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HYDRODYNAMICAL EQUATION FOR ATTRACTIVE PARTICLE SYSTEMS ON $\mathbb{Z}^d$

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We prove conservation of local equilibrium, away from the shock, for some attractive asymmetric particle systems on $\mathbb{Z}^d$. The method applies to a class of particle processes which includes zero-range and simple exclusion processes. The main point in the proof is to exploit attractiveness. The hydrodynamic equation obtained is a first-order nonlinear partial differential equation which presents shock waves.

Introduction. The asymmetric zero-range process is one of the simplest infinite particle systems. It describes the behavior of infinitely many indistinguishable particles on $\mathbb{Z}^d$. The particles move according to the following law: If a site x is occupied by k particles, the rate at which a particle leaves site x is $g(k)$. Once a particle leaves x, it goes to y with probability $P(x, y)$. It is proved in [1] that the product measures $\{\nu_\rho, \rho \geq 0\}$ given by (1.2) are invariant for this process.

On the other hand, we will consider the first-order nonlinear partial differential equation

$$\frac{\partial \rho(x,t)}{\partial t} + \sum_{j=1}^{d} \gamma_j \frac{\partial}{\partial x_j}[\phi(\rho(x,t))] = 0, \tag{0.1}$$

where ϕ is a function which depends, in a simple way, on the process and which throughout this paper will be concave. The solution of (0.1) may develop singularities even when the initial condition is smooth. Therefore, (0.1) has to be interpreted in a weak sense. Moreover, uniqueness of weak solutions does not hold in general. However, we know that (0.1) has a unique entropy solution (we refer the reader to [8] for the definition and the proof of the existence and uniqueness of an entropy solution of this equation). This solution can be obtained as the limit when ε goes to 0 of the solution of the equation

$$\frac{\partial \rho(x,t)}{\partial t} + \sum_{j=1}^{d} \gamma_j \frac{\partial}{\partial x_j}[\phi(\rho(x,t))] = \varepsilon\, \Delta\phi.$$

The reader is referred to [8] for more details on equation (0.1).

In the last years, some authors [2–5, 7, 11] proved conservation of local equilibrium, in the terminology of [2], for asymmetric particle processes on $\mathbb{Z}$.

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In this article, we investigate the hydrodynamical behavior of asymmetric particle systems on $\mathbb{Z}^d$. More precisely, let τ_a denote the shift by a on $X = \mathbb{N}^{\mathbb{Z}^d}$ and S_t the semigroup of the zero-range process. We prove that, for some measures μ on X [see (1.4) for the measure considered in this article], there exists a function $\rho(v, t)$, which is the entropy solution of (0.1) with initial condition depending on μ, such that

$$\lim_{\varepsilon \to 0} \tau_{[v\varepsilon^{-1}]} S_{t\varepsilon^{-1}} \mu = \nu_{\rho(v,t)}$$

for every $(v, t) \in \mathbb{R}^d \times \mathbb{R}_+$ away from the shock, where the limit is to be considered in the sense of weak convergence of measures and where $[a]$ denotes the integer part of $a \in \mathbb{R}^d$.

The method we use to prove preservation of local equilibrium was introduced in [3]. This article is divided as follows. In Section 1 we describe the model and state the theorems. In Section 2 we present the steps we will follow in the proofs of the theorems. In Sections 3 to 6 we prove the theorems and in Section 7 we extend some results stated in Section 1.

1. Results and notation. Let (η_t) be the zero-range process. This is the strongly continuous Markov process on $X = \mathbb{N}^{\mathbb{Z}^d}$ whose generator acts on cylindrical functions as

$$Lf(\eta) = \sum_{x, y \in \mathbb{Z}^d} g(\eta(x)) P(x, y) [f(\eta^{x,y}) - f(\eta)], \tag{1.0}$$

where

$$\eta^{x,y}(z) = \begin{cases} \eta(z), & \text{if } z \neq x, y, \\ \eta(x) - 1, & \text{if } z = x, \\ \eta(y) + 1, & \text{if } z = y. \end{cases}$$

Throughout this article, we will make the following assumptions on g and P.

ASSUMPTION 1.1.

(i) The function g is nondecreasing and bounded; $0 = g(0) < g(1)$.
(ii) $P(x, y) = P(0, y - x) = p(y - x)$ and there exists $\chi \geq d$, such that

$$\sum_{y \in \mathbb{Z}^d} \|y\|^{\chi} p(y) < \infty,$$

where, throughout this article,

$$\|y\| = \sum_{j=1}^{d} |y_j|.$$

In the proofs of Theorems 3 and 4, we will need more stringent hypotheses on the process.

ASSUMPTION 1.2. There exists $A \in \mathbb{N}$, such that, $p(y) = 0$ if $\|y\| > A$. In other words, the process is of finite range.

The existence and the construction of this Markov process, under more general hypotheses than our assumptions, is proved in [1]. From now on, we will denote by (S_t) the semigroup of this process. Before proceeding we introduce some notation.

NOTATION 1.3.

(a) $\{\tau_y, y \in \mathbb{Z}^d\}$ will denote the shifts on X: $\tau_y\eta(z) = \eta(z + y)$ for every y, z in $\mathbb{Z}^d$, η in X. We extend the shift to the functions and to the measures in the natural way: We define the function $\tau_y f$ by $\tau_y f(\eta) = f(\tau_y\eta)$ and the measure $\mu\tau_y$ by $\int f d(\mu\tau_y) = \int \tau_y f d\mu$. We observe that Assumption 1.1 implies that S_t and τ_y commute.
(b) $\mathscr{I}$ will be the set of invariant measures for the semigroup (S_t) and $\mathscr{S}$ the set of probability measures invariant under $\{\tau_y, y \in \mathbb{Z}^d\}$.
(c) For r in $\mathbb{R}$, $[r]$ denotes the integer part of r.
(d) $G = \sup_k[g(k + 1) - g(k)]$.
(e) $\gamma_j = \sum_{(z_1,\dots,z_d)\in \mathbb{Z}^d} z_j p(z)$, $\gamma = (\gamma_1, \dots, \gamma_d)$.
(f) H will be a closed cone with nonempty interior H^0.
(g) $\mathscr{P}(\mathbb{N}^{\mathbb{Z}^d})$ will be the set of probabilities on X.

We introduce in X the partial order defined by $\eta \le \xi$ if $\eta(x) \le \xi(x)$ for every x in $\mathbb{Z}^d$. We will denote by $\mathscr{M}$ the class of continuous functions on X, which are monotone in the sense that $f(\eta) \le f(\xi)$ whenever $\eta \le \xi$. If μ and ν are two probabilities on X, we shall say that $\mu \le \nu$, provided that $\int f d\mu \le \int f d\nu$ for all f in $\mathscr{M}$. A Feller process, with semigroup S_t, is said to be *attractive* if $\mu \le \nu$ implies $\mu S_t \le \nu S_t$ for all $t \ge 0$. It is proved in [1] that the monotonicity of g implies the attractiveness of the zero-range process. This property of the process, as we will see later, is the crucial point in the proof of the theorems. As in Corollary 2.2.8 of [10], if $\mu \le \nu$, in order to prove that $\mu = \nu$, we only have to show that

$$\mu[\eta(x_1,\dots,x_d)] = \nu[\eta(x_1,\dots,x_d)] \tag{1.1}$$

for every $(x_1,\dots,x_d)$ in $\mathbb{Z}^d$.

It is also proved in [1] that the set of extremal measures in $\mathscr{I} \cap \mathscr{S}$ is the weakly continuous family of product measures $\{\nu_\rho, \rho \in [0,\infty)\}$, such that

$$\nu_\rho[\eta, \eta(x) = k] = \begin{cases} \dfrac{[\phi(\rho)]^k}{g(1)\cdots g(k)}\dfrac{1}{\chi(\rho)}, & \text{if } k \ge 1, \\ \dfrac{1}{\chi(\rho)}, & \text{if } k = 0, \end{cases}$$

where $\chi(\rho)$ is a normalizing factor and

$$\phi(\rho) = \nu_\rho[g(\eta(0))]. \tag{1.2}$$

Therefore, every measure in $\mathscr{I} \cap \mathscr{S}$ can be written as

$$\int_0^\infty \nu_\rho \lambda(d\rho), \tag{1.3}$$

where λ is a probability measure on $\mathbb{R}_+$.

As is stated in the Introduction, we will prove conservation of local equilibrium for some initial profiles for the zero-range process under Assumption 1.1. To state the theorems, we define the product measures $\mu_{\alpha,\beta}$ and $\mu^*_{\alpha,\beta}$ by

$$\begin{aligned} \mu_{\alpha,\beta}[\eta,\, \eta(x) = k] &= \begin{cases} \nu_\alpha[\eta,\, \eta(x) = k], & \text{if } x_1 < 0, \\ \nu_\beta[\eta,\, \eta(x) = k], & \text{otherwise}, \end{cases} \\ \mu^*_{\alpha,\beta}[\eta,\, \eta(x) = k] &= \begin{cases} \nu_\alpha[\eta,\, \eta(x) = k], & \text{if } x \notin H, \\ \nu_\beta[\eta,\, \eta(x) = k], & \text{otherwise}. \end{cases} \end{aligned} \tag{1.4}$$

In Sections 3 and 4, we will prove the following theorems. To fix ideas, we will suppose throughout that $\gamma_1 \geq 0$.

THEOREM 1. *Under Assumption* 1.1, *suppose that* ϕ *given by* (1.2) *is concave and that* $0 \leq \alpha \leq \beta < \infty$. *Then, for* $\mu_{\alpha,\beta}$ *given by* (1.4),

$$\lim_{t\to\infty} \mu_{\alpha,\beta} S_t \tau_{([v_1 t], 0, \dots, 0)} = \begin{cases} \nu_\alpha, & \textit{if } v_1 < v_c, \\ \nu_\beta, & \textit{if } v_1 > v_c, \end{cases}$$

where

$$v_c = \gamma_1 \frac{\phi(\beta) - \phi(\alpha)}{\beta - \alpha}. \tag{1.5}$$

THEOREM 2. *Under Assumption* 1.1, *suppose that* ϕ *given by* (1.2) *is strictly concave, that* $\gamma_1 > 0$ *and that* $0 \leq \beta \leq \alpha < \infty$. *Then, for* $\mu_{\alpha,\beta}$ *given by* (1.4),

$$\lim_{t\to\infty} \mu_{\alpha,\beta} S_t \tau_{([v_1 t], 0, \dots, 0)} = \begin{cases} \nu_\alpha, & \textit{if } v_1 \leq \gamma_1 \phi'(\alpha), \\ \nu_{[\phi']^{-1}(v_1/\gamma_1)}, & \textit{if } \gamma_1\phi'(\alpha) < v_1 < \gamma_1 \phi'(\beta), \\ \nu_\beta, & \textit{if } \gamma_1 \phi'(\beta) \leq v_1. \end{cases}$$

REMARK 1.1. We shall observe that in Theorem 1, nothing is said for $v_1 = v_c$. On the other hand, in Section 7, we will see that these two theorems can be proved for a larger class of attractive processes.

REMARK 1.2. Let $(w_1, \dots, w_d)$ be in $\mathbb{Z}^d$ and let $\tilde{\mu}_{\alpha,\beta}$ be the product measure given by

$$\tilde{\mu}_{\alpha,\beta}[\eta,\, \eta(x) = k] = \begin{cases} \nu_\alpha[\eta,\, \eta(x) = k], & \text{if } \langle x, w\rangle < 0, \\ \nu_\beta[\eta,\, \eta(x) = k], & \text{otherwise}, \end{cases} \tag{1.6}$$

where $\langle\ ,\ \rangle$ denotes the inner product in $\mathbb{R}^d$. With a suitable change of variables, we obtain the following corollaries of Theorems 1 and 2.

COROLLARY 1. *Under the hypotheses of Theorem* 1, *but with* $\langle \gamma, w\rangle \geq 0$, *instead of* $\gamma_1 \geq 0$,

$$\lim_{t\to\infty} \tilde{\mu}_{\alpha,\beta} S_t \tau_{([v_1 t],\dots,[v_d t])} = \begin{cases} \nu_\alpha, & \text{if } \langle v - c\gamma, w\rangle < 0, \\ \nu_\beta, & \text{if } \langle v - c\gamma, w\rangle > 0, \end{cases}$$

where

$$c = \frac{\phi(\beta) - \phi(\alpha)}{\beta - \alpha}.$$

COROLLARY 2. *Under the hypotheses of Theorem* 2, *but with* $\langle \gamma, w\rangle > 0$, *instead of* $\gamma_1 > 0$,

$$\lim_{t\to\infty} \tilde{\mu}_{\alpha,\beta} S_t \tau_{([v_1 t],\dots,[v_d t])}$$
$$= \begin{cases} \nu_\alpha, & \text{if } \langle v, w\rangle \leq \phi'(\alpha)\langle \gamma, w\rangle, \\ \nu_{[\phi']^{-1}(\langle v, w\rangle / \langle \gamma, w\rangle)}, & \text{if } \phi'(\alpha)\langle \gamma, w\rangle < \langle v, w\rangle < \phi'(\beta)\langle \gamma, w\rangle, \\ \nu_\beta, & \text{if } \phi'(\beta)\langle \gamma, w\rangle \leq \langle v, w\rangle. \end{cases}$$

REMARK 1.3. Theorems 1 and 2 and their corollaries prove conservation of local equilibrium for some initial profiles. Indeed, if μ^ε is the product measure given by (1.4) or by (1.6), then they state that

$$\lim_{\varepsilon\to 0} \mu^\varepsilon S_{t\varepsilon^{-1}} \tau_{([v_1\varepsilon^{-1}],\dots,[v_d\varepsilon^{-1}])} = \nu_{\rho(v,t)},$$

where ρ is the entropy solution of (0.1) with initial condition $\rho(v,0) = \alpha 1_{[v_1 < 0]} + \beta 1_{[v_1 \geq 0]}$ [respectively $\rho(v,0) = \alpha 1_{[\langle v, w\rangle < 0]} + \beta 1_{[\langle v, w\rangle \geq 0]}$].

The fact that the initial measure is translation invariant in $(d-1)$ dimensions is essential to the proof of these two theorems. Nevertheless, for the zero-range process under Assumptions 1.1 and 1.2, we can prove conservation of local equilibrium for measures which do not have this property. This is the content of Theorems 3 and 4.

THEOREM 3. *Under Assumptions* 1.1 *and* 1.2, *suppose that* ϕ *given by* (1.2) *is concave, that* $0 \leq \alpha \leq \beta < \infty$ *and that* $\gamma = (\gamma_1, 0, \dots, 0)$ *is in* H^0. *Then, for* $\mu^*_{\alpha,\beta}$ *given by* (1.4),

$$\lim_{t\to\infty} \mu^*_{\alpha,\beta} S_t \tau_{([v_1 t],\dots,[v_d t])} = \begin{cases} \nu_\alpha, & \text{if } (v_1 - v_c, v_2, \dots, v_d) \notin H, \\ \nu_\beta, & \text{if } (v_1 - v_c, v_2, \dots, v_d) \in H^0, \end{cases}$$

where

$$v_c = \gamma_1 \frac{\phi(\beta) - \phi(\alpha)}{\beta - \alpha}.$$

THEOREM 4. *Under Assumptions* 1.1 *and* 1.2, *suppose that* ϕ *given by* (1.2) *is strictly concave, that* $0 \le \beta \le \alpha < \infty$ *and that* $\gamma = (\gamma_1, 0, \dots, 0)$ *is in* H^0. *Then, for* $\mu^*_{\alpha,\beta}$ *given by* (1.4),

$$\lim_{t\to\infty} \mu^*_{\alpha,\beta} S_t \tau_{([v_1 t], \dots, [v_d t])}$$

$$= \begin{cases} \nu_\alpha, & \text{if } (v_1 - \gamma_1\phi'(\alpha), v_2, \dots, v_d) \notin H, \\ \nu_{[\phi']^{-1}(\theta)}, & \text{if } (v_1 - \gamma_1\phi'(\theta), v_2, \dots, v_d) \in \partial H \text{ for } \beta \le \theta \le \alpha, \\ \nu_\beta, & \text{if } (v_1 - \gamma_1\phi'(\beta), v_2, \dots, v_d) \in H. \end{cases}$$

REMARK 1.4. Once more, we shall notice that in Theorem 3 nothing is said for $(v_1 - v_c, v_2, \dots, v_d) \in \partial H$. On the other hand, the hypothesis that H^0 is nonempty can be removed. In this case, we prove the theorem by approximating the cone H by a sequence of cones H_n with nonempty interiors.

REMARK 1.5. The proofs of Theorems 3 and 4 work if $-\gamma = (-\gamma_1, 0, \dots, 0)$ is in H^0. The important point is that either the surface ∂H propagates in all directions or it diffuses in all directions. This is the case when γ or $-\gamma$ is in H^0.

REMARK 1.6. As in Remark 1.2, if γ or $-\gamma$ is in H^0 and if γ can be written as $\gamma = \tilde{\sigma}(\sigma_1, \dots, \sigma_d)$, where $\tilde{\sigma} \in \mathbb{R}$ and $(\sigma_1, \dots, \sigma_d) \in \mathbb{Z}^d$, then, from Theorems 3 and 4, by a change of variables, we obtain corollaries analogous to those obtained in that remark.

In Section 7, we will see that in dimension 2 the hypothesis that $\gamma = \tilde{\sigma}(\sigma_1, \sigma_2)$ can be weakened in order to include Lebesgue-almost all vectors of $\mathbb{R}^2$.

We will prove all the theorems for dimension 2 and, when necessary, make some comments on the proof in higher dimensions. Observe that in dimension 2, a closed cone with nonempty interior which contains $(1, 0)$ and such that H^c is not a cone can be written in polar coordinates as $\{(\varphi, r) \in (-\pi, \pi] \times \mathbb{R}_+; h_2 \le \varphi \le h_1\}$, where $-\pi < h_2 < 0 < h_1 < \pi$, $h_1 - h_2 < \pi$.

2. Preliminary lemmas.

LEMMA 2.1. *Let* μ *be a probability on* $\mathbb{N}^{\mathbb{Z}^2}$. *Suppose that there are* $0 \le \theta_1 \le \theta_2 < \infty$ *and* $(a, b), (c, d) \in \mathbb{Z}^2$ *with* $a, b, c, d \ge 0$ *and with* $\gcd(d, b) = 1$, *such that we have*

either (i)
$$\mu \le \mu\tau_{(1,0)},$$
$$\mu \le \mu\tau_{(a,b)},$$
$$\mu \le \mu\tau_{(c,-d)}$$

or (i′) $\mu \geq \mu\tau_{(1,0)}$,

$$\mu \geq \mu\tau_{(a,b)},$$

$$\mu \geq \mu\tau_{(c,-d)}$$

and (ii) $\nu_{\theta_1} \leq \mu \leq \nu_{\theta_2}$.

Then any sequence $T_N \uparrow \infty$ has a subsequence T_{N_k} for which there exists a dense and countable subset D of $\mathbb{R}$, such that for every $v \in D$,

$$\lim_{k\to\infty} \frac{1}{T_{N_k}} \int_0^{T_{N_k}} \mu S_t \tau_{([vt],0)}\, dt = \mu_v \quad \textit{for some } \mu_v \in \mathscr{I} \cap \mathscr{S}.$$

PROOF. Almost all the proof is omitted since it is similar to the one of Lemma 3.1 of [3]. We will only show that the measure μ_v obtained in this lemma is in $\mathscr{S}$ for every $v \in D$.

Suppose that hypotheses (i) hold and fix $v \in D$. First, we observe that by attractiveness inequalities (i) extend to μ_v. For the rest of the proof, we will denote by (j) inequalities (i) for the measure μ_v instead of μ.

On the one hand, we know from the proof of Lemma 3.1 of [3] that $\mu_v[\eta(z,0)] = \mu_v[\eta(0,0)]$ for every $z \in \mathbb{Z}$. On the other hand, from the hypothesis $\gcd(b,d) = 1$, it is easy to see from (j) that for every $(x,y) \in \mathbb{Z}^2$ there exists $z_1, z_2 \in \mathbb{Z}$ (which depend on x and y), such that $\mu_v[\eta(z_1,0)] \leq \mu_v[\eta(x,y)] \leq \mu_v[\eta(z_2,0)]$. This proves that

$$\mu_v[\eta(x,y)] = \mu_v[\eta(0,0)] \quad \text{for every } (x,y) \in \mathbb{Z}^2. \tag{2.1}$$

Thus, since by (j), $\mu_v \leq \mu_v\tau_{(1,0)}$; according to (1.1), assertion (2.1) implies that

$$\mu_v = \mu_v\tau_{(1,0)}. \tag{2.2}$$

On the other hand, since $\gcd(b,d) = 1$, there exists $p, q \in \mathbb{N}$, such that $pb - qd = 1$. Hence, by hypothesis (j), $\mu_v \leq \mu_v\tau_{(pa+qc,1)}$. Thus, once more according to (1.1), equality (2.1) implies that $\mu_v = \mu_v\tau_{(pa+qc,1)}$. This together with equality (2.2) proves that $\mu_v \in \mathscr{S}$. □

REMARK 2.1. From (1.3) and Lemma 2.1, for every $v \in D$, there is a probability measure on $[\theta_1, \theta_2]$ which will be denoted by λ_v, such that

$$\mu_v = \int_{\theta_1}^{\theta_2} \nu_\rho \lambda_v(d\rho).$$

LEMMA 2.2. *With the notation of Lemma 2.1 and with $\mu = \mu_{\alpha,\beta}$ or $\mu = \mu^*_{\alpha,\beta}$ defined by (1.4), there exist $\bar{v}, \underline{v} \in \mathbb{R}$, such that*

$$\mu_v = \begin{cases} \nu_\alpha, & \textit{if } v < \underline{v}, \\ \nu_\beta, & \textit{if } v > \bar{v}. \end{cases}$$

PROOF. This lemma is a generalization to higher dimensions of Lemma 3.3 of [3]. We will only prove the existence of $\bar{v}$, since the proof for $\underline{v}$ is similar. Suppose that $\mu = \mu_{\alpha,\beta}$ and that $\alpha \le \beta$. Fix v in D. Since $\mu_{\alpha,\beta} \le \nu_\beta$, from the attractiveness of the process, we obtain $\mu_v \le \nu_\beta$. Hence, according to (1.1) and in order to prove that $\mu_v = \mu_\beta$, we only have to show that $\mu_v[\eta(x,y)] = \beta$ for every $(x,y) \in \mathbb{Z}^2$. This will be done by a coupling argument.

First, take η particles distributed according to $\mu_{\alpha,\beta}$ and suitably add ξ particles on the sites $\{(x,y) \in \mathbb{Z}^2; x < 0\}$ so that $\eta + \xi$ is distributed according to ν_β. Then we let (η_t, ξ_t) evolve according to the Markov process on $\mathbb{N}^{\mathbb{Z}^2} \times \mathbb{N}^{\mathbb{Z}^2}$, whose generator $\bar{L}$ is defined on cylindrical functions by

$$\begin{aligned}\bar{L}f(\eta,\xi) = &\sum_{x,y\in\mathbb{Z}^2} g(\eta(x))P(x,y)[f(\eta^{x,y},\xi) - f(\eta,\xi)]\\ &+ \sum_{x,y\in\mathbb{Z}^2}[g(\eta(x)+\xi(x)) - g(\eta(x))]\\ &\times P(x,y)[f(\eta,\xi^{x,y}) - f(\eta,\xi)].\end{aligned}$$

The generator $\bar{L}$ describes a process with two different kinds of particles η and ξ, the first having priority over the second. This coupling was introduced in [2]. It is clear that (η_t) and $(\eta_t + \xi_t)$ are zero-range processes with generator given by (1.0) and with initial distributions $\mu_{\alpha,\beta}$ and ν_β, respectively. We will denote by π the initial distribution on $\mathbb{N}^{\mathbb{Z}^2} \times \mathbb{N}^{\mathbb{Z}^2}$ of (η,ξ).

Thus, by the coupling we have just constructed and by Lemma 2.1,

$$\begin{aligned}\mu_v[\eta(x,y)] &= \mu_v[\eta(0,0)]\\ &= \lim_k \frac{1}{T_{N_k}}\int_0^{T_{N_k}} \mu_{\alpha,\beta}[\eta_t([vt],0)]\,dt\\ &= \beta - \lim_k \frac{1}{T_{N_k}}\int_0^{T_{N_k}} \pi[\xi_t([vt],0)]\,dt.\end{aligned}$$

Hence, to prove the theorem, we have only to show that $\pi[\xi_t([vt],0)] \to 0$ as $t\uparrow\infty$, for v sufficiently large. In order to do this, we will couple the ξ particles with another system of ζ particles. First, we label the ξ particles with superscript indices and we add a second superscript when we want to indicate the coordinates of the particle. To each $\xi^k = (\xi^{k,1}, \xi^{k,2})$ particle, we associate a ζ^k particle. The ζ particles move independently as translation invariant continuous-time random walks on $\mathbb{Z}$, whose holding times have rate $G = \sup_k[g(k+1) - g(k)]$ and whose transition probabilities are

$$r(0,k) = \begin{cases} 0, & \text{if } k \le 0,\\ \sum_{\|y\|=k} p(y), & \text{if } k > 0.\end{cases} \tag{2.3}$$

If there are n ξ-particles on the site x, we can consider that each one jumps at rate $[g(\eta(x)+n) - g(\eta(x))]/n$ which is at most G. On the other hand, the jumps of the ζ particles are stochastically larger than those of the ξ particles.

Therefore, we can couple them in such a way that

$$\begin{aligned} \zeta_0^k &= \xi_0^{k,1} - |\xi_0^{k,2}|, \\ \sum_{s \le t} \|\xi^k(s) - \xi^k(s-)\| &\le \zeta_t^k - \zeta_0^k. \end{aligned} \tag{2.4}$$

Let

$$\begin{aligned} m &= \sum_{k \ge 0} k r(k) \\ &= \sum_{y \in \mathbb{Z}^2} \|y\| p(y), \end{aligned} \tag{2.5}$$

which is finite by Assumption 1.1(ii). Let $\bar{v} = m + 1$ and (A_t) be a continuous-time random walk on $\mathbb{Z}$, with holding time rate equal to G, transition probability given by (2.3) and with $A_0 = 0$. For $v > \bar{v}$, we have

$$\pi[\xi_t([vt], 0)] \le \frac{K'}{([vt] - [\bar{v}t])^2} \sum_{y \in \nabla([\bar{v}t], [vt])} \pi[\xi_t(y)], \tag{2.6}$$

where

$$\nabla(j, k) = \{(x, y) \in \mathbb{Z}^2; j \le x \le k, |y| \le x - j\}.$$

From the coupling and from (2.4), we see that

$$\begin{aligned} \sum_{y \in \nabla([\bar{v}t], [vt])} \pi[\xi_t(y)] &\le \sum_{k \ge [\bar{v}t]} E[\zeta_t(k)] \\ &\le K''(\beta - \alpha) \sum_{k \ge 1} k P[A_t \ge [\bar{v}t] + k - 1] \\ &\le K''' \sum_{k \ge [\bar{v}t]} k^2 P[A_t = k], \end{aligned} \tag{2.7}$$

where the first inequality is obtained by the coupling, the second by Wald's lemma and the third by a change of variables. Thus, from (2.6) and (2.7),

$$\pi[\xi_t([vt], 0)] \le K E\left[\left(\frac{A_t}{t}\right)^2 \mathbf{1}_{A_t \ge [\bar{v}t]}\right]$$

and, by Assumption 1.1, the definition of $\bar{v}$ and (2.5), the right-hand side of this expression converges to 0 when t goes to ∞ by standard arguments. The proof of the existence of $\bar{v}$ when $\beta \le \alpha$ is similar.

The proof for the case $\mu = \mu^*_{\alpha,\beta}$ is similar. Inequality (2.6) is the unique point where extra arguments are needed. Keeping in mind the definition of H, let $M \in \mathbb{N}$, such that $1/M < (\tan(h_1))^+ \wedge (-\tan(h_2))^+$, that is, such that $(M, 1)$ and $(M, -1)$ are in H. From the definition of the measure π, $\pi[\xi(x - z_1, y - z_2)] \le \pi[\xi(x, y)]$ for every $(x, y) \in \mathbb{Z}^2$, $(z_1, z_2) \in \mathbb{Z}^2 \cap H$. By attractiveness, this inequality extends for every time t: $\pi[\xi_t(x - z_1, y - z_2)] \le \pi[\xi_t(x, y)]$ for every $(x, y) \in \mathbb{Z}^2$, $(z_1, z_2) \in \mathbb{Z}^2 \cap H$. Consider $v > \bar{v}$ and let $w < (v + M\bar{v})/(M + 1)$. A simple computation shows that for every sufficiently large t,

$\pi[\xi_t(\vartheta)] \le \pi[\xi_t([vt], 0)]$ for every $\vartheta \in \nabla([\bar{v}t], [wt])$. In this way, we obtain an inequality similar to (2.6). The rest of the proof follows in the same way. □

PROPOSITION 2.1. *With the notation and hypotheses of Lemma 2.1, if there exist v_0 and v_1, such that for every initial sequence (T_N) the measures μ_v obtained in the lemma satisfy*

$$\mu_v = \begin{cases} \nu_{\theta_1}, & \text{if } v < v_0, \\ \nu_{\theta_2}, & \text{if } v > v_1, \end{cases}$$

then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mu S_t \tau_{([vt],0)}\, dt = \begin{cases} \nu_{\theta_1}, & \text{if } v < v_0, \\ \nu_{\theta_2}, & \text{if } v > v_1. \end{cases}$$

PROOF. Suppose that $v > v_1$. From the attractiveness of the process and the inequality $\mu \le \nu_{\theta_2}$, we know that

$$\left\{ \frac{1}{T}\int_0^T \mu S_t \tau_{([vt],0)}\, dt,\ T \ge 0 \right\}$$

is relatively compact. To show that ν_{θ_2} is the unique cluster point, we suppose that

$$\lim_{N\to\infty} \frac{1}{T_N}\int_0^{T_N} \mu S_t \tau_{([vt],0)}\, dt = \tilde{\mu}$$

for a sequence $T_N \uparrow \infty$ and $\tilde{\mu}$ in $\mathscr{P}(\mathbb{N}^{\mathbb{Z}^2})$. We will show that $\tilde{\mu} = \nu_{\theta_2}$. Applying Lemma 2.1 to the measure μ and to the sequence T_N, we obtain a subsequence T_{N_k} and a dense subset D of $\mathbb{R}$. Take $v_1 < u < v$, $u \in D$. By hypothesis, $\mu_u = \nu_{\theta_2}$. Since $\mu \le \mu\tau_{(1,0)}$, by attractiveness, we have

$$\frac{1}{T_{N_k}}\int_0^{T_{N_k}} \mu S_t \tau_{([ut],0)}\, dt \le \frac{1}{T_{N_k}}\int_0^{T_{N_k}} \mu S_t \tau_{([vt],0)}\, dt \le \nu_{\theta_2}.$$

Letting $k \uparrow \infty$, we obtain that $\tilde{\mu} = \nu_{\theta_2}$, which proves the lemma for $v > v_1$. The proof for $v < v_0$ is similar. □

REMARK 2.2. With the same hypotheses of Proposition 2.1, if the measures μ_v obtained in Lemma 2.1 satisfy

$$\mu_v = \begin{cases} \nu_{\theta_1}, & \text{if } v > v_0, \\ \nu_{\theta_2}, & \text{if } v < v_1, \end{cases}$$

then by the same arguments

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mu S_t \tau_{([vt],0)}\, dt = \begin{cases} \nu_{\theta_1}, & \text{if } v > v_0, \\ \nu_{\theta_2}, & \text{if } v < v_1. \end{cases}$$

Observe that the measures $\mu_{\alpha,\beta}$ and $\mu^*_{\alpha,\beta}$ satisfy the hypotheses of Lemma 2.1. In the next four sections, we will prove the theorems. The idea will be always the same. First, we prove two lemmas which allow us to apply Proposition 2.1 for good v_0's and v_1's. Then we use a technical argument to obtain the desired convergence instead of the convergence in the Cesaro sense. For the case with diffusion, there will be another technical argument to obtain convergence for every velocity.

3. Proof of Theorem 1. In this section, we will suppose that $\alpha \le \beta$ and that ϕ given by (1.2) is concave.

We will now state the main lemma for the proof of Theorems 1 and 2. Under the hypotheses of Lemma 2.1 and supposing that the measure μ is translation invariant in the y-direction, this lemma enables us to compute the one-dimensional asymptotic density of particles between two macroscopic velocities. We will then be able to prove, with a beautiful argument taken from [3], that the measures $\{\mu_v,\ v \in D\}$ are indeed equal to ν_α for $v < v_c$ and ν_β for $v > v_c$ if $\alpha \le \beta$ [ν_α for $v < \gamma_1\phi'(\alpha)$ and ν_β for $v > \gamma_1\phi'(\beta)$ if $\beta \le \alpha$]. The proof of this lemma relies on a long computation on the generator and is deferred to the Appendix. The translation by $(\tilde{x}, 0)$ which appears in the statement will be necessary later.

LEMMA 3.1. *Let μ be a probability on $\mathbb{N}^{\mathbb{Z}^2}$, such that*

$$\text{(i)}\quad \nu_{\theta_1} \le \mu \le \nu_{\theta_2},$$

either

$$\text{(ii)}\quad \mu \le \mu\tau_{(1,0)},\qquad \mu = \mu\tau_{(0,1)}$$

or

$$\text{(ii}')\quad \mu \ge \mu\tau_{(1,0)},\qquad \mu = \mu\tau_{(0,1)}$$

and

$$\text{(iii)}\quad \lim_{N\to\infty} \frac{1}{T_N}\int_0^{T_N} \mu S_t \tau_{([wt],0)}\,dt = \mu_w \quad \text{for } w = u, v,$$

where

$$\mu_w = \int_{\theta_1}^{\theta_2} \nu_\rho \lambda_w(d\rho).$$

Then, for every $\tilde{x}$ in $\mathbb{Z}$,

$$\lim_{N\to\infty} \frac{1}{T_N} \mu S_{T_N}\tau_{(\tilde{x},0)}\left(\sum_{x=[uT_N]+1}^{[vT_N]} \eta(x,0)\right) = F(v) - F(u), \tag{3.1}$$

where

$$F(w) = w\int_{\theta_1}^{\theta_2} \rho\lambda_w(d\rho) - \gamma_1\int_{\theta_1}^{\theta_2} \phi(\rho)\lambda_w(d\rho). \tag{3.2}$$

REMARK 3.1. With the notation of Lemma 2.1, if we take $u, v \in D$, $\mu = \mu_{\alpha,\beta}$ given by (1.4) and T_{N_k} the subsequence obtained in that lemma, by Remark 2.1, the hypotheses of Lemma 3.1 are satisfied.

We can now state the following lemma.

LEMMA 3.2. *With the notation of Lemma* 2.1, *with* $\theta_1 = \alpha$, $\theta_2 = \beta$ *and with* $\mu = \mu_{\alpha,\beta}$ *defined by* (1.4) *and* v_c *by* (1.5), *for* $v \in D$, *we have*

$$\mu_v = \begin{cases} \nu_\alpha, & \text{if } v < v_c, \\ \nu_\beta, & \text{if } v > v_c. \end{cases}$$

PROOF. We know from Lemma 2.2 that there exist $\underline{v}$ and $\bar{v}$, such that $\mu_v = \nu_\beta$ (respectively ν_α), provided $v > \bar{v}$ (respectively $v < \underline{v}$). Let $u \in D$, $v_c < u \le \bar{v}$ and take $v > \bar{v}$, $v \in D$. By Remark 3.1, we can apply Lemma 3.1 with $\mu = \mu_{\alpha,\beta}$ and with the subsequence obtained in Lemma 2.1. Then by attractiveness we have from (3.1)

$$F(v) - F(u) \le \beta(v - u).$$

Thus, since $v > \bar{v}$,

$$u \int_\alpha^\beta [\beta - \rho] \lambda_u(d\rho) \le \gamma_1 \int_\alpha^\beta [\phi(\beta) - \phi(\rho)] \lambda_u(d\rho).$$

Since ϕ is concave,

$$[\phi(\beta) - \phi(\rho)] \le \left(\frac{\phi(\beta) - \phi(\alpha)}{\beta - \alpha} \right) [\beta - \rho].$$

Therefore, keeping in mind that $v_c = \gamma_1[[\phi(\beta) - \phi(\alpha)]/[\beta - \alpha]]$, we obtain

$$u \int_\alpha^\beta [\beta - \rho] \lambda_u(d\rho) \le v_c \int_\alpha^\beta [\beta - \rho] \lambda_u(d\rho).$$

As we took $u > v_c$, we get $\lambda_u(d\rho) = \delta_\beta(d\rho)$ and $\mu_u = \nu_\beta$. The other assertion of the lemma is proved in the same way. □

We are now ready to prove the convergence of the Cesaro means. Indeed, Lemma 3.2 and the attractiveness of the process allow us to extend the convergence in the Cesaro sense in Lemma 2.1 for velocities in D, to convergence for velocities in $\mathbb{R} - \{v_c\}$. The compactness of the set $\{\mu \in \mathscr{P}(\mathbb{N}^{\mathbb{Z}^2});\ \mu \le \nu_\beta\}$ will do the rest.

REMARK 3.2. Lemma 3.2 allows us to apply Proposition 2.1 with $\mu = \mu_{\alpha,\beta}$, $v_0 = v_1 = v_c$, $\nu_{\theta_1} = \nu_\alpha$ and $\nu_{\theta_2} = \nu_\beta$. With this result we can prove Theorem 1.

PROOF OF THEOREM 1. Let $v > v_c$. From the attractiveness of the process and the inequality $\mu_{\alpha,\beta} \le \nu_\beta$, we know that

$$\{\mu_{\alpha,\beta} S_t \tau_{([vt],0)},\ t \ge 0\}$$

is relatively compact. Thus, we only have to show that ν_β is the unique cluster point. Take $v_c < u < v$, a cluster point $\tilde{\mu}$ and a sequence $T_N \uparrow \infty$, such that

$$\lim_{N\to\infty} \mu_{\alpha,\beta} S_{T_N} \tau_{([vT_N],0)} = \tilde{\mu}.$$

Once more, by attractiveness, we have $\tilde{\mu} \le \nu_\beta$. According to (1.1) and in order to prove that $\tilde{\mu} = \nu_\beta$, we have to show that

$$\tilde{\mu}[\eta(x,y)] = \beta \quad \text{for every } (x,y) \in \mathbb{Z}^2.$$

On the other hand, since we know by Remark 3.2 that

$$\lim_{N\to\infty} \frac{1}{T_N}\int_0^{T_N} \mu_{\alpha,\beta} S_t \tau_{([wt],0)}\, dt = \nu_\beta \quad \text{for } w = u, v,$$

we can apply Lemma 3.1. But

$$\tilde{\mu}[\eta(x,y)] = \lim_N \mu_{\alpha,\beta}\big[\eta_{T_N}([vT_N] + x, y)\big] \tag{3.3}$$

which, by attractiveness and since $\mu_{\alpha,\beta} = \mu_{\alpha,\beta}\tau_{(0,1)}$, is greater than or equal to

$$\lim_N \frac{1}{[vT_N] - [uT_N]} \sum_{z=[uT_N]+1}^{[vT_N]} \mu_{\alpha,\beta} S_{T_N} \tau_{(x,0)}[\eta(z,0)]$$

and this last expression is equal to β by Lemma 3.1. □

4. Proof of Theorem 2. In this section, we will assume that $\beta < \alpha$ and that ϕ given by (1.2) is strictly concave. First, we state a lemma which corresponds to Lemma 3.2.

LEMMA 4.1. *With the notation of Lemma* 2.1, *with* $\theta_1 = \beta$, $\theta_2 = \alpha$ *and with* $\mu = \mu_{\alpha,\beta}$ *defined by* (1.4), *for* $v \in D$, *we have*

$$\mu_v = \begin{cases} \nu_\alpha, & \text{if } v < \gamma_1\phi'(\alpha), \\ \nu_\beta, & \text{if } v > \gamma_1\phi'(\beta). \end{cases}$$

PROOF. The proof is the same as the one of Claim 1, page 280 of [3] and is similar to the one of Lemma 3.2. □

Now, we can apply Remark 2.2 with $\mu = \mu_{\alpha,\beta}$, $v_1 = \gamma_1\phi'(\alpha)$, $v_0 = \gamma_1\phi'(\beta)$, $\nu_{\theta_2} = \nu_\alpha$ and $\nu_{\theta_1} = \nu_\beta$. This together with Lemma 3.1 will prove Theorem 2.

PROOF OF THEOREM 2. First, arguing just as in the proof of Theorem 1, we can prove that

$$\lim_{t\to\infty} \mu_{\alpha,\beta} S_t \tau_{([vt],0)} = \begin{cases} \nu_\alpha, & \text{if } v < \gamma_1\phi'(\alpha), \\ \nu_\beta, & \text{if } v > \gamma_1\phi'(\beta). \end{cases} \tag{4.1}$$

Now, we will consider the case $\gamma_1\phi'(\alpha) < v < \gamma_1\phi'(\beta)$. As in the proof of Theorem 1, we know that $\{\mu_{\alpha,\beta} S_t \tau_{([vt],0)},\ t \ge 0\}$ is relatively compact. Let $\tilde{\mu}$ be a weak limit point and take a sequence $T_N \uparrow \infty$, such that

$$\lim_{N\to\infty} \mu_{\alpha,\beta} S_{T_N} \tau_{([vT_N],0)} = \tilde{\mu}. \tag{4.2}$$

Since ϕ is C^1 and strictly concave, there exists $\omega_0 \in (\beta, \alpha)$ with $\gamma_1 \phi'(\omega_0) = v$. We will show that $\nu_{\omega_0} \le \tilde{\mu}$. Consider $\beta < \omega < \omega_0$. Since ϕ is strictly concave, $\phi'(\omega_0) < \phi'(\omega)$ and therefore $v < \gamma_1 \phi'(\omega)$. Hence, by (4.1), with the measure $\mu_{\omega,\beta}$ instead of $\mu_{\alpha,\beta}$, we have

$$\lim_{t\to\infty} \mu_{\omega,\beta} S_t \tau_{([vt],0)} = \nu_\omega. \tag{4.3}$$

From the inequality $\mu_{\omega,\beta} \le \mu_{\alpha,\beta}$, the attractiveness of the process, (4.2) and (4.3), we obtain $\nu_\omega \le \tilde{\mu}$. Letting $\omega \uparrow \omega_0$ (since $\{\nu_\rho, \rho \ge 0\}$ is a weakly continuous family), we get $\nu_{\omega_0} \le \tilde{\mu}$. In the same way, we show that $\tilde{\mu} \le \nu_{\omega_0}$, which proves the theorem for $\gamma_1\phi'(\alpha) < v < \gamma_1\phi'(\beta)$. The cases $v = \gamma_1\phi'(\alpha)$ and $v = \gamma_1\phi'(\beta)$ follow using the strict concavity of ϕ, the inequality $\mu\tau_{(1,0)} \le \mu$ and the attractiveness of the process. □

5. Proof of Theorem 3. In this section we will suppose that Assumptions 1.1 and 1.2 hold, that ϕ is concave and that $\alpha \le \beta$. Comparing the measure $\mu^*_{\alpha,\beta}$ with measures which are translation invariant in one direction, we will prove the following proposition.

PROPOSITION 5.1. *Under the hypotheses of Theorem* 3,

$$\lim_{t\to\infty} \mu^*_{\alpha,\beta} S_t \tau_{([v_1 t],[v_2 t])} = \nu_\alpha \quad \text{if } (v_1 - v_c, v_2) \notin H.$$

PROOF. Once again, we have only to show that ν_α is the unique cluster point of the sequence $(\mu^*_{\alpha,\beta} S_t \tau_{([v_1 t],[v_2 t])})_{t\ge 0}$. Let $\tilde{\mu}$ be a cluster point. Since $\nu_\alpha \le \mu^*_{\alpha,\beta}$, by attractiveness,

$$\nu_\alpha \le \tilde{\mu}. \tag{5.1}$$

Let $(r_k^j)_{k\in N}$, $j = 1,2$, be two sequences, such that, $r_k^1 \downarrow h_1$, $r_k^2 \uparrow h_2$ and $\tan r_k^j \in \mathbb{Q}$. For $j = 1,2$, let $(\mu_k^j)_{k\in N}$ be two sequences of product measures where μ_k^j is given by

$$\mu_k^j[\eta, \eta(x) = 1] = \begin{cases} \nu_\beta[\eta, \eta(x) = 1], & \text{if } (-1)^{j+1}\big(x_1 \sin(r_k^j) - x_2 \cos(r_k^j)\big) \ge 0, \\ \nu_\alpha[\eta, \eta(x) = 1], & \text{otherwise.} \end{cases}$$

We have that $\mu^*_{\alpha,\beta} \le \mu_k^j$, for k sufficiently large and for $j = 1,2$. Fix $j = 1$. Since we know by Corollary 1 that

$$\lim_{t\to\infty} \mu_k^1 S_t \tau_{([v_1 t],[v_2 t])} = \nu_\alpha \quad \text{provided } \big[(v_1 - v_c)\sin(r_k^1) - v_2\cos(r_k^1)\big] < 0,$$

we obtain

$$\tilde{\mu} \le \nu_\alpha \quad \text{if } \big[(v_1 - v_c)\sin(r_k^1) - v_2\cos(r_k^1)\big] < 0.$$

Letting $k \uparrow \infty$, we obtain

$$(5.2) \qquad \tilde{\mu} \le \nu_\alpha \quad \text{if } \left[(v_1 - v_c)\sin h_1 - v_2 \cos h_1\right] < 0.$$

In the same way, with the measure μ_k^2 instead of μ_k^1 we can show that

$$(5.3) \qquad \tilde{\mu} \le \nu_\alpha \quad \text{if } \left[-(v_1 - v_c)\sin h_2 + v_2 \cos h_2\right] < 0.$$

Finally, (5.1), (5.2) and (5.3) together prove the proposition. □

To complete the proof of the theorem, we will follow the steps of Section 3. Initially, we observe that the hypotheses of Lemmas 2.1 and 2.2 hold. As we do not have the one-dimensional translation invariance of the measure $\mu^*_{\alpha,\beta}$, we cannot compute the asymptotic density of particles between two macroscopic velocities anymore. Nevertheless, we can state a lemma which almost gives the asymptotic density.

LEMMA 5.1. *Let μ be a probability on $\mathbb{N}^{\mathbb{Z}^2}$, such that, for $a, b, c, d \ge 0$ and* $\gcd(b, d) = 1$,

(i) $\nu_{\theta_1} \le \mu \le \nu_{\theta_2}$,

either (ii) $\mu \le \mu\tau_{(1,0)}$, $\mu \le \mu\tau_{(a,b)}$, $\mu \le \mu\tau_{(c,-d)}$

or (ii′) $\mu \ge \mu\tau_{(1,0)}$, $\mu \ge \mu\tau_{(a,b)}$, $\mu \ge \mu\tau_{(c,-d)}$

and (iii) $\lim_N \dfrac{1}{T_N}\displaystyle\int_0^{T_N} \mu S_t \tau_{([wt],0)}\, dt = \mu_w$ *for* $w = u, v$,

where

$$\mu_w = \int_{\theta_1}^{\theta_2} \nu_\rho \lambda_w(dp).$$

Then, for every $(\tilde{x}, \tilde{y})$ in $\mathbb{Z}^2$ and for R sufficiently large,

$$(5.4) \qquad \begin{aligned} &\limsup \sum_{y=-R}^{R} \frac{1}{T_N} \mu S_{T_N} \tau_{(\tilde{x},\tilde{y})}\left(\sum_{x=[uT_N]+1}^{[vT_N]} \eta(x,y)\right) \\ &\qquad \le (2R+1)[F(v) - F(u)] + K_2, \\ &\liminf \sum_{y=-R}^{R} \frac{1}{T_N} \mu S_{T_N} \tau_{(\tilde{x},\tilde{y})}\left(\sum_{x=[uT_N]+1}^{[vT_N]} \eta(x,y)\right) \\ &\qquad \ge (2R+1)[F(v) - F(u)] + K_1, \end{aligned}$$

where F is given by (3.2) *and the constants depend only on A, θ_2, h_1, h_2 and ϕ.*

As in Lemma 3.1, the proof relies on a long computation on the generator. This time it strongly uses the assumption that $\gamma_2 = 0$ and that the process is of finite range. The proof is deferred to the Appendix. The important fact is that the constants appearing in formula (5.4) do not depend on R. In higher dimensions, the constants are multiplied by $(2R+1)^{d-2}$ and the rest of the proof follows as in dimension 2. This allows us to prove Lemma 3.2 by letting $R \uparrow \infty$ in our setting (i.e., with $\mu^*_{\alpha,\beta}$ instead of $\mu_{\alpha,\beta}$). Therefore, we can apply Proposition 2.1 with $\mu = \mu^*_{\alpha,\beta}$, $v_0 = v_1 = v_c$, $\nu_{\theta_1} = \nu_\alpha$ and $\nu_{\theta_2} = \nu_\beta$, to obtain

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mu^*_{\alpha,\beta} S_t \tau_{([vt],0)}\, dt = \nu_\beta \quad \text{if } v > v_c.$$

Then, to replace the convergence in the Cesaro sense by the desired one, we follow the proof of Theorem 1 until formula (3.3), where we have

$$\begin{aligned}\tilde{\mu}[\eta(x,y)] &= \lim_N \mu^*_{\alpha,\beta}\big[\eta_{T_N}([vT_N]+x,y)\big]\\ &\geq \frac{1}{(2R+1)} \lim_N \frac{1}{([vT_N]-[uT_N])}\\ &\quad\times \sum_{z_1=[uT_N]+1}^{[vT_N]} \sum_{z_2=-R}^{R} \mu^*_{\alpha,\beta} S_{T_N} \tau_{(x-RM,y)}\big[\eta(z_1,z_2)\big]\end{aligned}$$

for every $R \in \mathbb{N}$ sufficiently large and where $M \in \mathbb{N}$ is such that $1/M < (\tan(h_1))^+ \wedge (-\tan(h_2))^+$. By Lemma 5.1 and Proposition 2.1, the right-hand side of the last inequality is bounded below by

$$\beta + \frac{K}{2R+1}.$$

So, letting $R \uparrow \infty$, we obtain that $\tilde{\mu}[\eta(x,y)] = \beta$ for every $(x,y) \in \mathbb{Z}^2$ which proves that

$$\lim_{t\to\infty} \mu^*_{\alpha,\beta} S_t \tau_{([vt],0)} = \nu_\beta \quad \text{if } v > v_c.$$

To complete the proof of the theorem, it is not hard to see (since H is a cone) that if $(v_1 - v_c, v_2) \in H^0$ there exists $v_0 > v_c$, such that

$$\mu^*_{\alpha,\beta} S_t \tau_{([v_1 t],[v_2 t])} \geq \mu^*_{\alpha,\beta} S_t \tau_{([v_0 t],0)},$$

and the right-hand side of this inequality converges to ν_β.

6. Proof of Theorem 4. In this section, we will suppose that Assumptions 1.1 and 1.2 hold, that ϕ is strictly concave and that $\beta < \alpha$. As in the last section, we first state a proposition.

PROPOSITION 6.1. *Under the hypotheses of Theorem* 4,

$$\lim_{t\to\infty} \mu^*_{\alpha,\beta} S_t \tau_{([v_1 t],[v_2 t])} = \nu_\alpha \quad \textit{if } (v_1 - \gamma_1 \phi'(\alpha), v_2) \notin H.$$

The proof is the same as the one of Proposition 5.1.

As in the last section, we can prove Lemma 4.1 in our setting using Lemma 5.1 instead of Lemma 3.1. Then, applying Remark 2.2 with $\mu = \mu^*_{\alpha,\beta}$, $v_0 = \gamma_1\phi'(\beta)$ and $\nu_{\theta_1} = \nu_\beta$, we obtain that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mu^*_{\alpha,\beta} S_t \tau_{([vt],0)}\,dt = \nu_\beta \quad \text{if } v > \gamma_1\phi'(\beta).$$

Then, repeating the arguments used at the end of Section 5, we obtain

$$\lim_{t\to\infty} \mu^*_{\alpha,\beta} S_t \tau_{([v_1 t],[v_2 t])} = \nu_\beta \quad \text{if } (v_1 - \gamma_1\phi'(\beta), v_2) \in H^0.$$

Finally, arguing as in the proof of Theorem 2, we complete the proof of Theorem 4.

7. Extensions.

REMARK 7.1. Remark 5.1 of [3] is also valid in our context for all four theorems. On the other hand, Remark 5.3 of the same reference applies only for the first two theorems.

REMARK 7.2. As we have said in Section 1, in dimension 2 the hypothesis of Theorems 3 and 4 that $\gamma = \tilde{\sigma}(\sigma_1, \sigma_2)$ where $(\sigma_1, \sigma_2) \in \mathbb{Z}^2$ and $\tilde{\sigma} \in \mathbb{R}$ can be weakened. Suppose we are under the hypotheses of Theorem 3 but with $\gamma = (\gamma_1, \gamma_2) \in \mathbb{R}^2 \cap H^0$, such that γ_2/γ_1 is a real of nonconstant type (see [9], page 24 for the definition). Under this hypothesis, we can adapt the proofs of the theorem. We will leave the details to the reader and just point out the main steps. To fix ideas, suppose that $\gamma_1, \gamma_2 > 0$ and $\alpha < \beta$. Since γ_2/γ_1 is of nonconstant type, we can obtain a sequence $((p_1^k, p_2^k))_k \in \mathbb{Z}^2$, such that $\gcd(p_1^k, p_2^k) = 1$ and

$$\left|\frac{p_2^k}{p_1^k} - \frac{\gamma_2}{\gamma_1}\right| < \frac{1}{(p_1^k)^2 a_k}, \tag{7.1}$$

where $a_k \uparrow \infty$.

Suppose $p_2^k/p_1^k - \gamma_2/\gamma_1 > 0$. Then we choose q_1^k and q_2^k, such that $p_1^k q_2^k - p_2^k q_1^k = 1$, $1/2 \le p_j^k/q_j^k \le 1$, $j = 1,2$. The fact that $q_2^k/q_1^k \ge p_2^k/p_1^k$ will be important in the proof. This says that (γ_1, γ_2) is not in the cone generated by the vectors (p_1^k, p_2^k) and (q_1^k, q_2^k).

The first step in the proof is to change variables with the linear transformation T_k which sends (p_1^k, p_2^k) to $(1,0)$ and (q_1^k, q_2^k) to $(0,1)$. By the construction of (p_1^k, p_2^k) and (q_1^k, q_2^k), T_k sends $\mathbb{Z}^2$ onto $\mathbb{Z}^2$. $T_k(\eta_t)$ will be called in what follows the transformed process. It is clear that the transformed process satisfies Assumptions 1.1 and 1.2. At this stage, we follow the proof of Theorem 3 for the transformed process until Lemma 5.1. With the hypotheses of this lemma, we can prove that, for every $(\tilde{x}, \tilde{y})$ in $\mathbb{Z}^2$ and for R sufficiently large

$$\liminf \sum_{y=-R}^{R} \frac{1}{T_N} \mu S_{T_N} \tau_{(\tilde{x},\tilde{y})} \left(\sum_{x=[uT_N]+1}^{[vT_N]} \eta(x,y) \right) \geq (2R+1)[F(v) - F(u)] + K,$$

where F is given by (3.2) with $\tilde{\gamma}_1^k$ instead of γ_1, if we denote by $\tilde{\gamma}_1^k$ the first coordinate of $T_k(\gamma)$. This enables us to prove, with the very arguments used in Section 5, that

$$\lim_{t\to\infty} \tilde{\mu}^*_{\alpha,\beta} \tau_{([vt],0)} = \nu_\beta \quad \text{if } v > \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \tilde{\gamma}_1^k,$$

where $\tilde{\mu}^*_{\alpha,\beta}$ is the measure $\mu^*_{\alpha,\beta}$ transformed by T_k.

The proof of this result is similar to the one of Lemma 5.1. Nevertheless, it strongly uses that $T_k(\gamma)$ is in the fourth quadrant and that the first quadrant is contained in $T_k(H^0)$. Returning to the original process, we have that

$$(7.2) \quad \lim_{t\to\infty} \mu^*_{\alpha,\beta} \tau_{(p_1^k[vt],\, p_2^k[vt])} = \nu_\beta \quad \text{if } v > \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \left(q_2^k\gamma_1 - q_1^k\gamma_2\right) = v_0^k.$$

Let

$$\Delta_k = \left(q_2^k\gamma_1 - q_1^k\gamma_2\right)\left(p_1^k, p_2^k\right).$$

From (7.2), we obtain that

$$(7.3) \quad \lim_{t\to\infty} \mu^*_{\alpha,\beta} \tau_{([v_1t],[v_2t])} = \nu_\beta \quad \text{if } (v_1,v_2) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \Delta_k \in H^0.$$

Indeed, fix k in $\mathbb{N}$ and let (v_1, v_2) be in $\mathbb{R}^2$, such that

$$(v_1,v_2) - \frac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \Delta_k = (v_1,v_2) - v_0^k\left(p_1^k,p_2^k\right) \in H^0.$$

Since H^0 is open, there exists $\tilde{v}_0^k > v_0^k$, such that $(v_1, v_2) - \tilde{v}_0^k(p_1^k, p_2^k) = (w_1, w_2)$ still belongs to H^0. A simple computation shows that

$$\mu^*_{\alpha,\beta} \tau_{([v_1t],[v_2t])} = \mu^*_{\alpha,\beta} \tau_{([v_0^kt]p_1^k,\,[v_0^kt]p_2^k)} \tau_{([w_1t]+\kappa_1,\,[w_2t]+\kappa_2)},$$

where $\kappa_1, \kappa_2 \in \mathbb{Z}$ and $|\kappa_1| \vee |\kappa_2| \leq 1 + (p_1^k \vee p_2^k)$. Since H is a cone and $(w_1, w_2) \in H^0$, $([w_1t] + \kappa_1, [w_2t] + \kappa_2) \in H^0$ for all sufficiently large t. On the other hand, $\mu^*_{\alpha,\beta}\tau_{(z_1,z_2)} \geq \mu^*_{\alpha,\beta}$ for every $(z_1, z_2) \in H \cap \mathbb{Z}^2$. Therefore,

$$\mu^*_{\alpha,\beta} \tau_{([v_1t],[v_2t])} \geq \mu^*_{\alpha,\beta} \tau_{([v_0^kt]p_1^k,\,[v_0^kt]p_2^k)},$$

and by (7.2), the right-hand side of this expression converges to ν_β, proving (7.3).

To conclude the proof of the theorem, we only have to show that $\Delta_k \to (\gamma_1, \gamma_2)$ and this is done using (7.1) and the fact that $a_k \uparrow \infty$.

The proof of Theorem 4 under the hypothesis that γ_2/γ_1 is a real of nonconstant type is the same. We also know from [9] that the reals of constant type are of Lebesgue measure 0.

APPENDIX

PROOF OF LEMMA 3.1. Suppose that the hypotheses (ii) are satisfied. Let

$$\begin{aligned}(8.1)\qquad G(t) &= \mu S_t \tau_{(y,0)}\left[\sum_{x=[ut]+1}^{[vt]} \eta(x,0)\right]\\ &\quad + ([ut]+1-ut)\mu S_t\tau_{(y,0)}[\eta([ut],0)]\\ &\quad + (vt-[vt])\mu S_t\tau_{(y,0)}[\eta([vt]+1,0)].\end{aligned}$$

Then, since the last two terms are positive and bounded above by θ_2, the limit of the left-hand side of (3.1) is equal to $\lim G(T_N)/T_N$. For $ut, vt \notin \mathbb{Z}$, G is differentiable in t and for every $s \in \mathbb{R}_+$, $G(s) = \int_0^s G'(t)\,dt$. So, we have to compute

$$\begin{aligned}(8.2)\qquad \lim_N \int_0^{T_N} dt \Bigg\{ &\mu S_t \tau_{(y,0)} L\left[\sum_{x=[ut]+1}^{[vt]} \eta(x,0)\right]\\ &+ v\mu S_t\tau_{(y,0)}[\eta([vt]+1,0)] - u\mu S_t\tau_{(y,0)}[\eta([ut],0)]\\ &+ (vt-[vt])\mu S_t\tau_{(y,0)} L[\eta([vt]+1,0)]\\ &+ ([ut]+1-ut)\mu S_t\tau_{(y,0)} L[\eta([ut],0)]\Bigg\}.\end{aligned}$$

First, by hypothesis (iii), the second and the third terms of (8.2) converge to

$$v\mu_v[\eta(0,0)] - u\mu_u[\eta(0,0)] = v\int_{\theta_1}^{\theta_2}\rho\lambda_v(d\rho) - u\int_{\theta_1}^{\theta_2}\rho\lambda_u(d\rho).$$

Let

$$q(z) = \sum_{x\in\mathbb{Z}} p(z,x) \quad \text{for } z \in \mathbb{Z}.$$

For the first term of (8.2), we proceed in the following way. First, we compute

$$(8.3)\qquad \mu S_t\tau_{(y,0)}\left[\sum_{x=[ut]+1}^{[vt]} \eta(x,0)\right].$$

Then, using the hypothesis that $\mu = \mu\tau_{(0,1)}$, we project every term appearing in the computation on the sites $\{(x,y);\ y=0\}$. At this point we see that (8.3) is equal to

$$\sum_{x=[ut]+1}^{[vt]}\left(\sum_{z\le[ut]} + \sum_{z>[vt]}\right)\{\mu S_t\tau_{(y,0)}[g(\eta(z,0))]q(x-z) - \mu S_t\tau_{(y,0)}[g(\eta(x,0))]q(z-x)\}.$$

Since by Assumption 1.1 $\sum|z|q(z) < \infty$, we can add to the sum $\sum_{z\le[ut]}$ the terms $\sum_{x>[vt]}$ and to the sum $\sum_{z>[vt]}$ the terms $\sum_{x\le[ut]}$, without changing the limit. After this, we change the variables x by $x' = x-[ut]$ $(x-[vt])$ and $z' = z-[ut]$ $(z-[vt])$. We now reverse the order of the sum and the limit using the dominated convergence theorem. Finally, we let $N \uparrow \infty$ and use the hypothesis $\mu_w = \mu_w \tau_{(1,0)}$ for $w = u, v$, to obtain that the limit of the first term in (8.2) is equal to

$$(8.4)\quad \big(\mu_u[g(\eta(0,0))] - \mu_v[g(\eta(0,0))]\big)\Big(\sum_{z\ge 0} P[Y > z] - \sum_{z>0} P[-Y \ge z]\Big),$$

where Y is a r.v., such that $P[Y = z] = q(z)$. Thus, (8.4) is equal to

$$\gamma_1[\mu_u[g(\eta(0,0))] - \mu_v[g(\eta(0,0))]].$$

We handled the last two terms of (8.2) in the same way and we prove that they converge to 0 which concludes the proof. □

PROOF OF LEMMA 5.1. Suppose we are under hypotheses (ii). As in the proof of Lemma 3.1, we define

$$\begin{aligned} G(t) = \sum_{y=-R}^{R} \Bigg\{ & \mu S_t \tau_{(\tilde{x},\tilde{y})}\Bigg[\sum_{x=[ut]+1}^{[vt]} \eta(x,y)\Bigg] \\ & + ([ut] + 1 - ut)\mu S_t \tau_{(\tilde{x},\tilde{y})}[\eta([ut],y)] \\ & + (vt - [vt])\mu S_t \tau_{(\tilde{x},\tilde{y})}[\eta([vt]+1,y)]\Bigg\}. \end{aligned}$$

We observe that the difference between $G(T_N)/T_N$ and the left-hand side of (5.4) converges to 0 when N goes to ∞. Computing $G'(t)$, we obtain an expression similar to (8.2). As in the proof of Lemma 3.1, the terms corresponding to the second and third terms in the expression (8.2) converge by hypothesis (iii) to

$$(8.5)\qquad (2R+1)[v\mu_v[\eta(0,0)] - u\mu_u[\eta(0,0)]].$$

For the corresponding first term, we proceed in the following way. Initially, we compute

$$\mu S_t \tau_{(\tilde{x},\tilde{y})} L\Bigg[\sum_{y=-R}^{R} \sum_{x=[ut]+1}^{[vt]} \eta(x,y)\Bigg]$$

and obtain

$$(8.6)\quad \begin{aligned} \mu S_t \tau_{(\tilde{x},\tilde{y})} \sum_{x_1=[ut]+1}^{[vt]} \sum_{x_2=-R}^{R} & \Bigg[\sum_{y_2\in\mathbb{Z}}\Bigg(\sum_{y_1\le[ut]} + \sum_{y_1\ge[vt]+1}\Bigg) \\ & + \sum_{y_2\notin\{-R,\dots,R\}} \sum_{y_1=[ut]+1}^{[vt]}\Bigg] \\ & \{g(\eta(y_1,y_2))p(x_1-y_1,x_2-y_2) \\ & -g(\eta(x_1,x_2))p(y_1-x_1,y_2-x_2)\}. \end{aligned}$$

Let

$$r(z) = \sum_{x\in\mathbb{Z}} p(x,z).$$

Since $\gamma_2 = 0$,

$$\sum_{z\in\mathbb{Z}} zr(z) = \sum_{(x,z)\in\mathbb{Z}^2} zp(x,z) = 0.$$

Let t be sufficiently large. For the first two terms in the brackets in (8.6), we change variables. Then, since the process is of finite range, we observe that there are only a finite number of terms different from 0 in the sums. Therefore, we can exchange the order of the sums and the limit. Letting $N\uparrow\infty$, we see that they converge to

$$(2R+1)\gamma_1\big[\mu_u[g(\eta(0,0))] - \mu_v g[\eta(0,0)]\big]. \tag{8.7}$$

For the last term in the brackets in (8.6), we divide it in two sums: the first with the terms $R+1\le y_2\le R+A$ and the second with the terms $-R-A\le y_2\le -R-1$. We will only consider the first one, since the second is handled in the same way. Once more, we divide the remaining sum into three other sums, the first one with the terms $[ut]+1\le x_1\le [ut]+A$, the second one with $[ut]+A+1\le x_1\le [vt]-A$ and the last one with $[vt]-A+1\le x_1\le [vt]$. The absolute values of the first and the last sums are bounded by $2\phi(\theta_2)A^2$. There remains a sum, which, after changing variables, is equal to

$$\mu S_t\tau_{(\tilde{x},\tilde{y})}\sum_{x_1=[ut]+A+1}^{[vt]-A}\ \sum_{x_2=-A+1}^{0}\ \sum_{y_2=1}^{A}\big[g(\eta(x_1,y_2+R))r(x_2-y_2)$$
$$-g(\eta(x_1,x_2+R))r(y_2-x_2)\big].$$

Let $M\in\mathbb{N}$, such that, $1/M<(\tan h_1)^+\wedge(-\tan h_2)^+$. We now use hypothesis (i) to control this sum and obtain that it is bounded above by

$$\mu S_t\tau_{(\tilde{x},\tilde{y})}\sum_{x_1=[ut]+A+1}^{[vt]-A}\ \sum_{x_2=-A+1}^{0}\ \sum_{y_2=1}^{A}\big[g(\eta(x_1+AM,R))r(x_2-y_2)$$
$$-g(\eta(x_1-AM,R))r(y_2-x_2)\big]$$

and below by

$$\mu S_t\tau_{(\tilde{x},\tilde{y})}\sum_{x_1=[ut]+A+1}^{[vt]-A}\ \sum_{x_2=-A+1}^{0}\ \sum_{y_2=1}^{A}\big[g(\eta(x_1-AM,R))r(x_2-y_2)$$
$$-g(\eta(x_1+AM,R))r(y_2-x_2)\big].$$

Then, computing the two last sums, integrating, taking the limit and using the fact that $\gamma_2=\sum_{z>0}zr(z)+\sum_{z<0}zr(z)=0$, we obtain that the absolute value of the remaining sum is less that $2A^2M\phi(\theta_2)$. Hence, the lim sup of the corresponding first term in (8.2) is at most

$$(2R+1)\gamma_1[\mu_u[g(\eta(0,0))] - \mu_v g[(\eta(0,0))]] + 8\phi(\theta_2)A^2 + 4A^2M\phi(\theta_2) \tag{8.8}$$

and the lim inf is at least

$$(2R+1)\gamma_1[\mu_u g[(\eta(0,0))] - \mu_v g[(\eta(0,0))]] - 8\phi(\theta_2)A^2 - 4A^2M\phi(\theta_2).$$

For the corresponding last two terms in (8.2), we can show that they converge to 0 using the fact that the process is of finite range and hypotheses (ii) and (iii). This in addition to (8.5), (8.7) and (8.8) proves the lemma. □

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