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TANAKA FORMULAE AND RENORMALIZATION FOR TRIPLE INTERSECTIONS OF BROWNIAN MOTION IN THE PLANE

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We develop explicit stochastic integral representations for the renormalized triple intersection local time of planar Brownian motion. Our representations involve a new type of double stochastic integral, the bilateral stochastic integral, which is developed in detail.

1. Introduction. It is well known that Brownian motion W_t in the plane has (many) triple intersections. In an effort to study the set

$$L_0 \doteq \{(r,s,t) | W_r = W_s = W_t\}, \tag{1.1}$$

we are led naturally to try to give meaning to the formal expression

$$\iint_\Gamma \int \delta(W_t - W_s)\delta(W_s - W_r)\, dr\, ds\, dt \tag{1.2}$$

as a measure on L_0. Here $\delta(\cdot)$ signifies the Dirac delta "function."

A fruitful way to approach (1.2) is via the occupation density of the mapping

$$(r,s,t) \to (W_s - W_r, W_t - W_s).$$

More precisely, it can be shown that for any bounded Borel set $\Gamma \subseteq R^3_{\leq} \doteq \{(r,s,t)|0 \leq r \leq s \leq t\}$ there exists a measurable function $\alpha(x,y,\Gamma)$ such that

$$\iint_\Gamma \int f(W_s - W_r, W_t - W_s)\, dr\, ds\, dt = \iint_{\mathbb{R}^4} f(x,y)\alpha(x,y,\Gamma)\, d^2x\, d^2y \tag{1.3}$$

for all bounded Borel functions $f\colon \mathbb{R}^4 \to \mathbb{R}$ [Rosen (1988)]. If we formally take $f(x,y) = \delta(x)\delta(y)$, we recover (1.2) expressed as $\delta(0,0,\Gamma)$. For this formal substitution to make any sense however, we would need to have $\alpha(x,y,\Gamma)$ continuous at $x = y = 0$. This is not always the case: For general Γ's, this is only true for a "renormalization" of (1.2), which is the main topic of this paper.

Let us first recall a few facts from Rosen (1988). For $x, y \neq 0$, we can find a version of $\alpha(x,y,\cdot\,)$ which is a measure supported on

$$L_{x,y} = \{(r,s,t) | W_s - W_r = x, W_t - W_s = y\}. \tag{1.4}$$

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For this reason, we refer to $\alpha(x, y, \Gamma)$ as an intersection local time. Moreover,

$$(x,y) \to \alpha(x,y;\cdot), \qquad (x,y) \in \left(\mathbb{R}^2 - \{0\}\right)^2,$$

can be chosen to be weakly continuous. This will also hold for $x = 0$ and $y = 0$ provided the measures $\alpha(x, y; \cdot\,)$ are restricted to regions of $\mathbb{R}^3_{\le}$ which lie away from the diagonals.

Let $\Gamma_T = \{0 \le r \le s \le t \le T\} \subseteq \mathbb{R}^3_{\le}$. Then $\alpha(x, y, \Gamma_T)$ diverges as x or $y \to 0$. The purpose of this paper is to study the asymptotics of $\alpha(x, y, \Gamma_T)$ as $x, y \to 0$. We will find that after subtracting off certain explicit "infinite parts," the remainder, $\gamma(x, y, T)$, called the renormalized intersection local time, can be expressed in terms of stochastic integrals. We will refer to such an expression as a Tanaka formula. Our Tanaka formulae lead naturally to a proof of the joint continuity of $\gamma(x, y, T)$. This continuity has been established in Rosen (1988). The novelty of this paper lies in the explicit representation for $\gamma(x, y, T)$—our Tanaka formulae.

Let $\alpha(x, y, T) \doteq \alpha(x, y, \Gamma_T)$, write $p_s(x)$ for the density of W_s and

$$U_\varepsilon(x) = \int_\varepsilon^1 p_s(x)\,ds, \qquad U \doteq U_0. \tag{1.5}$$

It is known that for $x \neq 0$,

$$\alpha(x,T) = \lim_{\varepsilon \to 0} \int_0^T \int_0^t p_\varepsilon(W_t - W_s - x)\,ds\,dt$$

exists and that for all $x \in \mathbb{R}^2$,

$$\hat{\alpha}(x,T) = \lim_{\varepsilon \to 0} \left(\int_0^T \int_0^t p_\varepsilon(W_t - W_s - x)\,ds\,dt - TU_\varepsilon(x) \right) \tag{1.6}$$

exists and admits a jointly continuous version in (x, T). $\alpha(x, T)$ [resp. $\hat{\alpha}(x, T)$] is the intersection local time (resp. the renormalized intersection local time) for double intersections; see Rosen (1986a), Yor (1986a), Le Gall (1985) and Dynkin (1988). (These properties follow easily from the techniques of the present paper.)

THEOREM 1.

$$\begin{aligned} \gamma(x,y,T) \doteq \alpha(x,y,T) &- TU(x)U(y) - U(x)\hat{\alpha}(y,T) \\ &- U(y)\hat{\alpha}(x,T) \end{aligned} \tag{1.7}$$

has a jointly continuous extension to all x, y, T.

An easy, but interesting, variant of Theorem 1 is obtained as follows: Remark that $V(x) = U(x) - (1/\pi)\log(1/|x|)$, $x \neq 0$, can be extended by continuity to all of $\mathbb{R}^2$. Therefore, from the discussion preceding Theorem 1,

$$\tilde{\alpha}(x,T) = \alpha(x,T) - \frac{T}{\pi}\log\frac{1}{|x|}, \qquad x \neq 0,$$

also admits a jointly continuous extension in (x, T). We now deduce from Theorem 1 the following:

COROLLARY.

$$\tilde{\gamma}(x,y,T) \doteq \alpha(x,y,T) - \frac{T}{\pi^2}\log\left(\frac{1}{|x|}\right)\log\left(\frac{1}{|y|}\right) - \frac{1}{\pi}\left(\log\frac{1}{|x|}\right)\tilde{\alpha}(y,T) - \frac{1}{\pi}\left(\log\frac{1}{|y|}\right)\tilde{\alpha}(x,T) \tag{1.8}$$

has a jointly continuous extension to all (x, y, T).

Theorem 1 is derived via our Tanaka formulae. In fact, we develop two distinct versions. One in Section 3 follows Rosen (1986a), while the other in Section 4 follows Yor (1986b). Either can be used to prove the joint continuity of $\gamma(x, y, T)$.

Our Tanaka formulae contain a new type of double stochastic integral, which we refer to as a bilateral stochastic integral. Section 2 is devoted to the study of this integral.

The renormalization (1.6) is due to Varadhan (1969). For the renormalization of higher order multiple points, see Rosen (1986b) and Dynkin (1986). It remains an open problem to find Tanaka formulae for such renormalized intersection local times.

See also Le Gall (1987, 1990), where a different renormalization is used.

2. Definition of bilateral stochastic integrals. It will be convenient, in the sequel of the paper, to express a number of random variables as double stochastic integrals with respect to Brownian motion, one of which is taken in the forward direction and the other in the backward direction.

Therefore we need to define precisely these integrals, which we shall denote by

$$\int_0^t d_+B_v \int_0^v d_-B_w\, H(w,v),$$

where d_+B_v stands for the forward Itô differential and d_-B_w for the backward Itô differential [cf. McKean (1969), page 35].

For simplicity, we assume in this paragraph that B is real-valued, the extension to two-dimensional Brownian motion being immediate.

We first recall very briefly the definition of both the forward and backward stochastic integrals:

$$\int_0^t d_+B_w\, H(w) \quad \text{and} \quad \int_0^v d_-B_w\, H(w)$$

for suitable integrands H. From now on, we fix $t > 0$.

If $(H(w),\ w \leq t)$ is continuous and adapted to the family of σ-fields $\mathscr{B}_w^+ = \sigma\{B_u;\ u \leq w\}$ (resp. $\mathscr{B}_w^- = \sigma\{B_u - B_t;\ w \leq u \leq t\}$), we take

$$\int_0^t d_+B_w\, H(w) = \operatorname*{P-lim}_{n\to\infty} \sum_{\zeta_n} \left(B_{w_{i+1}^{(n)}} - B_{w_i^{(n)}}\right) H\left(w_i^{(n)}\right)$$

(resp.

$$\int_0^t d_-B_w\, H(w) \equiv \operatorname*{P-lim}_{n\to\infty} \sum_{\zeta_n} \left(B_{w_i^{(n)}} - B_{w_{i+1}^{(n)}}\right) H\left(w_{i+1}^{(n)}\right)),$$

where $\zeta_n = (0 = w_0^{(n)} < w_1^{(n)} < \cdots < w_{p_n}^{(n)} = t)$ is a sequence of subdivisions of $[0,t]$ such that $\sup_i (w_{i+1}^{(n)} - w_i^{(n)}) \to 0$ as $n \to \infty$.

We note the identity

$$(*) \qquad \int_0^t d_-B_w\, H(w) = -\int_0^t d\left({}_+B_w^{(t)}\right) H(t-w),$$

where $B_w^{(t)} = B_t - B_{t-w}$, $w \leq t$, is the Brownian motion obtained by time reversal of B at time t.

For both integrals, H satisfying the above hypothesis, we have the isometry properties

$$E\left[\left(\int_0^t d_\pm B_w\, H(w)\right)^2\right] = E\left[\int_0^t dw\, H^2(w)\right],$$

which allows us to extend the definition of $\int_0^t d_\pm B_w\, H(w)$ to processes H belonging to $L^2([0,t]\times\Omega, \mathscr{P}_\pm, dw\, dP)$, where $\mathscr{P}_\pm$ is the σ-field generated by the continuous processes which are adapted to the filtration $(\mathscr{B}_w^\pm,\ w \leq t)$.

In a similar manner, we define $\mathscr{P}_{-,+}$ as the σ-field on $\mathbb{R}^2_\leq \times \Omega$, which is generated by the processes

$$H(w,v) = 1_{(w<a)} h\left(B_{u_1} - B_a, B_{u_2} - B_{u_1}, -, B_b - B_{u_n}\right) 1_{(b<v)},$$

where $a < u_1 < \cdots < u_n < b$ and $h\colon \mathbb{R}^{n+1} \to \mathbb{R}$ is a bounded Borel function. In the sequel, we shall write, for simplicity, h_* instead of $h(B_{u_1} - B_a, B_{u_2} - B_{u_1}, -, B_b - B_{u_n})$.

We call the above processes H elementary and we then define

$$(2.1) \qquad \int_0^t d_+B_v \int_0^v d_-B_w\, H(w,v) = -h_* B_a (B_{t\vee b} - B_b),$$

where $t \vee b = \max(t,b)$.

A justification for this definition and, in particular, for the minus sign in the right-hand side of formula (2.1) is given by the following remarks: For every v, we have $\int_0^v d_-B_w\, H(w,v) = -h_* B_a 1_{(b<v)}$ so that

$$\int_0^t d_+B_v \int_0^v d_-B_w\, H(w,v) = \int_0^t d_+B_v\left(\int_0^v d_-B_w\, H(w,v)\right)$$

and, likewise, both sides of this identity are equal to

$$\int_0^t d_-B_w\left(\int_w^t d_+B_v\, H(w,v)\right).$$

The definition (2.1) may be extended by linearity to linear combinations of elementary processes and since for such combinations, the isometry identity

$$E\left[\left(\int_0^t d_+B_v \int_0^v d_-B_w\, H(w,v)\right)^2\right] = E\left[\int_0^t dv \int_0^v dw\, H^2(w,v)\right]$$

still holds, we can again extend the definition of the bilateral stochastic integral to all processes H in $L^2(\mathbb{R}^2_\leq(t) \times \Omega, \mathscr{P}_{-,+}, dv\,dw\,dP)$ and the above isometry property is still satisfied. Moreover, if H is such a process, then dv a.s., one has

$$E\left[\int_0^v dw\, H^2(w,v)\right] < \infty$$

and therefore the backward Itô integral $\int_0^v d_-B_w\, H(w,v)$ is well defined. Furthermore, there exists a process K_+ in $L^2([0,t] \times \Omega, \mathscr{P}_+, dv\,dP)$ such that dv a.s., $K_+(v) = \int_0^v d_-B_w\, H(w,v)$ and the identity

$$\int_0^t d_+B_v\, K_+(v) = \int_0^t d_+B_v \int_0^t d_-B_w\, H(w,v) \tag{2.2}$$

is satisfied.

Likewise, there exists a process K_- in $L^2([0,t] \times \Omega, \mathscr{P}_-, dw\,dP)$ such that dw a.s., $K_-(w) = \int_w^t d_+B_v\, H(w,v)$ and the identity

$$\int_0^t d_-B_w\, K_-(w) = \int_0^t d_+B_v \int_0^v d_-B_w\, H(w,v) \tag{2.3}$$

is satisfied.

Both identities (2.2) and (2.3) are Fubini-type statements and allow us to compute bilateral stochastic integrals by successively performing a forward and then a backward stochastic integral, or vice versa.

Bilateral stochastic integrals play an essential role in the following representation result.

PROPOSITION. *Let $t > 0$. Every r.v. Φ in $L^2(\mathscr{B}_t^+)$ may be represented in a unique way as*

$$\Phi = c + \int_0^t d_+B_v\, \phi(v) + \int_0^t d_+B_v \int_0^v d_-B_w\, \psi(w,v),$$

where $c \in \mathbb{R}$, $\phi \in L^2([0,t]; dv)$ and

$$\psi \in L^2\big(\mathbb{R}^2_\leq(t) \times \Omega; \mathscr{P}_{-,+}; dv\,dw\,dP\big).$$

PROOF. It suffices to show that such a representation holds for

$$\Phi = \varepsilon_t^f \doteq \exp\left(\int_0^t f(v)\, d_+B_v - \tfrac{1}{2}\int_0^t f^2(v)\, dv\right),$$

where $f \in C_c^\infty(\mathbb{R}_+)$, since the variables (ε_t^f) are total in $L^2(\mathscr{B}_t^+)$. From Itô's formula, we obtain

$$\varepsilon_t^f = 1 + \int_0^t d_+B_v\, f(v)\varepsilon_v^f.$$

Then, we have

$$\varepsilon_v^f = \exp\left(-\int_0^v f(w)\, d_-B_w - \tfrac{1}{2}\int_0^v f^2(w)\, dw\right)$$

and

$$\varepsilon_v^f - 1 = -\int_0^v d_-B_w\, f(w)\exp\left(-\int_0^w f(u)\, d_-B_u - \tfrac{1}{2}\int_0^w f^2(u)\, du\right).$$

The representation we are looking for now is obtained with $c = 1$, $\phi = f$ and

$$\psi(w,v) = -f(v)\, f(w)\exp\left(-\int_0^w f(u)\, d_-B_u - \tfrac{1}{2}\int_0^w f^2(u)\, du\right). \qquad \square$$

REMARK 1. For fixed $t > 0$, the space of bilateral stochastic integrals

$$\int_0^t d_+B_v \int_0^v d_-B_w\, \psi(w,v),$$

where $\psi \in L^2(\mathbb{R}^2_{\le}(t) \times \Omega, \mathscr{P}_{-,+}, dv\, dw\, dP)$, is precisely the orthogonal complement in $L^2(\mathscr{B}_t^+)$ of the vector space $\mathbb{R} \oplus G_t$, where G_t is the Gaussian space generated by $(B_u, u \le t)$.

REMARK 2. Any bilateral stochastic integral could also be written in the form of a double forward integral

$$\int_0^t d_+B_v \int_0^v d_+B_w\, \tilde{\psi}(w,v),$$

where $\tilde{\psi}(w,v)$ is measurable with respect to $\mathscr{P}_+ \otimes \mathscr{B}(\mathbb{R}_+)$; that is, $\tilde{\psi}$ is $(\mathscr{B}_w^+)$ predictable and jointly Borel in v and satisfies

$$E\left[\int_0^t dv \int_0^v dw\, \tilde{\psi}^2(w,v)\right] < \infty.$$

However, for a given r.v. Φ, the integrand ψ may be relatively simple while $\tilde{\psi}$ is much more complicated, or vice versa. This is the reason why, for the particular examples we are dealing with below, we have preferred the representation involving bilateral stochastic integrals instead of double forward integrals.

In the following we will often use stochastic Fubini theorems such as

$$(2.4)\qquad \int_a^b\left(\int_c^d f(x,t,\omega)\,d_+B_t\right)dx=\int_c^d\left(\int_a^b f(x,t,\omega)\,dx\right)d_+B_t,$$

with an analogous statement for the bilateral stochastic integral.

For our purposes, it is enough to have (2.4) for

$$F\in\Lambda\doteq L^2([a,b]\times[c,d]\times\Omega,\mathscr{B}[a,b]\times\mathscr{P}_+,dx\,dt\,dP).$$

To see that (2.4) is true for such f, we first note that (2.4) is trivial if f is the product of a bounded measurable function of x, with a bounded previsible function of (t,ω). However, such functions are total in Λ, and it is easy to check that both sides of (2.4) are continuous from Λ to $L^2(\Omega, dP)$.

3. First Tanaka formula and joint continuity. We use the notation $f_{\varepsilon,x}(y)=f_\varepsilon(y-x)$. By subtracting and adding $U_\varepsilon(x)$ from $\int_0^s p_{\varepsilon,x}(W_s-W_r)\,dr$ [resp. $U_\varepsilon(y)$ from $\int_s^T p_{\varepsilon,y}(W_t-W_s)\,dt$], we obtain

$$\iint_{\Gamma_T}\!\int p_{\varepsilon,x}(W_s-W_r)p_{\varepsilon,y}(W_t-W_s)\,dr\,ds\,dt$$

$$=\int_0^T\left(\int_0^s p_{\varepsilon,x}(W_s-W_r)\,dr\right)\left(\int_s^T p_{\varepsilon,y}(W_t-W_s)\,dt\right)ds$$

$$(3.1)\quad =\int_0^T\left(\int_0^s p_{\varepsilon,x}(W_s-W_r)\,dr-U_\varepsilon(x)\right)\left(\int_s^T p_{\varepsilon,y}(W_t-W_s)\,dt-U_\varepsilon(y)\right)ds$$

$$+\,U_\varepsilon(x)\left(\int_0^T\int_0^t p_{\varepsilon,y}(W_t-W_s)\,ds\,dt-TU_\varepsilon(y)\right)$$

$$+\,U_\varepsilon(y)\left(\int_0^T\int_0^s p_{\varepsilon,x}(W_s-W_r)\,dr\,ds-TU_\varepsilon(x)\right)+TU_\varepsilon(x)U_\varepsilon(y).$$

For $x,y\neq 0$,

$$(3.2)\qquad \iint_{\Gamma_T}\!\int p_{\varepsilon,x}(W_s-W_r)p_{\varepsilon,y}(W_t-W_s)\,dr\,ds\,dt$$

converges, as $\varepsilon\to 0$, to $\alpha(x,y,T)$. Recalling (1.6) and (1.7) we see that for $x,y\neq 0$,

$$(3.3)\qquad \begin{aligned}\gamma(x,y,T)=\lim_{\varepsilon\to 0}\int_0^T&\left(\int_0^s p_{\varepsilon,x}(W_s-W_r)\,dr-U_\varepsilon(x)\right)\\&\times\left(\int_s^T p_{\varepsilon,y}(W_t-W_s)\,dt-U_\varepsilon(y)\right)ds.\end{aligned}$$

We will obtain an expression for (3.3), involving stochastic integrals—our first Tanaka formula—and use this to prove Theorem 1.

We first apply the forward Itô formula to smooth, nonanticipating $U_{\varepsilon,y}(\cdot - W_s)$ on the interval $s \le t \le T$. This gives

$$U_{\varepsilon,y}(W_T - W_s) = U_\varepsilon(y) + \int_s^T \nabla U_{\varepsilon,y}(W_t - W_s)\cdot d_+W_t + \tfrac{1}{2}\int_s^T \Delta U_{\varepsilon,y}(W_t - W_s)\,dt. \tag{3.4}$$

Since

$$\tfrac{1}{2}\Delta U_\varepsilon(x) = \int_\varepsilon^1 \tfrac{1}{2}\Delta p_s(x)\,ds = \int_\varepsilon^1 \frac{\partial}{\partial s}p_s(x)\,ds = p_1(x) - p_\varepsilon(x),$$

we find from (3.4) that

$$\begin{aligned}\int_s^T p_{\varepsilon,y}(W_t - W_s)\,dt - U_\varepsilon(y) &= \int_s^T \nabla U_{\varepsilon,y}(W_t - W_s)\cdot d_+W_t \\ &\quad + \int_s^T p_{1,y}(W_t - W_s)\,dt - U_{\varepsilon,y}(W_T - W_s).\end{aligned} \tag{3.5}$$

Similarly, applying the backward Itô formula to $U_{\varepsilon,x}(W_s - \cdot\,)$ on $0 \le r \le s$, we find that

$$\begin{aligned}\int_0^s p_{\varepsilon,x}(W_s - W_r)\,dr - U_\varepsilon(x) &= -\int_0^s \nabla U_{\varepsilon,x}(W_s - W_r)\cdot d_-W_r \\ &\quad + \int_0^s p_{1,x}(W_s - W_r)\,dr - U_{\varepsilon,x}(W_s).\end{aligned} \tag{3.6}$$

Let $\gamma_\varepsilon(x,y,T)$ denote the right-hand side of (3.3) before taking the $\varepsilon \to 0$ limit. We see that

$$\begin{aligned}\gamma_\varepsilon(x,y,T) = &-\int_0^T\int_0^t\left(\int_r^t \nabla U_{\varepsilon,x}(W_s - W_r)\,\nabla U_{\varepsilon,y}(W_t - W_s)\,ds\right)d_-W_r\,d_+W_t \\ &-\int_0^T\left(\int_s^T\int_r^T \nabla U_{\varepsilon,x}(W_s - W_r)p_{1,y}(W_t - W_s)\,ds\,dt\right)\cdot d_-W_r \\ &+\int_0^T\left(\int_0^t\int_0^s p_{1,x}(W_s - W_r)\,\nabla U_{\varepsilon,y}(W_t - W_s)\,dr\,ds\right)\cdot d_+W_t \\ &+\int_0^T\left(\int_r^T \nabla U_{\varepsilon,x}(W_s - W_r)U_{\varepsilon,y}(W_T - W_s)\,ds\right)\cdot d_-W_r \\ &-\int_0^T\left(\int_0^t U_{\varepsilon,x}(W_s)\,\nabla U_{\varepsilon,y}(W_t - W_s)\,ds\right)\cdot d_+W_t \\ &+\int_0^T\left(\int_0^s p_{1,x}(W_s - W_r)\,dr - U_{\varepsilon,x}(W_s)\right) \\ &\qquad\times\left(\int_s^T p_{1,y}(W_t - W_s)\,dt - U_{\varepsilon,y}(W_T - W_s)\right)ds.\end{aligned} \tag{3.7}$$

We will show that each of the six terms on the right-hand side of (3.7) converges as $\varepsilon \to 0$ to a limit which can be chosen to be jointly continuous on (x, y, T). This will prove Theorem 1 and yield our first Tanaka formula, Theorem 2.

We do this explicitly for the first term in (3.7), the bilateral stochastic integral

$$
\begin{aligned}
&I(\varepsilon, x, y, T) \\
(3.8)\qquad &\doteq \int_0^T \int_0^t \left(\int_r^t \nabla U_{\varepsilon, x}(W_s - W_r) \nabla U_{\varepsilon, y}(W_t - W_s)\, ds \right) d_- W_r\, d_+ W_t.
\end{aligned}
$$

The other terms can be handled similarly, in fact, more easily.

We will show that

$$
\begin{aligned}
&E(I(\varepsilon, x, y, T) - I(\varepsilon', x', y', T'))^{2m} \\
(3.9)\qquad &\le C_m |(\varepsilon, x, y, T) - (\varepsilon', x', y', T')|^{m\gamma}, \quad \varepsilon, \varepsilon' \neq 0
\end{aligned}
$$

for some $\gamma > 0$, independent of m.

The multiparameter version of Kolmogorov's lemma [see Meyer (1980)] shows then that

$$|I(\varepsilon, x, y, T) - I(\varepsilon', x', y', T')| \le C |(\varepsilon, x, y, T) - (\varepsilon', x', y', T')|^{\bar{\gamma}}$$

for rational arguments, which allows us to obtain a jointly continuous limit as $\varepsilon \to 0$. From (3.8), this limit can be represented, for every (x, y) as the bilateral stochastic integral

$$\int_0^T \int_0^t \left(\int_r^t \nabla U_x(W_s - W_r) \nabla U_y(W_t - W_s)\, ds \right) d_- W_r\, d_+ W_t.$$

The following lemma will be used to prove (3.9).

LEMMA 1. *If*

$$|f(x)| \le \frac{c}{|x - a|^{1+\alpha}}, \qquad |g(x)| \le \frac{c}{|x - b|^{1+\alpha}}$$

and

$$I = \int_0^T \int_0^t \left(\int_r^t f(W_s - W_r) g(W_t - W_s)\, ds \right) d_- W_r\, d_+ W_t,$$

then for $\alpha \ge 0$ sufficiently small, we have

$$E(I^{2m}) \le \bar{c}$$

uniformly in a, b.

PROOF. Set

$$H(r,s,t) = f(W_s - W_r)g(W_t - W_s).$$

Then

$$\begin{aligned}
E(I^{2m}) &= E\left[\left(\int_0^T\left(\int_r^T\int_r^t H(r,s,t)\,ds\,d_+W_t\right)d_-W_r\right)^{2m}\right]\\
&\le cE\left[\left(\int_0^T\left(\int_r^T\int_r^t H(r,s,t)\,ds\,d_+W_t\right)^2 dr\right)^m\right]\\
&\le cE\left[\int_0^T\left(\int_r^T\int_r^t H(r,s,t)\,ds\,d_+W_t\right)^{2m} dr\right] \qquad (3.10)\\
&\le c\int_0^T E\left[\left(\int_r^T\int_r^t H(r,s,t)\,ds\,d_+W_t\right)^{2m}\right]dr\\
&\le c\int_0^T E\left[\left(\int_r^T\left(\int_r^t H(r,s,t)\,ds\right)^2 dt\right)^m\right]dr
\end{aligned}$$

where, to get from line 2 to line 3 we used Hölder's inequality. (We keep T in some finite interval, say $T \le 1$.)

We now see that

$$\begin{aligned}
E(I^{2m}) \le k\int_{\mathbb{R}^{6m+2}}\prod_{i=1}^m &\left|f(x_0 - x_{\bar{\pi}_i})\right|\left|f(x_0 - x_{\tilde{\pi}_i})\right|\left|g(x_{\bar{\pi}_i} - x_{\pi_i})\right| \qquad (3.11)\\
&\times\left|g(x_{\tilde{\pi}_i} - x_{\pi_i})\right|U(0 - x_0)\cdots U(x_{3m-1} - x_{3m}),
\end{aligned}$$

where π, $\tilde{\pi}$ and $\bar{\pi}$ are three complementary injections of $\{1,\dots,m\}$ into $\{1,\dots,3m\}$ such that for each i, $\bar{\pi}_i < \tilde{\pi}_i < \pi_i$.

We bound (3.11) by integrating successively starting from x_{3m}.

We encounter three types of integrals:

1. $i \in \operatorname{range}(\pi)$. Using Hölder's inequality,

$$\begin{aligned}
&\int \frac{1}{|x_i - x_j - a|^{1+\alpha}}\frac{1}{|x_i - x_k - b|^{1+\alpha}}U(x_{i-1} - x_i)\,d^2x_i\\
&\le c\left(\int \frac{1}{|x_i - x_j - a|^{(1+\alpha)3/2}}\frac{1}{|x_i - x_k - b|^{(1+\alpha)3/2}}\,d^2x_i\right)^{2/3}\\
&\le c\frac{1}{|x_j + a - x_k - b|^{2/3+2\alpha}} \quad \text{for } \alpha \ge 0 \text{ small.}
\end{aligned}$$

2. $i \in \operatorname{range}(\tilde{\pi})$. Using Hölder's inequality,

$$\int \frac{1}{|x_i - x_j - a|^{2/3+2\alpha}} \frac{1}{|x_i - x_0 - b|^{1+\alpha}} U(x_{i-1} - x_i)\, d^2x_i$$

$$\leq \left(\int \frac{1}{|x_i - x_j - a|^{5/3+5\alpha}} U(x_{i-1} - x_i)\, d^2x_i \right)^{2/5}$$

$$\times \left(\int \frac{1}{|x_i - x_0 - b|^{5/3(1+\alpha)}} U(x_{i-1} - x_i)\, d^2x_i \right)^{3/5}$$

$\leq c$, independent of a, b, x_j, x_0.

3. $i \in \operatorname{range}(\bar{\pi})$,

$$\int \frac{1}{|x_i - x_0 - b|^{1+\alpha}} U(x_{i-1} - x_i)\, d^2x_i \leq c,$$

independent of b, x_0 and, of course, $\int U(x_i)\, d^2x_i \leq c$.

We now apply this to prove (3.9). Note that

$$|\nabla U_{\varepsilon, z}(x - y)| \leq \frac{c}{|x - y - z|}. \tag{3.12}$$

To handle the ε variation we use

$$\begin{aligned} &|\nabla U_{\varepsilon', z}(x - y) - \nabla U_{\varepsilon, z}(x - y)| \\ &\leq \int_{\varepsilon}^{\varepsilon'} |\nabla p_s(x - y - z)|\, ds \\ &\leq |\varepsilon' - \varepsilon|^{\gamma} \left(\int_0^{\infty} e^{-s} |\nabla p_s(x - y - z)|^{1/1-\gamma}\, ds \right)^{1-\gamma} \\ &\leq c|\varepsilon' - \varepsilon|^{\gamma} \frac{1}{|x - y - z|^{1+2\gamma}}, \end{aligned} \tag{3.13}$$

with $\gamma > 0$ small. Here we use the fact that

$$|\nabla p_s(x)| \leq \frac{c}{s^{3/2}} e^{-Mx^2/s}. \tag{3.14}$$

The x, y variations are handled similarly, using

$$|\nabla p_s(x) - \nabla p_s(x')| \leq c|x - x'|^{\gamma} \left[\left(\frac{e^{-Mx^2/s}}{s^{(3+\gamma)/2}} \right) + \left(\frac{e^{-Mx'^2/s}}{s^{(3+\gamma)/2}} \right) \right]. \tag{3.15}$$

[See Rosen (1987), (2.4f).]

Finally, to handle the T variation, we note that the factors $U(x_i - x_{i+1})$ in (3.11) came from integrating the factors $p_u(x_i - x_{i+1})$ [from the expectation in

(3.10)] with respect to u. We use Hölder's inequality as in (3.13),

$$\begin{aligned}\int_\Gamma p_u(x)\,du &\le C|\Gamma|^\gamma\left(\int_0^\infty e^{-u}p_u(x)^{1/1-\gamma}\,du\right)^{1-\gamma}\\ &\le C|\Gamma|^\gamma\frac{1}{|x|^{2\gamma}}\quad\text{for } x \text{ small,}\end{aligned}\tag{3.16}$$

with exponential falloff as $x \to \infty$. It is easily checked that (3.11) is uniformly bounded if the u factors are replaced by (3.16). This completes our proof of (3.9), hence Theorem 1. These calculations also justify the existence and regularity of the stochastic integrals appearing in our next theorem. □

THEOREM 2 (First Tanaka formula).

$$\begin{aligned}\gamma(x,y,T) = &-\int_0^T\int_0^t\int_r^t \nabla U_x(W_s - W_r)\nabla U_y(W_t - W_s)\,ds\,d_-W_r\,d_+W_t\\ &-\int_0^T\int_s^T\int_r^T \nabla U_x(W_s - W_r)p_{1,y}(W_t - W_s)\,ds\,dt\cdot d_-W_r\\ &+\int_0^T\int_0^t\int_0^s p_{1,x}(W_s - W_r)\nabla U_y(W_t - W_s)\,dr\,ds\cdot d_+W_t\\ &+\int_0^T\int_r^T \nabla U_x(W_s - W_r)U_y(W_T - W_s)\,ds\cdot d_-W_r\\ &-\int_0^T\int_0^t U_x(W_s)\nabla U_y(W_t - W_s)\,ds\cdot d_+W_t\\ &+\int\int_{\Gamma_T}\int p_{1,x}(W_s - W_r)p_{1,y}(W_t - W_s)\,dr\,ds\,dt\\ &-\int_0^T\int_0^s p_{1,x}(W_s - W_r)U_y(W_T - W_s)\,dr\,ds\\ &-\int_0^T\int_0^t U_x(W_s)p_{1,y}(W_t - W_s)\,ds\,dt\\ &+\int_0^T U_x(W_s)U_y(W_T - W_s)\,ds.\end{aligned}\tag{3.17}$$

4. The second Tanaka formula. Let $f(\cdot)$ be a symmetric C^∞ function on $\mathbb{R}^2$ with compact support and $\int f(x)\,d^2x = I$.

We set $f_\varepsilon(x) = (1/\varepsilon^2)f(x/\varepsilon)$.

We also introduce the basic function, for $u > 0$,

$$q(x,u) = \begin{cases} -\dfrac{x}{\pi|x|^2}e^{-|x|^2/2u}, & |x| \neq 0,\\ 0, & x = 0\end{cases}\tag{4.1}$$

and set $F_\varepsilon(x,u) = f_\varepsilon * (q(\cdot,u))(x)$.

If Y is any random variable, we let

$$\{Y\} = Y - E(Y).$$

Writing $f = \{f\} + E(f)$,

$$\begin{aligned}
&\iint_{\Gamma_T}\int f_{\varepsilon,x}(W_s - W_r)\, f_{\varepsilon,y}(W_t - W_s)\, dr\, ds\, dt \\
&\quad = \iint_{\Gamma_T}\int \{f_{\varepsilon,x}(W_s - W_r)\}\{f_{\varepsilon,y}(W_t - W_s)\}\, dr\, ds\, dt \\
&\qquad + \iint_{\Gamma_T}\int E(f_{\varepsilon,x}(W_s - W_r))\{f_{\varepsilon,y}(W_t - W_s)\}\, dr\, ds\, dt \\
&\qquad + \iint_{\Gamma_T}\int \{f_{\varepsilon,x}(W_s - W_r)\}E(f_{\varepsilon,y}(W_t - W_s))\, dr\, ds\, dt \\
&\qquad + \iint_{\Gamma_T}\int E(f_{\varepsilon,x}(W_s - W_r))E(f_{\varepsilon,y}(W_t - W_s))\, dr\, ds\, dt.
\end{aligned} \tag{4.2}$$

Since, e.g.,

$$E(f_{\varepsilon,x}(W_s - W_r)) = f_\varepsilon * p_{s-r}(x) \tag{4.3}$$

it will turn out that the last three terms in (4.2) are close to the renormalization terms in $\gamma(x,y,T)$.

We return to this point later, but concentrate now on the first term:

$$\begin{aligned}
&\iint_{\Gamma_T}\int \{f_{\varepsilon,x}(W_s - W_r)\}\{f_{\varepsilon,y}(W_t - W_s)\}\, dr\, ds\, dt \\
&\quad = \int_0^T \left(\int_0^s \{f_{\varepsilon,x}(W_s - W_r)\}\, dr\right)\left(\int_s^T \{f_{\varepsilon,y}(W_t - W_s)\}\, dt\right) ds.
\end{aligned} \tag{4.4}$$

By Yor (1986b) we have

$$\int_s^T \{f_{\varepsilon,y}(W_t - W_s)\}\, dt = \int_s^T F_{\varepsilon,y}(W_t - W_s; T - t)\, d_+ W_t$$

and similarly

$$\int_0^s \{f_{\varepsilon,y}(W_s - W_r)\}\, dr = \int_0^s F_{\varepsilon,x}(W_s - W_r; r)\, d_- W_r$$

so that (4.4) can be written

$$\begin{aligned}
&\int_0^T \int_r^T \left(\int_r^t F_{\varepsilon,x}(W_s - W_r; r) F_{\varepsilon,y}(W_t - W_s; T - t)\, ds\right) d_+ W_t\, d_- W_r \\
&\quad = g_\varepsilon * J(x,y,T),
\end{aligned} \tag{4.5}$$

where $g_\varepsilon(u,v) = f_\varepsilon(u) f_\varepsilon(v)$ and

$$J(x,y,T)$$

$$(4.6)\qquad = \int_0^T \int_r^T \left(\int_r^t q_x(W_s - W_r; r) q_y(W_t - W_s; T-t)\, ds \right) d_+ W_t\, d_- W_r.$$

The justification for (4.5) will come from proving

$$(4.7)\quad E(J(x,y,T) - J(x',y',T'))^{2m} \le c_m |(x,y,T) - (x',y',T')|^{m\gamma}$$

for some $\gamma > 0$, independent of m.

We will then be able to take a jointly continuous version of $J(x,y,T)$, so that by (4.5), the $\varepsilon \to 0$ limit of the first term in (4.2) will be precisely the bilateral stochastic integral $J(x,y,T)$.

The proof of (4.7) is quite similar to that of (3.9). In place of (3.12) we use

$$(4.8)\qquad |q(x,u)| \le \frac{c}{|x|},$$

while for the y variation we use

$$(4.9)\quad |q(x,u) - q(x',u)| \le c \left| \frac{x}{|x|^2} - \frac{x'}{|x'|^2} \right| + \frac{c}{|x'|} \left| e^{-|x|^2/2u} - e^{-|x'|^2/2u} \right|.$$

The first term in (4.9) can be bounded as follows. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, we have

$$(4.10)\qquad \left| \frac{x}{|x|^2} - \frac{x'}{|x'|^2} \right| = \left| \frac{1}{x} - \frac{1}{x'} \right| = \frac{|x - x'|}{|x|\,|x'|}.$$

Assume $|x| \ge |x'|$. Then, since $|x - x'| \le 2|x|$, we have

$$\left| \frac{1}{x} - \frac{1}{x'} \right| \le 2^{1-\gamma} |x - x'|^\gamma \left(\frac{1}{|x'|^{1+\gamma}} + \frac{1}{|x|^{1+\gamma}} \right).$$

By symmetry, this also holds for $|x| \le |x'|$, hence for all couples (x, x').

For the second term we use

$$\left| e^{-x^2/2u} - e^{-x'^2/2u} \right| \le \frac{c|x - x'|^\gamma}{u^{\gamma/2}}.$$

Notice that in controlling the y variation, $u = T - t_i$ and the factor $1/(T - t_i)^{\gamma/2}$ is controlled as in (3.16). For the x variation, we rewrite J so that the outer integral is $d_+ W_t$ [which is completely justified since we have already shown the uniform L^{2m} bounds for $J(x,y,T)$] and proceed as above. This completes the proof of (4.7), hence the analysis of the first term in (4.2).

We now return to the other terms in (4.2). The second term is

$$\begin{aligned}
&\int_0^T\int_0^t f_\varepsilon * U^s(x)\{f_{\varepsilon,y}(W_t - W_s)\}\,ds\,dt \\
&\qquad = f_\varepsilon * U(x)\int_0^T\int_0^t\{f_{\varepsilon,y}(W_t - W_s)\}\,ds\,dt \\
&\qquad\quad - \int_0^T\int_0^t f_\varepsilon * U_s(x)\{f_{\varepsilon,y}(W_t - W_s)\}\,ds\,dt,
\end{aligned}\tag{4.11}$$

where $U^s(x) = \int_0^s p_u(x)\,du$.

We note that

$$\begin{aligned}
\int_0^T\int_0^t\{f_{\varepsilon,y}(W_t - W_s)\}\,ds\,dt &= \int f_{\varepsilon,y}(z)\alpha(z,T)\,d^2z - \int_0^T f_\varepsilon * U^t(y)\,dt \\
&= f_\varepsilon * \hat{\alpha}(y,T) + \int_0^T f_\varepsilon * U_t(y)\,dt,
\end{aligned}\tag{4.12}$$

where $\alpha(z,T)$ is the intersection local time at z for $W_t - W_s$ on the time interval $0 \le s \le t \le T$ and $\hat{\alpha}$ is given by (1.6).

Summarizing, the second term in (4.2) is

$$\begin{aligned}
&g_\varepsilon * U(x)\left(\hat{\alpha}(y,T) + \int_0^T U_t(y)\,dt\right) \\
&\quad - \int_0^T\int_0^t f_\varepsilon * U_s(x)\{f_{\varepsilon,y}(W_t - W_s)\}\,ds\,dt.
\end{aligned}\tag{4.13}$$

Similarly the third term of (4.2) is

$$\begin{aligned}
&g_\varepsilon * U(y)\left(\hat{\alpha}(x,T) + \int_0^T U_s(x)\,ds\right) \\
&\quad - \int_0^T\int_0^s\{f_{\varepsilon,x}(W_s - W_r)\}f_\varepsilon * U_{T-s}(y)\,dr\,ds.
\end{aligned}\tag{4.14}$$

The fourth term of (4.2) is

$$\begin{aligned}
&\int_0^T(f_\varepsilon * U^s(x))(f_\varepsilon * U^{T-s}(y))\,ds \\
&\quad = g_\varepsilon * \int_0^T U^s(x)U^{T-s}(y)\,ds \\
&\quad = g_\varepsilon * \int_0^T(U(x) - U_s(x))(U(y) - U_{T-s}(y))\,ds \\
&\quad = Tg_\varepsilon * U(x)U(y) - g_\varepsilon * U(x)\int_0^T U_{T-s}(y)\,ds \\
&\quad\quad - g_\varepsilon * U(y)\int_0^T U_s(x)\,ds + g_\varepsilon * \int_0^T U_s(x)U_{T-s}(y)\,ds.
\end{aligned}\tag{4.15}$$

Combining all this we can rewrite (4.2) as

$$
\begin{aligned}
g_\varepsilon * \gamma(x,y,T) &= g_\varepsilon * J(x,y,T) - \int_0^T \int_0^t f_\varepsilon * U_s(x)\{f_{\varepsilon,y}(W_t - W_s)\}\,ds\,dt \\
&\quad - \int_0^T \int_0^s \{f_{\varepsilon,x}(W_s - W_r)\} f_\varepsilon * U_{T-s}(y)\,dr\,ds \qquad (4.16)\\
&\quad + g_\varepsilon * \int_0^T U_s(x) U_{T-s}(y)\,ds.
\end{aligned}
$$

By the formulae following (4.4), the second and third terms in (4.16) can be written as

$$
\begin{aligned}
&\int_0^T \left(\int_0^t f_\varepsilon * U_s(x) F_{\varepsilon,y}(W_t - W_s; T-t)\,ds \right) d_+ W_t \\
&\quad + \int_0^T \left(\int_r^T F_{\varepsilon,x}(W_s - W_r; r) f_\varepsilon * U_{T-s}(y)\,ds \right) d_- W_r \qquad (4.17)\\
&= g_\varepsilon * \int_0^T \left(\int_0^t U_s(x) q_y(W_t - W_s; T-t)\,ds \right) d_+ W_t \\
&\quad + g_\varepsilon * \int_0^T \left(\int_r^T q_x(W_s - W_r; r) U_{T-s}(y)\,ds \right) d_- W_r .
\end{aligned}
$$

We can now state

THEOREM 3 (Second Tanaka formula).

$$
\begin{aligned}
\gamma(x,y,T) &= \int_0^T \int_r^T \left(\int_r^t q_x(W_s - W_r; r) q_y(W_t - W_s; T-t)\,ds \right) d_+ W_t\, d_- W_r \\
&\quad - \int_0^T \left(\int_0^t U_s(x) q_y(W_t - W_s; T-t)\,ds \right) d_+ W_t \\
&\quad - \int_0^T \left(\int_r^t q_x(W_s - W_r; r) U_{T-s}(y)\,ds \right) d_- W_r \\
&\quad + \int_0^T U_s(x) U_{T-s}(y)\,ds.
\end{aligned}
$$

PROOF. For $x, y \neq 0$, the continuity of γ, J and the last term allow us to take the $\varepsilon \to 0$ limit. To handle the middle terms we can easily prove an analogue of (4.7), for example,

$$
\begin{aligned}
E&\left(\left[\int_0^T\left(\int_0^t U_s(x)q_y(W_t - W_s; T-t)\,ds\right)d_+W_t\right]^{2m}\right)\\
&\le cE\left[\left(\int_0^T\left(\int_0^t U_s(x)q_y(W_t - W_s; T-t)\,ds\right)^2 dt\right)^m\right]\\
&\le \int c\prod_{i=1}^m \frac{1}{|x_{\pi_i} - x_{\bar{\pi}_i} - y|}\,\frac{1}{|x_{\pi_\iota} - x_{\tilde{\pi}_\iota} - y|}\,dx\\
&\qquad\times\left(\int\prod_{j=1}^{2m} U_{s_j}(x)\prod_{i=1}^{3m} p_{r_i}(x_i - x_{i-1})\,ds\,dt\right),
\end{aligned}
$$

where the r_i are successive increments in the ordered set $(s_1, \dots, s_{2m}, t_1, \dots, t_m)$. We use Hölder's inequality in the $ds\,dt$ integral as in (3.16), since

$$U_s(x) \le c\log(s),$$

so it is integrable to any power on bounded intervals. □

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