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ORDERED SKOROKHOD STOPPING FOR A SEQUENCE OF MEASURES

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Let X be a transient right (Markov) process on a compact metric space including a death point. Let μ and ν_n be finite measures whose potentials satisfy $\mu U \geq \cdots \geq \nu_n U \geq \cdots \geq \nu_1 U$. We prove that there exists a right-continuous stochastic process $Y = (\tilde{\Omega}, \mathscr{M}, \tilde{\mathscr{M}}_t, Y_t, Q)$ that is a version of X with initial measure $\nu_\infty(\cdot) = Q(Y_0 \in \cdot)$ and in which there are $(\tilde{\mathscr{M}}_t)$-stopping times $\tilde{\tau}_n \downarrow 0$ with $Q(Y(\tilde{\tau}_n) \in \cdot, \tilde{\tau}_n < \infty) = \nu_n(\cdot)$. Furthermore, a canonical representation of Y and $(\tilde{\tau}_n)$ is given in which one has a better understanding of the tail behavior of the sequence $\tilde{\tau}_n$. Based on this representation an open question is posed whose answer in the positive would permit defining in X, assuming it admits a continuous real random variable independent of the path, *decreasing* stopping times T_n such that $P^\mu(X(T_n) \in \cdot, T_n < \infty) = \nu_n(\cdot)$. These T_n would satisfy the Markov property $T_n = T_{n+1} + S_n \circ \theta(T_{n+1})$, S_n a stopping time linking ν_n and ν_{n+1}. Fitzsimmons has now proved the existence of a desired decreasing sequence T_n in X for any given μ and ν_n as above, using a very different approach. His T_n, however, do not satisfy the Markov property.

1. Introduction and main results. Consider a transient right process $X = (\Omega, \mathscr{M}, \mathscr{M}_t, X_t, \theta_t, P^x)$ on a compact metric space $E_\Delta = E \cup \{\Delta\}$, where Δ is the usual adjoined death point. See [6], [11] and [3] for definitions and notation of Markov processes and right processes. For the meaning and relevent implications of transience see [7]. Let $U = U(x, A)$ denote the potential kernel of X. By the transience assumption, if μ is a finite measure on E, its potential μU is a σ-finite measure (on E). The following fact (general Skorokhod stopping theorem) is well known. Let μ, ν be finite measures on E with $\mu U \geq \nu U$; then there exists a stopping time T such that $\nu(\cdot) = \mu P_T(\cdot) = P^\mu(X_T \in \cdot, T < \zeta)$, ζ the lifetime $T_{\{\Delta\}}$, provided that the $\mathscr{M}_t$ are sufficiently rich, in particular that there exists a continuous real random variable in $\mathscr{M}_0$ independent of (X_t) (under any initial measure). There are various schemes to construct such a stopping time; see, e.g., [9], [8], [1], [10] and [4]. In this article we study the following question raised by Fitzsimmons. Let μ and ν_n, $n \geq 1$, be finite measures on E with

$$\mu U \geq \cdots \geq \nu_n U \geq \cdots \geq \nu_1 U. \tag{1.1}$$

Does there exist a *decreasing* sequence of stopping times T_n such that $\nu_n = \mu P_{T_n}$ for all n, assuming the $\mathscr{M}_t$ are sufficiently rich? [The converse that the existence of such a sequence implies (1.1) is of course trivially valid.] The following result (see Theorem 3.4) is obtained. There exists a right-continuous

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stochastic process $Y = (\tilde{\Omega}, \tilde{\mathscr{M}}, \tilde{\mathscr{M}}_t, Y_t, Q)$, $\tilde{\mathscr{M}}_t$ a right-continuous filtration to which (Y_t) is adapted, that is a version of the right process X with initial measure $\nu_\infty(\cdot) = Q(Y_0 \in \cdot)$, and such that there are $(\tilde{\mathscr{M}}_t)$-stopping times $\tilde{\tau}_n$ decreasing to 0 with $Q(Y_{\tilde{\tau}_n} \in \cdot, \tilde{\tau}_n < \infty) = \nu_n(\cdot)$ for all n. The $\tilde{\tau}_n$ satisfy a certain Markov property. Note that $\nu_\infty U = \lim_n \nu_n U \le \mu U$; so in X there exists a stopping time T_∞ with $\nu_\infty = \mu P_{T_\infty}$. If the process Y could be regarded as the part of X, under P^μ, after time T_∞, the above question would be completely answered in the positive, with $T_n = T_\infty + \tilde{\tau}_n \circ \theta(T_\infty)$. Furthermore, the T_n satisfy the Markov property $T_n = T_{n+1} + S_n \circ \theta(T_{n+1})$, where S_n is a stopping time such that the $P^{\nu_{n+1}}$-distribution of $X(T_n)$ is ν_n. However, attaching Y to X under P^μ after time T_∞ involves changing the filtration $(\mathscr{M}_t)$, and this change depends on μ and ν_n. In other words the result obtained does not imply that one can define in a given X, i.e., with fixed $(\mathscr{M}_t)$, a desired sequence T_n for arbitrary μ and ν_n satisfying (1.1). We have only partially answered the question.

But it seems this partial answer is still of interest. Furthermore, we give what we call a canonical representation of Y and the times $\tilde{\tau}_n$ (see Theorem 4.2), in which one has a better understanding of how the tail of $(\tilde{\tau}_n)$ may be determined. With that an open question is posed (see Remark 2 after Theorem 2.4) whose answer in the positive would completely resolve the problem being studied. This open question is perhaps interesting in its own right.

Fitzsimmons [5] has now proved that if X contains an independent continuous randomization variable, then for any μ, ν_n satisfying (1.1) there exist decreasing stopping times T_n with ν_n as the P^μ-distribution of $X(T_n)$. The approach is very different; he uses a theorem of Baxter and Chacon (see [5] for reference) on compactness of stopping times. However, the stopping times T_n obtained in [5] do not satisfy the Markov property. Let us remark, incidentally, that using the ordered stopping times $\tilde{\tau}_n$ in our process Y and based on the representation of randomized stopping times by measures in $[0,1] \times \hat{\Omega}$ ($\hat{\Omega}$ the path space as defined below; see [5], especially Lemma 2) and vice versa, which is the basic observation of Baxter and Chacon, one can also define a decreasing sequence τ_n in X with initial measure ν_∞, and then obtain $T_n = T_\infty + \tau_n \circ \theta(T_\infty)$ as desired. However, the τ_n and therefore T_n thus obtained again do not satisfy the Markov property.

The process Y is constructed in Section 2. In Section 3 the behavior of Y_t as $t \to 0$ is studied. We present in Section 4 the canonical representation of Y and $\tilde{\tau}_n$ mentioned above. In Section 5 examples are given to illustrate some possibilities of the behavior of the tail of $(\tilde{\tau}_n)$.

2. The process *Y*. We assume that X admits a continuous real random variable independent of (X_t). Thus we can let $\Omega = \hat{\Omega} \times \mathbb{R}$, where $\hat{\Omega}$ is the space of right-continuous functions from $[0, \infty)$ into E_Δ and $\mathbb{R}$ the reals; $X_t(\hat{\omega}, r) = \hat{\omega}_t$; $\theta_t(\hat{\omega}, r) = (\theta_t \hat{\omega}, r)$, where $\theta_t \hat{\omega}$ is the usual shifted path $\hat{\omega}'$ with $\hat{\omega}'_\cdot = \hat{\omega}_{t+\cdot}$; $R(\hat{\omega}, r) = r$ be the random variable independent of (X_t) and with a continuous distribution $\lambda(dr)$ under any P^x. $\mathscr{M}$ and $\mathscr{M}_t$ are the usual completions of the σ-algebras $\sigma(X_t, t \ge 0) \vee \sigma(R)$ and $\bigcap_{\varepsilon > 0} \sigma(X_s, s < t + \varepsilon) \vee \sigma(R)$

with respect to all measures P^μ, of course with the X_t regarded as taking values in $(E_\Delta, \mathscr{E}_\Delta^*)$, $\mathscr{E}_\Delta^*$ the σ-algebra of universally measurable sets of E_Δ. Let Z_n, $n \geq 0$, be independent continuous real random variables depending only on R.

Consider now fixed finite measures μ and ν_n, $n \geq 1$, on E satisfying (1.1). For each n there exists a stopping time S_n satisfying $\nu_n = \nu_{n+1}P_{S_n}$ (using any Skorokhod stopping scheme), which we can require to be one relative to the filtration $(\mathscr{G}_t \vee \sigma(Z_n))$. Here $\mathscr{G}_t = \bigcap_{\varepsilon>0}\sigma(X_s,\ s < t + \varepsilon)$ with the X_t regarded as taking values in $(E_\Delta, \mathscr{E}_\Delta)$, $\mathscr{E}_\Delta$ the Borel σ-algebra; Z_n serves as a randomization variable for S_n that may be needed. For convenience all stopping times T on Ω are required to satisfy $T = \infty$ if $T \geq \zeta$. Next define stopping times T_{kn}, $1 \leq n \leq k < \infty$, as follows. Let T_{kk} be a stopping time relative to $(\mathscr{G}_t \vee \sigma(Z_0))$ satisfying $\nu_k = \mu P_{T_{kk}}$, and for $n < k$ let

$$T_{kn} = T_{k,n+1} + S_n \circ \theta(T_{k,n+1}) \tag{2.1}$$

with the understanding $T_{kn} = \infty$ if $T_{k,n+1} = \infty$. Thus $\nu_n = \mu P_{T_{kn}}$, and obviously the distribution of $(T_{kn} - T_{k,n+1},\ 1 \leq n < n_1)$ is independent of $k \geq n_1$ under P^μ (with the convention $\infty - \infty = 0$).

Proposition 2.1. *For any $\delta > 0$, $P^\mu(T_{kn} - T_{km} > \delta) \to 0$ as $m > n \to \infty$ (note the probability is independent of $k \geq m$).*

Proof. Let $\varepsilon > 0$. By replacing ν_1 by some ν_j we may assume $\sup \nu_k(E) = \lim \nu_k(E) < \nu_1(E) + \varepsilon/4$. By the transience of X there exists a transient nearly Borel or even compact $B \subset E$ such that $\nu_1(E - B) < \varepsilon/4$. Since the last exit time $L_B = \sup\{t\colon X_t \in B\}$ is finite a.s., $P^\mu(L_B > t_0) < \varepsilon/4$ for some $t_0 < \infty$. It follows that

$$\begin{aligned} &P^\mu(T_{kk} < \infty, T_{k1} > t_0) \\ &\quad\leq P^\mu(T_{kk} < \infty, T_{k1} = \infty) + P^\mu(T_{k1} < \infty, X(T_{k1}) \notin B) \\ &\qquad + P^\mu(t_0 < T_{k1} < \infty, X(T_{k1}) \in B) \\ &\quad\leq (\nu_k(E) - \nu_1(E)) + \nu_1(E - B) + P^\mu(L_B > t_0) < 3\varepsilon/4. \end{aligned}$$

By the independence of k of the distribution $(T_{kn} - T_{k,n+1},\ n < n_1)$ mentioned above, if $m > n$ are sufficiently large, $P^\mu(T_{kn} - T_{km} > \delta, T_{k1} \leq t_0) < \varepsilon/4$, and so

$$\begin{aligned} P^\mu(T_{kn} - T_{km} > \delta) &= P^\mu(T_{kn} - T_{km} > \delta, T_{kk} < \infty) \\ &\geq P^\mu(T_{kn} - T_{km} > \delta, T_{k1} \leq t_0) \\ &\quad + P^\mu(T_{kk} < \infty, T_{k1} > t_0) < \varepsilon. \end{aligned}$$

□

Consider now the sequence of processes $Y^k = (X_t, P^{\nu_k'})$, where ν_k' is the measure on E_Δ with $\nu_k'(B) = \nu_k(B)$ for $B \subset E$, $\nu_k'\{\Delta\} = \mu(E) - \nu_k(E)$. They are considered as defined on $(\Omega, \mathscr{M}^0)$, where $\mathscr{M}^0 = \mathscr{G} \vee \sigma(R)$, $\mathscr{G} = \sigma(X_t,\ t \geq 0)$ with the X_t regarded as taking values in $(E_\Delta, \mathscr{E}_\Delta)$. Note that $(\Omega, \mathscr{M}^0)$ is a Radon space (see [11] for a definition and relevent facts), a fact needed in defining the space $(\tilde{\Omega}, \tilde{\mathscr{M}}, Q)$ below. In Y^k we (re)-define T_{kn}, $1 \le n \le k$, as follows: $T_{kk} = 0$ if $X_0 \in E$, $= \infty$ if $X_0 = \Delta$; and T_{kn}, $n < k$, satisfy (2.1). Note that $((X_t), (T_{kn} - T_{kk}))$ under $P^{\nu'_k}$, where T_{kn} are defined as before, is equivalent to $((X_{T_{kk}+t}), (T_{kn} - T_{kk}))$ under P^{μ}; here we use the convention $X_\infty = \Delta$. We will define Y as a projective limit process of the sequence Y^k. Let $\Omega_k = \Omega$ denote the sample space of Y^k and define a mapping φ_k: $(\Omega_{k+1}, \mathscr{M}^0) \to (\Omega_k, \mathscr{M}^0)$ by

$$\varphi_k(\omega) = \omega' \quad \text{iff } R(\omega) = R(\omega') \text{ and } X_\cdot(\omega') = X_{T_{k+1,k}+\cdot}(\omega).$$

Clearly φ_k embeds Y^k in Y^{k+1} as $(X_{T_{k+1,k}+t}, P^{\nu'_{k+1}})$; i.e., we have $P^{\nu'_k} = P^{\nu'_{k+1}} \circ \varphi_k^{-1}$. By a well-known theorem, there is a (projective limit) space $(\tilde{\Omega}, \tilde{\mathscr{M}}, Q)$ that has all $(\Omega_k, \mathscr{M}^0, P^{\nu'_k})$ embedded in it, where

$$\tilde{\Omega} = \{\tilde{\omega} = (\omega^1, \dots, \omega^k, \dots) : \omega^k \in \Omega_k,\ \varphi_k(\omega^{k+1}) = \omega_k \text{ for all } k\},$$

$$\tilde{\mathscr{M}} = \sigma\left(\bigcup_k \pi_k^{-1}\mathscr{M}^0\right), \quad \text{where } \pi_k(\tilde{\omega}) = \omega^k,$$

$$Q = \text{the unique measure on } \tilde{\mathscr{M}} \text{ satisfying } Q \circ \pi_k^{-1} = P^{\nu'_k} \quad \text{for all } k.$$

To define Y_t, first let

$$\tilde{\tau}_n(\tilde{\omega}) = \lim_k \left(T_{kn}(\omega^k) - T_{kk}(\omega^k)\right)$$

an increasing limit. Note the expression on the right is either $T_{kn}(\omega^k)$ or $\infty - \infty = 0$; its value remains the same if one thinks of Y^k as $(X_{T_{kk}+t}, P^{\mu})$ (and so ω^k as a point in Ω under P^{μ}). From Proposition 2.1 we have $\tilde{\tau}_n \downarrow 0$ a.s. Q. We assume $\tilde{\tau}_n \downarrow 0$ for all $\tilde{\omega}$. Also delete all $\tilde{\omega}$ where $T_{kk}(\omega^k) = \infty$ for all k; consequently $Q(\tilde{\Omega}) = \lim \nu_k(E)$, which may be smaller than $\mu(E) = \nu'_k(E_\Delta)$. Now define

$$Y_t(\tilde{\omega}) = X_{T_{kk}(\omega^k)+t-\tilde{\tau}_k(\tilde{\omega})}(\omega^k)$$

if $T_{kk}(\omega^k) = 0$ [$T_{kk}(\omega^k) < \infty$ in case one thinks of ω^k as in Ω under P^{μ}] and $\tilde{\tau}_k(\tilde{\omega}) \le t$, the right-hand side being independent of such k. By the right continuity of X_t we also have

$$Y_t(\tilde{\omega}) = \lim_k X_{T_{kk}(\omega^k)+t}(\omega^k) = \lim_k X_t(\omega^k).$$

Denote

$$\Lambda_0 = \{\tilde{\tau}_n > 0 \text{ for all } n\} = \{\tilde{\omega}: T_{kk}(\omega^k) \ne T_{k,k-1}(\omega^k) \text{ for infinitely many } k\}.$$

The above does not define Y_0 on Λ_0. Define on Λ_0, $Y_0 = \lim_{t\to 0} Y_t$ if this limit exists; $= \Delta$ otherwise.

3. Behavior of Y_t at $t = 0$. In studying Y_t as $t \to 0$ on Λ_0, Y^k will be regarded as $(X_{T_{kk}+t}, P^{\mu})$; thus the sets Λ_k in the following proofs are subsets of the (same) space Ω under P^{μ}.

PROPOSITION 3.1. *$\lim_{t\to 0} Y_t$ exists a.s. on Λ_0.*

PROOF. Suppose not. Then by restricting $\tilde{\tau}_n$ to a subsequence one may assume that there exists a (compact) transient $B \subset E$, $\varepsilon > 0$ and constants $b_n \downarrow 0$, $c_n > 0$ such that the set

$$\Lambda_0 \cap \{Y(\tilde{\tau}_1) \in B; \text{for all } n \text{ there are } t_n, t'_n \text{ with } c_n < t_n - \tilde{\tau}_n < t'_n - \tilde{\tau}_n < b_n \text{ and } d(Y(t_n), Y(t'_n)) > \varepsilon\}$$

has Q-measure greater than ε; here d is the metric on E_Δ. Thus the sets

$$\Lambda_k = \{X(T_{k1}) \in B; \text{for all } n \le k \text{ there exist } s_{kn}, s'_{kn} \text{ with } c_n < s_{kn} - T_{kn} < s'_{kn} - T_{kn} < b_n \text{ and } d(X(s_{kn}), X(s'_{kn})) > \varepsilon\}$$

all have P^μ-measure greater than ε, and so $\Lambda_\infty = \limsup \Lambda_k$ has P^μ-measure greater than or equal to ε. Define S on Λ_∞ as follows: If Λ_{k_j} is the entire subsequence of Λ_k containing ω, let $S(\omega) = \liminf_j T_{k_j k_j}(\omega)$. If $t_1 = S(\omega) < \infty$, $X_t(\omega)$ cannot be right continuous at t_1 because for any $\delta > 0$ there exist s, s' in $(t_1, t_1 + \delta)$ with $d(X_s(\omega), X_{s'}(\omega)) > \varepsilon$; so $S(\omega) = \infty$ for all $\omega \in \Lambda_\infty$. On the other hand, let $t_0 < \infty$ be such that $P^\mu(L_B > t_0) < \varepsilon/2$. Then $S(\omega) \le \sup_j T_{k_j 1}(\omega) \le t_0$ if ω is in $\Lambda_\infty - \{L_B > t_0\}$, which has P^μ-measure greater than $\varepsilon/2$. So we have a contradiction. □

PROPOSITION 3.2. $\lim_{t \to 0} Y_t \in E$ *a.s. on* Λ_0.

PROOF. Suppose not. Then by restricting $\tilde{\tau}_n$ to a subsequence we may assume that there exist a transient $B \subset E$, $\varepsilon > 0$, and $b_n \downarrow 0$, $c_n > 0$ such that the set

$$\Lambda_0 \cap \{Y(\tilde{\tau}_1) \in B; c_n < \tilde{\tau}_n - \tilde{\tau}_{n+1} \le \tilde{\tau}_n < b_n \text{ and } d(Y(\tilde{\tau}_n), \Delta) < 1/n \text{ for all } n\}$$

has Q-measure greater than ε. Thus the sets

$$\Lambda_k = \{X(T_{k1}) \in B; c_n < T_{kn} - T_{k,n+1} \le T_{kn} - T_{kk} < b_n \text{ and } d(X(T_{kn}), \Delta) < 1/n \text{ for all } n < k\}$$

all have P^μ-measure greater than ε. Let $\Lambda_\infty = \limsup \Lambda_k$ and defines on Λ_∞ as in the preceding proof. Again $S < \infty$ on a subset of Λ_∞ of positive P^μ-measure. But if $t_1 = S(\omega) < \infty$ then for any $\delta > 0$ there exist $T_{kn}(\omega) \in (t_1, t_1 + \delta)$ with $d(X_{T_{kn}}(\omega), \Delta) < 1/n$ and n arbitrary large; so $X_{t_1}(\omega) = \Delta$. Since the above $T_{kn}(\omega)$ are finite and so $X_{T_{kn}}(\omega) \neq \Delta$ and since Δ is the death point, such ω are in a null set. We thus have a contradiction. □

We have established that a.s. Y_t is right continuous at $t = 0$ and $Y_0 \in E$.

PROPOSITION 3.3. *For any $a > 0$ and bounded f in $\mathscr{E}_\Delta$, $U^a f(Y_t)$ is right continuous at $t = 0$ a.s. on Λ_0, where U^a denotes the a-potential of X.*

PROOF. $U^a f(Y_t)$ is of course right continuous on $[\tilde{\tau}_n, \infty)$ for all n a.s. From this and the upcrossing lemma applied to the supermartingales

$\{e^{-at}U^a f(Y(\tilde{\tau}_n + t)),\ t \geq 0\}$, $n \geq 1$, when $f \geq 0$, we have $\lim_{t \to 0} U^a f(Y_t)$ exists a.s. Suppose the proposition is false. Then there exist $\varepsilon > 0$, constants b_1, b_2 and a compact $C \subset E$ such that (say) $b_1 < b_2$, $C \subset \{U^a f < b_1\}$ and

$$\Lambda_0 \cap \left\{Y_0 \in C,\ \lim_{t \to 0} U^a f(Y_t) > b_2\right\}$$

has Q-measure greater than ε (or such that the above holds with all inequalities except the last one reversed). Then again by using a subsequence of $\tilde{\tau}_n$ we may assume that there exist a transient $B \subset E$ and $c_n \downarrow 0$, $c_n' > 0$ such that

$$\Lambda_0 \cap \{Y(\tilde{\tau}_1) \in B;\ U^a f(Y(\tilde{\tau}_n)) > b_2,\ c_n' < \tilde{\tau}_n - \tilde{\tau}_{n+1} \leq \tilde{\tau}_n < c_n \text{ and } d(Y(\tilde{\tau}_n), C) < 1/n \text{ for all } n\}$$

has Q-measure greater than $\varepsilon/2$; here $d(x, C)$ denotes the distance from x to C. Thus the sets

$$\Lambda_k = \{X(T_{k1}) \in B;\ U^a f(X(T_{kn})) > b_2,\ c_n' < T_{kn} - T_{k,n+1} \leq T_{kn} - T_{kk} < c_n \text{ and } d(X(T_{kn}), C) < 1/n \text{ for all } n < k\}$$

all have P^μ-measure greater than $\varepsilon/2$. Let $\Lambda_\infty = \limsup \Lambda_k$ and define S on Λ_∞ as before. Again $S < \infty$ on a subset of Λ_∞ of positive measure. But if $t_1 = S(\omega) < \infty$, there exists a sequence of $t \downarrow t_1$ such that $d(X_t(\omega), C) \to 0$ and $U^a f(X_t(\omega)) > b_2$. It follows that $X_{t_1}(\omega) \in C$ and so $U^a f(X_t(\omega))$ is not right continuous at t_1. This contradicts the fact that a.s. $P^\mu(d\omega)$, $U^a f(X_t(\omega))$ is right continuous. □

Define $\mathscr{M}_t^0 = \mathscr{G}_t \vee \sigma(R)$, where $\mathscr{G}_t$ was defined early in Section 2. Then define

$$\tilde{\mathscr{M}}_t^0 = \bigcap_n \sigma\left(\bigcup_{k \geq n} \pi_k^{-1} \mathscr{M}^0_{T_{kn}+t}\right), \qquad \tilde{\mathscr{M}}_t = \tilde{\mathscr{M}}^0_{t+}.$$

It is easy to see that $\pi_k^{-1}\mathscr{M}^0_{T_{kn}+t}$ increases with k and $\sigma(\bigcup_{k \geq n} \pi_k^{-1} \mathscr{M}^0_{T_{kn}+t})$ decreases with n.

THEOREM 3.4. (i) *$Y = (\tilde{\Omega}, \tilde{\mathscr{M}}, \tilde{\mathscr{M}}_t, Y_t, Q)$ is a version of the right process X under measure P^{ν_∞} where $\nu_\infty(\cdot) = Q(Y_0 \in \cdot)$.* (ii) *The $\tilde{\tau}_n$ are $(\tilde{\mathscr{M}}_t)$-stopping time with $\tilde{\tau}_n \downarrow 0$ and $Q(Y(\tilde{\tau}_n) \in \cdot,\ \tilde{\tau}_n < \infty) = \nu_n(\cdot)$.*

PROOF. (i) Using the Markov property of Y^k at times $T_{kn} + t$, and the monotonicity stated in the sentence just before the theorem, it is routine to show the desired Markov property of (Y_t) relative to the σ-algebra $\tilde{\mathscr{M}}_t^0$ when $t > 0$. The desired strong Markov property of (Y_t) relative to the filtration $(\tilde{\mathscr{M}}_t)$ follows from this and the right continuity of $U^a f(Y_t)$ on $[0, \infty)$ for all $a > 0$, bounded f in $\mathscr{E}_\Delta$. (ii) It is easy to verify that $\tilde{\tau}_n$ is a stopping time relative to $(\tilde{\mathscr{M}}_t)$. We have already seen $\tilde{\tau}_n \downarrow 0$. Finally from the fact $\nu_n = \mu P_{T_{kn}}$ it is clear that $Q(Y(\tilde{\tau}_n) \in \cdot,\ \tilde{\tau}_n < \infty) = \nu_n(\cdot)$. □

We note in passing that ν_∞ is concentrated on E and is the weak limit of ν_n. Let $\tilde{Z}_n(\tilde{\omega}) = Z_n(\omega^k)$. Then the $\tilde{Z}_n$ depend only on $\tilde{R}(\tilde{\omega}) = R(\omega^k)$, and (Y_t) and $(\tilde{Z}_n)_{n\ge 1}$ under Q have the same joint distribution as (X_t) and $(Z_n)_{n\ge 1}$ under P^{ν_∞}. Also, $\tilde{\tau}_n - \tilde{\tau}_{n+1}$ is a stopping time relative to the filtration $(\bigcap_{\varepsilon>0}\sigma(Y(\tilde{\tau}_n + s), s < t + \varepsilon) \vee \sigma(\tilde{Z}_n))$; this follows from (2.1) and the fact S_n is a stopping time relative to $(\mathscr{G}_t \vee \sigma(Z_n))$.

4. A canonical representation of Y and the sequence $\tilde{\tau} = (\tilde{\tau}_n)$. We will need to consider the following spaces and σ-algebras. Let $V = \{v = (v_n)_{n\ge 1}: 0 \le v_n \le \infty,\ v_n \downarrow 0\}$, $\mathscr{V}$ be the σ-algebra on V generated by the coordinate mappings v_n, and $\mathscr{V}_\infty$ the tail σ-algebra $\bigcap_m \sigma(v_n, n \ge m)$. Let V_∞ be the space of tails of elements v in V, i.e., space of equivalence classes of V induced by $v \sim v'$ iff $v_n = v'_n$ for all large n; $\mathscr{V}_\infty$ is also regarded as a σ-algebra on V_∞. Note $(V, \mathscr{V})$ is a Lusin space but $(V_\infty, \mathscr{V}_\infty)$ is not. Let $\hat{\Omega}_{0+}$ be the space of infinitesimal initial parts of elements $\hat{\omega}$ in $\hat{\Omega}$, i.e., space of equivalence classes of $\hat{\Omega}$ induced by $\hat{\omega} \sim \hat{\omega}'$ iff $\hat{\omega}_t = \hat{\omega}'_t$ for all small t. Denote $\mathbb{R}^\infty = \{z = (z_n)_{n\ge 1}: z_n \in \mathbb{R}$ for all $n\}$, $\mathscr{B}^\infty = \sigma(z_n, n \ge 1)$. Let $\mathbb{R}^\infty_\infty$ be the space of tails of elements z in $\mathbb{R}^\infty$. Let $\mathscr{H} = \bigcap_m \hat{\mathscr{G}}_{1/m} \times \sigma(z_n, n \ge m)$, a sub-$\sigma$-algebra of $\hat{\mathscr{G}} \times \mathscr{B}^\infty$ on $\hat{\Omega} \times \mathbb{R}^\infty$; here $\hat{\mathscr{G}} = \sigma(\hat{\omega}_t, t \ge 0)$ and $\hat{\mathscr{G}}_t = \bigcap_{\varepsilon>0}\sigma(\hat{\omega}_s, s < t + \varepsilon)$. $\mathscr{H}$ is also regarded as a σ-algebra on $\hat{\Omega}_{0+} \times \mathbb{R}^\infty_\infty$. Elements of V_∞ (resp. of $\mathbb{R}^\infty_\infty$) are denoted u (resp. w), and the tail of $v \in V$ (resp. of $z \in \mathbb{R}^\infty$) is denoted $v_{\infty-}$ (resp. $z_{\infty-}$); elements of $\hat{\Omega}_{0+}$ are denoted ξ, and the infinitesimal initial part of $\hat{\omega} \in \hat{\Omega}$ is denoted $\hat{\omega}_{0+}$.

Let $\tilde{\tau} = (\tilde{\tau}_n)$. We now choose a regular conditional distribution (r.c.d.)

$$\alpha(\hat{\omega}, z, dv) = Q(\tilde{\tau} \in dv \mid Y = \hat{\omega}, \tilde{Z} = z),$$

where of course $Y = (Y_t)$, $\tilde{Z} = (\tilde{Z}_n)$. α is a (transition) kernel in $\mathscr{V}/\hat{\mathscr{G}} \times \mathscr{B}^\infty$ (the meaning of this notation being obvious). The existence of α is due to the fact that $(V, \mathscr{V})$ is Lusinian. Denote

$$q(d\hat{\omega}, dz) = Q(Y \in d\hat{\omega}, Z \in dz).$$

By (2.1) we have

$$v_n = v_{n+1} + \hat{S}_n(z_n) \circ \theta_{v_{n+1}}(\hat{\omega}), \qquad n \ge 1, \tag{4.1}$$

a.e. $Q(Y \in d\hat{\omega},\ \tilde{Z} \in dz,\ \tilde{\tau} \in dv) = q(d\hat{\omega}, dz)\alpha(\hat{\omega}, z, dv)$, where $\hat{S}_n(a)$ is the value of $S_n(\omega) = S_n(\hat{\omega}, r)$ in (2.1) when $Z_n(\omega) = a$. Note $\hat{S}_n(a)$ is a $(\hat{\mathscr{G}}_t)$-stopping time on $\hat{\Omega}$.

PROPOSITION 4.1. *There exists $\Gamma \in \hat{\mathscr{G}} \times \mathscr{B}^\infty$ with $q(\Gamma) = 0$ such that for all $H \in \mathscr{V}_\infty$, the restriction of $\alpha(\cdot, \cdot, H)$ to Γ^c is in $\mathscr{H}$, i.e., in $\mathscr{H} \cap \Gamma^c$.*

PROOF. For $m \ge 1$, $t > 0$ denote $\tilde{\tau}^{m,t} = (\tilde{\tau}_n \wedge \tilde{\tau}_m \wedge t)_{n\ge 1}$ and choose a r.c.d.

$$\alpha_{m,t}(\hat{\omega}, z, dv) = Q(\tilde{\tau}^{m,t} \in dv \mid Y = \hat{\omega}, \tilde{Z} = z)$$

as a kernel in $\mathscr{V}/\hat{\mathscr{G}}_t \times \sigma(Z_n,\ n \geq m)$. Similar to (4.1) we have

$$(4.2) \qquad v_n = \left[v_{n+1} + \hat{S}_n(z_n) \circ \theta_{v_{n+1}}(\hat{\omega})\right] \wedge v_m \wedge t, \qquad n \geq 1,$$

a.e. $q(d\hat{\omega}, dz)\alpha_{m,t}(\hat{\omega}, z, dv)$. Define $\alpha^{m,t}(\hat{\omega}, z, dv)$ to be the image measure of $\alpha_{m,t}(\hat{\omega}, z, dv')$ under the mapping $v' \to v$ defined by: If v' satisfies (4.2) then v satisfies (4.1) and $v_{\infty-} = v'_{\infty-}$; otherwise $v \equiv 0$. Clearly $\alpha^{m,t}(\hat{\omega}, z, dv)$ is another version of $Q(\tilde{\tau} \in dv | Y = \hat{\omega},\ \tilde{Z} = z)$. So $\alpha^{m,t}(\hat{\omega}, z, dv) = \alpha(\hat{\omega}, z, dv)$ a.e. $q(d\hat{\omega}, dz)$. Let $\mathscr{V}_{m,t} = \sigma(v_n \wedge v_m \wedge s,\ n \geq 1 \text{ and } s < t)$. Clearly, for $H \in \mathscr{V}_{m,t}$, $\alpha^{m,t}(\hat{\omega}, z, H) = \alpha_{m,t}(\hat{\omega}, z, H)$ and is therefore a function in $\hat{\mathscr{G}}_t \times \sigma(z_n, n \geq m)$. It follows that $\alpha(\hat{\omega}, z, H)$ is in the q-completion of $\hat{\mathscr{G}}_t \times \sigma(z_n, n \geq m)$ relative to $\hat{\mathscr{G}} \times \mathscr{B}^\infty$. Since $\mathscr{V}_{m,t}$ is countably generated, there exists $\Gamma_{m,t} \in \hat{\mathscr{G}} \times \mathscr{B}^\infty$ with $q(\Gamma_{m,t}) = 0$ such that for all $H \in \mathscr{V}_{m,t}$, $\alpha(\cdot, \cdot, H)$ restricted to $\Gamma^c_{m,t}$ is in $\hat{\mathscr{G}}_t \times \sigma(z_n,\ n \geq m)$. Since $\mathscr{V}_\infty \subset \mathscr{V}_{m,t}$ for all m and $t > 0$ (because $v_n \downarrow 0$) and since $\mathscr{H} = \bigcap_m \hat{\mathscr{G}}_{1/m} \times \sigma(z_n,\ n \geq m)$, the proposition follows, with $\Gamma = \bigcup_m \Gamma_{m,1/m}$. □

DEFINITION. Let $\Gamma_0 = \{(\xi, w) \in \hat{\Omega}_{0+} \times \mathbb{R}^\infty_\infty$: There exist no $(\hat{\omega}, z) \in \Gamma^c$ with $(\hat{\omega}_{0+}, z_{\infty-}) = (\xi, w)\}$. Define for each $(\xi, w) \in \hat{\Omega}_{0+} \times \mathbb{R}^\infty_\infty$ a measure $\beta(\xi, w, \cdot)$ on $\mathscr{V}_\infty$ as follows:

$$\beta(\xi, w, \cdot) = \alpha(\hat{\omega}, z, \cdot) \quad \text{if } (\hat{\omega}, z) \in \Gamma^c \text{ and } (\hat{\omega}_{0+}, z_{\infty-}) = (\xi, w)$$
$$= \text{point mass at } v^0_{\infty-}, \quad \text{where } v^0 \equiv 0, \text{ if } (\xi, w) \in \Gamma_0.$$

Let $\mathscr{H}^* = \sigma(\mathscr{H}, \Gamma_0)$. Then $\beta(\xi, w, du)$ is a kernel in $\mathscr{V}_\infty/\mathscr{H}^*$. Note that the mapping $(\hat{\omega}, z) \to (\hat{\omega}_{0+}, z_{\infty-})$ is in $\mathscr{H}^*/\hat{\mathscr{G}} \times \mathscr{B}^\infty$.

We now proceed to define a representation of Y and $\tilde{\tau}$ on the following space $(\overline{\Omega}, \overline{\mathscr{M}}, \overline{P})$: $\overline{\Omega} = \Omega \times V_\infty$, $\overline{\mathscr{M}} = \mathscr{M} \times \mathscr{V}_\infty$, and with $P = P^{\nu_\infty}$,

$$\overline{P}(d\omega, du) = P(d\omega)\beta(\hat{\omega}_{0+}, z_{\infty-}, du),$$

where of course $\omega = (\hat{\omega}, r)$, $z = Z(\omega) = (Z_n(\omega))$, X_t, θ_t, Z_n, S_n are regarded as defined on $\overline{\Omega}$ by $X_t(\omega, u) = X_t(\omega)$, $\theta_t(\omega, u) = (\theta_t\omega, u)$, etc. Also $\mathscr{G}_t = \bigcap_{\varepsilon > 0}\sigma(X_s,\ s < t + \varepsilon)$ are regarded as σ-algebras on $\overline{\Omega}$. Let $\overline{\mathscr{M}}_t = \mathscr{M}_t \times \mathscr{V}_\infty$. Finally, let $U(\omega, u) = u$ (this U is not to be confused with the potential kernel U).

Define

$$v(\hat{\omega}, z, u) = \text{the (unique) } v \text{ satisfying (4.1) and } v_{\infty-} = u \quad \text{if such a } v \text{ exists}$$
$$= v^0 \equiv 0 \text{ otherwise}.$$

It is easy to see that $v(\hat{\omega}, z, u)$ is in $\mathscr{V}/\hat{\mathscr{G}} \times \mathscr{B}^\infty \times \mathscr{V}_\infty$. From (4.1) and Proposition 4.1

$$\alpha(\hat{\omega}, z, dv) = \int_{V_\infty} \alpha(\hat{\omega}, z, du)\varepsilon_{v(\hat{\omega}, z, u)}(dv)$$
$$= \int_{V_\infty} \beta(\hat{\omega}_{0+}, z_{\infty-}, du)\varepsilon_{v(\hat{\omega}, z, u)}(dv)$$

a.e. $q(d\hat{\omega}, dz)$, where $\varepsilon_{\cdot}(\cdot)$ denotes a point mass. Now define τ on $\overline{\Omega}$ by

$$\tau = v(X, Z, U),$$

where of course $X = (X_t)$, $Z = (Z_n)_{n \geq 1}$.

THEOREM 4.2. (i) (X, Z, τ) *under* $\bar{P}$ *has the same distribution as* $(Y, \tilde{Z}, \tilde{\tau})$ *under* Q. *In particular*, $\bar{P}(X(\tau_n) \in \cdot,\ \tau_n < \infty) = Q(Y(\tilde{\tau}_n) \in \cdot,\ \tilde{\tau}_n < \infty) = \nu_n(\cdot)$ *for all* n. (ii) *Each* τ_n *is a stopping time relative to the filtration* $(\mathscr{G}_t \vee \sigma(Z_m, m \geq n) \vee \sigma(U))$ [*and therefore relative to* $(\bar{\mathscr{M}}_t)$], *and* $\tau_n = \tau_{n+1} + S_n \circ \theta(\tau_{n+1})$ *a.s.* $\bar{P}$.

PROOF. (i) By the definition of τ,

$$\begin{aligned}
&\bar{P}(X \in d\hat{\omega}, Z \in dz, \tau \in dv)\\
&\quad = \int_{u \in V_\infty} \bar{P}(X \in d\hat{\omega}, Z \in dz, U \in du)\varepsilon_{v(\hat{\omega}, z, u)}(dv)\\
&\quad = P(X \in d\hat{\omega}, Z \in dz)\int_{V_\infty} \beta(\hat{\omega}_{0+}, z_{\infty -}, du)\varepsilon_{v(\hat{\omega}, z, u)}(dv)\\
&\quad = Q(Y \in d\hat{\omega}, \tilde{Z} \in dz)\alpha(\hat{\omega}, z, dv)\\
&\quad = Q(Y \in d\hat{\omega}, \tilde{Z} \in dz, \tilde{\tau} \in dv).
\end{aligned}$$

The second assertion in (i) follows from the first and the right continuity of Y_t and X_t. (ii) Define for fixed n and $t > 0$: $v^{n,t}(\hat{\omega}, z, u) =$ the (unique) v satisfying (4.2) with m, n interchanged and $v_{\infty -} = u$ if such v exists; $= v^0 \equiv 0$ otherwise. Then $(\tau_m \wedge \tau_n \wedge t)_{m \geq 1} = v^{n,t}(X, Z, U)$ and $v^{n,t} \in \mathscr{V}/\hat{\mathscr{G}}_t \times \sigma(z_m, m \geq n) \times \mathscr{V}_\infty$. It follows that $\tau_n \wedge t \in \mathscr{G}_t \vee \sigma(Z_m, m \geq n) \vee \sigma(U)$. This proves the first assertion of (ii). The second follows from (4.1) and the definition of $v(\hat{\omega}, z, u)$. □

REMARK 1. The above representation is still valid if $Y = (\tilde{\Omega}, \tilde{\mathscr{M}}, \tilde{\mathscr{M}}_t, Y_t, Q)$ is any right-continuous stochastic process, with the $\tilde{Z}_n$ not necessarily satisfying the independence conditions, and of course assuming that $\tilde{\tau}_n - \tilde{\tau}_{n+1}$ is a stopping time relative to the filtration $(\sigma(\tilde{Z}_n) \vee \bigcap_{\varepsilon > 0}\sigma(Y(\tilde{\tau}_n + s), s < t + \varepsilon))$. (X_t) and (Z_n) can be defined on $\Omega = \hat{\Omega} \times \mathbb{R}^\infty$ (with the Z_n as coordinates on $\mathbb{R}^\infty$) or on $\Omega = \hat{\Omega} \times \mathbb{R}$ [with the Z_n as functions depending on $R(\hat{\omega}, r) = r$] to have the same joint distribution under a measure P as that of (Y_t) and $(\tilde{Z}_n)$. The rest is the same.

REMARK 2. The following is an open question. Consider the kernel $\beta(\xi, w, du)$ defined above. Let Z_0 be a continuous real random variable. Does there exist a function

$$u = u(\xi, w, z_0): \hat{\Omega}_{0+} \times \mathbb{R}_\infty^\infty \times \mathbb{R} \to V_\infty$$

in $\mathscr{V}_\infty/\mathscr{H}^* \times \mathscr{B}$ (where $\mathscr{B}$ is the Borel σ-algebra of $\mathbb{R}$) such that $u(\xi, w, Z_0)$ has distribution $\beta(\xi, w, du)$ for all (ξ, w)? Suppose the answer to this question is yes. Then, recalling there is in the process X a continuous real random variable Z_0 (which is also independent of everything else), one can directly define on Ω (rather than on $\bar{\Omega} = \Omega \times V_\infty$) stopping times τ_n by

$$\tau = v(X, Z, u), \quad \text{where } u = u(X_{0+}, Z_{\infty -}, Z_0).$$

The τ_n would satisfy the assertions in Theorem 4.2, with Z_0 playing the role of U in defining the relevent σ-algebras. Furthermore, assuming the above is true, if one includes in the right process X another continuous real random variable Z_{-1} (independent of everything else), one could define a stopping time T_∞ relative to $(\mathscr{G}_t \vee \sigma(Z_{-1}))$ satisfying $\nu_\infty = \mu P_{T_\infty}$; then the stopping times $T_n = T_\infty + \tau_n \circ \theta(T_\infty)$ would satisfy $\nu_n = \mu P_{T_n}$ [and one would also have $T_n = T_{n+1} + S_n \circ \theta(T_{n+1})$ a.s. P^μ, with T_{n+1} a stopping time relative to $(\mathscr{G}_t \vee \sigma(Z_m, m \geq n+1) \vee \sigma(Z_0, Z_{-1}))$ and S_n a stopping time relative to $(\mathscr{G}_t \vee \sigma(Z_n))$]. Thus Fitzsimmons' question would be completely answered in the positive.

REMARK 3. Suppose a r.c.d. $p(\xi, w, d\hat{\omega}, dz) = Q(Y \in d\hat{\omega},\ \tilde{Z} \in dz | Y_{0+} = \xi,\ z_{\infty-} = w)$ (which exists) satisfies $p(\xi, w, \{\hat{\omega}_{0+} = \xi,\ z_{\infty-} = w\}) = 1$ for all (ξ, w) (in the terminology of [2] p is said to be "proper"). (Note this may sometimes be the case in the more general setting stated in Remark 1.) We show that there exists a function $u(\xi, w, z_0)$ satisfying the condition in Remark 2. Choose a r.c.d. $\gamma(\xi, w, dv) = Q(\tilde{\tau} \in dv | Y_{0+} = \xi,\ \tilde{Z}_{\infty-} = w)$. Then a.e. $Q(Y_{0+} \in d\xi,\ \tilde{Z}_{\infty-} \in dw)$,

$$
\begin{aligned}
\gamma(\xi, w, dv) &= \int p(\xi, w, d\hat{\omega}, dz) Q\big(\tilde{\tau} \in dv | Y_{0+} = \xi,\ \tilde{Z}_{\infty-} = w,\ Y = \hat{\omega},\ \tilde{Z} = z\big) \\
&= \int p(\xi, w, d\hat{\omega}, dz) Q(\tilde{\tau} \in dv | Y = \hat{\omega},\ \tilde{Z} = z) \\
&= \int p(\xi, w, d\hat{\omega}, dz) \alpha(\hat{\omega}, z, dv),
\end{aligned}
$$

using the property of p in the second equality. Using this property once more and the definition of $\beta(\xi, w, du)$, we have $\gamma(\xi, w, du) = \beta(\xi, w, du)$ on $\mathscr{V}_\infty$ a.e. $Q(Y_{0+} \in d\xi,\ \tilde{Z}_{\infty-} \in dw)$. Now since $(V, \mathscr{V})$ is Lusinian, there exists $v(\xi, w, z_0)$ in $\mathscr{V}/\mathscr{H}^* \times \mathscr{B}$ such that $v(\xi, w, Z_0)$ has distribution $\gamma(\xi, w, dv)$ for all (ξ, w). Let $u(\xi, w, z_0)$ be the tail (i.e., the projection to V_∞) of $v(\xi, w, z_0)$. Then $u(\xi, w, z_0) \in \mathscr{V}_\infty/\mathscr{H}^* \times \mathscr{B}$ and $u(\xi, w, Z_0)$ has distribution $\beta(\xi, w, du)$ for all (ξ, w) as desired in Remark 2. Note, however, that the assumption in this remark fails if either the Z_n are independent (so that the Kolmogorov zero–one law applies) or if, given Y_0, the σ-algebra $\tilde{\mathscr{G}}_0 = \bigcap_{\varepsilon > 0} \sigma(Y_t, t < \varepsilon)$ is trivial but not atomic (as is typical for the Y in this article because of the Blumenthal zero–one law). If no $\tilde{Z}_n$ are involved (in the determination of $\tilde{\tau}_n - \tilde{\tau}_{n+1}$), and a r.c.d. $p(\xi, d\hat{\omega}) = Q(Y \in d\hat{\omega} | Y_{0+} = \xi)$ satisfies $p(\xi, \{\hat{\omega}_{0+} = \xi\}) = 1$ for all ξ, then of course as in the above a desired function $u(\xi, z_0)$ exists. This is the situation in Example 2 of Section 5.

5. Examples. The following examples are given only to illustrate some possibilities of the kernel $\beta(\xi, w, du)$ or $\beta(\xi, du)$.

EXAMPLE 1. Let X be Brownian motion on the interval $[-1, 1]$ absorbed at -1 and 1, which are identified as Δ. Let x_{in}, $n \geq 1$, $i \in I$, where I is countable, be points in $(-1, 1) - \{0\}$ with $x_{in} \downarrow 0$ or $x_{in} \uparrow 0$ for each i, and with x_{in}, $i \in I$, distinct for each n. Let $c_{in} = P^0(T_{\{x_{in}\}} < \infty)$ where T_A denote the first hitting time of A, and let $p(i) > 0$ with $\sum p(i) = 1$. Define $\nu_n = \sum_i p(i)c_{in}\varepsilon_{x_{in}}$, $\mu = \varepsilon_0$. Then μ and ν_n satisfy (1.1). Here there exist (obvious) nonrandomized stopping times S_n such that $\nu_n = \nu_{n+1}P_{S_n}$; so randomization variables Z_n are not needed. Clearly $\nu_\infty = \mu$. The process Y can be regarded as the process X starting at 0 together with an independent (randomization) variable W with values in I satisfying $Q(W = i) = p(i)$; if $W = i$, $\tilde{\tau}_n$ is the first time $Y_t = X_t$ hits x_{in}. Obviously, each measure $\beta(\xi, du)$ is supported by a countable set which has the same cardinality as I and varies with ξ. In this example one can let I be (say) $[0, 1]$ and x_{in}, c_{in} be as above, $p(i)$ be a positive density and define $\nu_n = \int_0^1 p(i)c_{in}\varepsilon_{x_{in}}\,di$ and $\mu = \varepsilon_0$. Then each $\beta(\xi, du)$ is supported by an uncountable set which varies with ξ.

EXAMPLE 2. Let X be uniform motion to the right on the internal $[0, 1]$ with $\Delta = 1$, possibly with premature death. Let x_{in}, $n \geq 1$, $i \in I$, I countable, be points in $(0, 1)$ with $x_{in} \downarrow 0$ for each i, and with x_{in}, $i \in I$, distinct for each n. Let $c_{in}, p(i), \nu_n, \mu$ be the same as in Example 1. Again there exist (obvious) nonrandomized stopping times S_n with $\nu_n = \nu_{n+1}P_{S_n}$; so no Z_n enter the picture. Y can be described in the same way as in Example 1. Each measure $\beta(\xi, du)$ is supported by a set having the same cardinality as I, but this time it is independent of ξ a.e. $Q(Y_{0+} \in d\xi)$. This is the situation mentioned at the end of Remark 3 above because $Q(Y_{0+} \in d\xi)$ is atomic (in this case a unit mass). In this and the next example, as in Example 1, one can let $I = [0, 1]$ to obtain an uncountable supporting set for $\beta(\xi, du)$ or $\beta(\xi, w, du)$.

EXAMPLE 3. In the above example let x_{in} be the same, and let $0 < c_{in} \uparrow 1$ for each i satisfy

$$c_{in} \leq \min\left\{P^0\left(T_{\{x_{in}\}} < \infty\right), c_{i,n+1}P^{x_{i,n+1}}\left(T_{\{x_{in}\}} < \infty\right)\right\}.$$

Let $p(i), \nu_n, \mu$ be the same. There exist randomized stopping times S_n with $\nu_n = \nu_{n+1}P_{S_n}$; so there are Z_n's involved as in Section 2. It is easy to describe for each i stopping times T_{in}, with $T_{in} = T_{i,n+1} + S_n \circ \theta(T_{i,n+1})$, such that $P^0(X(T_{in}) = x_{in},\ T_{in} < \infty) = c_{in}$. Then Y can be described in a similar way (using W) as in Example 1, with $\tilde{\tau}_n$ satisfying: If $W = i$, then $\tilde{\tau}_n$ "equals" T_{in}. Here $\beta(\xi, w, du)$ is again supported by a set with the same cardinality as I; however, this set is independent of ξ but varies with w.

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