The Annals of Probability
1992, Vol. 20, No. 4, 1867–1876

SOME ASYMPTOTIC AND LARGE DEVIATION RESULTS IN POISSON APPROXIMATION

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Let $X_{n1}, \dots, X_{nn}$, $n \geq 1$, be independent random variables with $P(X_{ni} = 1) = 1 - P(X_{ni} = 0) = p_{ni}$ such that $\max\{p_{ni}: 1 \leq i \leq n\} \to 0$ as $n \to \infty$. Let $W_n = \sum_{1 \leq k \leq n} X_{nk}$ and let Z be a Poisson random variable with mean $\lambda = EW_n$. Poisson approximation for the distribution of W_n dates back to 1960, when Le Cam obtained upper bounds for the total variation distance $d(W_n, Z) = \sum_{k \geq 0} |P(W_n = k) - P(Z = k)|$. Barbour and Hall (in 1984) and Deheuvels and Pfeifer (in 1986) investigated the asymptotic behavior of $d(W_n, Z)$ as $n \to \infty$ for small, moderate and large λ. Their results imply that the orders of the bounds obtained by Le Cam are best possible. Chen proved (in 1974 and 1975) that the more general variation distance $d(h, W_n, Z) = \sum_{k \geq 0} h(k)|P(W_n = k) - P(Z = k)|$, $h \geq 0$, converges to 0 as $n \to \infty$, provided λ remains bounded and $Eh(Z) < \infty$. We investigate the asymptotic behavior of: (i) $Eh(W_n) - Eh(Z_\lambda)$ for real h; and (ii) $d(h, W_n, Z)$ for $h \geq 0$ as $n \to \infty$ for small and moderate λ, thus generalizing the corresponding results of Barbour and Hall and of Deheuvels and Pfeifer. Our method also yields a large deviation result and holds promise for successful application in the case when $X_{n1}, \dots, X_{nn}$ are dependent.

1. Introduction. Let $X_{n1}, \dots, X_{nn}$, $n = 1, 2, 3, \dots$, be a triangular array of independent Bernoulli random variables with $P(X_{ni} = 1) = 1 - P(X_{ni} = 0) = p_{ni}$ such that $\tilde{p}_n \to 0$ as $n \to \infty$, where $\tilde{p}_n = \max\{p_{ni}: 1 \leq i \leq n\}$. Let $W_n = \sum_{1 \leq k \leq n} X_{nk}$ and Z_μ be a Poisson random variable with mean μ. Approximating the distribution of W_n by that of Z_λ, where $\lambda = EW_n$, dates back to Le Cam (1960), who obtained upper bounds for the total variation distance $d(W_n, Z_\lambda) = \sum_{k \geq 0} |P(W_n = k) - P(Z_\lambda = k)|$ between $\mathscr{L}(W_n)$ and $\mathscr{L}(Z_\lambda)$. The results of Le Cam can be summarized as $d(W_n, Z_\lambda) \leq C(1 \wedge \lambda^{-1}) \sum_{i=1}^n p_{ni}^2$, where C is an absolute constant. The magnitude of C has been successively improved by Kerstan (1964), Chen (1975a) and Barbour and Hall (1984). Barbour and Hall (1984) and Deheuvels and Pfeifer (1986) investigated the asymptotic behavior of $d(W_n, Z_\lambda)$ as $n \to \infty$ for small, moderate and large λ. Their results imply that the order of the bound $(1 \wedge \lambda^{-1}) \sum_{i=1}^n p_{ni}^2$ is best possible. Chen (1974, 1975b) showed that the more general variation distance $d(h, W_n, Z_\lambda)$ converges to 0 as $n \to \infty$ provided λ remains bounded and $Eh(Z) < \infty$, where $d(h, W_n, Z_\lambda) = \sum_{k \geq 0} h(k)|P(W_n = k) - P(Z_\lambda = k)|$ and $h \geq 0$.

In this paper, we investigate the asymptotic behavior of (i) $Eh(W_n) - Eh(Z_\lambda)$ for real h and (ii) $d(h, W_n, Z_\lambda)$ for $h \geq 0$, as $n \to \infty$ for small and moderate λ,

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Received November 1990; revised May 1991.

AMS 1980 *subject classifications*. Primary 60F05, 60F10, 60G50.

Key words and phrases. Poisson approximation, asymptotic results, large deviations, Stein's method.

thus generalizing the corresponding results of Barbour and Hall and of Deheuvels and Pfeifer. Our method also yields a large deviation result. In particular, our method for large deviation is different from the heuristic treatment of Stein [(1986), Chapter 5] and that of Barbour and Jensen (1989), who used Stein's method for asymptotic expansions and the method of conjugate distributions. It also holds promise for successful application in the case when $X_{n1}, \dots, X_{nn}$ are dependent. Poisson approximation for dependent Bernoulli random variables has been found to have many applications. See, for example, Arratia, Goldstein and Gordon (1989, 1990), Barbour and Holst (1989) and Barbour, Holst and Janson (1988, 1992). The ideas used in this paper also provide an approach to Poisson approximation for unbounded functions which is different from, and perhaps also neater than, that considered by Barbour (1987).

2. Notation and main results. We denote the set of nonnegative integers by $\mathbb{Z}^+$ and the statement that Z_λ has a Poisson distribution with mean λ by $Z_\lambda \sim \mathrm{Pois}(\lambda)$. To simplify notation, we use p_i for p_{ni}, X_i for X_{ni}, W for W_n and Z for Z_λ when ambiguity does not arise. Throughout this paper, h is a real-valued function defined on $\mathbb{Z}^+$. For a given h, let $U_\lambda h$ denote a solution to the difference equation $\lambda f(w+1) - wf(w) = h(w) - Eh(Z_\lambda)$, $w \in \mathbb{Z}^+$. Note that the solution to this difference equation is unique except at $w = 0$ and that the value of any solution at $w = 0$ does not enter into our calculations at all. Let $V_\lambda h(w) = U_\lambda h(w+2) - U_\lambda h(w+1)$, $w \ge 0$, that is, $V_\lambda h(w) = \Delta U_\lambda h(w+1)$. Let I_A denote the indicator function of A, a subset of $\mathbb{Z}^+$. In the case $A = \{r\}$, $r \in \mathbb{Z}^+$, we will use I_r instead of $I_{\{r\}}$. We denote $f(n)/g(n) \to 1$ as $n \to \infty$ by $f(n) \sim g(n)$. Finally, we denote $P(Z_\lambda = r)$ by $p(\lambda; r)$.

THEOREM 2.1. *Suppose* $E|h(Z_\gamma + 4)| < \infty$ *for some* $\gamma > 0$, *where* $Z_\gamma \sim \mathrm{Pois}(\gamma)$. *If* $\lambda \to 0$ *as* $n \to \infty$, *then the following hold*:

(i) $Eh(W) - Eh(Z) \sim -2^{-1}(\sum_{i=1}^n p_i^2)[h(2) - 2h(1) + h(0)]$, *provided that* $\Delta^2 h(0) = h(2) - 2h(1) + h(0) \neq 0$;

(ii) $d(h, W, Z) \sim 2^{-1}(\sum_{i=1}^n p_i^2)[h(2) + 2h(1) + h(0)]$, *where* h *is a nonnegative function on* $\mathbb{Z}^+$ *and* $\sum_{k=0}^2 h(k) > 0$.

THEOREM 2.2. *Let* $a > 0$. *Suppose* $E|h(Z_\gamma + 4)| < \infty$ *for some* $\gamma > a$. *If* $\lambda \to a$ *and* $\tilde{p}_n \to 0$ *as* $n \to \infty$, *then the following hold*:

(i) $Eh(W) - Eh(Z) \sim 2^{-1}(\sum_{i=1}^n p_i^2) E\,\Delta^2 h(Z_a)$, *provided that* $E\,\Delta^2 h(Z_a) \neq 0$;

(ii) $d(h, W, Z) \sim (2a^2)^{-1}(\sum_{i=1}^n p_i^2) E|Z_a^2 - (2a+1)Z_a + a^2| h(Z_a)$, *where* h *is a nonnegative function on* $\mathbb{Z}^+$ *and* $E|Z_a^2 - (2a+1)Z_a + a^2| h(Z_a) > 0$.

To state our next results on large deviation, we let $h(x)$ be a polynomial of degree N in the form $a_0 + \sum_{k=1}^N a_k \prod_{j=0}^{k-1}(x - j)$. Let A be a subset of $\mathbb{Z}^+$, and let a denote $\min\{k : k \in A\}$.

THEOREM 2.3. *Let $H(x) = h(x)I_A(x)$. Suppose λ remains bounded, $\tilde{p}_n \to 0$ and $a = o(\lambda[\sum_{i=1}^n p_i^2]^{-1/2})$ as $n \to \infty$. Then*

$$\frac{EH(W)}{EH(Z)} - 1 \sim \frac{-a^2}{2\lambda^2}\left(\sum_{i=1}^n p_i^2\right).$$

By putting $A = \{z+1, z+2, \dots\}$ and $h(x) \equiv 1$, we immediately obtain the following corollary.

COROLLARY 2.4. *Suppose λ remains bounded, $\tilde{p}_n \to 0$ and $z = o(\lambda[\sum_{i=1}^n p_i^2]^{-1/2})$ as $n \to \infty$. Then*

$$\frac{P(W > z)}{P(Z > z)} - 1 \sim \frac{-z^2}{2\lambda^2}\left(\sum_{i=1}^n p_i^2\right).$$

REMARKS. (i) The special cases where $h(n) = 1$, $n \geq 0$, in parts (ii) of Theorems 2.1 and 2.2 have been proved by Deheuvels and Pfeifer (1986) by a semigroup approach. They have also been obtained by Barbour and Hall (1984).

(ii) It is perhaps interesting to note that in Theorem 2.1, the asymptotic values of $Eh(W) - Eh(Z)$ and $d(h, W, Z)$ involve $h(0)$, $h(1)$ and $h(2)$.

(iii) Corollary 2.4 can also be proved by using the results in Barbour and Jensen (1989).

3. Preliminary lemmas. In this section, we establish some basic identities and inequalities which are needed in the proofs of the theorems. The first lemma can be easily proved by induction.

LEMMA 3.1. *Let $k, z \in \mathbb{Z}^+$ and $1 \leq k \leq z$. Then*

$$\sum_{i=1}^{z-k+1} i(i+1)\cdots(i+k-2) = k^{-1}\prod_{i=0}^{k-1}(z-i),$$

where the summand on the left-hand side is taken to be 1 *for $k = 1$.*

LEMMA 3.2. *For $k \geq 1$ and $Z \sim$* Pois(λ), *we have*

$$E|U_\lambda h(Z+k)| \leq k^{-1}|Eh(z)| + k^{-1}\lambda^{-k}EZ(Z-1)\cdots(Z-k+1)|h(Z)|$$
$$= k^{-1}\{|Eh(Z)| + E|h(Z+k)|\}.$$

PROOF. For $w \geq 1$, $U_\lambda h(w) = -\sum_{l=w}^\infty (w-1)!\lambda^{l-w}[h(l) - Eh(Z)]/l!$ [see (18) on page 84 of Stein (1986)], which can be rewritten as

$$U_\lambda h(w) = -\frac{E[h(Z) - Eh(Z)]I(Z \geq w)}{\lambda p(\lambda; w-1)}.$$

Therefore

$$\begin{aligned}
E|U_\lambda h(Z+k)| &= \sum_{w=0}^{\infty} \frac{|E[h(Z)-Eh(Z)]I(Z \ge w+k)|}{\lambda p(\lambda; w+k-1)} p(\lambda; w) \\
&\le \lambda^{-k} \sum_{w=0}^{\infty} (w+k-1) \\
&\qquad \cdots (w+1) E(|h(Z)| + |Eh(Z)|) I(Z \ge w+k) \\
&= \lambda^{-k} E\left\{ (|h(Z)| + |Eh(Z)|) \sum_{w=1}^{Z-k+1} w(w+1) \right. \\
&\qquad \left. \cdots (w+k-2) \right\} \\
&= k^{-1} \lambda^{-k} E(|h(Z)| + |Eh(Z)|) Z(Z-1) \cdots (Z-k+1) \\
&= k^{-1} \{ |Eh(Z)| + E|h(Z+k)| \},
\end{aligned}$$

where we have used Lemma 3.1 and the following identity [Stein (1986), Theorem 1, page 81]:

$$EZf(Z) = \lambda Ef(Z+1). \tag{3.1}$$

□

REMARK. It is not difficult to see that if $EZ^l|h(Z+k-l)| < \infty$ for some $l = 0, 1, \dots, k$, then $EZ^l|h(Z+k-l)| < \infty$ for all $l = 0, 1, \dots, k$.

LEMMA 3.3. *Let* $V_\lambda h(w) = U_\lambda h(w+2) - U_\lambda h(w+1)$, $w \ge 0$. *We have*

$$\begin{aligned}
E|V_\lambda h(Z)| &\le (2\lambda^2)^{-1} EZ(Z-1)|h(Z)| + \lambda^{-1} EZ|h(Z)| + (3/2)|Eh(Z)| \\
&= 2^{-1} E|h(Z+2)| + E|h(Z+1)| + (3/2)|Eh(Z)|.
\end{aligned}$$

PROOF. Since $E|V_\lambda h(Z)| \le E|U_\lambda h(Z+2)| + E|U_\lambda h(Z+1)|$, the lemma follows from Lemma 3.2 and (3.1). □

REMARK. In fact, for $k \ge 0$, we have

$$\begin{aligned}
E|V_\lambda h(Z+k)| &\le 2^{-1} E|h(Z+k+2)| \\
&\qquad + E|h(Z+k+1)| + (3/2) E|h(Z+k)|.
\end{aligned}$$

LEMMA 3.4. *We have*

$$\begin{aligned}
E|V_\lambda^2 h(Z)| &\le (8\lambda^4)^{-1} EZ(Z-1)(Z-2)(Z-3)|h(Z)| \\
&\quad + (2\lambda^3)^{-1} EZ(Z-1)(Z-2)|h(Z)| \\
&\quad + 5(4\lambda^2)^{-1} EZ(Z-1)|h(Z)| \\
&\quad + 3(2\lambda)^{-1} EZ|h(Z)| + \tfrac{27}{8}|Eh(Z)| \\
&= 8^{-1}\{ E|h(Z+4)| + 4E|h(Z+3)| + 10E|h(Z+2)| \\
&\qquad + 12E|h(Z+1)| + 27|Eh(Z)| \}.
\end{aligned}$$

PROOF. Replacing h by $V_\lambda h$ in Lemma 3.3, we have

$$\begin{aligned} E|V_\lambda^2 h(Z)| &\le 2^{-1}E|V_\lambda h(Z+2)| + E|V_\lambda h(Z+1)| + (3/2)E|V_\lambda h(Z)| \\ &\le 2^{-1}\{E|U_\lambda h(Z+4)| + 3E|U_\lambda h(Z+3)| \\ &\qquad + 2E|U_\lambda h(Z+2)| + 3E|V_\lambda h(Z)|\}. \end{aligned}$$

Applying Lemmas 3.2 and 3.3 and (3.1), we obtain Lemma 3.4. □

LEMMA 3.5. *Suppose* $EZ^2|h(Z)| < \infty$. *Then* $EV_\lambda h(Z) = -E\,\Delta^2 h(Z)/2$.

PROOF. Denote $U_\lambda h$ by f. Then for $k \ge 0$,

$$\lambda f(w+k+2) - (w+k+1)f(w+k+1) = h(w+k+1) - Eh(Z),$$

and

$$\lambda f(w+k+1) - (w+k)f(w+k) = h(w+k) - Eh(Z).$$

Taking the difference, we get

$$\begin{aligned} &\lambda[f(w+k+2) - f(w+k+1)] - w[f(w+k+1) - f(w+k)] \\ &\quad + kf(w+k) - (k+1)f(w+k+1) = h(w+k+1) - h(w+k). \end{aligned}$$

Taking the expectation of both sides with respect to the Poisson distribution, we get

$$\begin{aligned} &kEf(Z+k) - (k+1)Ef(Z+k+1) \\ &\qquad = E(h(Z+k+1) - h(Z+k)). \end{aligned} \tag{3.2}$$

Putting $k = 0$ and $k = 1$ into (3.2), we have

$$-Ef(Z+1) = E(h(Z+1) - h(Z)) \tag{3.3}$$

and

$$Ef(Z+1) - 2Ef(Z+2) = E(h(Z+2) - h(Z+1)). \tag{3.4}$$

Therefore, $2E(f(Z+2) - f(Z+1)) = -E\,\Delta^2 h(Z)$ and Lemma 3.5 follows. □

Putting $h = I_r$, we immediately obtain the following lemma.

LEMMA 3.6. *Let* $r \in \mathbb{Z}^+$. *Then*

$$EV_\lambda I_r(Z) = -(2\lambda^2)^{-1} p(\lambda; r)\left[r^2 - (2\lambda+1)r + \lambda^2\right].$$

4. Proof of Theorems 2.1 and 2.2. We will only prove Theorem 2.1(i) because the method of proof can be carried over to Theorem 2.1(ii) and Theorem 2.2. We start with the following identity [Stein (1986), page 86]:

$$Eh(W) - Eh(Z) = \sum_{i=1}^{n} p_i^2 EV_\lambda h(W^{(i)}), \tag{4.1}$$

where $W^{(i)} = W - X_i$. Rewriting (4.1), we get

$$\begin{aligned} Eh(W) - Eh(Z) &= \left(\sum_{i=1}^n p_i^2\right) EV_\lambda h(W) + \sum_{i=1}^n p_i^2\{EV_\lambda h(W^{(i)}) - EV_\lambda h(W)\} \\ &= \left(\sum_{i=1}^n p_i^2\right) EV_\lambda h(W) \\ &\quad + \sum_{i=1}^n p_i^3 E\{V_\lambda h(W^{(i)}) - V_\lambda h(W^{(i)}+1)\}. \end{aligned}$$

Applying (4.1) to $EV_\lambda h(W)$, we obtain

$$\begin{aligned} Eh(W) - Eh(Z) &= \left(\sum_{i=1}^n p_i^2\right) EV_\lambda h(Z) + \left(\sum_{i=1}^n p_i^2\right)\left(\sum_{j=1}^n p_j^2 EV_\lambda^2 h(W^{(j)})\right) \\ &\quad + \sum_{i=1}^n p_i^3 E\{V_\lambda h(W^{(i)}) - V_\lambda h(W^{(i)}+1)\}. \end{aligned} \tag{4.2}$$

From Lemma 3.5,

$$\begin{aligned} \left(\sum_{i=1}^n p_i^2\right) EV_\lambda h(Z) &= -\left(\sum_{i=1}^n p_i^2\right) E\,\Delta^2 h(Z)/2 \\ &\to -\left(\sum_{i=1}^n p_i^2\right) \Delta^2 h(0)/2 \\ &= -\left(\sum_{i=1}^n p_i^2\right)[h(2) - 2h(1) + h(0)]/2 \quad \text{as } \lambda \to 0. \end{aligned}$$

Therefore, Theorem 2.1(i) will be proved if we can show that each of

$$\sum_{j=1}^n p_j^2 EV_\lambda^2 h(W^{(j)}), \qquad \sum_{i=1}^n p_i^3 EV_\lambda(W^{(i)})\Big/\left(\sum_{i=1}^n p_i^2\right)$$

and

$$\sum_{i=1}^n p_i^3 EV_\lambda h(W^{(i)}+1)\Big/\left(\sum_{i=1}^n p_i^2\right)$$

tends to 0 as $n \to \infty$. From a result of Chen [(1975b), page 998] $P(W^{(i)} = k) \le 2Sp(\lambda; k)$ for all $k \ge 0$ and for sufficiently large n, where $S = \{1 + \lambda^{-1}\sum_{i=1}^n p_i^2/(1-p_i)\}^t$ and t is the integral part of $\lambda + 1 + \sum_{i=1}^n p_i^2/(1-p_i)$. So,

$$\begin{aligned} \left|\sum_{i=1}^n p_i^2 EV_\lambda^2 h(W^{(i)})\right| &\le 2S\left(\sum_{i=1}^n p_i^2\right) E\left|V_\lambda^2 h(Z)\right| \\ &\le 4^{-1} S\tilde{p}_n \lambda\{E|h(Z+4)| + 4E|h(Z+3)| \\ &\qquad + 10E|h(Z+2)| + 12E|h(Z+1)| + 27|Eh(Z)|\} \\ &\le 27MS\tilde{p}_n\lambda/2 \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

where, for $\lambda < \gamma$,

$$M = \max\{E|h(Z+i)|: 0 \le i \le 4\} \le e^{\gamma} \max\left\{E\left|h(Z_\gamma + i)\right|: 0 \le i \le 4\right\} < \infty.$$

Similarly,

$$\begin{aligned}\left|\sum_{i=1}^{n} p_i^3 E V_\lambda h(W^{(i)})\right| \Big/ \left(\sum_{i=1}^{n} p_i^2\right) &\le 2S\tilde{p}_n E|V_\lambda h(Z)| \\ &\le S\tilde{p}_n\{E|h(Z+2)| + 2E|h(Z+1)| \\ &\qquad + 3|Eh(Z)|\} \\ &\le 6MS\tilde{p}_n \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

In the same way, we can show that $\sum_{i=1}^n p_i^3 E V_\lambda h(W^{(i)} + 1)/(\sum_{i=1}^n p_i^2) \to 0$. This completes the proof of part (i) of Theorem 2.1. □

5. Proof of Theorem 2.3. Before we prove Theorem 2.3, we need the following lemma.

LEMMA 5.1. *Suppose* $h(x) = a_0 + \sum_{k=1}^N a_k \prod_{j=0}^{k-1}(x-j)$, $a_N \ne 0$. *For any* $m \in \mathbb{Z}^+$, *we have*

$$Eh(Z+m)I_A(Z+m) \sim a_N \lambda^N p(\lambda; a-N-m) \quad \text{as } a \to \infty, \tag{5.1}$$

where $A \subseteq \mathbb{Z}^+$ *and* $a = \min\{k: k \in A\}$. *Indeed, we have*

$$E|h(Z+m)I_A(Z+m)| \sim |a_N|\lambda^N p(\lambda; a-N-m).$$

PROOF. Rewriting $h(x+m)$ as $b_0 + \sum_{k=1}^N b_k \prod_{j=0}^{k-1}(x-j)$, where $b_N = a_N$, we have

$$\begin{aligned}Eh(Z+m)I_A(Z+m) &= b_0 P(Z+m \in A) \\ &\quad + \sum_{k=1}^{N} b_k EZ(Z-1)\cdots(Z-k+1)I_A(Z+m) \\ &= b_0 P(Z+m \in A) + \sum_{k=1}^{N} b_k \lambda^k P(Z+m+k \in A) \\ &= b_N \lambda^N P(Z+N+m \in A) \\ &\quad \times \left\{1 + \sum_{k=0}^{N-1} \frac{b_k \lambda^{k-N} P(Z+m+k \in A)}{b_N P(Z+m+N \in A)}\right\}.\end{aligned}$$

We will show that:

(i) $$P(Z+N+m \in A) \sim P(Z = a-N-m)$$

and

(ii) $$\frac{\lambda^{k-N} P(Z+m+k \in A)}{P(Z+m+N \in A)} = o(1)$$

as $a \to \infty$, $0 \le k \le N-1$. Now,

$$\begin{aligned} P(Z = a - N - m) &\le P(Z + N + m \in A) \le P(Z + N + m \ge a) \\ &\le P(Z = a - N - m)\left\{1 + \sum_{k=1}^{\infty}\left[\frac{\lambda}{a - N - m + 1}\right]^k\right\} \\ &= P(Z = a - N - m)\left[1 - \frac{\lambda}{a - N - m + 1}\right]^{-1}, \end{aligned}$$

so (i) follows easily. For $0 \le k \le N-1$,

$$\begin{aligned} \frac{\lambda^{k-N}P(Z + m + k \in A)}{P(Z + m + N \in A)} &\le \lambda^{k-N}\frac{P(Z \ge a - m - k)}{P(Z = a - m - N)} \\ &\le \frac{\lambda^{k-N}P(Z = a - m - k)}{P(Z = a - m - N)}\left[1 - \frac{\lambda}{a - k - m + 1}\right]^{-1} \\ &= \frac{(a - m - N)!}{(a - m - k)!}\left[1 - \frac{\lambda}{a - k - m + 1}\right]^{-1} \to 0 \\ &\qquad \text{as } a \to \infty. \end{aligned}$$

This proves (ii). Lemma 5.1 then follows immediately from (i), (ii) and the fact that $b_N = a_N$. □

Proof of Theorem 2.3. By putting H into (4.2) and by Lemma 3.5, we obtain

$$\begin{aligned} EH(W) - EH(Z) = &-\tfrac{1}{2}\left(\sum_{i=1}^{n} p_i^2\right)E\,\Delta^2 H(Z) + \left(\sum_{i=1}^{n} p_i^2\right)\left(\sum_{j=1}^{n} p_j^2 E V_\lambda^2 H(W^{(j)})\right) \\ &+ \sum_{i=1}^{n} p_i^3 E\left[V_\lambda H(W^{(i)}) - V_\lambda H(W^{(i)} + 1)\right]. \end{aligned}$$

This implies

$$\begin{aligned} \frac{EH(W)}{EH(Z)} - 1 = &-\frac{1}{2}\left(\sum_{i=1}^{n} p_i^2\right)\frac{E\,\Delta^2 H(Z)}{EH(Z)} + \left(\sum_{i=1}^{n} p_i^2\right)\frac{\sum_{j=1}^{n} p_j^2 E V_\lambda^2 H(W^{(j)})}{EH(Z)} \\ &+ \frac{\sum_{i=1}^{n} p_i^3 E\left[V_\lambda H(W^{(i)}) - V_\lambda H(W^{(i)} + 1)\right]}{EH(Z)}. \end{aligned}$$

Theorem 2.3 will be proved if we can show that

$$\frac{E\,\Delta^2 H(Z)}{EH(Z)} \sim \frac{a^2}{\lambda^2}, \tag{5.2}$$

$$\frac{\left|\sum_{j=1}^{n} p_j^2 E V_\lambda^2 H(W^{(j)})\right|}{\left|E\,\Delta^2 H(Z)\right|} \to 0, \tag{5.3}$$

$$\frac{\left|\sum_{j=1}^{n} p_j^3 E V_\lambda H(W^{(j)})\right|}{\left[\left(\sum_{i=1}^{n} p_i^2\right)\left|E\,\Delta^2 H(Z)\right|\right]} \to 0 \tag{5.4}$$

and

$$\frac{\left|\sum_{j=1}^{n} p_j^3 E V_\lambda H(W^{(j)}+1)\right|}{\left[\left(\sum_{i=1}^{n} p_i^2\right)\left|E\,\Delta^2 H(Z)\right|\right]} \to 0. \tag{5.5}$$

Since

$$\begin{aligned} E\,\Delta^2 H(Z) &= EH(Z+2) - 2EH(Z+1) + EH(Z) \\ &= a_N \lambda^N \{ p(\lambda; a-N-2)(1+o(1)) \\ &\qquad -2p(\lambda; a-N-1)(1+o(1)) + p(\lambda; a-N)(1+o(1))\}, \end{aligned}$$

this implies $E\,\Delta^2 H(Z) = a_N \lambda^N p(\lambda; a-N-2)(1+o(1))$. Therefore,

$$\frac{E\,\Delta^2 H(Z)}{EH(Z)} = \frac{p(\lambda; a-N-2)}{p(\lambda, a-N)}(1+o(1)) \sim \frac{a^2}{\lambda^2}.$$

This shows (5.2).

Next,

$$\frac{\left|\sum_{j=1}^{n} p_j^2 E V_\lambda^2 H(W^{(j)})\right|}{\left|E\,\Delta^2 H(Z)\right|} \le 2S\left(\sum_{j=1}^{n} p_j^2\right)\frac{E\left|V_\lambda^2 H(Z)\right|}{\left|E\,\Delta^2 H(Z)\right|}.$$

By Lemma 3.4, the left-hand side of (5.3) is

$$\begin{aligned} &\le \frac{1}{4} S\left(\sum_{i=1}^{n} p_i^2\right)\{E|H(Z+4)| + 4E|H(Z+3)| + 10E|H(Z+2)| \\ &\qquad\qquad + 12E|H(Z+1)| + 27|EH(Z)|\}\left(\left|E\,\Delta^2 H(Z)\right|\right)^{-1} \\ &\le \frac{S}{4}\left(\sum_{i=1}^{n} p_i^2\right)\frac{|a_N|\lambda^N p(\lambda; a-N-4)(1+o(1))}{|a_N|\lambda^N p(\lambda; a-N-2)(1+o(1))} \\ &= \frac{1}{4} S\left(\sum_{i=1}^{n} p_i^2\right)\frac{a^2}{\lambda^2}(1+o(1)) \to 0, \quad \text{because } a = o\left(\lambda\left[\sum_{i=1}^{n} p_i^2\right]^{-1/2}\right). \end{aligned}$$

This proves (5.3). To show (5.4) and (5.5) we proceed similarly and apply Lemma 3.3 as follows:

$$\begin{aligned} \frac{\left|\sum_{j=1}^{n} p_j^3 E V_\lambda H(W^{(j)})\right|}{\left(\sum_{i=1}^{n} p_i^2\right)\left|E\,\Delta^2 H(Z)\right|} &\le 2S\frac{\left(\sum_{j=1}^{n} p_j^3\right)}{\left(\sum_{j=1}^{n} p_j^2\right)}\frac{E\left|V_\lambda H(Z)\right|}{\left|E\,\Delta^2 H(Z)\right|} \\ &\le S\frac{\left(\sum_{j=1}^{n} p_j^3\right)}{\left(\sum_{j=1}^{n} p_j^2\right)}(1+o(1)) \le S\tilde{p}_n(1+o(1)) \to 0. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\left|\sum_{j=1}^{n} p_j^3 E V_\lambda H(W^{(j)}+1)\right|}{\left(\sum_{i=1}^{n} p_i^2\right)\left|E\,\Delta^2 H(Z)\right|} &\le S\frac{\left(\sum_{j=1}^{n} p_j^3\right)}{\left(\sum_{j=1}^{n} p_j^2\right)}\frac{a}{\lambda}(1+o(1)) \\ &\le S\left(\sum_{i=1}^{n} p_i^2\right)^{1/2}\frac{a}{\lambda}(1+o(1)) \to 0, \end{aligned}$$

since

$$\left(\sum_{i=1}^{n} p_i^3\right)^2 \le \left(\sum_{i=1}^{n} p_i^4\right)\left(\sum_{i=1}^{n} p_i^2\right) \le \left(\sum_{i=1}^{n} p_i^2\right)^3 \quad \text{and} \quad a = o\left(\lambda\left[\sum_{i=1}^{n} p_i^2\right]^{-1/2}\right).$$

This completes the proof of Theorem 2.3. □

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