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LOCALIZATION AND SELECTION IN A MEAN FIELD BRANCHING RANDOM WALK IN A RANDOM ENVIRONMENT

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We consider a continuous time branching random walk on the finite set $\{1, 2, \dots, N\}$ with totally symmetric diffusion jumps and some site-dependent i.i.d. random birth rates which are unbounded. We study this process as the time t and the space size N tend to infinity simultaneously. In the classical law of large numbers setup for spatial branching models, the growth of the population obeys an exponential limit law due to the localization of the overwhelming portion of particles in the record point of the medium. This phenomenon is analyzed further: The historical path (in space) of a typical particle picked at time t (selection) is of a rather simple and special nature and becomes in the limit singular (in distribution) to the path of the underlying mean field random walk. In general, the properties of the typical path depend on the relation in which t and N tend to infinity.

1. Introduction.

1.*A*. *Motivation and background*. In the last years quite some work has been done studying *infinite particle systems* evolving in *random media* (*environments*). Many new phenomena have been discovered. Among them are new types of phase transitions: see Bramson, Durrett and Schonmann (1991), Greven (1985, 1986), Greven and den Hollander (1992), Baillon, Clemént, Greven and den Hollander (1990); or changes in critical dimensions: see Dawson and Fleischmann (1983, 1985); or changes in transport properties of particle systems: see Greven (1990); or new clustering phenomena: see Ferreira (1988), Dawson and Fleischmann (1991); or changes in the structure of the set of extremal invariant measures: see Liggett (1991).

For other models it is shown that results hold comparable with classical ones; see, for instance, Dawson, Fleischmann and Gorostiza (1989).

If we run a particle system of the branching type in a random medium, we may observe that the particles start *clumping* and are as a rule *located in a very small subset of the space*. This phenomenon of *localization* of the branching system caused by the random medium involving, in particular, *unbounded* birth rates was studied on the level of the *mean value* equation

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(concerning the *expected* number of particles) in Gärtner and Molchanov (1990) and in Fleischmann and Molchanov (1990). In the latter case even a *total localization* (of the normalized mean of the population) in the *record point* of the random medium was exhibited.

In Greven and den Hollander (1992) it was found that in a branching process with *bounded* random birth rates where the underlying walk has a sufficiently small *drift*, a particle present at time t originates from a particle at time 0, which is only within distance $o(t)$ (localization). Furthermore, the *typical path* of a particle present at time t is of a very special type. Only particles whose ancestors traveled in a specific way through the medium will have a substantial number of descendants at time t. Here the term specific refers to properties of the process of local times created by the typical path; see also Theorem 3 in Baillon, Clemént, Greven and den Hollander (1990).

The fact that the typical path is of a very special nature reflects a *selection mechanism* corresponding to a *survival of the fittest principle*. Only particles whose descendants stay at the good places have the chance of making up a substantial portion of the population as time tends to infinity.

Both phenomena, localization and selection, seem to be new in the mathematical literature. They occur in models with a (randomly) spatially varying evolution mechanism and are not present in the classical spatially homogeneous models.

The purpose of the present paper is to contribute first of all to a better understanding of localization and selection and its impact on the normalized population growth, and second, to clarify the role of the *unboundedness* of the random birth rates.

Ideally, one would like to treat models not necessarily of a *mean field character*. But as a first step it seems to be reasonable to exploit this simplification, to keep the arguments transparent. Compare also with the role of the Curie–Weiss model in statistical mechanics; see, for instance, Ellis (1985).

1.*B*. *The model*. Here we shall introduce the model in an informal way; for a rigorous approach, see sections 2.A and 2.B below.

For a given (large) natural number $N \geq 1$, let particles move and split in the *finite* space $I_N := \{1, 2, \dots, N\}$ according to the following rules.

Each particle may jump with rate $\kappa \geq 0$ to a site $z \in I_N$ chosen at random, that is, according to the uniform distribution on I_N (*mean field random walk, totally symmetric random walk*). In particular, the new position is chosen independently from the present one. Additionally, each particle situated at $y \in I_N$ is replaced with rate $\xi_N(y) \geq 0$ by two particles (binary fission) both located at the same site y.

This means that the birth rate $\xi_N(y)$ possibly changes from site to site (*varying medium*). Moreover, these birth rates will be realized from independent identically distributed (i.i.d.) random variables $\xi_N(1), \dots, \xi_N(N)$ (*random medium*). Here we will focus on the case of exponentially distributed variables. (For a discussion of this assumption, see Section 1.D below.)

We would like to stress that always first the random medium $\xi_N := [\xi_N(1), \dots, \xi_N(N)]$ is realized and then given ξ_N we define, via the evolution rules described above, a continuous-time Markov branching process $\{\Phi_N^\psi(t, \cdot \,|\xi_N); \, t \geq 0\}$ with initial state ψ. Here $\Phi_N^\psi(t, y|\xi_N)$ denotes the number of particles at time t at site y.

This process $\Phi_N^\psi(\cdot\,|\xi_N)$ we call *mean field branching random walk in the random medium* ξ_N. We mention that for simplicity of notation we will often drop the dependence on ξ_N, for instance, we write Φ_N^ψ instead of $\Phi_N^\psi(\cdot\,|\xi_N)$, if no confusion is possible. Also, we write Φ_N^y if the process starts with exactly one particle situated at y.

1.*C*. *Results*. The paper is devoted to the following four questions. What is the approximate size of the total population $\Sigma_{z=1}^N \Phi_N^\psi(t, z|\xi_N) := \Phi_N^\psi(t, I_N|\xi_N)$ at time t? At which points will the main portion of the total population be located? From which initial particles will this population mainly be created? What path will a typical particle picked at time t have followed during the time interval $[0, t]$ in space? To answer those questions, we let t and N both tend to infinity (rescaling of time and space), formally simply expressed by $t, N \to \infty$.

(I) EXPONENTIAL LIMIT LAW; LOCALIZATION. If we only look for the expected number of particles (given ξ_N), then we can use results from Fleischmann and Molchanov (1990). There the exact asymptotics of these expectations as $t, N \to \infty$ were obtained. Moreover, it was shown that there is a *complete localization in the record point* $y_{1,N}$ of the random medium, that is, at that site y where $\xi_N(y)$ takes its maximum.

In this paper we shall try to investigate in more detail the meaning and extent of this localization phenomenon. If we assume for the moment that $\xi_N(y)$ does not depend on y (constant medium), then it can be shown that $\Phi_N^{\mathbf{1}}(t, I_N)/\mathbb{E}\{\Phi_N^{\mathbf{1}}(t, I_N)\}$ converges stochastically to 1 as $t, N \to \infty$, that is, to its expectation (law of large numbers). Here $\mathbf{1}(y) \equiv 1$ refers to the population which has exactly one particle at each site. Such a law of large numbers (LLN) is expected to remain valid also for a *bounded* random medium if we replace the expectation by expectation conditioned with respect to the medium; compare the LLN in Greven (1986).

One effect of localization in our model with *unboundedly varying* birth rates is now that this law of large numbers setup leads to an *exponential limit law* which is similar to the situation in a classical (i.e., without spatial motion) supercritical branching process, namely for the Yule process (pure birth process).

Recall that $y_{1,N}$ denotes the record point of the medium ξ_N and that $t, N \to \infty$ means that both t and N tend to infinity. Let $\to_{\mathscr{D}}$ denote convergence in distribution and let W be an *exponentially* distributed random variable with mean 1.

THEOREM 1 (Exponential limit law; localization). *For almost all* ξ *we have*

$$\Phi_N^{\mathbf{1}}(t, I_N|\xi_N)/\mathbb{E}\{\Phi_N^{\mathbf{1}}(t, I_N)|\xi_N\} \xrightarrow[t, N\to\infty]{\mathscr{D}} W, \tag{1.1}$$

$$\Phi_N^{\mathbf{1}}(t, y_{1,N}|\xi_N)/\mathbb{E}\{\Phi_N^{\mathbf{1}}(t, I_N)|\xi_N\} \xrightarrow[t, N\to\infty]{\mathscr{D}} W, \tag{1.2}$$

$$\Phi_N^{y_{1,N}}(t, I_N|\xi_N)/\mathbb{E}\{\Phi_N^{\mathbf{1}}(t, I_N)|\xi_N\} \xrightarrow[t, N\to\infty]{\mathscr{D}} W. \tag{1.3}$$

Consequently, the complete localization of the means mentioned above implies the corresponding effects for the particle system itself: The system at time t essentially consists of particles situated at the record point $y_{1,N}$ and originated from the particle also placed at $y_{1,N}$.

REMARK 1.4. Note that $\mathbb{E}\{\Phi_N^{\mathbf{1}}(t, I_N)|\xi_N\}$ grows *superexponentially* (under t, $N \to \infty$) due to the (a.s) unboundedness of the birth rates $\xi_N(y)$ (as $N \to \infty$); see Fleischmann and Molchanov (1990), Remark 5.14. Compare this also with the bounded case in Greven and den Hollander (1992). The position of $y_{1,N}$ in I_N where the localization takes place is of course uniformly distributed.

For some *speed regions*, that is, relations in which t and N tend to infinity, we will determine, which historical path a typical particle taken from $\Phi_N^{\mathbf{1}}(t,\cdot|\xi_N)$ will follow in space. Before describing it, we will first point out heuristically how this is related to the occurrence of the exponentially distributed variable W in Theorem 1 (instead of a nonrandom limit 1).

We start with the simplest relation in which t and N tend to infinity, namely such that $t = o(N)$. Here a typical particle starts at time 0 at the record point $y_{1,N}$ and does not make any diffusion jump until time t. The process started with exactly one particle placed at $y_{1,N}$ and conditioned on the event that no diffusion jump occurs is a (nonspatial) pure birth process with birth rate $\xi_N(y_{1,N})$. For this Yule process with parameter $\xi_N(y_{1,N})$, it is simple to see by explicit calculation that in the limit $t \to \infty$ an exponentially distributed limit variable W (with parameter 1) occurs, uniformly in N. This explains the occurrence of W in the case of the speed region $t = o(N)$.

If t now has a faster growth than $o(N)$, then additionally certain diffusion jumps occur in a typical path. On the other hand, we can write $t = s + (t - s)$ with $s = o(N)$. By the reasons just explained, at time s we have about $W\mathbb{E}\{\Phi_N^{\mathbf{1}}(s, I_N)|\xi_N\}$ many particles situated in $y_{1,N}$. By the Markov and branching property, in the remaining time $t - s$, all these particles give rise to i.i.d. branching processes starting with exactly one particle at $y_{1,N}$. Then, roughly speaking, for those family sizes, the LLN for i.i.d. random variables holds. Thus, the limit variable W reflects only the events occurring already during the time interval $[0, o(N)]$. Therefore it is not surprising that for all speed regions, we get the same exponential limit law in Theorem 1.

(II) TYPICAL PATH; SELECTION. Now we turn more seriously to our second question, which is concerned with the path of a *typical particle*. The latter is simply a particle taken at random from the population at time t. For a rigorous definition of a typical path, we use the construction of the whole historical process with the complete family tree. This object is widely studied in mathematical literature; see, for instance, Harris (1963), Chapter VI. We shall not indulge in carrying out the details of such constructions.

Label every particle which is present at time t by $i = 1, \dots, \Phi_N^1(t, I_N|\xi_N)$ (the order of numbering will not play any role below). With the ith particle, we can associate a spatial path $Z_N^i(t) := \{Z_N^i(t, s);\ s \in [0, t]\}$ which says that the ancestor of particle i, which is alive at time s is at position $Z_N^i(t, s)$. This object is by construction a càdlàg function, that is, it belongs to the space $\mathbf{D}[[0, t], I_N]$ of maps from $[0, t]$ to I_N which are right continuous and have left limits.

Consider for a given realization of the historical process the empirical distribution

$$[\Phi_N^1(t, I_N|\xi_N)]^{-1} \sum_{i=1}^{\Phi_N^1(t, I_N|\xi_N)} \delta_{Z_N^i(t)} \tag{1.5}$$

of a path, that is, the path distribution of a particle taken at random at time t. Now integrate this object (1.5) over the law of the historical process (for a fixed medium ξ_N) to obtain a distribution defined on the Borel σ-field of $\mathbf{D}[[0, t], I_N]$ and denoted by $\vartheta_N(t, \cdot\ |\xi_N)$.

DEFINITION 1.6. The typical path $Z_N(t) := \{Z_N(t, s);\ s \in [0, t]\}$ is a $\mathbf{D}[[0, t], I_N]$-valued random variable with law $\vartheta_N(t, \cdot\ |\xi_N)$.

Our second main result can be formulated as follows. Here $a \ll b$ means that $a/b \to 0$ holds, whereas $a \sim b$ stands for $a/b \to 1$.

THEOREM 2 (Typical path; selection). *For almost all ξ we have the following situation*:

(a) *For every relation in which t and N tend to infinity*:

$$\mathbb{P}\{Z_N(t, 0) = Z_N(t, t) = y_{1,N}|\xi_N\} \xrightarrow[t, N \to \infty]{} 1, \tag{1.7}$$

$$t^{-1} \int_0^t ds \mathbf{1}\{Z_N(t, s) = y_{1,N}\} \xrightarrow[t, N \to \infty]{\mathscr{D}} 1. \tag{1.8}$$

(b) *The finer structure of the typical path depends on the relation in which t and N tend to infinity.*

If $t \ll N \log N$, then

$$\mathbb{P}\{Z_N(t, s) = y_{1,N} \text{ for all } s \in [0, t]|\xi_N\} \xrightarrow[t, N \to \infty]{} 1. \tag{1.9}$$

If $t \sim cN \log N$ for some constant $c > 0$, then

$$\mathbb{P}\{\text{in } Z_N(t) \text{ there occurs a jump from } x \text{ to } y|\xi_N\} \xrightarrow[t, N \to \infty]{} 0, \tag{1.10}$$

provided that $x, y \in I_N \setminus \{y_{1,N}\}$ *and* $x \neq y$. *But*

$$\mathbb{P}\{Z_N(t) \text{ makes at least one jump away from } y_{1,N} | \xi_N\} \underset{t,N\to\infty}{\longrightarrow} 1 - \exp[-c\rho\kappa^2]. \tag{1.11}$$

Consequently, in all speed regions, the typical particle starts and ends up at the record point $y_{1,N}$, and the fraction of time it spends outside $y_{1,N}$ tends to 0 in distribution. Under $t \ll N \log N$, the typical particle has a path which stays all its time at $y_{1,N}$. In the regime $t \sim cN \log N$, a typical particle will also visit other points besides $y_{1,N}$. But if it jumps from $y_{1,N}$ to a site $y \neq y_{1,N}$, then after that it is not allowed to jump to any other site than $y_{1,N}$.

REMARK 1.12. The results in Theorem 2(a) are related in spirit to Theorem 3 in Baillon, Clemént, Greven and den Hollander (1990), since we can conclude that the law of the typical path, respectively, various functionals of this path, become in the limit singular to the corresponding laws (properly rescaled) of the underlying mean field random walk.

1.*D*. *Remarks on the role of the exponential distribution*. The results presented above naturally raise the question whether the restriction to exponentially distributed birth rates is essential to obtain the localization and selection phenomena.

A comparison with the model in Greven and den Hollander (1992) where one has bounded birth rates shows that, of course, there is a big difference: As discussed in Section 1.A, in the bounded case the localization is much weaker. But even if the $\xi_N(y)$ are unbounded (above), a picture different from Theorem 2 may occur. For example, let the birth rates $\xi_N(y)$ be distributed with a density proportional to $\exp[-r^2]$, $r \geq 0$. Then with the methods from Section 5 we can easily see that also in the simplest speed region $t = o(N)$, a typical particle will not necessarily stick completely to the record point $y_{1,N}$ of the random medium ξ_N since the difference between the two highest peaks of the medium will become arbitrarily small as $N \to \infty$.

On the other hand, one expects that distributions with still fatter tails than the exponential distribution should localize at least as strongly as expressed in Theorem 2. This however is slightly more subtle than one might expect at first sight, and therefore we decided to focus on the exponential case; see also Astrauskas and Molchanov (1991).

1.*E*. *Organization*. The paper is organized as follows: We start in Section 2 with constructing the medium, the process and analyzing properties of the generating function and moments. Section 3 is devoted to properties of the mean field operator in a random potential, in particular to asymptotic properties as $N \to \infty$. In Section 4 we prove the exponential limit law (Theorem 1) and in Section 5 the assertions on the typical path of a particle (Theorem 2).

2. Construction of the process and basic tools.

2.*A*. *The random medium*. For a fixed constant $\rho > 0$, we consider the exponential distribution function $F_\rho(r) := 1 - e^{-\rho r}$, $r \geq 0$.

Fix $N \geq 1$. Let $\xi_N := [\xi_N(1), \dots, \xi_N(N)]$ be a vector with independent components distributed with F_ρ. With probability 1, the sites y in I_N may be strictly ordered by means of the order statistics of ξ_N. In fact, we introduce $y_{K,N}$ and $\xi_{K,N}$ as follows:

$$\xi_N(y_{1,N}) =: \xi_{1,N} > \cdots > \xi_N(y_{N,N}) =: \xi_{N,N}. \tag{2.1}$$

Moreover, we use the convention $\xi_{0,N} := \infty$.

By a classical property of the exponential distribution, the increments $\{\xi_{M,N} - \xi_{M+1,N};\ 1 \leq M < N\}$ are independent and $\xi_{M,N} - \xi_{M+1,N}$ has the exponential distribution function $F_{M\rho}$ which is independent of N [see, for instance, Feller (1966), Section 1.6].

From now on we suppose that all random quantities appearing are defined on a common probability space $[\Omega, \mathscr{F}, \mathbb{P}]$: We construct our medium in such a way that the series scheme $\{\xi_N;\ N = 1, 2, \dots\} =: \xi$ has the property that for fixed $M \geq 1$ the increments $\xi_{M,N} - \xi_{M+1,N}$ are the same for all $N > M$. According to the previous remark this is achieved as follows. For each $N \geq 1$, the order statistics (2.1) can be realized by $[\eta_1 + \cdots + \eta_N, \dots, \eta_{N-1} + \eta_N, \eta_N]$ with $\eta_1, \eta_2, \dots$ independent and where η_K has distribution function $F_{K\rho}$, $K \geq 1$. Then from the order statistics a version of the random medium ξ_N is constructed.

This construction implies the following.

Lemma 2.2. *With probability* 1,

$$\sum_{M=2}^{N} (\xi_{1,N} - \xi_{M,N})^{-1} \sim \rho N/\log N \quad \text{as } N \to \infty.$$

Proof. First of all, note that

$$\xi_{1,N} - \xi_{M,N} \equiv \eta_1 + \cdots + \eta_{M-1} \sim \rho^{-1} \log M \quad \text{a.s. as } M \to \infty,$$

[see Fleischmann and Molchanov (1990) formula line (4.8)]. Since the logarithm is a monotone function, the sum in the lemma can be bounded above and below by integrals. But by l'Hôpital's rule,

$$\int_e^N ds (\log s)^{-1} / (N/\log N) \sim 1 \quad \text{as } N \to \infty,$$

and we are done. □

2.*B*. *Definition of the process for given medium*. We start by explaining some terminology. Fix $N \geq 1$. Let φ belong to $\mathbb{Z}_+^N := \{0, 1, 2, \dots\}^N$. We interpret φ as a population: $\varphi(y)$ describes the number of particles situated at y. In particular, $\varphi = \delta_y$ (Kronecker symbol) will refer to the population consisting of exactly one particle at site y.

For each fixed ξ_N, we introduce a generator G_{N,ξ_N} defined by

$$G_{N,\xi_N} g(\varphi) := \kappa \sum_{y=1}^{N} \varphi(y) \left[N^{-1} \sum_{z=1}^{N} g(\varphi + \delta_z - \delta_y) - g(\varphi) \right]$$

$$+ \sum_{y=1}^{N} \varphi(y) \xi_N(y) \left[g(\varphi + \delta_y) - g(\varphi) \right],$$

where g is any function on $\mathbb{Z}_+^N$.

For every fixed ξ_N, there exists a family of continuous time Markov processes $\Phi_N^\psi = \Phi_N^\psi(\cdot|\xi_N)$ with state space $\mathbb{Z}_+^N$, generator G_{N,ξ_N}, initial state $\psi \in \mathbb{Z}_+^N$ and right continuous trajectories [cf. Ethier and Kurtz (1986), Chapter 4], which we call *mean field branching random walk* in the *random medium* ξ_N.

In fact, the first term in the definition of G_{N,ξ_N} refers to a mean field diffusion jump from y to z, whereas the second one represents a binary splitting of a particle situated at y. Of course, for fixed ξ_N, these processes have the branching property:

$$\Phi_N^{\psi+\psi'} =_{\mathscr{D}} \Phi_N^\psi + \Phi_N^{\psi'}, \qquad \psi, \psi' \in \mathbb{Z}_+^N,$$

where Φ_N^ψ and $\Phi_N^{\psi'}$ are independent.

As remarked in Section 1.C, in order to define the typical path we used the richer structure of the historical process. But to carry out the proofs of the announced results, we finally operate only in the semigroup context just introduced.

CONVENTION. The conditional expectation of quantities with respect to the branching process $\Phi_N^\psi(\cdot|\xi_N)$ given ξ_N will be denoted by $\mathbb{E}_\psi\{\cdot|\xi_N\}$. In the case $\psi = \delta_y$ we simply write Φ_N^y and $\mathbb{E}_y\{\cdot|\xi_N\}$. We also omit the upper index (ψ or y) if the initial state is obvious from the context.

2.*C*. *The transition probability generating functions and moments.* For branching processes the most convenient way to study the law of the process at a time t is via generating functions and moment equations. This route will be taken here, too. For future reference, in this section we collect the necessary facts on the equations governing the evolution of generating functions and moments.

We will use the following notation:

$$\theta^\varphi := \prod_{z=1}^{N} [\theta(z)]^{\varphi(z)}, \qquad \theta \in [0,1]^N, \varphi \in \mathbb{Z}_+^N.$$

We also need the mean field operator Δ_N acting on functions f on I_N as follows:

$$\Delta_N f(y) := N^{-1} \sum_{z=1}^{N} f(z) - f(y), \qquad y \in I_N.$$

LEMMA 2.3. *For fixed $\theta \in [0,1]^N$ and for given ξ_N, the transition probability generating functions*

$$\mathbb{E}_y\{\theta^{\Phi_N(t)} | \xi_N\} =: u_N(t, y | \theta, \xi_N), \qquad t \geq 0, y \in I_N,$$

give the unique solution of the equation

$$\dot{u} = \kappa \Delta_N u - \xi_N u(1-u) \tag{2.4}$$

(*the shorthand notation for the vector-valued differential equation*

$$\frac{\partial}{\partial t} u(t,y) = \kappa \Delta_N u(t,y) - \xi_N(y) u(t,y)[1 - u(t,y)], \qquad t \geq 0, y \in I_N)$$

with initial condition $u(0,y) = \theta(y)$, $y \in I_N$.

PROOF. Given ξ_N, let $T_N := \{T_N(t);\ t \geq 0\}$ denote the semigroup corresponding to the generator G_{N,ξ_N}. Then for $g_\theta(\varphi) := \theta^\varphi$,

$$u_N(t, y|\theta, \xi_N) = T_N(t) g_\theta(\delta_y).$$

The branching property implies

$$T_N(t) g_\theta(2\delta_y) = u_N^2(t, y|\theta, \xi_N).$$

Hence, inserting this in the right-hand side of

$$\frac{\partial}{\partial t} T_N(t) = G_{N,\xi_N} T_N(t),$$

we see that the function u_N satisfies equation (2.4). The initial condition and uniqueness are obvious since N is fixed at this point. □

An important tool in studying the mean number of particles is the linear equation appearing in the following lemma.

LEMMA 2.5. *Fix ξ_N and $z \in I_N$. The (conditional) expectations*

$$\mathbb{E}_y\{\Phi_N(t,z) | \xi_N\} =: m_N(t, y|z, \xi_N), \qquad t \geq 0, y \in I_N,$$

are the unique solution of the equation

$$\dot{m} = \kappa \Delta_N m + \xi_N m \tag{2.6}$$

with initial condition $m(0, \cdot\,) = \delta_z$.

PROOF. Let again $\theta \in [0,1]^N$ and recall that $\mathbf{1}(y) \equiv 1$. We have

$$m_N(t, y|z, \xi_N) = \mathbb{E}_y\{\Phi_N(t,z)|\xi_N\} = \frac{\partial}{\partial \theta(z)} u_N(t, y|\theta, \xi_N)\bigg|_{\theta = \mathbf{1}}.$$

If we differentiate (2.4) with respect to $\theta(z)$, we see that the functions $m_N(\cdot,\cdot\,|z,\xi_N)$ fulfill the equation (2.6) with initial condition $m(0,y) = \delta_z(y)$, since

$$u_N(t,y|\theta,\xi_N)|_{\theta=1} \equiv 1. \tag{2.7}$$

Due to the fact that N is fixed here, the uniqueness is trivial. □

Equation (2.6) can be used to derive the following symmetry property.

LEMMA 2.8. *For fixed* ξ_N,

$$\mathbb{E}_y\Phi\{_N(t,z)|\xi_N\} = \mathbb{E}_z\{\Phi_N(t,y)|\xi_N\}, \qquad t \ge 0,\, y,z \in I_N.$$

PROOF. By Lemma 2.5, the expectations fulfill (2.6). Therefore we may represent them with the help of the Feynman–Kac formula:

$$m_N(t,y|z,\xi_N) = \mathbb{E}\left\{\delta_z(w_t^y)\exp\left[\int_0^t ds\, \xi_N(w_s^y)\right]\middle|\xi_N\right\}. \tag{2.9}$$

Here $w^y := \{w_s^y;\ s \ge 0\}$ is a mean field random walk in I_N, that is, a Markov process in I_N with generator $\kappa\Delta_N$, starting in $y \in I_N$. If in the integral in (2.9), we use the substitution $s \to t - s$, the symmetry of that random walk implies our claim, since a path and the reversed path have the same probability. □

From Lemmas 2.5 and 2.8, together with linearity, we get by summation immediately the following *duality relation*:

LEMMA 2.10. *For fixed* ξ_N,

$$\mathbb{E}_y\{\Phi_N(t,I_N)|\xi_N\} = \mathbb{E}_1\{\Phi_N(t,y)|\xi_N\}, \qquad t \ge 0,\, y \in I_N.$$

In order to see whether the expected number of particles gives sufficient information on the number of particles themselves, we shall use later on as usual second moments.

To get an equation for the second factorial moment of $\Phi_N^y(t, I_N|\xi_N)$, we start with the generating function of $\Phi_N^y(t, I_N|\xi_N)$, which by Lemma 2.3 satisfies equation (2.4) with initial condition $u(0,y) \equiv \theta$, $\theta \in [0,1]$. Differentiate twice with respect to θ, set $\theta = 1$ and use (2.7) to obtain the following:

LEMMA 2.11. *Fix* ξ_N. *The second factorial moments*

$$\mathbb{E}_y\{\Phi_N(t,I_N)[\Phi_N(t,I_N) - 1]\,|\xi_N\} =: v_N(t,y|\xi_N), \qquad t \ge 0,\, y \in I_N,$$

are the unique solution of the equation

$$\dot v = \kappa\Delta_N v + \xi_N v + 2\xi_N m_N^2(\cdot\,|\xi_N)$$

with initial condition $v(0, y) \equiv 0$, *where (similarly as in the notation of Lemma* 2.5),

$$m_N(t, y|\xi_N) := \mathbb{E}_y\{\Phi_N(t, I_N)|\xi_N\}, \qquad t \geq 0, y \in I_N.$$

3. The mean field operator with random potential. In this section we derive asymptotics for the largest eigenvalue of the mean field operator with exponentially distributed random potential and deduce from there bounds for normalized first and second moments of the number of particles in $y_{1,N}$.

3.*A*. *Preparations.* We start by recalling in Lemma 3.1 and 3.2 some results from Fleischmann and Molchanov (1990), in particular Proposition 3.9 therein. Given ξ_N, consider the symmetric operator (i.e., $N \times N$-matrix)

$$\mathbf{H}_{N, \xi_N} := \kappa \Delta_N + \xi_N$$

(mean field operator with random potential).

Let $\lambda_{1,N} > \cdots > \lambda_{N,N}$ denote the eigenvalues of $\mathbf{H}_{N,\xi_N}$ which are strictly ordered with probability 1. For an asymptotic analysis, it is useful to introduce $\varepsilon_{M,N}$ and $\omega_{M,N}$ for $1 \leq M \leq N$ defined by

$$\lambda_{M,N} = \xi_{M,N} - \kappa + \varepsilon_{M,N}$$

and

$$\omega_{M,N}(y) := \kappa N^{-1}[\varepsilon_{M,N} + \xi_{M,N} - \xi_N(y)]^{-1}, \qquad y \in I_N.$$

In the case $\kappa = 0$, use the convention $\omega_{M,N} := \delta_{y_{M,N}}$.

LEMMA 3.1. *Let* $1 \leq M \leq N$. *The following assertions hold with probability* 1:

(a) $0 < \varepsilon_{M,N} < \xi_{M-1,N} - \xi_{M,N}$ *if* $\kappa > 0$ *and* $\varepsilon_{M,N} = 0$ *otherwise.*

(b) $\omega_{M,N}$ *is an eigenfunction of* $\mathbf{H}_{N,\xi_N}$ *associated with the eigenvalue* $\lambda_{M,N}$ *and normalized such that* $\sum_{y=1}^N \omega_{M,N}(y) \equiv 1$.

(c) $\sum_{K=1}^N \|\omega_{K,N}\|^{-2} = N$.

(Here $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^N$.)

The next lemma concerns the asymptotic behaviour of the eigenfunctions. In particular, they are localized.

LEMMA 3.2. *Fix* $K \geq 1$. *The following assertions hold with probability* 1:

(a) $\omega_{K,N}(y_{K,N}) \to 1$ *and* $\|\omega_{K,N}\| \to 1$ *as* $N \to \infty$.

(b) $\omega_{M,N}(y_{K,N}) = O(1/N)$ *as* $N \to \infty$ *uniformly in* $M \neq K$.

3.*B*. *Asymptotics for the largest eigenvalue of* $\mathbf{H}_{N,\xi_N}$. This section is devoted to an asymptotic expansion of the largest eigenvalue $\lambda_{1,N}$ of $\mathbf{H}_{N,\xi_N}$.

PROPOSITION 3.3. *With probability* 1, *as* $N \to \infty$,

$$\lambda_{1,N} = \xi_{1,N} - \kappa + \kappa/N + \rho\kappa^2/N \log N + o(1/N \log N).$$

PROOF. Because of Lemma 3.1(a) we may suppose that $\kappa > 0$. Write

$$\lambda_{1,N} = \xi_{1,N} - \kappa + \kappa/N + \gamma_N.$$

Note that $\kappa/N + \gamma_N = \varepsilon_{1,N} > 0$. By the defining equation for $\omega_{1,N}$ and Lemma 3.2(a),

$$\varepsilon_{1,N} \sim \kappa/N \quad \text{as } N \to \infty. \tag{3.4}$$

Furthermore, sum $\omega_{1,N}$ over the ordered sites $y_{M,N}$, insert $\xi_N(y_{M,N}) = \xi_{M,N}$ and use the normalization property in Lemma 3.1(b) to get

$$1 \equiv \kappa N^{-1} \sum_{M=1}^{N} (\xi_{1,N} - \xi_{M,N} + \varepsilon_{1,N})^{-1}.$$

Bringing the term with $M = 1$ to the left, we obtain

$$\gamma_N/\varepsilon_{1,N} = \kappa N^{-1} \sum_{M=2}^{N} (\xi_{1,N} - \xi_{M,N} + \varepsilon_{1,N})^{-1}. \tag{3.5}$$

From (3.4) we know that as $N \to \infty$:

$$[1 + \varepsilon_{1,N}/(\xi_{1,N} - \xi_{M,N})]^{-1} \to 1,$$

uniformly in $M = 2, \dots, N$ (recall that $\xi_{1,N} - \xi_{M,N} \equiv \eta_1 + \cdots + \eta_{M-1}$). Hence, taking the factor $(\xi_{1,N} - \xi_{M,N})^{-1}$ out in each summand in (3.5), we see that the sum may asymptotically be replaced by $\sum_{M=2}^{N}(\xi_{1,N} - \xi_{M,N})^{-1}$. Together with Lemma 2.2, we have

$$\gamma_N/\varepsilon_{1,N} \sim \kappa\rho/\log N \quad \text{as } N \to \infty.$$

Inserting (3.4) in the left-hand side of the latter relation, the asymptotics in Proposition 3.3 follow. □

REMARK 3.6. The log N-term in the asymptotic expansion of $\lambda_{1,N}$ reflects a property of the *exponential* distribution expressed in Lemma 2.2.

3.C. *Localization for the means, bounds for second moments.* The next task is now to apply those properties of the mean field operator with random potential to the evolution of the mean number of particles in our branching process. For this purpose we rely on the following lemma obtained in Fleischmann and Molchanov (1990), Theorems 2.1 and 2.3, combined with Lemma 2.8 above. Recall that $y_{1,N}$ is the record point of the random medium ξ_N.

LEMMA 3.7. *With probability* 1, *all three expressions*,

$$\mathbb{E}_{y_{1,N}}\{\Phi_N(t, I_N)|\xi_N\}, \qquad \mathbb{E}_1\{\Phi_N(t, y_{1,N})|\xi_N\} \quad \textit{and} \quad \mathbb{E}_1\{\Phi_N(t, I_N)|\xi_N\},$$

are asymptotically equivalent to $\exp[\lambda_{1,N}t]$ *as both* $t \to \infty$ *and* $N \to \infty$ (*in any relation*).

At this point we add one more technical fact needed later, which is not surprising in the light of the previous lemma.

LEMMA 3.8. *With probability* 1,

$$\sup\{\exp[-\lambda_{1,N}t]\mathbb{E}_y\{\Phi_N(t,I_N)|\xi_N\};\ t\geq 0,\ y\in I_N,\ N\geq 1\}$$

is finite.

PROOF. Since the initial particle of the process starting in $y_{1,N}$ has the largest birth rate, we get

$$\mathbb{E}_y\{\Phi_N(t,I_N)|\xi_N\}\leq \mathbb{E}_{y_{1,N}}\{\Phi_N(t,I_N)|\xi_N\}.$$

Using the well-known spectral representation for the solutions of the finite system (2.6) of linear ordinary differential equations with $m(0)\equiv 1$ [see Fleischmann and Molchanov (1990), formula (3.23)], we have

$$m_N(t,y_{1,N}|\xi_N)=\sum_{M=1}^{N}\exp(\lambda_{1,N}t)\,\omega_{M,N}(y_{1,N})\|\omega_{M,N}\|^{-2}.$$

Therefore we get

$$\exp(-\lambda_{1,N}t)\mathbb{E}_y\{\Phi_N(t,I_N)|\xi_N\}\leq\sum_{M=1}^{N}|\omega_{M,N}(y_{1,N})|\|\omega_{M,N}\|^{-2}.$$

Then Lemmas 3.2(b) and 3.1(c) applied to the terms with $M\neq N$ and Lemma 3.2(a) applied to the case $M=N$ yield the claim. □

Later on we shall use the following boundedness property of second moments.

LEMMA 3.9. *With probability* 1,

$$\limsup_{N\to\infty}\sup\{\exp[-2\lambda_{1,N}t]\mathbb{E}_y\{\Phi_N^2(t,I_N)|\xi_N\};\ t\geq 0,\ y\in I_N\}$$

is finite.

PROOF. In view of Lemma 3.8 it suffices to study the second factorial moments $v_N(t,y|\xi_N)$ introduced in Section 2.C. Because of Lemma 2.11, similarly to formula (2.9) we have the Feynman–Kac representation

$$v_N(t,y|\xi_N)=\mathbb{E}\left\{\int_0^t ds\,2\xi_N(w_s^y)\,m_N^2(t-s,w_s^y)\exp\left[\int_0^s dr\,\xi_N(w_r^y)\right]\bigg|\xi_N\right\},$$

$t\geq 0$, $y\in I_N$, $N\geq 1$. If we bound $\xi_N(w_s^y)$ by its maximal value $\xi_{1,N}$, we get the estimate

$$v_N(t,y|\xi_N)\leq 2\int_0^t ds\,\xi_{1,N}\,m_N^2(t-s,y_{1,N})\exp[\xi_{1,N}s].$$

By Lemma 3.8, we find a constant, denoted C, such that

$$m_N^2(t-s, y_{1,N}) \le C\exp[2\lambda_{1,N}(t-s)].$$

Inserting this in our estimate for $v_N(t, y|\xi_N)$, we end up with

$$\exp[-2\lambda_{1,N}t]v_N(t, y|\xi_N) \le C\int_0^t ds\, \xi_{1,N}\exp[-(2\lambda_{1,N}-\xi_{1,N})s].$$

Because of Proposition 3.3 and $\xi_{1,N} \to \infty$ as $N \to \infty$, we have for N sufficiently large: $2\lambda_{1,N} > \xi_{1,N}$. Thus, the right-hand side of last inequality is

$$\le C\xi_{1,N}(2\lambda_{1,N} - \xi_{1,N})^{-1}.$$

By Proposition 3.3, this term is bounded, so we are done. □

4. The exponential limit law.

4.*A*. *Preparations*: *Some simple minorants*. Recall that $\xi_{1,N}$ is the record value of the random medium ξ_N and that Φ_N^1 is the branching process starting with $\psi(y) \equiv 1$.

LEMMA 4.1. *Given* ξ_N, *the normalized total populations*

$$\Phi_N^1(t, I_N|\xi_N)\exp[-(\xi_{1,N}-\kappa)t]$$

and

$$\Phi_N^1(t, I_N|\xi_N)\exp[-(\xi_{1,N}-\kappa+\kappa/N)t]$$

have minorants which converge in distribution to an exponentially distributed variable W *with mean* 1, *as* $t \to \infty$ *and* $N \to \infty$ (*simultaneously*).

PROOF. To find the minorant, we first drop all the initial particles except the one in the record point $y_{1,N}$ of the random medium. Second, during the evolution, remove all particles which jump to another site. For the minorant in the first case in the lemma, remove also each particle that makes a diffusion jump to its own site; for the second case, leave such particles.

The remaining particles form a birth and death process, for simplicity denoted by $\beta_N = \{\beta_N(t);\, t \ge 0\}$, with birth rate $\mathscr{b} := \xi_{1,N}$ and death rate $\mathscr{d} := \kappa$ or $\mathscr{d} := \kappa - \kappa/N$, respectively. Here we may assume that $\mathscr{b} > \mathscr{d}$ since $\mathscr{b} \to \infty$ (with probability 1) and $\mathscr{d} \to \kappa$ as $N \to \infty$. For the generating functions

$$\mathbb{E}\{\theta^{\beta_N(t)}|\xi_N\} =: f(t,\theta), \qquad t \ge 0,\ 0 < \theta \le 1,$$

of the process β_N, we have

$$f(t,\theta) = \frac{(\mathscr{d}(1-\theta) + (\mathscr{b}\theta - \mathscr{d})\exp[-(\mathscr{b}-\mathscr{d})t])}{(\mathscr{b}(1-\theta) + (\mathscr{b}\theta - \mathscr{d})\exp[-(\mathscr{b}-\mathscr{d})t])}$$

[see, for instance, Athreya and Ney (1972), page 109]. In order to get from this formula the Laplace transforms corresponding to the normalized birth and death process $\beta_N(t)\exp[-(\mathscr{b}-\mathscr{d})t]$, we have to set $\theta = \exp(-r\exp[-(\mathscr{b}-\mathscr{d})t])$, $r \geq 0$. But since $\mathscr{b} > \mathscr{d}$, for $N \to \infty$ and $t \to \infty$, simultaneously,

$$(1-\theta)\exp[(\mathscr{b}-\mathscr{d})t] \to r.$$

Then

$$f(t,\theta) \to (0+1)/(r+1).$$

However, $1/(r+1)$ is the Laplace transform of the standard exponential distribution, which completes the proof. □

Roughly speaking, since $\mathscr{b} \gg d$, the birth and death processes β_N used as minorants behave like Yule processes with splitting rate $\mathscr{b}$.

4.*B*. *The exponential limit law in the case of the simplest speed region.* We will use the previous result to deal with the exponential limit law in the case of a particular relation in which t and N converge to infinity.

LEMMA 4.2. *Let t and N simultaneously tend to infinity but in such a way that $t \ll N$. Then, given ξ_N, all three expressions*

$$\Phi_N^1(t, I_N|\xi_N)\exp[-\lambda_{1,N}t], \qquad \Phi_N^{y_{1,N}}(t, I_N|\xi_N)\exp[-\lambda_{1,N}t],$$
$$\Phi_N^1(t, y_{1,N}|\xi_N)\exp[-\lambda_{1,N}t],$$

converge in distribution to W.

PROOF. The expectation of

$$\Phi_N^1(t, I_N|\xi_N)\exp[-(\xi_{1,N}-\kappa)t] \tag{4.3}$$

is related to

$$\mathbb{E}_1\{\Phi_N(t, I_N)|\xi_N\}\exp[-\lambda_{1,N}t] \tag{4.4}$$

by the factor

$$\exp[(\lambda_{1,N}-\xi_{1,N}+\kappa)t], \tag{4.5}$$

converging to 1 by Proposition 3.3 and the assumption $t \ll N$. But by Lemma 3.7, the expression (4.4) goes to 1, hence the expectation of (4.3) approaches 1, also. It is straightforward to define our branching processes and their minorants in Lemma 4.1 on a common probability space in such a way that

$$\Phi_N^1(t, I_N|\xi_N) \geq \beta_N(t)$$

with β_N taken from the proof of Lemma 4.1. Therefore, combining this with the facts above, the expectation of the difference between (4.3) and its first minorant in Lemma 4.1 tends to 0. Consequently, (4.3) itself converges in distribution to W. Using again the correction factor (4.5), we arrive at the first convergence relation claimed.

By standard coupling arguments, the two other expressions in the lemma can be understood as minorants of the first one. But by Lemma 3.7 these have the full expectation, and we are done. □

Note that the Lemmas 4.2 and 3.7 yield the exponential limit law in the situation $t \ll N$.

4.*C*. *Proof of Theorem* 1. We may exclude the case $t \ll N$. Fix ξ_N. First of all, we construct a coupling between $\Phi_N^{\mathbf{1}}$ and $\Phi_N^{y_1, N}$ such that

$$\Phi_N^{\mathbf{1}}(\cdot, I_N|\xi_N) \ge \Phi_N^{y_1, N}(\cdot, I_N|\xi_N),$$

by simply coupling the initial configurations such that this relation holds. In order to analyze the right-hand side, we study first the processes at some times $s_N = o(N)$ and use this to deal with $t > s_N$.

Choose a sequence $\{s_N; N \ge 1\}$ which increases to infinity in such a way that $s_N/N \to 0$ as $N \to \infty$. Trivially,

$$\Phi_N^{y_1, N}(s_N, I_N|\xi_N) \ge \Phi_N^{y_1, N}(s_N, y_{1,N}|\xi_N) =: A_N.$$

For $t > s_N$, we use the previous steps, the Markov property and the branching property to decompose and write

$$\Phi_N^{\mathbf{1}}(t, I_N|\xi_N) \ge \sum_{i=1}^{A_N} {}^i\Phi_N^{y_1, N}(t - s_N, I_N|\xi_N), \tag{4.6}$$

where ${}^i\Phi_N^{y_1, N}(\cdot|\xi_N)$ are independent copies of $\Phi_N^{y_1, N}(\cdot|\xi_N)$. This inequality can be written as

$$\Phi_N^{\mathbf{1}}(t, I_N|\xi_N)\exp[-\lambda_{1,N}t] \ge W_N S_N \tag{4.7}$$

with

$$W_N := A_N \exp[-\lambda_{1,N} s_N] = \Phi_N^{y_1, N}(s_N, y_{1,N}|\xi_N)\exp[-\lambda_{1,N} s_N]$$

and

$$S_N := A_N^{-1} \sum_{i=1}^{A_N} \left({}^i\Phi_N^{y_1, N}(t - s_N, I_N|\xi_N)\exp[-\lambda_{1,N}(t - s_N)]\right).$$

First note that by Lemma 4.2,

(4.8) W_N converges in distribution to W.

To show that $S_N \to 1$ in distribution, we first observe that $A_N \to \infty$ as $N \to \infty$ with probability 1. With Lemma 4.2 in mind, we may follow t along a sequence t_N such that $t_N - s_N \to \infty$ as $N \to \infty$. Let $X_{N,i}$ denote the ith term of the sum in S_N and let q_N be its Laplace transform. Now the Laplace transform of S_N at r is given by $(q_N(r/A_N))^{A_N}$. To analyze this expression use the Taylor expansion and write

$$q_N(r) = 1 - r[a_N - rR_N(r)], \qquad r \ge 0,$$

with (by Lemma 3.7)

$$a_N = \mathbb{E}\{X_{N,i}|\xi_N\} \to 1 \quad \text{as } N \to \infty \tag{4.9}$$

and

$$0 \le R_N(r) \le 2^{-1}\mathbb{E}\{X_{N,i}^2|\xi_N\}, \qquad r \ge 0. \tag{4.10}$$

According to Lemma 3.9, the latter expression is bounded as $N \to \infty$. Therefore the Laplace transform of S_N equals

$$(1 - r[a_N - (r/A_N)R_N(r/A_N)]/A_N)^{A_N}$$

and from (4.9) and (4.10), we then get convergence to e^{-r} as $N \to \infty$. In other words,

$$S_N \to 1 \quad \text{in distribution as } N \to \infty. \tag{4.11}$$

Combining (4.8) and (4.11), we see that $W_N S_N$ converges in distribution to W. Because of Lemma 3.7, the expectations of both sides of (4.7) have the same limit. Hence, the left-hand side of (4.7) converges in law to W. Again with Lemma 3.7, we arrive at (1.1).

Once more by coupling and Lemma 3.7, the minorants in the assertions (1.2) and (1.3) have the same limits in distribution. This completes the proof. □

5. The path of a typical particle.

5.*A*. *The asymptotics of a refined minorant.* In this subsection we shall deal with a sharper minorant than those in Lemma 4.1, and study its growth and localization properties.

To define the minorant, we omit all the particles which do not start at time 0 at $y_{1,N}$. Moreover, during the evolution, we drop all those particles which make a diffusion jump from $y_{K,N}$ to a different $y_{L,N}$, where $1 < K,\ L \le N$. The result is a version of a multitype Markov branching process $\nu_N = \{\nu_N(t);\ t \ge 0\}$ with states in $(\mathbb{Z}_+)^N$, where the type of a particle is K if the particle is located at $y_{K,N}$. Note that opposed to earlier notation, $\nu_N(t, K)$ denotes the number of particles present at time t at site $y_{K,N}$ (and not at K). We set

$$n_N(t, K) := \mathbb{E}\{\nu_N(t, K)|\xi_N\}, \qquad t \ge 0,\ K \in I_N.$$

Then the expectation vector $n_N(t) := [n_N(t,1), \dots, n_N(t,N)]$ satisfies the following linear equation:

$$\dot{n}_N = \mathbf{M}_N n_N, \qquad n_N(0) = \delta_1, \tag{5.1}$$

where $\mathbf{M}_N(\cdot|\xi_N) := \mathbf{M}_N$ is a symmetric $N \times N$-matrix defined by

$$\mathbf{M}_N(K, L) := \begin{cases} \xi_{K,N} - \kappa + \kappa/N, & \text{if } K = L, \\ \kappa/N, & \text{if } 1 = K < L \text{ or } K > L = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $l_{1,N} \geq \cdots \geq l_{N,N}$ denote the eigenvalues of $\mathbf{M}_N(\cdot|\xi_N)$ and $\sigma_{1,N}, \ldots, \sigma_{N,N}$ the corresponding eigenvectors. At least up to a certain order, $l_{1,N}$ has the same asymptotics as $\lambda_{1,N}$, which is shown in the lemma below.

LEMMA 5.2. *With probability* 1, *the largest eigenvalue* $l_{1,N}$ *of* $\mathbf{M}_N(\cdot|\xi_N)$ *satisfies as* $N \to \infty$,

$$l_{1,N} = \xi_{1,N} - \kappa + \kappa/N + \rho\kappa^2/N \log N + o(1/N \log N).$$

PROOF. First of all, we may restrict to $\kappa > 0$. By explicit calculation, one finds that the characteristic equation $|\mathbf{M}_N - l\,\mathrm{Id}_N| = 0$ (where Id_N is the identity matrix) can be written as

$$\prod_{K=1}^{N} [\mathbf{M}_N(K,K) - l] - (\kappa/N)^2 \sum_{M=2}^{N} \prod_{L \neq 1, M} [\mathbf{M}_N(L,L) - l] = 0.$$

In order to manipulate this equation, we want to exclude (with probability 1) that $l = \mathbf{M}_N(K,K)$ for some $K \neq 1$. We will get this via proof by contradiction. If with positive probability $l = \mathbf{M}_N(K,K)$ for some $K \neq 1$, then the equation above reduces to

$$\prod_{L \neq 1, K} [\mathbf{M}_N(L,L) - l] = 0.$$

Hence $\mathbf{M}_N(L,L) = \mathbf{M}_N(K,K)$, that is, $\xi_{L,N} = \xi_{K,N}$ for some $L \neq K$. But since the exponential distribution function is continuous, the latter event occurs only with probability 0, so we have a contradiction.

Consequently, with probability 1 each eigenvalue l of $\mathbf{M}_N(\cdot|\xi_N)$ satisfies the equation

$$(5.3) \qquad l - \xi_{1,N} + \kappa - \kappa/N = (\kappa/N)^2 \sum_{M=2}^{N} (l - \xi_{M,N} + \kappa - \kappa/N)^{-1}.$$

As a function in l with $l > \xi_{2,N} - \kappa + \kappa/N$, the right-hand side of (5.3) is positive, continuous and converges to 0 as $l \to \infty$. But the linear function on the left-hand side is positive if and only if $l > \xi_{1,N} + \kappa - \kappa/N$. Therefore the largest root $l_{1,N}$ of (5.3) satisfies

$$\wp_N := l_{1,N} - \xi_{1,N} + \kappa - \kappa/N > 0, \qquad N > 1.$$

Then from (5.3) we get

$$\wp_N N/\kappa = \kappa N^{-1} \sum_{M=2}^{N} (\xi_{1,N} - \xi_{M,N} + \wp_N)^{-1}.$$

If we argue now similarly as in the conclusions from (3.5), we arrive at

$$\wp_N N/\kappa \sim \kappa\rho/\log N \quad \text{as } N \to \infty.$$

This completes the proof. □

LEMMA 5.4. *With probability* 1, *the eigenvector* $\sigma_{1,N}$ *corresponding to* $l_{1,N}$ *is localized at the index of the maximal value, namely at* 1:

$$\sigma_{1,N}(1)/\|\sigma_{1,N}\| \to 1 \quad \text{as } N \to \infty.$$

PROOF. Again we may restrict to $\kappa > 0$. The defining equation for the eigenpair $l_{1,N}$, $\sigma_{1,N}$ implies for $1 < K \le N$,

$$\kappa N^{-1}\sigma_{1,N}(1) + [\xi_{K,N} - \kappa + \kappa/N]\sigma_{1,N}(K) = l_{1,N}\sigma_{1,N}(K),$$

that is,

$$\sigma_{1,N}(K) = \kappa N^{-1}\sigma_{1,N}(1)[\xi_{1,N} - \xi_{K,N} + \wp_N]^{-1},$$

with $\wp_N$ taken from the proof of Lemma 5.2. Since $\wp_N$ is positive, all components of the vector $\sigma_{1,N}$ have the same sign. Without loss of generality in this proof we may normalize by $\sigma_{1,N}(1) \equiv 1$. Then

$$0 < \sigma_{1,N}(K) < \kappa N^{-1}[\xi_{1,N} - \xi_{K,N}]^{-1},$$

and therefore

$$\sum_{K=2}^{N} \sigma_{1,N}(K) \to 0 \quad \text{as } N \to \infty,$$

by Lemma 2.2. Thus even

$$\sum_{K=2}^{N} \sigma_{1,N}^2(K) \to 0 \quad \text{as } N \to \infty$$

holds, that is, $\|\sigma_{1,N}\| \to 1$ as $N \to \infty$. This completes the proof. □

LEMMA 5.5. *With probability* 1, *we have*

$$\mathbb{E}\{\nu_N(t,1)|\xi_N\} \sim \exp[l_{1,N}t] \quad \text{as } t \to \infty \text{ and } N \to \infty,$$

where $l_{1,N}$ *is the largest eigenvalue of* $\mathbf{M}_N(\cdot|\xi_N)$.

PROOF. For convenience, we may assume here that the eigenvectors $\sigma_{1,N}, \dots, \sigma_{N,N}$ corresponding to $l_{1,N} \ge \cdots \ge l_{N,N}$ are orthonormal. By the spectral representation and $n_N(0,K) = \delta_1(K)$,

$$\begin{aligned} n_N(t,1) &= \sum_{M=1}^{N} \exp[l_{M,N}t]\langle\sigma_{M,N}, n_N(0)\rangle\sigma_{M,N}(1) \\ &= \sum_{M=1}^{N} \exp[l_{M,N}t]\sigma_{M,N}^2(1). \end{aligned} \tag{5.6}$$

For the first term of the sum, by Lemma 5.4 we get as $N \to \infty$

$$\exp[l_{1,N}t]\sigma_{1,N}^2(1) \sim \exp[l_{1,N}t].$$

The remaining sum can be estimated above by

$$(5.7)\qquad \begin{aligned} &\le \exp[l_{2,N}t] \sum_{M=2}^{N} \sigma_{M,N}^2(1) \le \exp[l_{2,N}t] n_N(0,1) \\ &= \exp[l_{2,N}t] = \exp[l_{1,N}t]\exp[-(l_{1,N}-l_{2,N})t], \end{aligned}$$

where we used (5.6) for the second inequality.

Since the right side of (5.3) is decreasing in l, the second largest root $l_{2,N}$ of (5.3) is smaller than $\xi_{2,N} - \kappa + \kappa/N$ (whereas $l_{1,N}$ is larger than $\xi_{1,N} \sim \kappa + \kappa/N$). Therefore,

$$l_{1,N} - l_{2,N} \ge \xi_{1,N} - \xi_{2,N} = \eta_1 > 0 \quad \text{a.s.}$$

Hence, the right-hand side of (5.7) is $o(\exp[l_{1,N}t])$ and the proof is complete. □

5.*B*. *Proof of Theorem* 2. We start with a general remark on how to prove properties of the typical path.

In order to verify that the typical particle has a property $Y_{t,N}$, by definition we have to count the number $\tau_N(t|\xi_N)$ of those particles alive at time t which inherit that property $Y_{t,N}$, and show that, for given ξ,

$$(5.8)\qquad \tau_N(t|\xi_N)/\Phi_N^1(t, I_N|\xi_N) \to_{\mathscr{D}} 1$$

as $t \to \infty$ and $N \to \infty$. Note that automatically $\tau_N(t|\xi_N) \le \Phi_N^1(t, I_N|\xi_N)$. The ratio $\tau_N(t|\xi_N)/\Phi_N^1(t, I_N|\xi_N)$ may be written as

$$1 - \frac{\exp(\lambda_{1,N}t)}{\Phi_N^1(t, I_N|\xi_N)} \cdot \frac{\Phi_N^1(t, I_N|\xi_N) - \tau_N(t|\xi_N)}{\exp(\lambda_{1,N}t)}.$$

In view of Theorem 1, it suffices in order to prove (5.8) to show that the expectation of the last ratio converges to 0. But in the light of Lemma 3.7, this holds provided that

$$(5.9)\qquad \liminf_{t,N\to\infty} \exp(-\lambda_{1,N}t)\mathbb{E}\{\tau_N(t)|\xi_N\} \ge 1.$$

Now we are in a position to carry out the proof of Theorem 2. In order to show that under *any* velocity a typical particle starts and ends in the record point, we set

$$\tau_N(t|\xi_N) = \Phi_N^{y_{1,N}}(t, y_{1,N}|\xi_N).$$

Its (conditional) expectation $m_N(t, y_{1,N}|y_{1,N}, \xi_N)$ satisfies equation (2.6) with initial condition $m(0) = \delta_{y_{1,N}}$. By the spectral representation of that solution of (2.6), we therefore have

$$\sum_{M=1}^{N} \exp[\lambda_{M,N}t]\omega_{M,N}^2(y_{1,N})\|\omega_{M,N}\|^{-2} \ge \exp[\lambda_{1,N}t]\omega_{1,N}^2(y_{1,N})\|\omega_{1,N}\|^{-2}.$$

But by Lemma 3.2(a), the latter expression is asymptotically equivalent to $\exp[\lambda_{1,N}t]$ as $N \to \infty$ and (5.9) follows. This proves (1.7).

To verify (1.8), let $\tau_N(t|\xi_N)$ denote the number of those particles alive at time t whose path at time αt is in $y_{1,N}$, where $0 < \alpha < 1$. By the Markov and branching property, similarly to (4.6) we get

$$\mathbb{E}\{\tau_N(t)|\xi_N\} \geq \mathbb{E}\{\Phi_N^1(\alpha t, y_{1,N})|\xi_N\}\mathbb{E}\{\Phi_N^{y_{1,N}}(t-\alpha t, I_N)|\xi_N\}.$$

But then (5.9) will follow with the help of Lemma 3.7. First, reformulate this lemma in terms of the typical path $Z_N(t)$ according to our Definition 1.6: With probability 1,

$$\mathbb{P}\{Z_N(\alpha t) = y_{1,N}|\xi_N\} \to_{t,N\to\infty} 1 \quad \text{for all } \alpha \in (0,1).$$

Setting $L_{t,N} := \int_0^1 d\alpha\, \mathbf{1}\{Z_N(\alpha t) = y_{1,N}\}$ we get, by dominated convergence,

$$\mathbb{E}\{L_{t,N}|\xi_N\} = \int_0^1 d\alpha\, \mathbb{P}\{Z_N(\alpha t) = y_{1,N}|\xi_N\} \to_{t,N\to\infty} 1 \quad \text{a.s.}$$

We conclude that for each $\varepsilon > 0$,

$$\mathbb{P}\{1 \geq L_{t,N} > 1-\varepsilon|\xi_N\} \to_{t,N\to\infty} 1 \quad \text{a.s.},$$

which is nothing else than (1.8). (In other words, the fraction of time the typical particle spends in $y_{1,N}$ tends stochastically to 1.)

Turning to (1.9), we substitute $\beta_N(t)$ for $\tau_N(t|\xi_N)$ from the proof of Lemma 4.1. Then

$$\mathbb{E}\{\tau_N(t)|\xi_N\} = \mathbb{E}\{\beta_N(t)|\xi_N\} = \exp[(\mathscr{b} - \mathscr{d})t],$$

where the birth rate $\mathscr{b}$ is given by $\xi_{1,N}$. Hence,

$$\exp(-\lambda_{1,N}t)\mathbb{E}\{\tau_N(t)|\xi_N\} = \exp[-(\lambda_{1,N} - \xi_{1,N} + \mathscr{d})t]. \tag{5.10}$$

Now we use the second case of the definition of the death rate $\mathscr{d}$, namely $\mathscr{d} = \kappa - \kappa/N$. Then by Proposition 3.3 under the assumption $t \ll N \log N$, for the expression (5.10) we get the desired limit 1. This already demonstrates the validity of (1.9) via (5.8).

To prove (1.10), we substitute $\tau_N(t|\xi_N)$ by $\nu_N(t,1)$, where ν_N is the minorant from Section 5.A. Then (1.10) follows from Lemmas 5.2 and 5.5.

Before we turn to the proof of (1.11), note that due to Proposition 3.3 and Lemma 5.2, we know that at times $t \approx N \log N$, our processes and the minorants are asymptotically equivalent. We are therefore left with the task of analyzing a typical path of the family of multitype processes ν_N which have a simpler structure than our original processes Φ_N^1. In other words, taking into account the result (1.7) as well, we may in (1.5) replace $\Phi_N^1(t, I_N|\xi_N)$ by $\nu_N(t,1)$.

In order to prove (1.11), we shall show that the probability of the event that no jump occurs will converge to $\exp[-c\rho\kappa^2]$. To evaluate this probability, we apply (1.5), then use the replacement explained above, to give us the quantity which is asymptotically equivalent to the desired probability, namely:

$$\mathbb{E}\left\{\nu_N^0(t,1)\big/\nu_N(t,1)\big|\xi_N\right\}, \tag{5.11}$$

where $\nu_N^0(t,1)$ is the number of paths without jumps. Next we introduce an additional time point $s = o(N \log N)$ tending to infinity. Using the Markov property and (1.7) we may restrict our attention to those paths which are at 1 at both times s and t. The next observation is that ν_N^0 is a version of the birth and death process β_N from the proof of Lemma 4.1 with death rate $\mathscr{d} = \kappa - \kappa/N$ and birth rate $\mathscr{b} = \xi_{1,N}$. Hence, both ν_N^0 and ν_N are branching processes. Therefore the integrand in (5.11) can be replaced by

$$\frac{\left[\nu_N^0(s,1)\right]^{-1}\sum_{i=1}^{\nu_N^0(s,1)} {}^i\nu_N^0(t-s,1)\exp\left[-(\xi_{1,N}-\kappa- + \kappa/N)(t-s)\right]}{\left[\nu_N(s,1)\right]^{-1}\sum_{i=1}^{\nu_N(s,1)} {}^i\nu_N(t-s,1)\exp\left[-l_{1,N}(t-s)\right]}$$
$$\cdot \frac{\nu_N^0(s,1)}{\nu_N(s,1)} \cdot \exp\left[-(l_{1,N}-\xi_{1,N}+\kappa-\kappa/N)(t-s)\right]$$

(where the index i refers to independent copies). We shall treat these three factors separately. By our assumption $t \sim cN \log N$ combined with the Lemma 5.2, the last factor is asymptotically equivalent to $\exp[-c\rho\kappa^2]$, which gives already the desired limiting probability. By (1.9) and $s = o(N \log N)$, the second factor tends to 1 in distribution. We are left to show that the first factor converges to 1 in distribution as well.

We will deal with the numerator and denominator separately and use a similar approach as in Section 4.C (proof of the exponential limit law), starting with the formula (4.8). In particular, we have to use $\nu_N(s,1) \to \infty$,

$$\mathbb{E}\nu_N(t-s,1)\exp\left[-l_{1,N}(t-s)\right] \to 1,$$

and that $\mathbb{E}\nu_N^2(t-s,1)\exp[-2l_{1,N}(t-s)]$ remains bounded as $t \sim cN \log N \to \infty$. Here we used Lemmas 5.5, 5.2 and 3.9 in connection with the coupling $\nu_N \le \Phi^{y_{1,N}}$.

This completes the proof of Theorem 2. □

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