2$, straightforward calculations give

$$\sup_{\mathscr{B}_t \in S_1} P\{Y_i(\mathscr{B}_t) = 1\} \le \tfrac{1}{3} + \tfrac{1}{3} \sup_{\mathscr{B}_s \in S_0} P\{Y_i(\mathscr{B}_s) = 1\}. \tag{9}$$

When $\mathscr{B}_t \in S_0$, the competing events are $D_i^*(\cdot)$ increments by ≤ -1 at rate $\ge 1/2d$, or $D_i^*(\cdot)$ does not increment and $\delta_i(\cdot)$ jumps from 0 to 1 at rate $\in [1/2d, d]$. The upper bound d of the latter rate occurs in the following sort of situation:



Any of the branchings shown will result in $\delta_2(\cdot)$ increasing from 0 to 1 without $D_2^*(\cdot)$ changing. Thus,

$$\sup_{\mathscr{B}_t \in S_0} P\{Y_i(\mathscr{B}_t) = 1\} \le \frac{d}{(2d)^{-1} + d} \sup_{\mathscr{B}_s \in S_1} P\{Y_i(\mathscr{B}_s) = 1\}. \tag{10}$$

Equations (9) and (10) give

$$\sup_{\mathscr{B}_t \in S_1} P\{Y_i(\mathscr{B}_t) = 1\} \le \frac{1 + 2d^2}{3 + 4d^2}.$$

Taking into account (10) the lemma follows. □

Now when $D_i(0) = 0$, (8) shows that $D_i(t)$ is the "queueing process" induced by $D_i^*(t)$, to use the terminology of Feller (1971), pages 194–198. For each $t > 0$ define $n(t)$ to be the number of jumps of $D_i^*(t)$ up to time t, so that

$$\tau_{n(t)} \le t < \tau_{n(t)+1}.$$

Put

$$S_k' = X_{n(t)} + X_{n(t)-1} + \cdots + X_{n(t)-k+1}.$$

Then, as shown in Feller (1971), pages 197 and 198,

$$D_i(t) = \max\left[0, S_1', \dots, S_{n(t)}'\right]. \tag{11}$$

Define a set of dummy random variables $X_0, X_{-1}, \dots$ with each X_i, $i \le 0$, independent of all X_j, $j \ne i$, and such that

$$P\{X_i = 1\} = p_d, \qquad P\{X_i = -1\} = 1 - p_d, \qquad i = 0, -1, \dots,$$

where $p_d = (1 + 2d^2)/(3 + 4d^2) < 1/2$.

It follows from (11) that

$$P\{D_i(t) \ge r\} \le \sum_{k=1}^{n(t)} P\{S_k' \ge r\} \le \sum_{k=1}^{\infty} P\{S_k' \ge r\}. \tag{12}$$

Now Lemma 3 implies that

$$P\{X_i = 1 | X_{i-1}, \dots, X_{n(t)-k+1}\} \le p_d,$$

for $i = n(t) - k + 2, \dots, n(t)$. A straightforward coupling argument shows, therefore, that

$$\begin{aligned} P\{S_k' \ge r\} &\le P\{B(k, p_d) \ge (k + r)/2\} \\ &= P\{B(k, p_d)/k - p_d \ge \left(\tfrac{1}{2} - p_d\right) + r/2k\}, \end{aligned}$$

where $B(k, p_d)$ is binomial with parameters k and p_d. The inequality (56.1) of Johnson and Kotz (1969), implies that

$$\begin{aligned} P\{S_k' \ge r\} &\le \exp\left\{-2k\left[\left(1/2 - p_d\right)^2 + r\frac{(1/2 - p_d)}{k} + \frac{r^2}{4k^2}\right]\right\} \\ &\le e^{-2\lambda_d^2 k} e^{-2\lambda_d r}, \end{aligned} \tag{13}$$

where $\lambda_d = (1/2 - p_d)$. Substituting into (12), we obtain

$$P\{D_i(t) \ge r | D_i(0) = 0\} \le \gamma_d e^{-2\lambda_d r}, \tag{14}$$

where $\gamma_d = (1 - e^{-2\lambda_d^2})^{-1}$.

Now, suppose $D_i(0) = a > 0$.

$$\begin{aligned} &P\{D_i(t) \ge r | D_i(0) = a\} \\ &\quad = P\{D_i(t) \ge r,\ D_i(s) = 0 \text{ for some } s \in [0, t] | D_i(0) = a\} \\ &\quad\quad + P\{D_i(t) \ge r,\ D_i(s) \ne 0 \text{ for } s \in [0, t] | D_i(0) = a\}. \end{aligned} \tag{15}$$

With $t - s$ playing the role of t, (14) gives a bound for the first term on the r.h.s. of (15). Further,

$$P\{D_i(s) \neq 0 \text{ for } s \in [0,t] | D_i(0) = a\} 2u$$

$$\leq P\{D_i^*(t) \geq -a\}$$

$$(16) \qquad \leq \sum_{n=n_0}^{\infty} P\left\{n(t) = n,\ \sum_{i=1}^{n} X_i \geq -a\right\} + P\{n(t) < n_0\}$$

$$\leq \sum_{n=n_0}^{\infty} P\left\{\sum_{i=1}^{n} X_i \geq -a\right\} + P\{P_{t/2d} < n_0\},$$

where $P_{t/2d} \sim \text{Poisson}(t/2d)$. Equation (13) gives a bound for the sum of k of the X_i so the first term is $< \gamma_d e^{-2\lambda_d^2 n_0} e^{2\lambda_d a}$. Johnson and Kotz (1969), (36.1), page 102, is an inequality due to Bohman:

$$P\{P_\theta \leq k\} < \Phi\big((k + 1 - \theta)/\sqrt{\theta}\big),$$

where Φ is the distribution function of the standard normal. So for $t > 2d(2n_0 + 2)$,

$$(17) \qquad P\{P_{t/2d} \leq n_0\} \leq \Phi\left(-\sqrt{\frac{n_0 + 1}{2}}\right) \leq e^{-n_0/8}.$$

Combining (15), (16) and (17), we obtain the following theorem.

THEOREM 5.

$$P\{D_i(t) \geq r | D_i(0) = a\} \leq \gamma_d\left[e^{-2\lambda_d r} + e^{-2\lambda_d^2 n_0} e^{2\lambda_d a} + e^{-n_0/8}\right],$$

for $n_0 > 0$, $t > 2d(2n_0 + 2)$.

7. The probability a site is occupied. For each point $x \in \mathbb{Z}^d$ we may define a stochastic process playing the role $Q(t)$ does for 0. We put

$$(18) \qquad Q_x(t) = Q_x(\mathscr{B}_t) = Q(\mathscr{B}_t^x),$$

where $\mathscr{B}_t^x = \{\beta_s - x,\ 0 \leq s \leq t\}$ and $y \in \beta_s - x$ iff $y + x \in \beta_s$. $Q_0(t) = Q(t)$.

Now consider the n points

$$(19) \qquad x^j = (2j\psi_n, 0, \dots, 0), \qquad j = 1, \dots, n,$$

where $\psi_n \geq n$ will be defined later. Let us designate $Q_{x^j}(t)$ as $Q^j(t)$. Let $D_i^j(t) = |Q_i^j(t)|$. We define

$$(20) \qquad m_n = \max_{\substack{1 \leq j \leq n \\ 1 \leq i \leq d}} |Q_i^j(0)|.$$

Then, putting $n_0 > \lambda_d^{-1}(\sqrt{n} + 2m_n)$, so that $-\lambda_d^2 n_0 + 2\lambda_d m_n < -\lambda_d\sqrt{n}$ in Theorem 5, we obtain

$$(21) \qquad P\{D_i^j(t) \geq \sqrt{n}\} \leq 3\gamma_d e^{-\lambda_d\sqrt{n}}, \qquad t > t_n,$$

for all i, j, where $t_n = 2d[2\lambda_d^{-1}(\sqrt{n} + 2m_n) + 2]$. The next lemma immediately follows.

LEMMA 4.

$$P\{D_i^j(t) < \sqrt{n}\,, i = 1,\dots,d;\, j = 1,\dots,n\}$$
$$> 1 - 3\gamma_d dn e^{-\lambda_d\sqrt{n}}, \qquad t > t_n.$$

Thus, as $n \to \infty$, for large enough t, the probability that the displacements of the nearest occupied positions to $x^1,\dots,x^n$ are all $< \sqrt{n} \to 1$. We now show that the probability that x^j is occupied is bounded away from 0 for large enough t.

LEMMA 5.

$$P\{Q^j(t + \rho\sqrt{n} + 1) = 0 \,|\, D_i^j(t) \le \sqrt{n}\,, i = 1,\dots,d\} > p > 0,$$

for n and t large enough, where p and ρ do not depend on n or t.

PROOF. Without loss of generality we prove the theorem for $Q(t)$. Once again we use Theorem 5. We take $n_0 > 3\lambda_d^{-1}\sqrt{n} - 1$ so that $-\lambda_d^2 n_0 + 2\lambda_d\sqrt{n} < -\lambda_d\sqrt{n} + \lambda_d^2$ and

$$P\{D_i(t + \rho\sqrt{n}) \ge r \,|\, D_i(t) < \sqrt{n}\} < \gamma_d\left(e^{-2\lambda_d r} + e^{\lambda_d^2}e^{-\lambda_d\sqrt{n}} + e^{(1-3\lambda_d^{-1}\sqrt{n})/8}\right),$$

where $\rho = 12d\lambda_d^{-1}$. Let r' be the smallest integer such that $\gamma_d e^{-2\lambda_d r'} < 1/2$. Then

$$\begin{aligned} P\{D_i(t + \rho\sqrt{n}) \le r', i = 1,\dots,d \,|\, D_i(t) \le \sqrt{n}\,, i = 1,\dots,d\} \\ \ge 1/2 - \gamma_d d\left(e^{\lambda_d^2}e^{-\lambda_d\sqrt{n}} + e^{(1-3\lambda_d^{-1}\sqrt{n})/8}\right) \\ \ge 1/3, \end{aligned}$$

for large enough n. Let $C_{r'} = \{x\colon |x_i| < r', i = 1,\dots,d\}$. Then, if

$$p' = \min_{x\in C_{r'}}\{P\{Q(t+1) = 0 \,|\, Q(t) = x\}\},$$

the result follows for $p = p'/3$, once we have shown $p' > 0$.

For any $x \in C_{r'}$ choose one of the shortest paths from x to 0 going from neighbour to neighbour. The length of the path is $\le dr'$. A sufficient condition for $Q(t+1) = 0 | Q(t) = x$ is for branching (at rate $1/2d$) along that path to take place within time interval 1, while the particle at $Q(s)$ is not annihilated by one of the $2d - 1$ neighbours not along the chosen path in the direction of the origin for $t \le s \le t + 1$. That is,

$$p' \ge e^{-1/2d}\frac{(1/2d)^{dr'}}{(dr')!}e^{-(2d-1)/2d}. \qquad \square$$

8. The number of occupied sites. The $Q^i(t)$ are not independent, but since $\|x^j - x^i\| \ge n$, $j \ne i$, and $P\{\|Q^j(t)\| < d\sqrt{n}\} \to 1$ as $n \to \infty$, it may be expected that in some sense within a time interval $O(\sqrt{n})$ the $Q^j(t)$ will act independently with probability approaching 1. We look for a condition such that no branching of a particle can affect both $Q^i(t)$ and $Q^j(t)$, $j \neq i$, in a time interval of $O(\sqrt{n})$.

Define the box of side $2l$, centred at x^j, as

$$A(j,l) = \{x: |x_i - x_i^j| \le l,\ i = 1,\dots,d\}. \tag{22}$$

We choose ψ_n, ϕ_n, $\psi_n > \phi_n > n$, large enough such that the probability of any paths in the percolation substructure of the $A(j,\sqrt{n})$ to $A^c(j,\phi_n)$ and from $A(j,\phi_n)$ to $A^c(j,\psi_n)$ can be made vanishingly small. In Appendix 1 it is shown that with $\phi_n > (n+1)e^{\rho\sqrt{n}+1}(2\sqrt{n}+1)^d$ and $\psi_n > (n+1)e^{\rho\sqrt{n}+1}(2\phi_n+1)^d$,

$$P\left\{N^{A(j,\phi_n)}_{[t,t+\rho\sqrt{n}+1]}(A^c(j,\psi_n)) > 0\right\} \le 1/n^2. \tag{23}$$

Define an event

$$F_n(t) \equiv N^{A^c(j,\psi_n)}_{[t,t+\rho\sqrt{n}+1]}(A(j,\phi_n)) = 0, \qquad j = 1,\dots,n. \tag{24}$$

It follows from (23) and (24) that

$$P\{F_n(t)\} \ge 1 - 1/n. \tag{25}$$

Define an event

$$H_n(t) \equiv N^{A^c(j,\phi_n)}_{[t+\rho\sqrt{n}+1,\,t]}\left(A(j,\sqrt{n})\right) = 0, \qquad j = 1,\dots,n, \tag{26}$$

where the path through the percolation substructure is in reverse time. If $Q^j(t) \in A(j,\sqrt{n})$, then a necessary condition for $Q^j(s) \in A^c(j,\phi_n)$ for some $t \le s \le t + \rho\sqrt{n} + 1$ is that there is a series of neighbouring sites $y^1,\dots,y^k$ and times $t \le s_1 < s_2 < \cdots < s_k \le t + \rho\sqrt{n} + 1$ such that $y^1 \to Q^j(t)$ at s_1, $y^2 \to y^1$ at $s_2,\dots,y^k \to y^{k-1}$ at s_k, where $y^k \in A^c(j,\phi_n)$. This implies, reversing time, that there is a path from $(y^k, t+\rho\sqrt{n}+1)$ to $(Q^j(t), t)$. Defining

$$G_n(t) \equiv Q^j(t) \in A(j,\sqrt{n}), \qquad j = 1,\dots,n, \tag{27}$$

it is clear that $G_n(t) \cap H_n(t)$ implies

$$Q^j(s) \in A(j,\phi_n), \qquad j = 1,\dots,n,\ t \le s \le t + \rho\sqrt{n} + 1.$$

If $E_n(t) \equiv F_n(t) \cap G_n(t) \cap H_n(t)$ occurs, then there is no way in which any event
in $[t, t+\rho\sqrt{n}+1]$ can affect both $Q^j(s)$ and $Q^i(s)$, $i \neq j$, within that time interval. Further, $F_n(t)$, $G_n(t)$ and $H_n(t)$ are independent as they are events disjoint either in time or space in the percolation substructure. In Appendix 1 it will be shown that

$$P\left\{N^{A(j,\sqrt{n})}_{[t,t+\rho\sqrt{n}+1]}(A^c(j,\phi_n)) = 0\right\} \le 1/n^2, \tag{28}$$

which with (26) gives

$$P\{H_n(t)\} \ge 1 - 1/n. \tag{29}$$

Using Lemma 4 for $G_n(t)$, we have

$$P\{E_n(t)\} > (1 - 1/n)\big(1 - 3\gamma_d\, dn\, e^{-\lambda_d\sqrt{n}}\big)(1 - 1/n) = 1 - \varepsilon_n, \qquad t > t_n, \tag{30}$$

where $\varepsilon_n \to 0$.

Now, if $\sigma_{i,n}(t)$ is the σ field generated by the process β in

$$\big[t, t + \rho\sqrt{n} + 1\big] A^c(i, \psi_n),$$

$$P\big\{Q^j(s) = x \big| E_n(t) \cap \sigma_{i,n}(t)\big\} = P\big\{Q^j(s) = x \big| E_n(t)\big\},$$

for $t \le s \le t + \rho\sqrt{n} + 1$. Since $Q^j(s)|E_n(t) \in \sigma_{i,n}(t) \cap E_n(t)$, $j \ne i$, $t \le s \le t + \rho\sqrt{n} + 1$,

$$\begin{aligned} &P\big\{Q^j(s) = x \big| E_n(t), Q^i(s), i = 1, \dots, j - 1\big\} \\ &\quad = P\big\{Q^i(s) = x \big| E_n(t)\big\}. \end{aligned} \tag{31}$$

Lemma 6 follows from (30), (31) and Lemma 5.

LEMMA 6.

$$P\big\{Q^j(t + \rho\sqrt{n} + 1) = 0 \big| E_n(t), Q^i(t + \rho\sqrt{n} + 1), i < j\big\} > p - \varepsilon_n, \qquad t > t_n.$$

Now, suppose that we have two independent BAPs β and $\hat{\beta}$. Define r.v.'s $I_0, I_1, \dots, I_n$ which are respectively the indicators of

$$E_n(t) \cap \hat{E}_n(t), \dots, Q^i(t + \rho\sqrt{n} + 1) = 0 \cap \hat{Q}^i(t + \rho\sqrt{n} + 1) = 0, \qquad i = 1, \dots, n,$$

where the superscript ^ implies that the events have been defined for the process $\hat{\beta}$ in a similar manner to those for β. It follows from Lemma 6 that

$$E\{I_i | I_0, \dots, I_{i-1}\} \ge (p - \varepsilon_n)^2 I_0.$$

We then use a theorem given in Loève (1955), page 387, that, with

$$I_i^* = I_i - E\{I_i | I_{i-1}, \dots, I_0\},$$

$$P\left\{\max_{1 \le k \le n} \left|\sum_{i=1}^{k} I_i^*\right| \ge \varepsilon\right\} \le \frac{1}{\varepsilon^2} \sum_{k=1}^{n} E\{I_k^{*2}\},$$

so that

$$P\left\{\left|\sum_{i=0}^{n} \big[I_i - E\{I_i | I_0, \dots, I_{i-1}\}\big]\right| \le \varepsilon\right\} > 1 - \frac{(n+1)}{\varepsilon^2}.$$

Putting $\varepsilon = (n+1)^{3/4}$, we have

$$P\left\{\sum_{i=0}^{n} E\{I_i | I_0, \dots, I_{i-1}\} - \sum_{i=0}^{n} I_i < (n+1)^{3/4}\right\} > 1 - \frac{1}{\sqrt{n+1}}$$

or

$$P\left\{\sum_{i=1}^{n} I_i > n(p-\varepsilon_n)^2 I_0 - (n+1)^{3/4} - 1\right\} > 1 - \frac{1}{\sqrt{n+1}}.$$

But

$$P\{I_0 = 1\} = (1-\varepsilon_n)^2, \qquad \varepsilon_n \to 0,$$

so that for n large enough

$$P\left\{\sum_{i=0}^{n} I_i > \frac{1}{2} np^2\right\} > (1-\varepsilon_n)^2\left(1 - \frac{1}{\sqrt{n+1}}\right),$$

for $t > \max(t_n, \hat{t}_n)$. Thus, we have that $|\beta_t \cap \hat{\beta}_t| \to_p \infty$.

At the end of Section 5 it was pointed out that this condition was sufficient to show that the only nonzero limiting distribution was the product density 1/2. We thus have Theorem 6.

THEOREM 6. *If β_t^A is a BAP on $\mathbb{Z}^d$ with $A \neq \varnothing$, then its limiting measure is $\nu_{1/2}$, the product measure with density* 1/2.

9. The double branching annihilating process (DBAP). In this section we consider a process on $\mathbb{Z}$ which is like the BAP except that particles always invade both neighbours simultaneously instead of one neighbouring site at a time. It is not a spin system. We call it the double branching annihilating process (DBAP). It is defined by its flip rates, $c(x-1, x+1, \beta)$, the rates at which the coordinates $\beta(x-1), \beta(x+1)$ simultaneously flip to $1-\beta(x-1), 1-\beta(x+1)$, respectively. We have

$$c(x-1, x+1, \beta) = \beta(x)$$

and the infinitesimal generator is

$$Lf(\beta) = \sum \beta(x)\left[f(\beta^{x-1,x+1}) - f(\beta)\right],$$

where $\beta^{x-1,x+1}$ is β with the values at $x-1$ and $x+1$ flipped.

THEOREM 7. *In a DBAP on $\mathbb{Z}$:*

(a) *if the initial number of particles is odd, then the limiting distribution is* $\nu_{1/2}$;

(b) *if the initial number of particles is even, then the number of particles remains bounded*;

(c) *if the initial measure is ν_p, $0 < p \leq 1$, then the limiting measure is* $\nu_{1/2}$;

(d) $\nu_{1/2}$ *is not the only limiting measure with density greater than* 0;

(e) *the stationary measures form a one-parameter family, the borders of product measures ν_p, $0 \leq p \leq 1/2$.*

First, we shall prove a lemma.

LEMMA 7. *The border of an exclusion process is a DBAP.*

PROOF. An exclusion process on $\mathbb{Z}$ has state space $\{0,1\}^{\mathbb{Z}}$. γ_t is the set of occupied sites at time t. Each pair of adjacent sites waits an exponential holding time with mean 1 and then swaps the particles. In other words, if a swap occurs at the pair $(x, x+1)$ at time t, $\gamma_t(x+1) = \gamma_{t-}(x)$, $\gamma_t(x) = \gamma_{t-}(x+1)$. Clearly a noticeable change only occurs when $\gamma_{t-}(x) \neq \gamma_{t-}(x+1)$. We define a coupled process ζ_t as

$$\zeta_t(x) = |\gamma_t(x) - \gamma_t(x+1)|. \tag{32}$$

If a swap occurs at time t at $(x, x+1)$, $\zeta_t(x)$ is unchanged, but if $\gamma_{t-}(x) \neq \gamma_{t-}(x+1)$ [i.e., $\zeta_{t-}(x) = 1$], then both $\zeta_t(x-1)$ and $\zeta_t(x+1)$ are changed. This is equivalent to the particle at x placing annihilating offspring on the sites $x-1$ and $x+1$, so ζ_t is a DBAP. □

PROOF OF THEOREM 7. We shall first consider the case for which the initial number of particles in the DBAP is finite. Let $\zeta_0(x_1) = \zeta_0(x_2) \cdots = \zeta_0(x_r) = 1$ and $\zeta_0(x) = 0$ otherwise. This is the border of an exclusion process which initially has 0's from $-\infty$ up to x_1, then 1's from x_1 up to x_2, 0's from x_2 up to x_3, etc. That is,

$$\gamma_0(x) = 0, \qquad x \le x_1, x_2 < x \le x_3, \dots,$$

$$\gamma_0(x) = 1, \qquad x_1 < x \le x_2, \dots .$$

(We could just as well have had 1's from $-\infty$ up to x_1, 0's from x_1 up to x_2, etc.) We have

$$\gamma_0(x) = 1, \qquad x > x_r \quad \text{for } r \text{ odd},$$

$$\gamma_0(x) = 0, \qquad x > x_r \quad \text{for } r \text{ even}.$$

Now as $n \to \infty$, $|\gamma_0 \cap [-n, n]|/2n \to 1/2$ for r odd, and it is well known [Liggett (1985), Chapter 8] that for such γ_0, $\nu_{1/2}$ is the limiting measure of γ_t. It follows that $\nu_{1/2}$ is also the limiting measure for ζ_t.

When r is even

$$|\gamma_0| = (x_2 - x_1) + (x_4 - x_3) + \cdots + (x_r - x_{r-1}).$$

Since 1's can neither be created nor destroyed $|\gamma_t| = |\gamma_0|$, and since this finite set of particles separates and wanders over the whole line, the probability that any two particles are adjacent approaches 0. This is shown in Appendix 2. We thus have that

$$P\{|\zeta_t| = 2|\gamma_0|\} \to 1.$$

To prove (c), suppose the initial measure of ζ_0 is ν_p, $0 < p \le 1$. γ_0 may be constructed uniquely by specifying $\gamma_0(0) = 0$ and having ζ_0 the border of γ_0.

γ_t then evolves as an exclusion process and ζ_t is its border so that

$$(33) \qquad \gamma_t(x+1) = \zeta_t(x) + \gamma_t(x) \pmod 2.$$

$\{\gamma_0(x)\}$ is a Markov chain with doubly stochastic transition matrix $\begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$. It follows that $P\{\gamma_0(x) = 1\} \to 1/2$ as $|x| \to \infty$ and by the strong law of large numbers for Markov chains $|\gamma_0 \cap [-n, n]|/2n \to 1/2$ so that, as in part (a), $\zeta_t \to \nu_{1/2}$.

Equations (32) and (33) define a one-to-one relationship between measures for γ_t and ζ_t. It is known that in the one-dimensional case the only invariant measures for exclusion processes are mixtures of the product measures, ν_α [see Liggett (1985), page 369]. If $\gamma_0 \sim \nu_\alpha$ then $\gamma_t \sim \nu_\alpha$ and $P\{\zeta_t(x) = 1\} = 2\alpha(1-\alpha)$, but ζ_t is not a product measure. The measures of the cylinder sets in these border measures of ν_α are simple to calculate [e.g., $\zeta_t(011) = \alpha^3(1-\alpha) + \alpha(1-\alpha)^3$], but do not seem to have a neat closed form. (d) and (e) follow from these observations. □

10. The double branching annihilating random walk (DBARW). In this process the annihilating random walk is added to the DBAP. In other words this is the model of Bramson and Gray with, however, branching always taking place simultaneously to both neighbours. Since it is known [see Schwartz (1978)] that the ARW can be viewed as a border of a voter model, it is no surprise that the DBARW is the border of a process which combines both swapping (exclusion) and the voter model. In the voter model each site waits an exponential holding time when it takes on the opinion (taking values 0 or 1) of one of its neighbours chosen at random. In the swapping voter model, η, we shall allow swapping at rate 1, but set the rate for the voter model at ρ. We will define η in terms of a percolation substructure as in Section 4.

Let $(M_z^-(t), M_z^+(t), N_z(t),\ t \ge 0,\ z \in \mathbb{Z})$ be independent Poisson processes with rates $\rho/2$, $\rho/2$ and 1, respectively. Let $T_{z,l}^-, T_{z,m}^+, U_{z,n}$ be the lth event in M_z^-, the mth event in M_z^+ and the nth event in N_z, respectively. At $T_{z,l}^-$, site z flips to be the same as site $z-1$. Similarly, at $T_{z,m}^+$, z flips to be the same as site $z+1$. At $U_{z,u}$ sites z and $z+1$ swap values. If there is no difference in the values at the sites there is no change in η. Given initial state η_0, M_z^-, M_z^+ and N_z define η_t.

The percolation substructure, P_t, is obtained by drawing the family of line segments $\{z\} \times [0, t]$, and drawing arrows from $(z-1, T_{z,l}^-)$ to $(z, T_{z,l}^-)$, from $(z+1, T_{z,m}^+)$ to $(z, T_{z,m}^+)$ and a two-headed arrow between $(z, U_{z,n})$ and $(z+1, U_{z,n})$. A path down from (z_1, t) to (z_2, s), $s \le t$, is a continuous path consisting of downward line segments and arrows in which, whenever an arrowhead is reached, the arrow is followed until its other end, when the adjacent downward line segment is followed. There is clearly at most one path from (z_1, t) to (z_2, s) and

$$(34) \qquad \eta_t(z_1) = \eta_s(z_2) \quad \text{if there is a path from } (z_1, t) \text{ to } (z_2, s).$$

We shall show the following theorem.

THEOREM 8. *If η_t is a swapping voter model on $\mathbb{Z}$, then $P\{\eta_t(z) = \eta_t(y)\} \to 1$ as $t \to \infty$.*

PROOF. We define a set of processes $\{\Pi_z^t(s),\ 0 \le s \le t,\ z \in \mathbb{Z}\}$ which will represent the position of particles Π_z. Π_z starts at z at time t and moves in reverse time. We define

$$\Pi_z^t(s) = x \quad \text{iff there is a path from } (z,t) \text{ to } (x,s), \qquad 0 \le s \le t.$$

This defines $\Pi_z^t(s)$ uniquely. Each Π_z thus executes a reverse time symmetric simple random walk with holding time $(2+\rho)^{-1}$. The jumps for Π_z and Π_y are independent unless they occupy adjacent sites. Then the next change in position for the two particles is either (a) a swap with probability $1/(3+2\rho)$, or (b) one of the particles jumps away with probability $(2+\rho)/(3+2\rho)$ or (c) they occupy the same site with probability $\rho/(3+2\rho)$. If they occupy the same site at time s, then $\Pi_z^t(u) = \Pi_y^t(u)$, $0 \le u \le s$. Since symmetric random walks in one dimension meet i.o., they will become adjacent i.o. and (c) will eventually occur. Thus Π_z and Π_y will join together in $[0,t]$ with probability 1 as $t \to \infty$. That is, $P\{\Pi_z^t(0) = \Pi_y^t(0)\} \to 1$, and since $\eta_t(z) = \eta_0(\Pi_z^t(0))$, and $\eta_t(y) = \eta_0(\Pi_y^t(0))$, the theorem follows. □

NOTE. This lemma only requires the associated random walks to meet i.o. It is thus true for any swapping voter process in one or two-dimensions which is symmetric with appropriate moment conditions on the random walk. The nearest-neighbour condition is unnecessary.

It is simple to check that the border process $\zeta_t(x) = |\eta_t(x) - \eta_t(x+1)|$ is a DBARW with double branching at rate 1 and the ARW at rate ρ. It follows from Theorem 8 that $P\{\zeta_t(x) = 1\} \to 0$, so that we have Theorem 9.

THEOREM 9. *The limiting measure for the DBARW is δ_ϕ.*

APPENDIX 1

We give here the justification for (23) and (29). Now the process defined by

$$W_t^A = \{x \colon N_t^A(x) > 0\}$$

is a Williams–Bjerknes process with $\kappa = \infty$. Every occupied position branches to one of its neighbours at unit rate, placing an offspring at a neighbouring site if it is unoccupied. It is thus a pure birth or Yule process inhibited by the spatial structure.

If $Y(t)$ is the number of particles in a Yule process with split rate 1 for each particle, then

$$E\{Y(t_0+t) \mid Y(t_0)\} = Y(t_0)e^t,$$

$$\operatorname{Var}\{Y(t_0+t) \mid Y(t_0)\} = Y(t_0)(e^{2t} - e^t).$$

So, using Chebyshev's inequality,

$$P\{Y(t_0 + t) > ke^t x | Y(t_0) = x\}$$
$$< \frac{xe^{2t}}{(ke^t x - e^t x)^2} = \frac{1}{(k-1)^2 x}.$$

Now, the number of sites in the set $A(j, l)$ is $\leq (2l + 1)^d$ so

$$P\{N^{A(j,l)}_{[t, t+\rho\sqrt{n}+1]}(A^c(j, m)) > 0\} < \frac{1}{n^2}$$

if $m \geq (n + 1)e^{\rho\sqrt{n}+1} \cdot (2l + 1)^d$. Equations (23) and (29) follow if we take ϕ_n to be the smallest integer greater than $(n + 1)e^{\rho\sqrt{n}+1}(2\sqrt{n} + 1)^d$, and ψ_n the smallest integer greater than $(n + 1)e^{\rho\sqrt{n}+1}(2\phi_n + 1)^d$.

ϕ_n and ψ_n are certainly far larger than is necessary. In fact $\psi_n = n$ is sufficient for (23) and (29) to be true.

APPENDIX 2

LEMMA. *If γ_t is an exclusion process with $|\gamma_0| < \infty$, then*

$$P\left\{\sum_x \gamma_t(x)\gamma_t(x+1) = 0\right\} \to 1.$$

PROOF. Consider two particles m_1, m_2 in an exclusion process. Each particle executes a continuous time simple random walk with jumps independent of each other except when they are adjacent, in which case they may swap positions. The distance between the particles is thus a zero-mean random walk reflected at 1. If $\text{dist}(m_1, m_2, t)$ is the distance between m_1 and m_2, then

$$P\{\text{dist}(m_1, m_2, t) = 1\} \to 0.$$

Since there are a finite number of pairs of particles, the lemma follows. □

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Theorem 6 is also contained in Durrett (1989), Theorem 2B.

REFERENCES

ARRATIA R. (1981). Limiting point processes for rescaling of coalescing and annihilating random walks on $\mathbb{Z}^d$. *Ann. Probab.* **9** 909–936.

BRAMSON, M. and GRAY, L. (1985). The survival of the branching annihilating random walk. *Z. Wahrsch. verw. Gebiete* **68** 447–460.

CLIFFORD, P. and SUDBURY, A. (1973). A model for spatial conflict. *Biometrika* **60** 581–588.

DURRETT, R. (1981). An introduction to infinite particle systems. *Stochastic Process. Appl.* **11** 109–150.

DURRETT, R. (1989). A new method for proving the existence of phase transitions. Technical Report, Cornell Univ.

FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications* **2**, 2nd ed. Wiley, New York.

GRIFFEATH, D. (1978). Limit theorems for non-ergodic set-valued Markov process. *Ann. Probab.* **6** 379–387.

GRIFFEATH, D. (1979). *Additive and Cancellative Interacting Particle Systems. Lecture Notes in Math.* **724**. Springer, New York.

HARRIS, T. E. (1978). Additive set-valued Markov processes and graphical methods. *Ann. Probab.* **6** 355–378.

JOHNSON, N. L. and KOTZ, S. (1969). *Discrete Distributions*. Houghton Mifflin, Boston.

LIGGETT, T. (1985). *Interacting Particle Systems*. Springer, New York.

LOÈVE, M. (1955). *Probability Theory*. Van Nostrand, New York.

SCHWARTZ, D. (1978). On hitting probabilities for an annihilating particle model. *Ann. Probab.* **6** 398–403.

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