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POLAR AND NONPOLAR SETS FOR A TREE INDEXED PROCESS[1]

BY STEVEN N. EVANS

University of California, Berkeley

We consider a class of stochastic processes of a type that was first introduced by Dubins and Freedman. These processes are indexed by the lines of descent through an infinite tree and take values in a space of sequences. Our main results concern necessary and sufficient conditions of a potential theoretic type for a subset of the state-space to be hit with positive probability by the sample paths of the process. We examine these conditions in some specific examples and also relate them to conditions expressed in terms of Hausdorff dimension. As well, we use similar techniques to investigate multiple points in the sample paths of the process.

1. Introduction. The following type of stochastic process was introduced in Section 8 of Dubins and Freedman (1967). Given an integer $a \geq 2$, let T denote the infinite rooted a-ary tree. A *path* or *line of descent* through T is a sequence $p = (v_0, v_1, \dots)$ of vertices such that v_i is at distance $i + 1$ from the root. Now consider another integer $b \geq 2$. For each vertex v of T, suppose that we have an independent random variable Z_v which is uniformly distributed on $E \equiv \{0, 1, \dots, b - 1\}$. We construct a stochastic process X indexed by the set of paths through T and taking values in E^∞ by associating with any such path $p = (v_0, v_1, \dots)$ the random sequence $X(p) = (Z_{v_0}, Z_{v_1}, \dots)$.

The main question we address in this paper is as follows. Given a (product) measurable subset B of E^∞, when is there positive probability that $X(p) \in B$ for at least one path p? In other words, we want to know whether or not the sample paths of X 'hit' B with positive probability. In the usual nomenclature, we will say that B is *nonpolar* or *polar* according to whether or not X hits B with positive probability.

Before we state our answer, we need to introduce some more notation. Make the set E^∞ into an Abelian group by doing addition coordinatewise modulo b. For $x = (x_0, x_1, \dots)$, $x \neq 0$, define $r(x) = \inf\{i\colon x_i \neq 0\}$. If we set $|x| = b^{-r(x)}$ for $x \neq 0$ and $|0| = 0$, then $(x, y) \mapsto |x - y|$ is a metric on E^∞ which gives the usual product topology.

THEOREM 1. *Suppose that $a > b$. Then the points are nonpolar.*

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THEOREM 2. *Suppose that $a = b$. Then a measurable set B is nonpolar if and only if there is a finite, nontrivial measure μ concentrated on B such that*

$$\int\int \log\left(\frac{1}{|x-y|}\right)\mu(dx)\mu(dy) < \infty,$$

where we adopt the convention that $\log(1/0) = \infty$.

THEOREM 3. *Suppose that $a < b$. Then a measurable set B is nonpolar if and only if there is a finite, nontrivial measure μ concentrated on B such that*

$$\int\int \frac{1}{|x-y|^{\delta}}\mu(dx)\mu(dy) < \infty,$$

where $\delta = (\log b - \log a)/\log b$ and we adopt the convention that $1/0 = \infty$.

We prove these results in Section 2. As a first consequence, we note here that if B is a singleton, say $B = \{(0, 0, \dots)\}$, then B is polar if and only if $a \le b$. A little thought shows that this is just the observation that a branching process with Binomial$(a, 1/b)$ offspring distribution becomes extinct almost surely if and only if $a \le b$. We give some more interesting examples in Section 3, where we also develop the connection between Hausdorff dimension and the conditions of Theorems 2 and 3. In Section 4, we use similar techniques to study multiple points in the sample paths of X. In particular, we show that X possesses k-tuple points if and only if $(\log a/\log b) > (k-1)/k$ (see Theorem 6).

As the integrands appearing in Theorems 2 and 3 will continually reappear in what follows, we adopt a more compact notation for them. If $a = b$, set $\kappa(z) = |\log z|$, $0 < z \le 1$, and put $\kappa(0) = \infty$; while if $a < b$, set $\kappa(z) = z^{-\delta}$, $0 < z \le 1$, where $\delta = (\log b - \log a)/\log b$, and put $\kappa(0) = \infty$.

2. Proofs of Theorems 1, 2 and 3. For ease of notation we only prove Theorem 2 in the special case $a = b = 2$. It will be seen that the proof of Theorem 2 in general and the proofs of Theorems 1 and 3 will only require fairly minor modifications.

We begin with some notation. Let $G = \prod_{k=0}^{\infty}\{0, 1\}$ and $G_n = \prod_{k=0}^{n}\{0, 1\}$, each thought of as groups with coordinatewise addition modulo 2. Give G the metric described in Section 1 and write $\mathscr{G}$ for the corresponding Borel σ-field (which is also just the product σ-field). Let λ denote normalized Haar measure on $(G, \mathscr{G})$, so that λ is just the product of the uniform measure on $\{0, 1\}$. Define π_n: $G \to G_n$ by $\pi_n(g_0, g_1, \dots, g_n, \dots) = (g_0, g_1, \dots, g_n)$. With a slight abuse of notation, we will also use the same notation for the similarly defined projection of G_m onto G_n when $m \ge n$.

Observe that we can use G_n in an obvious way to label the set of vertices of the binary tree which are at distance (in the usual graph theoretic sense) $n + 1$ from the root. Similarly, we can label the set of paths through the binary tree using G. Without further comment we will construct our process X as a G-indexed, G-valued process and continue to think of it that way.

Set $\Gamma = \bigcup_{k=0}^{\infty} G_k$ and $\Gamma_n = \bigcup_{k=0}^{n} G_k$. Put $\Omega = \prod_\Gamma \{0, 1\}$. For $\gamma \in \Gamma$, let ρ_γ: $\Omega \to \{0, 1\}$ denote the corresponding coordinate projection. Let $\mathscr{F}^0$ denote the product σ-field on Ω and let $\mathbf{P}^0$ denote the probability measure which is the product of the uniform measure on $\{0, 1\}$. Let $(\Omega, \mathscr{F}, \mathbf{P})$ be the completion of $(\Omega, \mathscr{F}^0, \mathbf{P}^0)$. For $(\omega, t) \in \Omega \times G$, define

$$X(\omega, t) = (\omega_{\pi_0(t)}, \omega_{\pi_1(t)}, \dots).$$

Similarly, for $(\omega, t) \in \Omega \times G_n$, define

$$X_n(\omega, t) = (\omega_{\pi_0(t)}, \omega_{\pi_1(t)}, \dots, \omega_{\pi_n(t)}).$$

Observe that X is a continuous process [in fact, $|X(s) - X(t)| \le |s - t|$ for all $s, t \in G$]. Note also that if we define measure-preserving bijections σ_t: $\Omega \to \Omega$ for $t \in G$ by $(\rho_\gamma \circ \sigma_t)(\omega) = \rho_{\pi_n(t)+\gamma}(\omega)$ when $\gamma \in G_n$, then $X(\sigma_t(\omega), s) = X(\omega, s + t)$. In particular, X is stationary.

We will first show the necessity of the stated condition. In doing so, we use a technique similar to that developed in Fitzsimmons and Salisbury (1989) for multiparameter Markov processes. We could follow Fitzsimmons and Salisbury quite closely and begin by showing that if $B \in \mathscr{G}$ is such that $\mathbf{P}(\exists\ t \in G$: $X(t) \in B) > 0$, then there exists a nontrivial homogeneous random measure concentrated on $\{t \in G$: $X(t) \in B\}$ which, by application of a suitable central projection operation, can be made adapted in some sense. Moreover, this adapted random measure has finite probabilistic energy. Finally, we could prove that the Revuz measure associated with this adapted random measure satisfies the finite analytic energy condition of Theorem 2. Such a course would, however, involve us in several unnecessary technicalities. Essentially, our approach involves carrying out an analogous program with X replaced by X_n and B replaced by $\pi_n B$, so that everything takes place in a much simpler discrete setting. Our constructions are suitably nested so that if μ_n denotes the Revuz measure of the adapted homogeneous random measure appearing at the nth stage, then $\mu_n = \mu \circ \pi_n^{-1}$ for some measure μ on G which is easily shown to satisfy the conditions of Theorem 2.

Suppose, then, that $B \in \mathscr{G}$ is such that $\mathbf{P}(\exists\ t \in G$: $X(t) \in B) > 0$. A Choquet capacity argument shows that there is a compact subset of B which is also nonpolar [cf. Lemma I.10.12 and Theorem I.10.6 of Blumenthal and Getoor (1968)], so we may suppose without loss of generality that B is compact.

Let $G^\Delta = G \cup \{\Delta\}$, where Δ is an abstract isolated point; and let $\mathscr{G}^\Delta$ be the corresponding Borel σ-field. It is easy to see that there is a $\mathscr{F}/\mathscr{G}^\Delta$-measurable map S: $\Omega \to G^\Delta$ such that

$$(\omega, S(\omega)) \in \{(\omega', t): X(\omega', t) \in B\} \Leftrightarrow X(\omega, G) \cap B \neq \emptyset$$

and

$$S(\omega) = \Delta \Leftrightarrow X(\omega, G) \cap B = \emptyset$$

(one can either appeal to a general cross-section theorem or give a simple direct construction using the fact that B is compact and X is continuous). Given $\omega \in \Omega$ and $A \in \mathscr{G}$, define $L(\omega, A) = \int_G \mathbf{1}_A((S \circ \sigma_t)(\omega) + t)\lambda(dt)$ for ω such that $X(\omega, G) \cap B \neq \emptyset$ and put $L(\omega, A) = 0$ otherwise. It is easy to check that L is a random measure with the following properties:

$$\mathbf{P}(L(G)) = \mathbf{P}(\exists\, t \in G\colon X(t) \in B), \tag{2.1}$$

$$L(\{t \in G\colon X(t) \notin B\}) = 0, \tag{2.2}$$

$$L(\sigma_t \omega, A) = L(\omega, t + A), \tag{2.3}$$

$$L(G) \le 1. \tag{2.4}$$

Fix an integer $n \ge 0$. For $t \in G_n$, let $\mathscr{F}_t^n = \sigma\{X_n(t)\} = \sigma\{\rho_{\pi_0(t)}, \dots, \rho_{\pi_n(t)}\}$. It is clear from (2.3) that there is a function $\mu_n\colon G_n \to [0, \infty[$ such that $\mathbf{P}(L(\pi_n^{-1}t) | \mathscr{F}_t^n) = \mu_n(X_n(t))$ for all $t \in G_n$.

Lemma 1. *The following hold*:

(i) $\sum_{x \in G_n} \mu_n(x) = \mathbf{P}(\exists\, t \in G\colon X(t) \in B)$.
(ii) $\sum_{x \notin \pi_n B} \mu_n(x) = 0$.
(iii) $\mathbf{P}([\sum_{t \in G_n} \mu_n(X_n(t))]^2) \le 16$.

Proof. Observe that

$$\begin{aligned} \mu_n(x) &= 2^{n+1}\mathbf{P}(X_n(0) = x)\mu_n(x) \\ &= \mathbf{P}\Bigg(\sum_{t \in G_n} \mathbf{1}(X_n(t) = x)\mu_n(X_n(t))\Bigg) \\ &= \mathbf{P}\Bigg(\sum_{t \in G_n} \mathbf{1}(X_n(t) = x)L(\pi_n^{-1}(t))\Bigg) \\ &= \mathbf{P}\bigg(\int_G \mathbf{1}(\pi_n \circ X(t) = x)L(dt)\bigg). \end{aligned} \tag{2.5}$$

(i) This is clear from (2.1) and (2.5).
(ii) From (2.5), we have

$$\begin{aligned} \sum_{x \notin \pi_n B} \mu_n(x) &= \mathbf{P}\bigg(\int_G \mathbf{1}(\pi_n \circ X(t) \notin \pi_n B)L(dt)\bigg) \\ &\le \mathbf{P}\bigg(\int_G \mathbf{1}(X(t) \notin B)L(dt)\bigg) \\ &= 0, \end{aligned}$$

by (2.2).

(iii) Define the usual lexicographic total order on G_n by declaring that $(\alpha_0, \dots, \alpha_n)$ is less than $(\beta_0, \dots, \beta_n)$ if and only if for some $0 \le m \le n$, we have $\alpha_i = \beta_i$ for $i < m$ and $\alpha_m = 0$, $\beta_m = 1$. Let $s(0), \dots, s(2^{n+1} - 1)$ be the listing of G_n in this order. So, for example, $s(0) = (0, \dots, 0)$, $s(1) = (0, \dots, 0, 1)$

and $s(2^{n+1}-1)=(1,\ldots,1)$. Set $\mathscr{H}_k=\mathscr{F}^n_{s(0)}\vee\cdots\vee\mathscr{F}^n_{s(k)}$ and $\mathscr{I}_k=\mathscr{F}^n_{s(k)}\vee\cdots\vee\mathscr{F}_{s(2^{n+1}-1)}$. Observe that if $0\le j<k<l\le 2^{n+1}-1$ and $0\le m\le n$, then

$$\pi_m s(j)=\pi_m s(l)\Rightarrow \pi_m s(j)=\pi_m s(k)=\pi_m s(l),$$

and hence

$$\left[\bigcup_{i=0}^{k}\bigcup_{m=0}^{n}\{\pi_m s(i)\}\right]\cap\left[\bigcup_{i=k}^{2^{n+1}-1}\bigcup_{m=0}^{n}\{\pi_m s(i)\}\right]=\bigcup_{m=0}^{n}\{\pi_m s(k)\}.$$

Thus, for each k there are σ-fields $\mathscr{E}_k$, $\mathscr{H}'_k$, $\mathscr{I}'_k$ such that $\mathscr{E}_k$, $\mathscr{F}^n_{s(k)}$, $\mathscr{H}'_k$, $\mathscr{I}'_k$ are independent, $\mathscr{F}=\mathscr{E}_k\vee\mathscr{F}^n_{s(k)}\vee\mathscr{H}'_k\vee\mathscr{I}'_k$, $\mathscr{H}_k=\mathscr{F}^n_{s(k)}\vee\mathscr{H}'_k$ and $\mathscr{I}_k=\mathscr{F}^n_{s(k)}\vee\mathscr{I}'_k$. From Lemma A.1 of the Appendix, we have $\mathbf{P}(L(\pi_n^{-1}s(k))|\mathscr{F}^n_{s(k)})=\mathbf{P}(\mathbf{P}(L(\pi_n^{-1}s(k))|\mathscr{I}_k)|\mathscr{H}_k)$ and the result follows from (2.4) and two applications of Lemma A.2 of the Appendix. □

Note from (2.5) that if $x=(x_0,\ldots,x_n)\in G_n$, then $\mu_n(x)=\mu_{n+1}(x_0,\ldots,x_n,0)+\mu_{n+1}(x_0,\ldots,x_n,1)$. An application of Kolmogorov's existence theorem gives that there is a measure μ on G such that $\mu_n(x)=\mu\circ\pi_n^{-1}(\{x\})$ for all n. It follows from parts (i) and (ii) of Lemma 1 that

$$\begin{aligned}\mu(B)&=\lim_{n\to\infty}\mu\left(\pi_n^{-1}\pi_n B\right)=\lim_{n\to\infty}\sum_{x\in\pi_n B}\mu_n(x)=\lim_{n\to\infty}\sum_{x\in G_n}\mu_n(x)\\&=\mu(G)=\mathbf{P}(\exists\, t\in G\colon X(t)\in B)>0\end{aligned}$$

[recall we are assuming that B is compact, so $\bigcap_{n=0}^{\infty}(\pi_n^{-1}\pi_n B)=\bigcap_{n=0}^{\infty}\{x\in G\colon \inf_{y\in B}|x-y|<2^{-n}\}=B$].

Moreover,

$$\begin{aligned}\mathbf{P}&\left(\left[\sum_{t\in G_n}\mu_n(X_n(t))\right]^2\right)\\&=\sum_{s\in G_n}\sum_{t\in G_n}\sum_{x\in G_n}\sum_{y\in G_n}\mu_n(x)\mu_n(y)\mathbf{P}(X_n(s)=x,X_n(t)=y)\\&=\sum_{x\in G_n}\sum_{y\in G_n}2^{n+1}\sum_{t\in G_n}\mathbf{P}(X_n(0)=x,X_n(t)=y)\mu_n(x)\mu_n(y)\\&=2^{n+1}\Bigg[\sum_x\mathbf{P}(X_n(0)=x)\mu\left(\pi_n^{-1}x\right)^2\\&\qquad+\sum_x\sum_{t\neq 0}\mathbf{P}(X_n(0)=x,X_n(t)=x)\mu\left(\pi_n^{-1}x\right)^2\\&\qquad+\sum_x\sum_{y\neq x}\sum_{t\neq 0}\mathbf{P}(X_n(0)=x,X_n(t)=y)\mu\left(\pi_n^{-1}x\right)\mu\left(\pi_n^{-1}y\right)\Bigg].\end{aligned}$$

Observe the following:

$$\mathbf{P}(X_n(0) = x) = 2^{-(n+1)},$$

$$\mathbf{P}(X_n(0) = x, X_n(t) = x) = 2^{m-2(n+1)}, \qquad t \neq 0, \qquad |\pi_n^{-1}t| = 2^{-m},$$

and

$$\mathbf{P}(X_n(0) = x, X_n(t) = y) = 2^{m-2(n+1)}\mathbf{1}(|\pi_n^{-1}(x-y)| \leq 2^{-m}),$$
$$x \neq y, \qquad t \neq 0, \qquad |\pi_n^{-1}t| = 2^{-m}$$

(note that if $u \in G_n \setminus \{0\}$, then $\{|v|: v \in \pi_n^{-1}u\}$ is a singleton and it makes sense to think of $|\pi_n^{-1}u|$ as a single number). Thus, after a straightforward computation,

$$\mathbf{P}\left(\left[\sum_{t \in G_n} \mu_n(X_n(t))\right]^2\right) = \sum_{x \in G_n} \mu(\pi_n^{-1}x)^2 + \frac{1}{2}\int_G\int_G\left(\log_2\left(\frac{1}{|x-y|}\right) + 1\right)$$
$$\wedge (n+1)\mu(dx)\mu(dy);$$

and we see from part (iii) of Lemma 1 that

$$\int\int \log\left(\frac{1}{|x-y|}\right)\mu(dx)\mu(dy) < \infty,$$

as required to complete the proof that the conditions of Theorem 2 are necessary for B to be nonpolar.

Suppose now that μ is a measure satisfying conditions of Theorem 2. Then the trace of μ on some compact subset of B also satisfies these conditions and so, without loss of generality, we may assume that B is compact. Put $\mu_n(x) = \mu \circ \pi_n^{-1}(\{x\})$ for $x \in G_n$ and set $M_n = \sum_{t \in G_n} \mu_n(X_n(t))$. From the calculations of the previous paragraph, we see that $\mathbf{P}(M_n) = \mu(G) > 0$ and $\limsup_{n \to \infty} \mathbf{P}(M_n^2) < \infty$. Thus, by the Cauchy–Schwarz inequality,

$$\begin{aligned}\mathbf{P}(\exists\, t \in G: X(t) \in B) &= \lim_{n \to \infty} \mathbf{P}(\exists\, t \in G_n: X_n(t) \in \pi_n B) \\ &\geq \liminf_{n \to \infty} \mathbf{P}(M_n > 0) \\ &\geq \liminf_{n \to \infty} \frac{[\mathbf{P}(M_n)]^2}{\mathbf{P}(M_n^2)} > 0,\end{aligned}$$

and so the conditions of Theorem 2 are also sufficient for B to be nonpolar. □

3. Some examples.

Proposition 1. *Suppose that $B = \prod_{k=0}^{\infty} B_k \neq \varnothing$, where $B_k \subset E$ has b_k elements. Then B is nonpolar if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{b}{a}\right)^n \prod_{k=0}^{n-1} \frac{1}{b_k} < \infty.$$

PROOF. We see from Theorem 1 that the result is certainly true when $a < b$, so we suppose that $a \le b$. There is an isometry of E^∞ which maps B to $\prod_{k=0}^{\infty}\{0, 1, \dots, b_k - 1\}$ and it is clear from inspection of the necessary and sufficient conditions of Theorems 2 and 3 that we may further suppose without loss of generality that $B_k = \{0, 1, \dots, b_k - 1\}$. Make B into an Abelian group with the addition operation $\oplus$ defined by performing addition in the kth coordinate modulo b_k. Let $\ominus$ be the corresponding subtraction operation. Note that $|x - y| = |x \ominus y|$ for $x, y \in B$. Let ν denote the normalized Haar measure on B.

Given two finite measures μ_1 and μ_2, define $\langle \mu_1, \mu_2 \rangle = \int\int \kappa(|x - y|)\mu_1(dx)\mu_2(dy)$. Observe that if μ_1 and μ_2 are both supported on B, then $\langle \mu_1, \mu_2 \rangle = \int\int \kappa(|x \ominus y|)\mu_1(dx)\mu_2(dy)$. Also, it is not too difficult to show that $\langle \mu_1, \mu_2 \rangle \le \langle \mu_1, \mu_1 \rangle^{1/2} \langle \mu_2, \mu_2 \rangle^{1/2}$ [cf. Section 3 of Evans (1988)]. If μ is a finite measure on B and $w \in B$, define the measure μ_w, which is also supported on B, by $\mu_w(C) = \mu(C \ominus w)$ and note that $\langle \mu_w, \mu_w \rangle = \langle \mu, \mu \rangle$. From the translation invariance of ν and Fubini's theorem we have

$$\begin{aligned} \mu(B)^2 \langle \nu, \nu \rangle &= \int\int \langle \nu_x, \nu_y \rangle \mu(dx)\mu(dy) \\ &= \int\int \langle \mu_x, \mu_y \rangle \nu(dx)\mu(dy) \\ &\le \int\int \langle \mu_x, \mu_x \rangle^{1/2} \langle \mu_y, \mu_y \rangle^{1/2} \nu(dx)\nu(dy) \\ &= \langle \mu, \mu \rangle. \end{aligned}$$

Applying Theorems 2 and 3, we see that B will be nonpolar if and only if $\langle \nu, \nu \rangle < \infty$. If we further observe that $\langle \nu, \nu \rangle = \int \kappa(|x|)\nu(dx)$ and $\nu(\{x\colon |x| \le b^{-n}\}) = (b_0 b_1 \cdots b_{n-1})^{-1}$, then the result follows from a straightforward summation by parts. □

PROPOSITION 2. *Suppose that $a = b$. Suppose that ξ is a probability measure on E that is not concentrated on a single point and ξ^∞ is the corresponding product measure on E^∞. If $B \subset E^\infty$ is a measurable set such that $\xi^\infty(B) > 0$, then B is nonpolar.*

PROOF. Let μ be the trace of ξ^∞ on B. Note that there exists a constant $c < 1$ such that $\sup_x \mu(\{y\colon |x - y| \le b^{-n}\}) \le c^n$, and hence

$$\int\int \log\left(\frac{1}{|x - y|}\right)\mu(dx)\mu(dy) \le \sup_x \int \log\left(\frac{1}{|x - y|}\right)\mu(dy) < \infty.$$

The result now follows from Theorem 2. □

Before we can state our next result, we need to recall some details regarding Hausdorff dimension [for a complete background see, e.g., Rogers (1970)].

Given $\alpha, \varepsilon > 0$ and a measurable set $B \subset E^\infty$, set

$$\Lambda_{\alpha,\varepsilon}(B) = \inf\left\{\sum_i \operatorname{diam}(G_i)^\alpha\right\},$$

where the infimum is over all countable collections $(G_i)_{i=0}^\infty$ of balls with diameter at most ε (in the metric $|\cdot - \cdot|$) that cover B. Put

$$\Lambda_\alpha(B) = \sup_{\varepsilon>0} \Lambda_{\alpha,\varepsilon}(B).$$

Then Λ_α is a measure on E^∞. The *Hausdorff dimension* of B is defined as

$$\dim B = \inf\{\alpha > 0 \colon \Lambda_\alpha(B) = 0\}.$$

It is easy to see that Λ_1 is just Haar measure on E^∞ and so, in particular, E^∞ has Hausdorff dimension 1.

By analogy with the Euclidean theory [see, e.g., Chapter 10 of Kahane (1985)], we also define the *capacity dimension* of a nonempty measurable set B as the supremum of the set of real numbers β such that it is possible to find a finite, nontrivial measure μ concentrated on B for which $\int\int |x-y|^{-\beta} \mu(dx)\mu(dy) < \infty$. An argument almost identical to the proof of Theorem (2.3) in Evans (1988) shows that the analogue of Frostman's theorem holds and these notions of dimension coincide [the proof in Evans (1988) is only for compact sets, but each of these methods for assigning a dimension to a measurable set has the property that the dimension of a set is the supremum of the dimensions of its compact subsets].

Proposition 3. *Suppose that* $B \neq \varnothing$ *is a measurable set. If* $\dim B > (\log b - \log a)/\log b$, *then* B *is nonpolar, whereas if* $\dim B < (\log b - \log a)/\log b$, *then* B *is polar.*

Proof. From Theorem 1, the result is trivial when $a > b$. When $a < b$, the result follows from Theorem 3 and the above observation that Hausdorff dimension and capacity dimension coincide. When $a = b$, the result follows from Theorem 2, the above observation and the fact that for all $\tau > 0$, we have $|\log z| \le z^{-\tau}$, $0 < z \le 1$. □

Given Proposition 3, one might conjecture that the Hausdorff dimension of $X(G)$ is $(\log a/\log b) \wedge 1$ almost surely. We remark without proof that, using standard techniques, it is not too difficult to show that this is indeed the case.

With Proposition 3 in hand we could give a large number of examples. This is especially so once we observe the fairly elementary fact that if for each measurable set $B \subset E^\infty$, we define $B^* \subset [0,1]$ by

$$B^* = \left\{\sum_{k=0}^{\infty} x_k b^{-(k+1)} \colon x = (x_0, x_1, \dots) \in B\right\},$$

then $\dim B = \dim B^*$ and hence the Hausdorff dimensions of many interesting sets may be found in the literature. However, we restrict ourselves to the following.

PROPOSITION 4. *Suppose that* $0 < \eta < (b-1)/2$. *Define*

$$B_{-} = \left\{ x = (x_0, x_1, \dots) \colon \liminf_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} x_k = \eta \right\},$$

$$B_0 = \left\{ x = (x_0, x_1, \dots) \colon \lim_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} x_k = \eta \right\}$$

and

$$B_{+} = \left\{ x = (x_0, x_1, \dots) \colon \limsup_{n \to \infty} n^{-1} \sum_{k=0}^{n-1} = \eta \right\};$$

and define C_-, C_0, C_+ *similarly, but with the* $=$ *sign replaced by the* $\le$ *sign. Let r be the unique root between* 0 *and* 1 *of the equation*

$$\sum_{j=1}^{b-1} r^j (j - \eta) = \eta,$$

and set

$$R = r^{-\eta} \sum_{i=0}^{b-1} r^i.$$

If $(b/a) < R$, *then* B_- *(respectively,* B_0, B_+, C_-, C_0, C_+*) is nonpolar, whereas if* $(b/a) > R$, *then* B_- *(respectively,* B_0, B_+, C_-, C_0, C_+*) is polar.*

PROOF. Given Proposition 3 and the above observation concerning the computation of Hausdorff dimension, the result follows from Theorem 14 of Eggleston (1951). □

Questions similar to those addressed by Proposition 4 were considered in Lemma 8.8 of Dubins and Freedman (1967) for the case when $a = b = 2$. In this setting, they consider a construction more general than that of X, in which the common distribution of the independent random variables (Z_v) need not be uniform on $\{0, 1\}$. They obtain a sufficient condition for C_+ to be polar. When the common distribution of the (Z_v) is uniform, their general condition becomes $\eta \log \eta + (1 - \eta)\log(1 - \eta) > 0$, which, of course, does not hold for any $0 < \eta < \frac{1}{2}$. This agrees with Proposition 4, which states that when $a = b = 2$, any of the sets considered in the statement of the proposition will be nonpolar.

4. Multiple points. Given an integer $k > 1$, we say that X has a k-tuple point if there exists $x \in E^\infty$ such that $\operatorname{card}(X^{-1})(x) \ge k$. As in the case of Lévy processes, the study of multiple points can be reduced to a study of a multiparameter process formed from independent copies of X [see Evans

(1987) or Fitzsimmons and Salisbury (1989) for examples of this technique for Lévy processes]. With this in mind, we state without proof the following result which can be proved by exactly the same means as Theorems 2 and 3.

THEOREM 4. *Suppose that* $a \le b$. *Define a* T^k*-indexed stochastic process* $\vec{X}$ *by setting*

$$\vec{X}(p) = \left(X^1(p^1), \dots, X^k(p^k)\right), \qquad p = (p^1, \dots, p^k) \in T^k,$$

where $X^1, \dots, X^k$ *are independent copies of* X. *Let* B *be a measurable subset of* $(E^\infty)^k$. *Then* B *is hit by* $\vec{X}$ *with positive probability if and only if there exists a finite, nontrivial measure* μ *concentrated on* B *such that*

$$\int\int \kappa(|x^1 - y^1|) \cdots \kappa(|x^k - y^k|)\mu(dx)\mu(dy) < \infty.$$

THEOREM 5. *Let* B *be a nonempty measurable subset of* E^∞. *If* $a > b$, *then the probability that there exists* $x \in B$ *such that* $\operatorname{card}(X^{-1})(x) \ge k$ *is positive. If* $a \le b$, *this probability is either positive or zero depending on whether there exists a finite, nontrivial measure* μ *concentrated on* B *such that*

$$\int\int \kappa(|x - y|)^k \mu(dx)\mu(dy) < \infty.$$

PROOF. For simplicity, we will carry out the proof for the case $a = b = 2$ and work with the notation established in Section 2. The proof in the general case requires only minor modifications.

For $n \ge 0$, define a mapping $\theta_n\colon G \to G$ by setting $\theta_n(g_0, g_1, \dots) = (g_{n+1}, g_{n+2}, \dots)$. Also, let $s_n(0), \dots, s_n(2^{n+1} - 1)$ be any listing of the elements of G_n. It is clear from the construction of X that the probability in question will be positive if and only if for some n with $2^{n+1} - 1 \ge k$, we have that

$$\left[\prod_{j=0}^{k-1} \theta_n \circ X\left(\pi_n^{-1} s_n(j)\right)\right] \cap \tilde{B}_n \ne \varnothing$$

with positive probability, where

$$\tilde{B}_n = \{(x, x, \dots, x) \colon x \in \theta_n B\}.$$

Applying Theorem 4, this is equivalent to the existence of a nontrivial finite measure μ_n concentrated on $\theta_n B$ for which $\int\int \kappa(|x - y|)^k \mu_n(dx)\mu_n(dy) < \infty$. This last condition, however, is readily seen to be equivalent to the one appearing in the statement of the result. □

THEOREM 6. *If* $(\log a/\log b) > (k - 1)/k$, *then almost surely* X *has* k*-tuple points. Otherwise,* X *almost surely does not have* k*-tuple points.*

PROOF. If we apply Theorem 5 with E^∞, then an argument similar to the one given in the proof of Proposition 1 shows that X has k-tuple points with positive probability if and only if $a > b$ or $a \le b$ and $\int\int \kappa(|x-y|)^k \nu(dx)\nu(dy) < \infty$, where ν is Haar measure on E^∞. An easy calculation shows that the latter condition holds if and only if $(\log a/\log b) > (k-1)/k$.

It therefore remains to show that if X has k-tuple points with positive probability, then X has k-tuple points almost surely. Once again we only carry out the proof in the case when $a = b = 2$. Obviously, X has k-tuple points if for all n, the restriction of X to the set $\pi_n^{-1}(0,\dots,0)$ has k-tuple points. In fact, from the construction of X, it can be seen that the probabilities of these two events are equal. Since the latter event is a tail event for the family of independent random variables $\{\rho_\gamma\colon \gamma \in \Gamma\}$, the result follows from Kolmogorov's zero-one law. □

APPENDIX

The following result follows from a straightforward monotone class argument.

LEMMA A.1. *Let $(\Sigma, \mathscr{A}, \mathbf{Q})$ be a probability space. Suppose that we have $\mathscr{A} = \mathscr{A}_1 \vee \mathscr{A}_2 \vee \mathscr{A}_3 \vee \mathscr{A}_4$, where $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3, \mathscr{A}_4$ are independent sub-σ-fields. For a $\mathbf{Q}$-integrable random variable Y, we have*

$$\mathbf{Q}(Y|\mathscr{A}_1) = \mathbf{Q}(\mathbf{Q}(Y|\mathscr{A}_1 \vee \mathscr{A}_2)|\mathscr{A}_1 \vee \mathscr{A}_3).$$

The next result, which we prove for the sake of completeness, is just a discrete time version of Meyer's energy inequality.

LEMMA A.2. *Let $(\Sigma, \mathscr{A}, \mathbf{Q})$ be a probability space. Suppose that $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \cdots \subset \mathscr{A}_n$ are sub-σ-fields of $\mathscr{A}$ and $Y_1, Y_2, \dots, Y_n$ are nonnegative, square-integrable random variables. Then*

$$\mathbf{Q}\left(\left[\sum_k \mathbf{Q}(Y_k|\mathscr{A}_k)\right]^2\right) \le 4\mathbf{Q}\left(\left[\sum_k Y_k\right]^2\right).$$

PROOF. We have

$$\begin{aligned}\mathbf{Q}\left(\left[\sum_k \mathbf{Q}(Y_k|\mathscr{A}_k)\right]^2\right) &\le 2\sum_k \sum_{l\ge k} \mathbf{Q}(\mathbf{Q}(Y_k|\mathscr{A}_k)\mathbf{Q}(Y_l|\mathscr{A}_l)) \\ &= 2\sum_k \sum_{l \ge k} \mathbf{Q}(\mathbf{Q}(Y_k|\mathscr{A}_k)Y_l) \\ &\le 2\mathbf{Q}\left(\sum_k \mathbf{Q}(Y_k|\mathscr{A}_k)\sum_l Y_l\right) \\ &\le 2\mathbf{Q}\left(\left[\sum_k \mathbf{Q}(Y_k|\mathscr{A}_k)\right]^2\right)^{1/2} \mathbf{Q}\left(\left[\sum_l Y_l\right]^2\right)^{1/2},\end{aligned}$$

where the last line follows from the Cauchy–Schwarz inequality. □

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DEPARTMENT OF STATISTICS
UNIVERSITY OF CALIFORNIA
367 EVANS HALL
BERKELEY, CALIFORNIA 94720