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SPECTRAL DECOMPOSITION FOR GENERALIZED DOMAINS OF ATTRACTION

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Regular variation is used to study the asymptotic behavior of norming operators for generalized domains of attraction. This leads to a powerful decomposition theorem. Applications include a complete, concise description of moment behavior, centering constants, convergence criteria and tail behavior for generalized domains of attraction.

1. Introduction. Suppose that $X_1, X_2, \dots$ are independent random vectors on $\mathbb{R}^k$ with common distribution μ and Y is a nondegenerate random vector on $\mathbb{R}^k$ with distribution ν. If there exist linear operators A_n on $\mathbb{R}^k$ and constants $b_n \in \mathbb{R}^k$ such that

$$A_n(X_1 + \cdots + X_n) - b_n \Rightarrow Y, \tag{1.1}$$

then we say that μ belongs to the generalized domain of attraction of ν and we write $\mu \in \mathrm{GDOA}(\nu)$. The class of all possible nondegenerate limit laws in (1.1) is called the operator-stable laws.

In this paper we will apply regular variation techniques to investigate the asymptotic behavior of the norming operators A_n in (1.1). In Section 2 we set the stage by deriving a decomposition result for the limit law ν in (1.1). This is called the spectral decomposition for operator-stable laws. We conclude Section 2 with a discussion of the open problem of characterizing operator-stable exponents and symmetries. In Section 3 we analyze the asymptotic behavior of $\langle A_n \rangle$ using regular variation. As a consequence we are able to state a result which completely characterizes the absolute moments of any $\mu \in \mathrm{GDOA}(\nu)$. The spectral decomposition for operator-stable laws plays a key role in this analysis. In Section 4 we apply the results of our regular variation analysis from Section 3 to obtain a spectral decomposition for generalized domains of attraction. This decomposition allows us to reduce the analysis of $\mu \in \mathrm{GDOA}(\nu)$ to the case where ν has a particularly simple form. We say that ν is spectrally simple in reference to the results of Section 2.

In Section 5 we present several applications of the spectral decomposition for generalized domains of attraction. A complete description of moments, centering constants and tail behavior is obtained by combining the spectral decomposition with previously known results. A similar approach yields a concise set of necessary and sufficient conditions for $\mu \in \mathrm{GDOA}(\nu)$ stated in

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terms of regular variation. Finally, in Section 6 we offer some concluding remarks, along with suggestions for future research.

2. Operator-stable laws. Operator-stable laws were characterized by Sharpe (1969). An operator-stable law ν on a finite-dimensional real vector space V is infinitely divisible with Lévy representation (a, Q, ϕ). Then we may define the t-fold convolution product ν^t as the infinitely divisible law with Lévy representation $(ta, tQ, t\phi)$. For every operator-stable law ν there is a linear operator B on V called an exponent of ν such that for all $t > 0$,

$$\nu^t = t^B \nu * \delta(b_t). \tag{2.1}$$

Here $\delta(a)$ is the unit mass at $a \in V$, $t^B \nu\{dx\} = \nu\{t^{-B}\, dx\}$ and $t^B = \exp(B \log t)$ where $\exp(A) = (I + A + A^2/2! + \cdots)$ the exponential operator. Since ν^t is a linear translation of $t^B \nu$, we have $tQ(x) = Q(t^{B^*}x)$ and $t\phi\{dx\} = \phi\{t^{-B}\, dx\}$ by equating the Lévy representations.

Let $\mathscr{E}(\nu)$ denote the set of exponents of ν operator-stable. An operator-stable law may have multiple exponents due to symmetry. Let $\mathscr{S}(\nu)$ denote the symmetry group of ν. A linear operator A on V belongs to $\mathscr{S}(\nu)$ if $\nu\{A\, dx\}$ is a linear translation of ν. It follows from Billingsley (1966) that $\mathscr{S}(\nu)$ is a compact subgroup of the group of invertible linear operators on V. (Recall from the definition of operator-stable that ν is full.) Exponents and symmetries of ν are related by

$$\mathscr{E}(\nu) = B + T\mathscr{S}(\nu), \tag{2.2}$$

where $B \in \mathscr{E}(\nu)$ is arbitrary and $T\mathscr{S}(\nu)$ is the tangent space of $\mathscr{S}(\nu)$ at the identity. Equation (2.2) is due to Holmes, Hudson and Mason (1982).

Choose $B \in \mathscr{E}(\nu)$ and factor the minimal polynomial of B into $f_1(x) \cdots f_m(x)$ such that all roots of $f_i(x)$ have real part equal to a_i and $a_i < a_j$ for $i < j$. The set $\{a_1 \cdots a_m\}$ will be called the real spectrum of B. Sharpe showed that the real spectrum is contained in the interval $[\frac{1}{2}, \infty)$. If we define $V_i = \operatorname{Ker}(f_i(B))$, then $V_1 \oplus \cdots \oplus V_m$ is a direct sum decomposition of V into B-invariant subspaces. We will call this the spectral decomposition of V relative to B. The idempotent operators P_i: $V \to V$ with $\operatorname{Im}(P_i) = V_i$ satisfy $P_1 + \cdots + P_m = I$ and $P_i P_j = 0$ if $i \neq j$. This is a special case of the primary decomposition theorem of linear algebra [see, e.g., Curtis (1974)]. Now define $\nu_i = P_i \nu$ (ν_i is a probability measure on V which is supported on the subspace V_i). We will call $(\nu_1, \dots, \nu_m)$ the spectral decomposition of ν. The restriction of ν_i to the B-invariant subspace V_i we will denote by $\bar{\nu}_i$. Since (2.1) is necessary and sufficient for ν to be operator-stable, by projecting onto V_i we see that $\bar{\nu}_i$ is also operator-stable with exponent $\overline{B}_i$ obtained by restricting $B_i = P_i B$ to V_i. The real spectrum of $\overline{B}_i$ consists of the single element a_i. We will say that $\bar{\nu}_i$ is spectrally simple. If $a_i = \frac{1}{2}$, then $\bar{\nu}_i$ is normal and otherwise ($a_i > \frac{1}{2}$) $\bar{\nu}_i$ is a nonnormal operator-stable law of an especially simple type.

LEMMA 2.1. *$V_1 \cdots V_m$ are A-invariant subspaces for every symmetry $A \in \mathscr{S}(\nu)$.*

PROOF. The essence of the proof is that every symmetry must preserve the tail behavior of ν and the tail behavior of ν is determined by the real spectrum of B. If ν is spectrally simple ($m = 1$), there is nothing to prove. Otherwise, suppose that $a_1 = \frac{1}{2}$ so that ν_1 is normal and $\nu_0 = (\nu_2, \dots, \nu_m)$ has no normal component. Since the (centered) normal and nonnormal components of infinitely divisible measures are uniquely determined, we get $\nu_1 = A\nu_1$ and $\nu_0 = A\nu_0 * \delta(a)$ for any $A \in \mathscr{S}(\nu)$. Since ν_1 is full on V_1 and ν_0 is full on $V_0 = V_2 \oplus \cdots \oplus V_m$, we obtain $AV_1 = V_1$ and $AV_0 = V_0$. Hence it suffices to prove the lemma in the case where ν has no normal component.

Suppose then that ν is strictly nonnormal operator-stable on V with Lévy representation $(a, 0, \phi)$. For $x \neq 0$, define

$$g(x) = \phi\{y: |\langle x, y\rangle| > 1\}. \tag{2.3}$$

Recall that $t\phi\{dx\} = \phi\{t^{-B}\,dx\}$. Also since A is a symmetry of ν, we have $\phi\{dx\} = \phi\{A^{-1}\,dx\}$. It follows that $tg(x) = g(t^{B^*}x)$ and $g(x) = g(A^*x)$, where $*$ denotes the transpose. We refer the reader to Meerschaert (1990) for a complete description of the orbit behavior of t^B. Essentially we just write down the real canonical form of B and compute the exponential $t^B = \exp(B \log t)$. Each V_i in the spectral decomposition is associated with an element a_i of the real spectrum of B. Taking duals we obtain $V^* = V_1^* \oplus \cdots \oplus V_m^*$ so that every $x \in V^*$ can be written uniquely in the form $x = x_1 + \cdots + x_m$, where $x_i \in V_i^*$. Define

$$\begin{aligned} \alpha^*(x) &= \min\{a_i: x_i \neq 0\}, \\ \beta^*(x) &= \max\{a_i: x_i \neq 0\}. \end{aligned} \tag{2.4}$$

In Meerschaert (1990) we showed that $R(t) = \|t^{B^*}x\|$ varies regularly with index $\beta^*(x)$. Let $t(r)$ denote the asymptotic inverse of $R(t)$ [cf. Seneta (1976)], a regularly varying function with index $1/\beta^*(x)$ such that $R(t(r)) \sim t(R(r)) \sim r$ as $r \to \infty$. Define $\theta_r = t(r)^{B^*}(x/r)$ and note that $g(x/r) = g(\theta_r)/t(r)$. Because $\|\theta_r\| = R(t(r))/r \to 1$ as $r \to \infty$ the set $\{\theta_r: r \geq r_0\}$ is a relatively compact subset of $V^* - \{0\}$ for $r_0 > 0$ sufficiently large. Also, it is shown in Meerschaert (1990) that g is bounded away from zero and infinity on compact sets. Hence $g(x/r) = O(1/t(r))$ so that $g(x/r) \to 0$ as $r \to \infty$ at the same rate as a regularly varying function with index $-1/\beta^*(x)$. Now it follows from the fact that $g(x) = g(A^*x)$ that $\beta^*(x) = \beta^*(A^*x)$ for all $x \neq 0$.

Now let $S_0(t) = \|t^{-B^*}x\|$ regularly varying with index $-\alpha^*(x)$. Then $S(t) = 1/S_0(t)$ varies regularly with index $\alpha^*(x)$. Let s denote the asymptotic inverse of $S(t)$, regularly varying with index $1/\alpha^*(x)$. Now we have $g(rx) = s(r)g(\theta_r)$, where $\theta_r = s(r)^{-B^*}(rx)$ and so $g(rx) = O(s(r))$, where the index of s depends on $\alpha^*(x)$. It follows that $\alpha^*(x) = \alpha^*(A^*x)$ for all $x \neq 0$. Since A^* preserves both α^* and β^*, we must have $A^*V_i^* = V_i^*$ for all $i = 1, \dots, m$ and this is equivalent to $AV_i = V_i$ for all $i = 1, \dots, m$. This concludes the proof of the lemma. Remark: It may be possible to simplify the above argument using a different norm,

$$|||x||| = \int_0^1 \int_{S(\nu)} \|gt^B x\| t^{-1} H(dg)\, dt,$$

introduced by Jurek (1984). Here H denotes Haar measure on the symmetry group $\mathscr{S}(\nu)$. The advantage of using this norm is that the functions R, S are monotone, so that we can take inverses instead of using asymptotic inverses. The disadvantage is that we can no longer compute $\|t^{B^*}x\|$, which we used to establish the regular variation of R and S. We have been unable to obtain a similar growth condition using Jurek's norm. □

THEOREM 2.2. *Every exponent $B \in \mathscr{E}(\nu)$ has the same real spectrum $\{a_1 \cdots a_m\}$ and leads to the same spectral decomposition $V = V_1 \oplus \cdots \oplus V_m$ and $\nu = (\nu_1, \dots, \nu_m)$.*

PROOF. Since $\{a_1 \cdots a_m\}$ and $V_1 \cdots V_m$ characterize the tail behavior of ν (and g), they must be the same for every exponent B. In particular, α^* and β^* cannot depend on B.

PROOF. Suppose Y is a random vector on V whose distribution ν is operator-stable. Apply the spectral decomposition and write $Y_i = P_i Y$, so that $Y =$
$Y_1 + \cdots + Y_m$. We have shown that any operator-stable random vector on V can be written as a sum of spectrally simple components and this representation is unique. We also have the following characterization of the exponents and symmetries of an operator-stable law. Recall that $\bar{\nu}_i$ is operator-stable on V_i so that $\mathscr{E}(\bar{\nu}_i)$ and $\mathscr{S}(\bar{\nu}_i)$ are collections of linear operators on V_i. If $T_1 \cdots T_m$ are linear operators on $V_1 \cdots V_m$, respectively, we can define the direct sum $T = T_1 \oplus \cdots \oplus T_m$ as follows. For any $x \in V$, let $x_i = P_i x$ so that $x = x_1 + \cdots + x_m$ and $x_i \in V_i$. Now let $Tx = T_1 x_1 + \cdots + T_m x_m$. Moreover if $\mathscr{T}_1 \cdots \mathscr{T}_m$ are collections of linear operators on $V_1 \cdots V_m$, respectively, we will denote by $\mathscr{T}_1 \oplus \cdots \oplus \mathscr{T}_m$ the collection of all such direct sums.

COROLLARY 2.3. *For ν operator-stable, we have*

$$\mathscr{E}(\nu) \subseteq \mathscr{E}(\bar{\nu}_1) \oplus \cdots \oplus \mathscr{E}(\bar{\nu}_m),$$
$$\mathscr{S}(\nu) \subseteq \mathscr{S}(\bar{\nu}_1) \oplus \cdots \oplus \mathscr{S}(\bar{\nu}_m). \tag{2.5}$$

PROOF. We have already shown that for every $B \in \mathscr{E}(\nu)$, the linear operator $\bar{B}_i$ obtained by restricting $B_i = P_i B$ to V_i is an exponent of $\bar{\nu}_i$. Since $V_1 \cdots V_m$ are B-invariant subspaces, we have $B = \bar{B}_1 \oplus \cdots \oplus \bar{B}_m$. In other words, every exponent of ν can be expressed as a direct sum of exponents of $\bar{\nu}_1 \cdots \bar{\nu}_m$. We have also shown that $V_1 \cdots V_m$ are A-invariant subspaces for every $A \in \mathscr{S}(\nu)$, which is to say that A commutes with each projection operator $P_1 \cdots P_m$. Hence $A\nu_i = AP_i\nu = P_i A\nu$, which can differ from ν_i by at most a linear translation and so A is also a symmetry of ν_i. Call $\bar{A}_i$ the restriction of $A_i = P_i A$ to V_i and note that $\bar{A}_i \in \mathscr{S}(\bar{\nu}_i)$ and $A = \bar{A}_1 \oplus \cdots \oplus \bar{A}_m$. Thus every symmetry of ν can be expressed as a direct sum of symmetries of $\bar{\nu}_1 \cdots \bar{\nu}_m$. □

The problem of classifying the exponents and symmetries of an operator-stable law is still open. The previous corollary allows us to restrict our attention to the case where ν is spectrally simple, that is, the real spectrum of $B \in \mathscr{E}(\nu)$ has only one element $a \in [\frac{1}{2}, \infty)$.

Example 1. Suppose ν is normal ($a = \frac{1}{2}$) and let Y be a random vector on V with distribution ν. Since Y and $Y - EY$ have the same exponents and symmetries, we may assume without loss of generality that $EY = 0$. As in Billingsley [(1979), page 336], construct a linear map $P\colon V \to V$ such that $X = PY$ has the standard normal distribution μ. It is well known that $\mathscr{S}(\mu)$ is the orthogonal group on V and $\mathscr{E}(\mu)$ the skew-symmetric operators on V. Since $\mu = P\nu$, we have $\mathscr{S}(\mu) = P\mathscr{S}(\nu)P^{-1}$ and $\mathscr{E}(\mu) = P\mathscr{E}(\nu)P^{-1}$. Hence both $\mathscr{E}(\nu)$ and $\mathscr{S}(\nu)$ are as large as possible.

Example 1 shows that the interesting cases are where ν is nonnormal. If ν is nonnormal and every eigenvalue of $B \in \mathscr{E}(\nu)$ has real part equal to $a \in (\frac{1}{2}, \infty)$, then there exists a basis $b_1 \cdots b_k$ for V such that the matrix of B with respect to this basis has block diagonal form. For each real eigenvalue (there may be multiplicities) there is a block with a's on the diagonal, 1's along the subdiagonal and zero entries elsewhere. To each complex conjugate pair $a \pm ib$, there is a block with matrices $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ on the diagonal, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ along the subdiagonal and zero entries elsewhere. This is called the real canonical form of B. See, for example, Hirsch and Smale (1974). Using the real canonical form of B we can compute t^B.

For any invertible linear operator T, the operator-stable law $T\nu$ has symmetry group $T\mathscr{S}(\nu)T^{-1}$. If $B \in \mathscr{E}(\nu)$, then $t^B\nu$ is a linear translation of ν^t and so it has the same symmetries. Consideration of Lévy representations shows that ν and ν^t have the same symmetries. Therefore, $\mathscr{S}(\nu) = t^B\mathscr{S}(\nu)t^{-B}$ for all $t > 0$. The fact that $\mathscr{S}(\nu)$ is compact means that $t^B\mathscr{S}(\nu)t^{-B}$ is bounded away from zero and infinity in the operator norm independent of $t > 0$. This fact can be used to compute $\mathscr{S}(\nu)$ for certain cases of interest and then (2.2) can be used to obtain $\mathscr{E}(\nu)$.

Example 2. Suppose that ν is nonnormal and $B \in \mathscr{E}(\nu)$ has a real canonical form consisting of one block corresponding to the real eigenvalue a. In other words, $Bb_k = ab_k$ and $V = \langle b_k \rangle$ the cyclic subspace containing b_k. Suppose $A = (a_{ij}) \in \mathscr{S}(\nu)$. Let $C = (c_{ij}) = t^B A t^{-B}$. Compute

$$c_{ij} = \sum_{p=0}^{i-1} \sum_{q=0}^{k-j} a_{i-p,j+q} s^p (-s)^q / p!q!, \tag{2.6}$$

where $s = \log t$ ranges over $(-\infty, \infty)$. Each c_{ij} is a polynomial in s and compactness of $\mathscr{S}(\nu)$ requires that the coefficients of this polynomial with the exception of the constant term a_{ij} must equal zero. Considering the linear term we see that $a_{i-1,j} - a_{i,j+1} = 0$ provided $i \neq 1$ and $j \neq k$. This shows that the elements down any left-to-right diagonal are all equal. Considering the case $i = 1$, we obtain $a_{i,j+1} = 0$ and so the matrix is lower triangular of the form $\operatorname{diag}(a_{11}, a_{21}, \dots, a_{k1})$ with a_{11} down the diagonal, a_{21} down the subdiagonal and so forth. Let $C = (c_{ij}) = A^n$. Compute $c_{21} = na_{11}^{n-1}a_{21}$. Since $Ab_k = a_{11}b_k$, we must have $|a_{11}| = 1$ and so $a_{21} = 0$. Repeat to show $a_{31} = 0$ and so forth. Then $A = \pm I$, where I is the identity. Hence either $\mathscr{S}(\nu) = \{I\}$ the trivial group or $\mathscr{S}(\nu) = \{I, -I\}$. Since $\mathscr{S}(\nu)$ is discrete, $T\mathscr{S}(\nu)$ is trivial and there is only one exponent $\mathscr{E}(\nu) = \{B\}$.

EXAMPLE 3. Suppose that ν is nonnormal and $B \in \mathscr{E}(\nu)$ has real canonical form consisting of a single block of the second kind. A slight modification of the argument used in Example 2 (now a_{ij} and c_{ij} are 2×2 matrices) yields that A has repetitions of the same 2×2 matrix A_0 down the diagonal and zeros elsewhere. The set of all such A_0, that is, the projection of $\mathscr{S}(\nu)$ onto the subspace $V_0 = \operatorname{Span}\{b_1, b_2\}$, is a compact subgroup of $\mathrm{GL}(V_0)$. Therefore $\mathscr{S}(\nu)$ is isomorphic to a compact subgroup of the orthogonal group on $\mathbb{R}^2$. If $\mathscr{S}(\nu)$ is discrete, then $\mathscr{E}(\nu)$ consists of the single exponent B. Otherwise $\mathscr{S}(\nu)$ is isomorphic to the entire orthogonal group on $\mathbb{R}^2$ and $\mathscr{E}(\nu)$ is one-dimensional.

As far as we know, Billingsley (1966) was the first to consider the problem of characterizing symmetry groups $\mathscr{S}(\mu)$ for nondegenerate probability measures μ on $\mathbb{R}^k$. He pointed out that $\mathscr{S}(\mu)$ must be conjugate to a closed subgroup of the orthogonal group, but that not every subgroup could occur. For example, on $\mathbb{R}^2$ if every rotation is a symmetry, then so are reflections.

There is a body of literature on exponents and symmetries of operator-stable laws beginning with two papers by Hudson and Mason. The result on $\mathscr{E}(\nu)$ for ν normal (Example 1) appeared as Theorem 4 in Hudson and Mason (1981a). The last assertion in Example 3 above follows from Theorem 1 and Hudson and Mason (1981b), together with the fact that the projection of ν onto V_0 is operator-stable. The paper by Holmes, Hudson and Mason (1982) from which we obtained (2.2) also contains a complete characterization of $\mathscr{E}(\nu)$ in the special case $\dim V = 3$.

One final remark concerning the spectral decomposition $\nu = (\nu_1, \dots, \nu_m)$. The following example due to J. A. Veeh (private communication) shows that the spectrally simple components $\bar{\nu}_1 \cdots \bar{\nu}_m$ need not be independent and furthermore the direct sum $\mathscr{S}(\bar{\nu}_1) \oplus \cdots \oplus \mathscr{S}(\bar{\nu}_m)$ may be strictly larger than $\mathscr{S}(\nu)$. Take $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ on $\mathbb{R}^2$ and let ν be the operator-stable measure with Lévy representation $(0, 0, \phi)$, where for each $r > 0$, we have $\phi\{t^B \begin{pmatrix} 1 \\ 1 \end{pmatrix}: t > r\} = 1/r$, $\phi\{t^B \begin{pmatrix} 1 \\ -1 \end{pmatrix}: t > r\} = 6/r$, $\phi\{t^B \begin{pmatrix} -1 \\ 1 \end{pmatrix}: t > r\} = 3/r$ and $\phi\{t^B \begin{pmatrix} -1 \\ -1 \end{pmatrix}: t > r\} = 4/r$. Then B is an exponent of ν, V_1 and V_2 are the coordinate axes and P_1

and P_2 are the corresponding orthogonal projections onto the axes. It is not hard to see that $\mathscr{S}(\nu)$ consists of the identity only (since ϕ is full, ν is too and they have the same symmetry group). However $\bar{\nu}_1$ is symmetric stable and so $\mathscr{S}(\bar{\nu}_1) = \{1, -1\}$. Also $\bar{\nu}_1$ and $\bar{\nu}_2$ are not independent.

3. Moments. Suppose that $\mu \in \mathrm{GDOA}(\nu)$ and (1.1) holds. Hudson, Veeh and Weiner (1988) show that (1.1) implies the existence of some moments of μ. In particular there is a real constant $\alpha \in (0, 2]$ depending only on ν such that

$$\int \|x\|^{\rho} \mu\{dx\} \tag{3.1}$$

exists for all $\rho < \alpha$ and all $\mu \in \mathrm{GDOA}(\nu)$. Furthermore for any $\rho > \alpha$, there is at least some $\mu \in \mathrm{GDOA}(\nu)$ for which (3.1) diverges.

In this section we will consider the moments

$$\int |\langle x, \theta \rangle|^{\rho} \mu\{dx\} \tag{3.2}$$

for $\mu \in \mathrm{GDOA}(\nu)$ and $\theta \in \mathbb{R}^k$. Of course, (3.1) exists if and only if (3.2) exists for all θ. But it usually happens that the range of $\rho > 0$ for which (3.2) exists depends on θ and in this case our approach yields additional information.

Michaliček (1972) showed that for any $B \in \mathscr{E}(\nu)$ and any $\lambda > 0$, we may write $A_{[n\lambda]} A_n^{-1} = \lambda^{-B} G_n I_n$, where $G_n \in \mathscr{S}(\nu)$ and $I_n \to I$ the identity in $\mathrm{GL}(\mathbb{R}^k)$. Let us agree to write $x_n \to \mathscr{S}$ to mean that the distance between the point x_n and the closed set $\mathscr{S}$ tends to zero as $n \to \infty$. Then we have

$$A_{[n\lambda]} A_n^{-1} \to \lambda^{-B} \mathscr{S}(\nu) \tag{3.3}$$

for all $\lambda > 0$ and $B \in \mathscr{E}(\nu)$. This is a regular variation condition on $\langle A_n \rangle$. By virtue of Corollary 2.3, we have that $V_1 \cdots V_m$ are $(\lambda^{-B} A)$-invariant subspaces for all $A \in \mathscr{S}(\nu)$.

We will also be concerned with the path behavior of the sequence $\langle (A_n^*)^{-1} \rangle$. From (3.3) we obtain

$$\left(A_{[n\lambda]}^*\right)^{-1} A_n^* \to \lambda^{B^*} \mathscr{S}(\nu)^*, \tag{3.4}$$

where $\mathscr{S}(\nu)^* = \{A^*\colon A \in \mathscr{S}(\nu)\}$. Certainly $\mathscr{S}(\nu)^*$ is also a compact subgroup of $\mathrm{GL}(\mathbb{R}^k)$. Note also that $V_1^* \cdots V_m^*$ are $(\lambda^{B^*} A^*)$-invariant subspaces for all $\lambda > 0$ and all $A^* \in \mathscr{S}(\nu)^*$. For the remainder of this section we will be examining the path behavior of $\langle (A_n^*)^{-1} \rangle$ using the tools of multivariate regular variation. All of our results could be restated for $\langle A_n \rangle$ or any other sequence of linear operators satisfying (3.3) and (2.5). In particular, see Theorem 4.1 of the next section.

THEOREM 3.1. *Convergence in* (3.4) *is uniform on compact subsets of* $\lambda > 0$.

PROOF. The proof is a straightforward extension of Theorem 2.2 in Meerschaert (1988) using the compactness of $\mathscr{S}(\nu)^*$. □

Now we will state our main result in this section. Let us write $A_t = A_{[t]}$ and $x_t = (A_t^*)^{-1}x$.

THEOREM 3.2. *For each* $i = 1, \dots, n$ *there is a subspace* $L_i^* \subseteq \mathbb{R}^k$ *such that*

(a) $\dim(L_i^*) = \dim(V_1^* \oplus \cdots \oplus V_i^*)$;
(b) *if* $x \in L_i^*$, *then* $x_t/\|x_t\| \to V_1^* \oplus \cdots \oplus V_i^*$;
(c) *if* $x \notin L_i^*$, *then* $x_t/\|x_t\| \to V_{i+1}^* \oplus \cdots \oplus V_n^*$;
(d) *if* $x \in L_i^*$, *then* $t^{-\rho}\|x_t\| \to 0$ *for all* $\rho > a_i$;
(e) *if* $x \notin L_i^*$, *then* $t^{-\rho}\|x_t\| \to \infty$ *for all* $\rho < a_{i+1}$.

PROOF. We will see that the subspace L_i^* is in a certain sense the limit of $A_t^*(V_1^* \oplus \cdots \oplus V_i^*)$ as $t \to \infty$. Fix i and let $V = V_{i+1}^* \oplus \cdots \oplus V_n^*$, $V' = V_1^* \oplus \cdots \oplus V_i^*$, $U = \mathbb{R}^k - V'$. Choose $\rho \in (a_i, a_{i+1})$ and let $C_t = t^{-\rho}(A_t^*)^{-1}$. Then

$$C_{\lambda t}C_t^{-1} \to \lambda^A \mathscr{S} \tag{3.5}$$

uniformly on compact subsets of $\lambda > 0$, where $A = B^* - \rho I$ and $\mathscr{S} = \mathscr{S}(\nu)^*$.

Now V, V' are A-invariant subspaces and in fact V is the direct sum of the generalized eigenspaces of A whose eigenvalues have positive real part, while V' is the direct sum of the generalized eigenspaces of A whose eigenvalues have negative real part. Furthermore both V, V' are G-invariant subspaces for every $G \in \mathscr{S}$.

The collection of linear operators $\{t^A: t > 0\}$ is simply a reparametrization of a one-parameter subgroup of $\mathrm{GL}(\mathbb{R}^k)$. The path behavior of t^A can be obtained easily by reference to standard results from the theory of liner differential equations in $\mathbb{R}^k$ [see, e.g., Hirsch and Smale (1974)]. For example, we have that $\|t^A x\| \to \infty$ uniformly on compact subsets of U and $\|t^A x\| \to 0$ uniformly on compact subsets of V'.

For any $0 < m < M < \infty$, we have for any $\lambda_0 > 0$ sufficiently large that

$$\begin{aligned} \|\lambda_0^A G x\| &\geq M\|x\|, \qquad \forall\, x \in V, \\ \|\lambda_0^A G x\| &\leq m\|x\|, \qquad \forall\, x \in V', \end{aligned} \tag{3.6}$$

for all $G \in \mathscr{S}$. Then given $\varepsilon > 0$, there exists $t_0 > 0$ such that $\forall\ t \geq t_0$, $\forall$ $\lambda \in [1, \lambda_0]$,

$$\min_{G \in \mathscr{S}} \| C_{\lambda t}C_t^{-1} - \lambda^A G\| < \varepsilon. \tag{3.7}$$

First suppose that $y_0 \in V$ and let $x = C_t^{-1}y_0$ for some $t \geq t_0$. Define $y_n = C_{\lambda_0^n t}x$ and write $y_n = r_n\theta_n + \rho_n\theta_n'$, where $r_n, \rho_n \in \mathbb{R}^+$ and θ_n, θ_n' are unit vectors in V, V', respectively. For $n = 1, 2, 3, \dots$, we have

$$C_{\lambda_0^n t}C_{\lambda_0^{n-1} t}^{-1} = \lambda_0^A G_n + E_n, \tag{3.8}$$

where $G_n \in \mathscr{S}$ and $\|E_n\| < \varepsilon$. Hence we have that $y_{n+1} = \lambda_0^A G_{n+1}(y_n) + E_{n+1}(y_n)$ and so

$$(3.9)\qquad \begin{aligned} r_{n+1}\theta_{n+1} &= \lambda_0^A G_{n+1}(r_n\theta_n) + \pi_i E_{n+1}(y_n), \\ \rho_{n+1}\theta'_{n+1} &= \lambda_0^A G_{n+1}(\rho_n\theta'_n) + \pi'_i E_{n+1}(y_n), \end{aligned}$$

where π_i, π'_i are projections onto V, V', respectively. It follows that $r_{n+1} \geq Mr_n - \varepsilon(r_n + \rho_n)$ and $\rho_{n+1} \leq m\rho_n + \varepsilon(r_n + \rho_n)$. If M is large and m, ε are small, then certainly $r_n \to \infty$ which shows that $\|y_n\| \to \infty$. To show that $\|C_s x\| \to \infty$ as $s \to \infty$ write $s = \lambda\lambda_0^n t \;\exists\; \lambda \in [1, \lambda_0]$. Then $C_s x = C_{\lambda\lambda_0^n t} C_{\lambda_0^n t}^{-1}(y_n) = \lambda^A G_s(y_n) + E_s(y_n)$, where $G_s \in \mathscr{S}$ and $\|E_s\| < \varepsilon$ for all large s. Hence $\|C_s x\| \geq \|\lambda^A G_s y_n\| - \varepsilon\|y_n\| \geq (d - \varepsilon)\|y_n\|$, where $d = \min\{\|\lambda^A G\theta\|: 1 \leq \lambda \leq \lambda_0, G \in \mathscr{S}, \|\theta\| = 1\}$. As long as $\varepsilon < d$, we are ensured that $\|C_s x\| \to \infty$ as $s \to \infty$.

Define $V_\delta = \{r\theta + \rho\theta': r/\rho > \delta\}$ for any $\delta > 0$, where as before $r, \rho \in \mathbb{R}^+$ and θ, θ' are unit vectors in V, V', respectively. Once again for M sufficiently large and m, ε sufficiently small we obtain (by essentially the same argument as before) that $\|C_s x\| \to \infty$ as $s \to \infty$ whenever $x = C_t^{-1}(y_0)$ for any $y_0 \in V_\delta$ and any $t \geq t_0$, where $t_0 = t_0(\delta)$ is from (3.7). In other words, we have $\|C_t x\| \to \infty$ as $t \to \infty$ for all $x \in S$, where

$$(3.10)\qquad S = \bigcup_{\delta > 0} \bigcup_{t \geq t_0(\delta)} C_t^{-1}(V_\delta).$$

By letting M increase to infinity in (3.6), we see that $C_t x/\|C_t x\| \to V$ for all $x \in S$. Furthermore, if x is any nonzero element of $\mathbb{R}^k$ for which $C_t x/\|C_t x\| \to V$, then $C_t x \in V_\delta$ for all large t [in particular for $t \geq t_0(\delta)$] so that $x \in S$. Hence S is the set of all $x \neq 0$ for which $C_t x/\|C_t x\| \to V$.

Define $L_i^* = \mathbb{R}^k - S$. Then assertions (c) and (e) of the theorem are certainly satisfied. Next we will prove (b). Let $x \neq 0$ be in L_i^* and as before write $C_t x = r_t\theta_t + \rho_t\theta'_t$. Suppose that $\limsup(r_t/\rho_t) > 0$ as $t \to \infty$. Then for some $\delta > 0$ and some $t_n \to \infty$, we have $r_{t_n}/\rho_{t_n} > \delta$ for all n and in particular $C_{t_n} x \in V_\delta$ for some $t_n \geq t_0(\delta)$. But then $x \in S$ which is a contradiction and this proves part (b).

By virtue of part (b), the set L_i^* (including zero) is certainly a linear subspace of $\mathbb{R}^k$. In order to prove part (d), all we need to do is go back to (3.9) and use the fact that $r_n/\rho_n \to 0$. As for part (a) we certainly have $\dim(L_i^*) \leq \dim(V') = k - \dim(V)$ since $C_t^{-1}V$ is contained in S for all t sufficiently large. The fact that $\dim(L_i^*) \geq \dim(V')$ will suffice to prove part (a). We will show this by proving that for every $y \in V'$ every limit point of $\omega_t = C_t^{-1}y/\|C_t^{-1}y\|$ belongs to L_i^*. Suppose not. Then for some $t_n \to \infty$, we have $\omega_{t_n} \to \omega \notin L_i^*$ and so $\omega \in C_t^{-1}(V_\delta)$ for some $\delta > 0$ and some $t \geq t_0(\delta)$. It follows that $\omega_{t_n} \in C_t^{-1}(V_\delta)$ open for all large n. Since C_t is linear, the convergence $C_t x/\|C_t x\| \to V$ is automatically uniform on compact subsets of S and hence we have $C_{t_n}\omega_{t_n}/\|C_{t_n}\omega_{t_n}\| \to V$. This contradicts the fact that $C_t\omega_t/\|C_t\omega_t\| = y/\|y\| \in V'$ for all t. Thus part (a) is established and we have completed the proof of Theorem 3.2. □

The following theorem extends the results of Hudson, Veeh and Weiner (1988) discussed previously.

THEOREM 3.3. *For any probability distribution* μ *in the generalized domain of attraction of an operator-stable law* ν *on* $\mathbb{R}^k$, *there exists an index function* $\rho(\theta) \in (0, 2]$ *defined for all* $\theta \neq 0$ *in* $\mathbb{R}^k$ *such that*

$$\int |\langle x, \theta \rangle|^{\rho} \mu\{dx\} \tag{3.11}$$

converges for all $0 < \rho < \rho(\theta)$ *and if* $\rho(\theta) < 2$, *then* (3.11) *diverges for all* $\rho > \rho(\theta)$.

PROOF. Invoke Theorem 3.2 to obtain a nested sequence of subspaces $\{0\} = L_0^* \subset L_1^* \subset L_2^* \cdots \subset L_n^* = \mathbb{R}^k$ and define $\rho(\theta) = 1/a_i$ for $\theta \in L_i^* - L_{i-1}^*$. Now let $F_\theta(r) = \mu\{x\colon |\langle x, \theta \rangle| \le r\}$ and define

$$\begin{aligned} U_\rho(r, \theta) &= \int_0^r t^\rho F_\theta\{dt\}, \\ V_\rho(r, \theta) &= \int_r^\infty t^\rho F_\theta\{dt\}. \end{aligned} \tag{3.12}$$

First, suppose that $\rho(\theta) = 2$, which means that $\langle Y, \theta \rangle$ is univariate normal. Without loss of generality, $E\langle X, \theta \rangle = 0$ whenever it is finite. In this case an application of the Schwarz inequality [as in Gnedenko and Kolmogorov (1968), page 173] yields that $U_1(r, \theta)^2 = o(U_2(r, \theta))$ as $r \to \infty$. Using the standard convergence criteria for triangular arrays of random vectors, we obtain

$$n \int_{\|x\| < 1} \langle x, \theta \rangle^2 \mu\{A_n^{-1}\, dx\} \to Q(\theta), \tag{3.13}$$

and this convergence is uniform on compact subsets of $L_1^* - L_0^*$.

For $r > 0$, define $n(r) = \max\{n\colon \|(A_n^*)^{-1}(\theta/r)\| \le 1\}$ so that $n = n(r) \to \infty$ as $r \to \infty$. Let $\omega_r = (A_n^*)^{-1}(\theta/r)$ and note that $\langle \omega_r \rangle$ is relatively compact where every limit point ω lies in V_1^* [so that $Q(\omega) > 0$]. By (3.13), if $\omega_r \to \omega$ along a subsequence, then $nr^{-2}U_2(r, \theta) \to Q(\omega)$. If $\lambda > 0$, we also have $n(\lambda r)^{-2} U_2(\lambda r, \theta) \to Q(\omega/\lambda)$ along the same subsequence. Since $Q(\omega/\lambda) = \lambda^{-2} Q(\omega)$, we conclude that

$$\lim_{r \to \infty} \frac{U_2(\lambda r, \theta)}{U_2(r, \theta)} = 1, \tag{3.14}$$

that is, $U_2(r, \theta)$ is slowly varying. As in the one variable case [see Feller (1971)] this implies that (3.11) is finite for all $\rho < 2$.

Now consider the case $\rho(\theta) < 2$ and recall that

$$n\mu\{A_n^{-1}\, dx\} \to \phi\{dx\} \tag{3.15}$$

by the standard convergence criteria for triangular arrays. Define $f(x) = \mu\{y\colon |\langle x, y \rangle| > 1\}$ and $g(x) = \phi\{y\colon |\langle x, y \rangle| > 1\}$ and conclude from (3.15) that

$nf(A_n^* x) \to g(x)$ uniformly on compact subsets of $\{x \in \mathbb{R}^k\colon \rho(x) < 2\}$. Also note that $V_0(r, \theta) = f(\theta/r)$. Define $n(r)$ and ω_r as before. Once again, $\langle \omega_r \rangle$ is relatively compact but now every limit point lies in $\{\omega \in \mathbb{R}^k\colon g(\omega) > 0\}$. If $\omega_r \to \omega$ along a subsequence, then

$$\frac{f(\theta/\lambda r)}{f(\theta/r)} = \frac{f(A_n^* \omega_r/\lambda)}{f(A_n^* \omega_r)} \to \frac{g(\omega/\lambda)}{g(\omega)}. \tag{3.16}$$

If $\rho(\theta) = 1/a_i$, then every limit point ω lies in V_i^*. This and the fact that $tg(x) = g(t^{B^*}x)$ in general implies that for any $\beta < \rho(\theta)$, we have $g(\omega/\lambda)/g(\omega) < \lambda^{-\beta}$ for all large λ [see, for example, Meerschaert (1990)] so that we have

$$\frac{V_0(\lambda r, \theta)}{V_0(r, \theta)} < \lambda^{-\beta} \tag{3.17}$$

for all λ sufficiently large. It follows that (by increasing β slightly if necessary) $V_0(r, \theta) < r^{-\beta}$ for all large r, all $\beta < \rho(\theta)$. Now if $\rho < \rho(\theta)$, integrate by parts in (3.12) to obtain

$$U_\rho(r, \theta) = -r^\rho V_0(r, \theta) + \int_0^r \rho t^{\rho-1} V_0(t, \theta)\, dt, \tag{3.18}$$

which is easily seen to remain bounded as $r \to \infty$ by taking β between ρ and $\rho(\theta)$ in the above inequality. Hence (3.11) converges for all $0 < \rho < \rho(\theta)$ in the case $\rho(\theta) < 2$.

Finally suppose that $\rho(\theta) < 2$ and $\rho > \rho(\theta)$. Argue as before that $V_0(r, \theta) > r^{-\beta}$ for large r for any $\beta > \rho(\theta)$ and also note that for λ_0 sufficiently large, we have $V_0(r\lambda_0, \theta) \le (\frac{1}{2})V_0(r, \theta)$, for r sufficiently large. We wish to show that $V_\rho(r, \theta) = \infty$ for $\rho > \rho(\theta)$. Choose r_0 large and let $J_n = [r_0\lambda_0^n, r_0\lambda_0^{n+1})$. The integral $I_n = \int t^\rho F_\theta\{dt\}$ taken over J_n is bounded below by $(\lambda_0^n r_0)^\rho (\frac{1}{2}) V_0(\lambda_0^n r_0, \theta)$ so by choosing β between $\rho(\theta)$ and ρ, we see that $\Sigma I_n = \infty$ as desired.

This concludes the proof of Theorem 3.3. The index function $\rho(\theta)$ which appears in the statement of the theorem is related to the function $\beta^*(\theta)$ defined in (2.4). See Section 5(b) for details. □

4. Spectral decomposition. Suppose that $\mu \in \mathrm{GDOA}(\nu)$ and (1.1) holds. In the presence of a large degree of symmetry in the limit, the norming operators A_n in (1.1) may exhibit wild behavior. For example suppose that μ is a mean zero finite covariance matrix probability distribution on $\mathbb{R}^k$. By the central limit theorem, (1.1) holds with $A_n = n^{-1/2}I$ and ν centered Gaussian. But (1.1) still holds for $A_n = n^{-1/2}U_n$ for any sequence of orthogonal transformations $\langle U_n \rangle$.

Convergence of types in $\mathbb{R}^k$ was discussed in Billingsley (1966). If (1.1) holds, we can also obtain convergence to the same limit law using the norming operators $S_n A_n$ for any sequence of symmetries $S_n \in \mathscr{S}(\nu)$. Alternatively we can replace A_n by any sequence of operators $B_n \sim A_n$ (i.e., $B_n A_n^{-1} \to I$). The fact that ν is nondegenerate ensures that A_n will be nonsingular for all large n.

Let $\nu = (\nu_1, \dots, \nu_m)$ denote the spectral decomposition of the limit in (1.1). Spectral decomposition for operator-stable laws was discussed in Section 2. As before we will denote by $\bar{\mu}_i$ the restriction of $\mu_i = P_i \mu$ to V_i. If P_i and A_n commute in general (i.e., if $V_1 \cdots V_m$ are A_n-invariant subspaces for all n), then $\bar{\mu}_i \in \text{GDOA}(\bar{\nu}_i)$ for all $i = 1, \dots, m$. This reduces the analysis of $\mu \in \text{GDOA}(\nu)$ to the case of a spectrally simple limit.

In general it is too much to expect that the norming sequence $\langle A_n \rangle$ in (1.1) is as well-behaved as in the preceding paragraph. For example, suppose $T \in \text{GL}(\mathbb{R}^k)$ and let $\mu_0 = T\mu$. Then $\mu_0 \in \text{GDOA}(\nu)$ and in fact

$$B_n \mu_0^n * \delta(b_n) \Rightarrow \nu \tag{4.1}$$

with $B_n = A_n T^{-1}$. We cannot decompose the sequence $\langle B_n \rangle$ as we did before. All we can say is that there is another direct sum decomposition $\mathbb{R}^k = W_1 \oplus \cdots W_m$ such that $B_n(W_i) \subseteq V_i$ for all $i = 1, \dots, m$ [take $W_i = T(V_i)$]. The main theorem of this section (Theorem 4.2) says that this kind of a decomposition result is always possible. In other words, for any $\mu_0 \in \text{GDOA}(\nu)$, there is a sequence of norming operators $\langle B_n \rangle$ and a direct sum decomposition $\mathbb{R}^k = W_1 \oplus \cdots \oplus W_n$ such that (4.1) holds and $B_n(W_i) \subseteq V_i$ for all $i = 1, \dots, m$. It follows that for any $\mu_0 \in \text{GDOA}(\nu)$, there exists $T \in \text{GL}(\mathbb{R}^k)$ such that $\mu = T^{-1}\mu_0$ decomposes into $(\mu_1, \dots, \mu_m)$, where $\bar{\mu}_i \in \text{GDOA}(\bar{\nu}_i)$ for all $i = 1, \dots, m$.

Meerschaert (1990) showed that the function $R(t) = \|t^{-B}x\|$ varies regularly with index $(-a_i)$ for all $x \in V_i$ and furthermore that if we define $\alpha(x) = \min\{a_i : P_i(x) \neq 0\}$, then $R(t)$ varies regularly with index $-\alpha(x)$ for all nonzero $x \in \mathbb{R}^k$. The regular variation condition (3.3) suggests that as $t \to \infty$, the linear operators $A_t = A_{[t]}$ behave like $t^{-B}S_t$ for some $S_t \in \mathscr{S}(\nu)$. Write $t^{-B}S_t x = r_t \theta_t$ with $r_t > 0$ and $\|\theta_t\| = 1$ (polar representation). If $\alpha(x) = a_i$, it is not hard to see that $\theta_t \to V_i$ and $\log(r_t)/\log t \to (-a_i)$ as $t \to \infty$. In other words, $t^{-B}S_t x$ is asymptotic to V_i and tends to zero in norm faster than $t^{-a_i+\varepsilon}$ and slower than $t^{-a_i-\varepsilon}$ for any $\varepsilon > 0$.

The sequence of norming operators $\langle A_t \rangle$ must indeed behave like $t^{-B}S_t$ but with one important difference: the set of x for which $\log(\|A_t x\|)/\log t \to (-a_i)$ may not be the same. Returning to the example, grant for the moment that $\log\|A_t x\|/\log t \to (-a_i)$ for $x \in V_i$. The correct norming operators for $\mu_0 = T\mu$ are $B_n = A_n T^{-1}$. We still have $B_{t\lambda} B_t^{-1} = A_{t\lambda} A_t^{-1} \to \lambda^{-B}\mathscr{S}(\nu)$, but now $\log\|B_t x\|/\log t \to (-a_i)$ on the set $W_i = T(V_i)$.

THEOREM 4.1. *Suppose ν is operator-stable on $\mathbb{R}^k$ with real spectrum $\{a_1 \cdots a_m\}$ and $\mathbb{R}^k = V_1 \oplus \cdots \oplus V_m$ is the spectral decomposition with respect to ν. For any $\mu \in \text{GDOA}(\nu)$, there exists a nested sequence of linear subspaces $\mathbb{R}^k = L_1 \supset L_2 \supset \cdots \supset L_m$ with $\dim(L_i) = \dim(V_i \oplus \cdots \oplus V_m)$ such that if $\langle A_n \rangle$ is any sequence of norming operators in* (1.1), *then for all*

$x \in L_i - L_{i+1}$, we have

$$A_n x / \|A_n x\| \to V_i, \tag{4.2a}$$

$$\log\|A_n x\| / \log n \to (-a_i). \tag{4.2b}$$

The proof is very similar to Theorem 3.2 and will be left to the reader. Notice that (4.2b) is equivalent to the condition that $t^{\rho}\|A_t x\| \to 0$ for all $\rho < a_i$ and $t^{\rho}\|A_t x\| \to \infty$ for all $\rho > a_i$. Suppose $\mu \in \mathrm{GDOA}(\nu)$. We will say that μ is spectrally compatible with ν if there is a sequence of norming transformations $\langle A_n \rangle$ such that (1.1) holds and $V_1 \cdots V_m$ are A_n-invariant subspaces for all n. In this case we must have $\bar{\mu}_i \in \mathrm{GDOA}(\bar{\nu}_i)$ for all $i = 1, \ldots, m$.

THEOREM 4.2 (Spectral decomposition theorem). *For any $\mu_0 \in \mathrm{GDOA}(\nu)$, there exists a linear operator T such that $\mu = T^{-1}\mu_0$ is spectrally compatible with ν. Equivalently, μ_0 is spectrally compatible with $T\nu$.*

PROOF. Apply Theorem 4.1 and choose $T \in \mathrm{GL}(\mathbb{R}^k)$ so that $L_i = T(V_i \oplus \cdots \oplus V_m)$ for all $i = 1, \ldots, m$. Let $W_i = T(V_i)$ so that $V_i = T^{-1}(W_i)$. If $\langle A_n \rangle$ is a sequence of norming operators for μ_0, let us define $B_n x = P_i A_n x$ for $x \in W_i$. Extend by linearity to define $B_n x$ for all $x \in \mathbb{R}^k$. If $x \in W_i$, then $B_n x \in V_i$ and so $B_n(W_i) \subseteq V_i$ for all i. It follows that $B_n(W_i) = V_i$ and $B_n^{-1}(V_i) = W_i$.

Now suppose x is a unit vector in V_i and let $y_n = B_n^{-1}x / \|B_n^{-1}x\|$. Convergence in (4.2a) is uniform on compact subsets of W_i and it follows that for all i,

$$\frac{A_n\theta}{\|A_n\theta\|} - \frac{B_n\theta}{\|B_n\theta\|} \to 0, \tag{4.3i}$$

$$\frac{\|A_n\theta\|}{\|B_n\theta\|} \to 1, \tag{4.3ii}$$

uniformly on compact subsets of $\theta \in W_i$. It follows easily that $A_n B_n^{-1}x = A_n y_n / \|B_n y_n\| \to x$ for every unit vector in V_i. By linearity, we have $A_n \sim B_n$ (i.e., $\|A_n B_n^{-1} - I\| \to 0$) which shows that $\langle B_n \rangle$ is another suitable sequence of norming operators for μ_0. In other words, (4.1) holds. Letting $\mu = T^{-1}\mu_0$, we certainly have $B_n T\mu^n * \delta(b_n) \Rightarrow \nu$. Since $T(V_i) = W_i$ and $B_n(W_i) = V_i$, each of $V_1 \cdots V_m$ are $(B_n T)$-invariant for all n. This shows that μ is spectrally compatible with ν. On the other hand, $TB_n\mu_0^n * \delta(Tb_n) \Rightarrow T\nu$ and $W_1 \cdots W_m$ are (TB_n)-invariant. Since $W_i = T(V_i)$, this must be the unique spectral decomposition relative to $T\nu$ and so μ_0 is spectrally compatible with $T\nu$. □

We remark that in the proof of Theorem 4.2, we can arrange that $W_1 \cdots W_m$ are mutually orthogonal. Simply let $W_m = L_m$ and then for each $i = m, m-1, \ldots, 2$, take W_{i-1} to be the orthogonal complement of W_i in L_{i-1}. Then the spectral decomposition $\mu_0 = (\mu_1, \ldots, \mu_m)$ is uniquely determined. The spectral decomposition for generalized domains of attraction is useful because it allows us to reduce to the case of a spectrally simple limit in (1.1). The next section contains several applications.

5. Applications. Suppose that μ_0 belongs to the generalized domain of attraction of ν operator-stable on $\mathbb{R}^k$ and that (1.1) holds. Apply the spectral decomposition to obtain $T \in \mathrm{GL}(\mathbb{R}^k)$ such that $\mu = T^{-1}\mu_0$ is spectrally compatible with ν. Let $\{a_1 \cdots a_m\}$ denote the real spectrum of any exponent $B \in \mathscr{E}(\nu)$, $V_1 \oplus \cdots \oplus V_m$ the spectral decomposition of $\mathbb{R}^k$ relative to B and $P_1 \cdots P_m$ the associated projections. Writing $\mu_i = P_i\mu$ and $\nu_i = P_i\nu$, we have $\mu = (\mu_1, \dots, \mu_m)$ and $\nu = (\nu_1, \dots, \nu_m)$. If $\bar{\mu}_i, \bar{\nu}_i$ denote the restriction of μ_i, ν_i, respectively, to the linear subspace V_i, then we have $\bar{\mu}_i \in \mathrm{GDOA}(\bar{\nu}_i)$ for all $i = 1, \dots, m$.

A. *Operator–stable laws.* If $a_i = \frac{1}{2}$, then $\bar{\nu}_i$ is nondegenerate normal on V_i and otherwise $\bar{\nu}_i$ is a nondegenerate operator-stable law on V_i having no normal component. In Meerschaert (1990) we describe the moments and tail behavior of probability distributions which belong to the domain of normal attraction of some operator-stable law. This is a special case of (1.1) with $A_n = n^{-B}$. Since every operator-stable law belongs to its own domain of normal attraction, we can specialize those results to the case of ν_i nonnormal. The tails of $\bar{\nu}_i$ are of the same order as a regularly varying function with index $(-1/a_i)$. Related to this is the fact that $\int \|x\|^\rho \bar{\nu}_i\{dx\} < \infty$ for all $\rho < 1/a_i$ and $\int |\langle x, \theta\rangle|^\rho \bar{\nu}_i\{dx\} = \infty$ for all $\rho \geq 1/a_i$ and all $\theta \neq 0$ in V_i. In terms of ν, we have that $\int |\langle x, \theta\rangle|^\rho \nu\{dx\} < \infty$ if and only if either $\beta^*(\theta) = \frac{1}{2}$ or $\beta^*(\theta) > \frac{1}{2}$ and $\rho < 1/\beta^*(\theta)$, where $\beta^*(\theta)$ is as in (2.4).

B. *Moments.* If $\mu \in \mathrm{GDOA}(\nu)$, then without loss of generality, μ is spectrally compatible with ν. Recall that a_1 and a_m are, respectively, the smallest and largest elements of the real spectrum of any $B \in \mathscr{E}(\nu)$. Hudson, Veeh and Weiner (1988) showed that $\int \|x\|^\rho \mu\{dx\} < \infty$ for all $\rho < 1/a_m$ and Meerschaert (1986b) showed that in the case where ν has no normal component ($a_1 > \frac{1}{2}$) we have $\int |\langle x, \theta\rangle|^\rho \mu\{dx\} = \infty$ for all $\theta \neq 0$ and all $\rho > 1/a_1$. Apply the spectral decomposition to obtain an alternative proof of Theorem 3.3. In this case $\rho(\theta) = 1/\beta^*(\theta)$, where β^* was defined in (2.4), and $L_i^* = V_1^* \oplus \cdots \oplus V_i^*$.

C. *Centering.* Meerschaert (1986b) contains the following characterization of the centering constants b_n in (1.1). If $a_1 > 1$, then we may take $b_n = 0$ for all n. If $a_m < 1$, then we may center to zero expectation. By applying these results to $\bar{\mu}_1 \cdots \bar{\mu}_m$, we see that as long as $a_i \neq 1$ for all i, the only centering required in (1.1) is to center to zero expectation for the components with $a_i < 1$. These are exactly the components for which the expectation exists. For the case $a_i = 1$, the form of the centering constant is given by Rvačeva (1962) as the truncated mean.

D. *Convergence criteria*. Hahn and Klass (1980) showed that a mean zero probability measure μ on $\mathbb{R}^k$ belongs to the generalized domain of attraction of a normal law if and only if the trimmed second moment $M(t,\theta) = \int\langle x,\theta\rangle^2 \wedge t^2\mu\{dx\}$ is slowly varying uniformly on the unit sphere $S = \{\theta \in \mathbb{R}^k: \|\theta\| = 1\}$. In other words, $M(\lambda t, \theta_t)/M(t,\theta_t) \to 1$ as $t \to \infty$ for all $\lambda > 0$ and any $\theta_t \in S$. As we pointed out in Application C, the assumption that $\int x\mu\{dx\} = 0$ entails no loss of generality. Meerschaert (1986a) showed that μ belongs to the generalized domain of attraction of some operator-stable law ν on $\mathbb{R}^k$ having no normal component if and only if μ satisfies the regular variation condition $n\mu\{A_n^{-1}\,dx\} \to \phi\{dx\}$ for some $\|A_n\| \to 0$, where ϕ is the Lévy measure of ν. In this case we say that μ is RV(B), where B is an exponent of ν.

In the general case, μ belongs to some generalized domain of attraction if and only if there exist mutually orthogonal subspaces W_1, W_2 with $\mathbb{R}^k = W_1 \oplus W_2$ and, writing $\bar{\mu}_i$ for the restriction of μ_i to W_i, we have that $\bar{\mu}_1$ belongs to the generalized domain of attraction of some normal law on W_1 and $\bar{\mu}_2$ belongs to the generalized domain of attraction of some operator-stable law on W_2 having no normal component. In other words, the trimmed second moment of $\bar{\mu}_1$ is slowly varying uniformly on the unit sphere in W_1 (after centering to zero expectation) and $\bar{\mu}_2$ is RV(B) for some linear operator B, whose real spectrum is contained in the interval $(\frac{1}{2}, \infty)$. The subspace W_1 can be characterized by the fact that $\int|\langle x,\theta\rangle|^\rho\mu\{dx\}$ exists for all $\rho < 2$ if and only if $\theta \in W_1$. The reverse implication that $\bar{\mu}_i \in \mathrm{GDOA}(\bar{\nu}_i)$ for $i = 1, 2$ implies $\mu \in \mathrm{GDOA}(\nu)$ was contained in Meerschaert (1986b). In that paper we also presented an alternative regular variation criterion for attraction to a normal law. For ν normal with Lévy measure $(a, Q, 0)$, we have $\mu \in \mathrm{GDOA}(\nu)$ if and only if (after centering to zero expectation) there exists $\|A_n\| \to 0$ such that $nF(A_n^* x) \to Q(x)$ for all x, where $F(x) = \int\langle x,y\rangle^2 I(|\langle x,y\rangle| \le 1)\mu\{dy\}$.

E. *Tails*. Let U_ρ, V_ρ denote the truncated moment functions defined in (3.12). If μ belongs to the generalized domain of attraction of a normal law, then $U_2(t,\theta) \sim M(t,\theta)$ as $t \to \infty$ and so $U_2(t,\theta)$ is slowly varying. Here M is the trimmed second moment of μ defined in Application D. It follows using Feller [(1971), page 283] that $U_\rho(t,\theta)/t^{\rho-2}U_2(t,\theta) \to 0$ for all $\rho > 2$ and that $t^{2-\rho}V_\rho(t,\theta)/U_2(t,\theta) \to 0$ for all $0 \le \rho < 2$. In particular, $V_0(t,\theta) = \mu\{x: |\langle x,\theta\rangle| > t\} \to 0$ at least as fast as t^{-2}.

Suppose now that $\mu \in \mathrm{GDOA}(\nu)$, where ν is purely nonnormal. Recall that a_1, a_m are, respectively, the smallest and largest elements of the real spectrum of any $B \in \mathscr{E}(\nu)$. Meerschaert (1986b) showed that for all $\rho > 1/a_1$, we have $U_\rho(t,\theta)$ R–O varying with $U_\rho(t,\theta)/t^\delta \to 0$ for all $\delta > \rho - 1/a_1$. For all $\rho < 1/a_m$, we have $V_\rho(t,\theta)$ R–O varying with $V_\rho(t,\theta)/t^\delta \to \infty$ for all $\delta < \rho - 1/a_m$. For a discussion of R–O variation, see Seneta (1976). Without loss of generality, μ is spectrally compatible with ν. By applying these results to each component in the spectral decomposition, we obtain

$$\log V_\rho(t,\theta)/\log t \to \rho - \rho(\theta) \tag{5.1}$$

for all $\rho < \rho(\theta)$ and

$$\log U_\rho(t,\theta)/\log t \to \rho - \rho(\theta) \tag{5.2}$$

for all $\rho > \rho(\theta)$. In particular, we have $\log \mu\{x: |\langle x, \theta\rangle| > t\}/\log t \to -\rho(\theta)$ as $t \to \infty$. In other words, the tails of $\bar{\mu}_i$ tend to zero about as fast as t^{-1/a_i}.

F. *Norming operators*. Suppose that $\mu_0 \in \mathrm{GDOA}(\nu)$ and apply Theorem 4.2 to obtain $T \in \mathrm{GL}(\mathbb{R}^k)$ such that $\mu = T^{-1}\mu_0$ is spectrally compatible with ν. Then μ_0 is spectrally compatible with $T\nu$ and as we remarked at the end of Section 4, we may choose $W_1 \cdots W_m$ to be mutually orthogonal. If $B \in \mathscr{E}(\nu)$, then $TBT^{-1} \in \mathscr{E}(T\nu)$ and $W_1 \oplus \cdots \oplus W_m$ is the unique spectral decomposition of $\mathbb{R}^k$ relative to TBT^{-1}. Each $W_1 \cdots W_m$ is TB_n-invariant for all $n = 1, 2, 3, \dots$, where $\langle B_n \rangle$ is the sequence of norming operators constructed in the proof of Theorem 4.2. Select an orthonormal basis $\{e_1 \cdots e_k\}$ for $\mathbb{R}^k$ such that $W_1 = \mathrm{Span}\{e_1 \cdots e_{k_1}\}, \dots, W_m = \mathrm{Span}\{e_{k_{m-1}+1} \cdots e_k\}$. Naturally, $\dim(W_1) = k_1$ and $\dim(W_i) = k_i - k_{i-1}$ for $i = 2, \dots, m$. The matrix for TB_n with respect to $\{e_1 \dots e_k\}$ has a block-diagonal form with blocks of size k_1, $(k_2 - k_1), \dots, (k_m - k_{m-1})$. Recall that $\langle TB_n \rangle$ is a suitable sequence of norming operators for $\mu_0 \in \mathrm{GDOA}(T\nu)$. Also $B_n T = T^{-1}(TB_n)T$ yields norming operators for $\mu \in \mathrm{GDOA}(\nu)$ and these operators have a block-diagonal matrix form with respect to the basis $\{T^{-1}e_1 \cdots T^{-1}e_k\}$. The crucial point here is that each element in the sequence of norming operators has the same block-diagonal form.

6. Remarks. The spectral decomposition allows us to reduce the analysis of operator-stable laws and generalized domains of attraction to the case of a spectrally simple limit. Spectrally simple operator-stable laws are analogous to one-dimensional stable laws. They have the same moments and essentially the same tail behavior. (The real spectrum $\{a\}$ of a spectrally simple operator-stable law and the characteristic exponent α of the analogous one-dimensional stable law are related by $a = 1/\alpha$.)

One-dimensional domains of attraction are characterized by tail behavior. If a probability distribution μ on $\mathbb{R}^1$ is attracted to a normal law, then $V_0(t) = \mu\{x: |x| > t\} \to 0$ faster than $t^{-\rho}$ for any $\rho < 2$. If μ is attracted to a nonnormal stable law with characteristic exponent $\alpha \in (0, 2)$, then $V_0(t) \to 0$ about as fast as $t^{-\alpha}$ and in fact V_0 varies regularly with index $(-\alpha)$.

Generalized domains of attraction are also characterized by tail behavior. If a probability distribution μ on $\mathbb{R}^k$ belongs to some nonnormal generalized domain of attraction, then $V_0(t,\theta) = \mu(x: |\langle x, \theta\rangle| > t\} \to 0$ about as fast as $t^{-\alpha}$ for some $\alpha \in (0, 2)$ and if μ is attracted to a normal limit, then $V_0(t, \theta) \to 0$ faster than $t^{-\alpha}$ for any $\alpha \in (0, 2)$. In the case of an operator-stable limit having both normal and nonnormal components, one of these two tail conditions must hold for all $\theta \neq 0$. Unlike the one variable case, the tails of $\mu \in \mathrm{GDOA}(\nu)$ with ν nonnormal need not vary regularly. In fact, see Meerschaert (1990) for an example of a nonnormal operator-stable law whose tails do not vary regularly.

The first set of necessary and sufficient conditions for $\mu \in \mathrm{GDOA}(\nu)$ was given by Rvačeva around 1950, actually in the more general case of triangular arrays. Another set of necessary and sufficient conditions appeared in Hahn and Klass (1985) emphasizing the construction of a suitable sequence of norming operators. Theorem 4.1 and the discussion in Application F represent new information about these norming operators. We are currently investigating the extent to which a combination of these methods can further our understanding of the complex behavior which the operators A_n in (1.1) are known to exhibit.

Beyond the results of Section 3 concerning moments of $\mu \in \mathrm{GDOA}(\nu)$ lies the question of existence and convergence of the so-called pseudomoments

$$\int_0^\infty \mu\{x: \|t^{-B/p}x\| \geq 1\}\, dt \tag{6.1}$$

introduced in Weiner (1987). Existence and convergence results for μ in the domain of normal attraction of ν [$A_n = n^{-B}$ in (1.1)] were established in that paper. We are also investigating the application of the spectral decomposition to pseudomoments.

Finally, we remark that, in the course of research into generalized domains of attraction on $\mathbb{R}^k$, we are seeing the emergence of a new multivariable analogue to the theory of regular variation [see, e.g., Meerschaert (1988)]. The one variable theory of regular variation was used by Feller (1971) to obtain an elegant treatment of domains of attraction and subsequently found numerous applications both in probability theory and elsewhere. It is not hard to imagine that our multivariate regular variation techniques will find applications in other contexts as well.

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REFERENCES

BILLINGSLEY, P. (1966). Convergence of types in k-space. *Z. Wahrsch. Verw. Gebiete* **5** 175–179.

BILLINGSLEY, P. (1979). *Probability and Measure*. Wiley, New York.

BINGHAM, N., GOLDIE, C. and TEUGELS, J. (1987). *Regular Variation. Encyclopedia of Mathematics and Its Applications* **27**. Cambridge Univ. Press.

CURTIS, C. (1974). *Linear Algebra*, 3rd ed. Allyn and Bacon, Boston.

FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications* **2**, 2nd ed. Wiley, New York.

GNEDENKO, B. and KOLMOGOROV, A. (1968). *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley, Reading, Mass.

HAHN, M. and KLASS, M. (1980). Matrix normalization of sums of random vectors in the domain of attraction of the multivariate normal. *Ann. Probab.* **8** 262–280.

HAHN, M. and KLASS, M. (1985). Affine normability of partial sums of i.i.d. random vectors: A characterization. *Z. Wahrsch. Verw. Gebiete* **69** 479–505.

HIRSCH, M. and SMALE, S. (1974). *Differential Equations, Dynamical Systems, and Linear Algebra*. Academic, New York.

HOLMES, J., HUDSON, W. and MASON, J. D. (1982). Operator-stable laws: Multiple exponents and elliptical symmetry. *Ann. Probab.* **10** 602–612.

HUDSON, W. and MASON, J. D. (1981a). Operator-stable laws. *J. Multivariate Anal.* **11** 434–447.

HUDSON, W. and MASON, J. D. (1981b). Operator-stable distributions on R^2 with multiple exponents. *Ann. Probab.* **9** 482–489.

HUDSON, W., VEEH, J. and WEINER, D. (1988). Moments of distributions attracted to operator-stable laws. *J. Multivariate Anal.* **24** 1–10.

JUREK, Z. J. (1984). Polar coordinates in Banach spaces. *Bull. Polish Acad. Sci. Math.* **32** 61–66.

MEERSCHAERT, M. (1986a). Domains of attraction of nonnormal operator-stable laws. *J. Multivariate Anal.* **19** 342–347.

MEERSCHAERT, M. (1986b). Regular variation and generalized domains of attraction in $\mathbb{R}^k$. Unpublished manuscript.

MEERSCHAERT, M. (1988). Regular variation in $\mathbb{R}^k$. *Proc. Amer. Math. Soc.* **102** 341–348.

MEERSCHAERT, M. (1990). Moments of random vectors which belong to some domain of normal attraction. *Ann. Probab.* **18** 870–876.

MICHALIČEK, J. (1972). Der Anziehungsbereich von Operator-Stabilen Verteilungen im R_2. *Z. Wahrsch. Verw. Gebiete* **25** 57–70.

RVAČEVA, E. (1962). On domains of attraction of multidimensional distributions. *Selected Translations in Mathematical Statistics and Probability* **2** 183–205. Amer. Math. Soc., Providence, R.I.

SENETA, E. (1976). *Regularly Varying Functions. Lecture Notes in Math.* **508**. Springer, New York.

SHARPE, M. (1969). Operator-stable probability distributions on vector groups. *Trans. Amer. Math. Soc.* **136** 51–56.

WEINER, D. (1987). On the existence and convergence of pseudomoments for variables in the domain of normal attraction of an operator-stable distribution. *Proc. Amer. Math. Soc.* **101** 521–528.

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