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p-VARIATION OF THE LOCAL TIMES OF SYMMETRIC STABLE PROCESSES AND OF GAUSSIAN PROCESSES WITH STATIONARY INCREMENTS[1]

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Let $\{L_t^x, (t, x) \in R^+ \times R\}$ be the local time of a real-valued symmetric stable process of order $1 < \beta \le 2$ and let $\{\pi(n)\}$ be a sequence of partitions of $[0, a]$. Results are obtained for

$$\lim_{n \to \infty} \sum_{x_i \in \pi(n)} |L_t^{x_i} - L_t^{x_{i-1}}|^{2/(\beta - 1)}$$

both almost surely and in L^r for all $r > 0$. Results are also obtained for a similar expression but where the supremum of the sum is taken over all partitions of $[0, a]$ and a function other than a power is applied to the increments of the local times. The proofs use a lemma of the authors' which is a consequence of an isomorphism theorem of Dynkin and which relates sample path behavior of local times with those of associated Gaussian processes. The major effort in this paper consists of obtaining results on the p-variation of the associated Gaussian processes. These results are of independent interest since the associated processes include fractional Brownian motion.

1. Introduction. Let $X = \{X(t), t \in R^+\}$ be a symmetric stable process of order $1 < \beta \le 2$, that is, a real-valued Lévy process with characteristic function

$$Ee^{i\lambda X(t)} = e^{-t|\lambda|^\beta}, \qquad -\infty < \lambda < \infty. \tag{1.1}$$

It follows from Boylan (1964) [see also Barlow (1988)], that X has an almost surely jointly continuous local time which we denote by $L = \{L_t^x, (t, x) \in R^+ \times R\}$.

The interest in the p-variation of stochastic processes was initiated, no doubt, by Lévy's elegant result on the quadratic, or 2-variation, of Brownian motion $\{B(t), t \in R^+\}$ [with $E(B(s) - B(t))^2 = |s - t|$], that is,

$$\lim_{n \to \infty} \sum_{i=0}^{2^n - 1} \left(B\left(\frac{i}{2^n}\right) - B\left(\frac{i+1}{2^n}\right)\right)^2 = 1 \quad \text{a.s.}$$

Generalizations of this result lead to complications. Let $\pi = \{0 = x_0 < x_1 \cdots < x_{k_\pi} = a\}$ denote a partition of $[0, a]$ and let $m(\pi) = \sup_{1 \le i \le k_\pi}(x_i - x_{i-1})$ denote the length of the largest interval in π. [$m(\pi)$ is called the mesh of

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π]. Let $Q_a(\delta) = \{\text{partitions } \pi \text{ of } [0,a] | m(\pi) \le \delta\}$. Dudley (1973) showed that for $\{\pi(n)\}$ any sequence of partitions of $[0,a]$ such that $m(\pi(n)) = o(1/\log n)$,

$$(1.2) \qquad \lim_{n\to\infty} \sum_{x_i \in \pi(n)} \left(B(x_i) - B(x_{i-1})\right)^2 = a \quad \text{a.s.}$$

However, de la Vega (1974) showed that this is no longer true if the condition on $m(\pi(n))$ is relaxed to $m(\pi(n)) = O(1/\log n)$. (In fact, he shows more, as we point out later on.) Taylor (1972) showed that

$$\lim_{\delta\to 0} \sup_{\pi \in Q_n(\delta)} \sum_{x_i \in \pi} \bar{\psi}(|B(x_i) - B(x_{i-1})|) = 1 \quad \text{a.s.},$$

where $\bar{\psi}(x) = |x/\sqrt{2\log^+ \log 1/x}\,|^2$, $(\log^+ u \equiv 1 \vee \log u)$. All of these results are generally referred to as results about the quadratic or 2-variation of Brownian motion. Similar results for other stochastic processes, with 2 replaced by p, are referred to as results about the p-variation of these processes. We will consider the p-variation of the local times of symmetric stable processes in the spatial variable. That these results are similar to the ones for Brownian motion is more than a coincidence. They are a consequence of work developed in Marcus and Rosen (1992), which gives relationships between Gaussian processes and the local times of strongly symmetric Markov processes, that is based on an isomorphism theorem of Dynkin (1983), (1984).

To clarify the notation in (1.2) and in all that follows, note that in the expression $\sum_{x_i \in \pi} f(x_{i-1}, x_i)$, for some function f, we mean that the sum is taken over all the terms in which both x_{i-1} and x_i are contained in π.

THEOREM 1.1. *Let $X = \{X(t), t \in R^+\}$ be a real-valued symmetric stable process of index $1 < \beta \le 2$ and let $\{L_t^x, (t,x) \in R^+ \times R\}$ be the local time of X.*

(i) *If $\{\pi(n)\}$ is any sequence of partitions of $[0,a]$ with $\lim_{n\to\infty} m(\pi(n)) = 0$, then*

$$(1.3) \qquad \lim_{n\to\infty} \sum_{x_i \in \pi(n)} |L_t^{x_i} - L_t^{x_{i-1}}|^{2/(\beta-1)} = c(\beta) \int_0^a |L_t^x|^{1/(\beta-1)}\, dx$$

in L^r uniformly in t on any bounded interval of R^+, for all $r > 0$, where

$$(1.4) \qquad c(\beta) = \frac{2^{2/(\beta-1)}}{\sqrt{\pi}} \Gamma\left(\frac{1}{\beta-1} + \frac{1}{2}\right) \left(\frac{1}{\Gamma(\beta)\sin((\pi/2)(\beta-1))}\right)^{1/(\beta-1)}.$$

(ii) *If $\{\pi(n)\}$ is any sequence of partitions of $[0,a]$ such that $m(\pi(n)) = o(1/\log n)^{1/(\beta-1)}$, then*

$$(1.5) \qquad \lim_{n\to\infty} \sum_{x_i \in \pi(n)} |L_t^{x_i} - L_t^{x_{i-1}}|^{2/(\beta-1)} = c(\beta) \int_0^a |L_t^x|^{1/(\beta-1)}\, dx$$

for almost all $t \in R^+$ almost surely.

(iii)

$$\lim_{\delta\to 0}\sup_{\pi\in Q_a(\delta)}\sum_{x_i\in\pi}|L_t^{x_i}-L_t^{x_{i-1}}|^{2/(\beta-1)}=\infty \tag{1.6}$$

almost surely for each $t\in R^+$.

(iv) *If* $\psi(x)=|x/\sqrt{2\log^+\log 1/x}\,|^{2/(\beta-1)}$, *then*

$$\lim_{\delta\to 0}\sup_{\pi\in Q_a(\delta)}\sum_{x_i\in\pi}\psi(|L_t^{x_i}-L_t^{x_{i-1}}|)=c'(\beta)\int_0^a|L_t^x|^{1/(\beta-1)}\,dx \tag{1.7}$$

almost surely for each $t\in R^+$, *where*

$$c'(\beta)=\left(\frac{2}{\Gamma(\beta)\sin((\pi/2)(\beta-1))}\right)^{1/(\beta-1)}.$$

Some results of this sort have already been obtained. We first note that if $\beta=1+(1/k)$, where k is an integer, then (1.3) is

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}(L_t^{x_i}-L_t^{x_{i-1}})^{2k}=c(\beta)\int_0^a|L_t^x|^k\,dx, \tag{1.8}$$

where the right-hand side is a k-fold self-intersection local time for intersections of the underlying stable process in $[0,a]$. In particular, for the local time of Brownian motion we have

$$\lim_{n\to\infty}\sum_{x_i\in\pi(n)}(L_t^{x_i}-L_t^{x_{i-1}})^2=4\int_0^a L_t^x\,dx. \tag{1.9}$$

Formula (1.9), but with convergence in probability, was obtained in Bouleau and Yor (1981) and Perkins (1982), and allows one to develop stochastic integration with respect to the space parameter of Brownian local time: see also Walsh (1983). The formula (1.8), with convergence in L^2, was established in Rosen (1990) by a complicated computational argument. Let us also note that it follows from Theorem 1.1 (ii) that we get convergence for almost all t almost surely in (1.8) and (1.9) as long as the condition on $m(\pi(n))$ is satisfied.

We show in Theorem 3.5 that the result in Theorem 1.1 (ii) is "best possible" in the sense that for all $b>0$, we can find a sequence of partitions $\{\pi(n)\}$ with $m(\pi(n))\le b/\log n$, such that (1.5) is false, whatever the value of $1<\beta\le 2$. Thus for $\beta=2$, (ii) is indeed "best possible" and for $1<\beta<2$ it is close.

To prove Theorem 1.1 we use Lemma 4.3 in Marcus and Rosen (1992), which enables us to obtain results for various types of p-variation of the local times of symmetric stable processes from analogous results about the p-variation of their associated Gaussian processes. The mean zero Gaussian process $\{G(x), x\in R\}$ with covariance $g(x,y)$ is said to be associated with the Markov process X if $g(x,y)$ is the 1-potential of X. In the case of symmetric stable processes of index β, the associated Gaussian processes are stationary with covariance given by

$$g(x,y) = \frac{1}{\pi}\int_0^\infty \frac{\cos\lambda(x-y)}{1+\lambda^\beta}\,d\lambda \tag{1.10}$$

[see, e.g., Barlow (1988)]. In general, for Gaussian processes with stationary increments, we define

$$\sigma^2(h) = E(G(x+h) - G(x))^2. \tag{1.11}$$

Results about various aspects of the p-variation of Gaussian processes appear in Kono (1969), Kawada and Kono (1973), Giné and Klein (1975), Jain and Monrad (1983), and Adler and Pyke (1990). We use the results of Kawada and Kono in the proofs of (iii) and (iv) of Theorem 1.1. For the other parts of the proof of Theorem 1.1 we obtain, in Theorem 1.2, some new results on the almost sure convergence of the p-variation of certain Gaussian processes for $p \geq 2$. In the case $p = 2$ they are similar to those in Giné and Klein (1975). In Marcus and Rosen (1992) we advanced the position that it is useful to study local times of symmetric Markov processes through their associated Gaussian processes because there are many tools available to us in the theory of Gaussian processes. This is also the case in this paper.

THEOREM 1.2. *Let $\{G(x), x \in R\}$ be a mean zero Gaussian process with stationary increments. If $\sigma^2(h)$ is concave for $h \in [0,\delta]$ for some $\delta > 0$ and satisfies $\lim_{h\to 0}\sigma(h)/h^{1/p} = \alpha$ for some $p \geq 2$ and $0 \leq \alpha < \infty$, then for any sequence of partitions $\{\pi(n)\}$ of $[0,a]$ such that $m(\pi(n)) = o(1/\log n)^{p/2}$,*

$$\lim_{n\to\infty} \sum_{x_i\in\pi(n)} |G(x_i) - G(x_{i-1})|^p = E|\eta|^p\alpha^p a \quad a.s., \tag{1.12}$$

where η is a normal random variable with mean 0 *and variance* 1. *Also*

$$\lim_{n\to\infty} \sum_{x_i\in\pi(n)} |G^2(x_i) - G^2(x_{i-1})|^p = E|\eta|^p(2\alpha)^p\int_0^a |G(x)|^p\,dx \quad a.s. \tag{1.13}$$

For Brownian motion, as discussed above, $p = 2$ and $\alpha = 1$ and we recover the result of Dudley (1973) mentioned in (1.2). The result in (1.12) for $p = 2$ has some overlap with Theorem 1 in Giné and Klein (1975). Actually we can also prove this theorem with the methods used here but this is not surprising since Borell's inequality, which we use, is sharper than the Hanson–Wright bound used in Giné and Klein (1975). We also show, in Theorem 2.6, that (1.12) and (1.13) are close to "best possible" in the same sense that the results in Theorem 1.1 (ii) are "best possible," as we mentioned above.

In Section 2 we give the results on Gaussian processes that we use, including the proof of Theorem 1.2. In Section 3 we prove Theorem 1.1. We are grateful to M. Yor for calling our attention to the question of the p-variation of local times.

2. Gaussian processes. In this section we will obtain results on the p-variation of Gaussian processes and give the proof of Theorem 1.2. We will use the following result on the norm of a special class of matrices. It is a simple statement about the behavior of symmetric operators on R^n but we will provide a more direct proof for the convenience of some readers.

LEMMA 2.1. *Let $B = (B_{ij})_{i,j=1}^n$ be an $n \times n$ positive definite symmetric matrix and let $\|B\|$ denote the operator norm of B as an operator from $l_2^n \to l_2^n$. Suppose that*

$$\sum_{j=1}^{n} |B_{ij}| \le C \qquad \forall\, 1 \le i \le n. \tag{2.1}$$

Then $\|B\| \le C$.

PROOF. We note that B is a symmetric positive definite matrix and therefore

$$\|B\| = \lim_{m \to \infty} (\operatorname{trace}(B^m))^{1/m}. \tag{2.2}$$

We have

$$\begin{aligned}
\operatorname{trace}(B^m) &= \sum_{i_1, i_2, \dots, i_m = 1}^{n} B_{i_1 i_2} B_{i_2 i_3} \cdots B_{i_m i_1} \\
&\le \sum_{i_1, \dots, i_m = 1}^{n} |B_{i_1 i_2}||B_{i_2 i_3}| \cdots |B_{i_m i_1}| \\
&\le \sum_{i_1, \dots, i_m, i_{m+1} = 1}^{n} |B_{i_1, i_2}| \cdots |B_{i_m i_{m+1}}| \\
&\le C^m n.
\end{aligned}$$

Since n is fixed, the lemma follows from (2.2). □

We will consider a slightly broader definition of partition than the one given in the introduction. We let $\pi = \{b_0 = x_0 < x_1 \cdots < x_{k_\pi} = b_1\}$ denote a partition of $[b_0, b_1]$, with the understanding that b_0 and b_1 can be different for the different partitions considered. For $G = \{G(x), x \in R\}$, a real-valued Gaussian process, we associate with a partition π the covariance matrix

$$\rho_{ij}(\pi) = E\big(G(x_i) - G(x_{i-1})\big)\big(G(x_j) - G(x_{j-1})\big) \qquad i, j = 1, \dots, k_\pi. \tag{2.3}$$

For a real-valued random variable Z we denote the median of Z by med(Z). The major result used in this paper, besides Lemma 4.3 in Marcus and Rosen (1992), is the following restatement of Borell's inequality.

LEMMA 2.2. *Let $G = \{G(x), x \in R\}$ be a real-valued Gaussian process and let $\{\pi(m)\}_{m=1}^{\infty}$ be partitions of $\{[b_0(m), b_1(m)]\}_{m=1}^{\infty}$. For $p > 1$ define*

$$\|\| G \|\|_{\pi(m), p} = \left(\sum_{x_i \in \pi(m)} |G(x_i) - G(x_{i-1})|^p \right)^{1/p}. \tag{2.4}$$

Then

$$P\left(\left| \sup_m \|\| G \|\|_{\pi(m), p} - \operatorname{med}\left(\sup_m \|\| G \|\|_{\pi(m), p} \right) \right| > t \right) \le 2e^{-t^2/(2\hat{\sigma}^2)}, \tag{2.5}$$

where

$$\hat{\sigma}^2 = \sup_m \sup_{\{\{a_k\}: \Sigma |a_k|^q \le 1\}} \sum_{i,j=1}^{k_{\pi(m)}} a_i a_j \rho_{i,j}(\pi(m)) \tag{2.6}$$

and $1/p + 1/q = 1$. If $p \ge 2$, then

$$\hat{\sigma}^2 \le \sup_m \| \rho(\pi(m)) \|, \tag{2.7}$$

where $\|\rho(\pi)\|$ denotes the operator norm of ρ as an operator from $l_2^{k_\pi} \to l_2^{k_\pi}$. Also

$$\left| E\left(\sup_m \|\| G \|\|_{\pi(m), p} \right) - \operatorname{med}\left(\sup_m \|\| G \|\|_{\pi(m), p} \right) \right| \le \hat{\sigma}\sqrt{2\pi}. \tag{2.8}$$

PROOF. The standard form of Borell's inequality is the following [see, e.g., Fernique (1985), Ledoux and Talagrand (1991)]: Let $\{H(u), u \in U\}$, U a countable index set, be a Gaussian process. Then

$$P\left(\left| \sup_{u \in U} H(u) - \operatorname{med}\left(\sup_{u \in U} H(u) \right) \right| > t \right) \le e^{-t^2/(2\nu^2)}, \tag{2.9}$$

where

$$\nu^2 = \sup_{u \in U} E(H^2(u)).$$

Let $U = \Pi \times B_q$, where B_q is a countable dense subset of the unit ball of l_q and $\Pi = \{\pi(m)\}_{m=1}^{\infty}$. For $\pi(m) \in \Pi$ and $\{a_i\} \in B_q$, set

$$H(\pi(m), \{a_i\}) = \sum_{x_i \in \pi(m)} a_i (G(x_i) - G(x_{i-1})).$$

We see that

$$\begin{aligned} \sup_{(\pi(m), \{a_i\}) \in U} H(\pi(m), \{a_i\}) &= \sup_{\pi(m) \in \Pi} \left(\sum_{i=1}^{k_{\pi(m)}} |G(x_i) - G(x_{i-1})|^p \right)^{1/p} \\ &= \sup_{\pi(m) \in \Pi} \|\| G \|\|_{\pi(m), p}. \end{aligned}$$

Also

$$\nu^2 = \sup_{(\pi(m),\{a_i\})\in U} \sum_{i,j=1}^{k_{\pi(m)}} a_i a_j \rho_{i,j}(\pi(m))$$

(2.10)

$$= \sup_m \sup_{\{\{a_i\}:\ \sum|a_i|^q \le 1\}} \sum_{i,j=1}^{k_{\pi(m)}} a_i a_j \rho_{i,j}(\pi(m)).$$

Combining these statements we get (2.5) and (2.6). When $p \ge 2$ we have $q \le 2$ and since, in this case, the unit ball of l_q is contained in the unit ball of l_2, we see that the last line of (2.10) is

$$\le \sup_m \sup_{\{\{a_i\}:\ \sum|a_i|^2 \le 1\}} \sum_{i,j=1}^{k_{\pi(m)}} a_i a_j \rho_{i,j}(\pi(m))$$

$$= \sup_m \| \rho(\pi(m)) \|$$

by the definition of $\|\rho(\pi(m))\|$. The statement in (2.8) follows from (2.5). □

We prove a slight generalization of (1.12) of Theorem 1.2 because we need it in the proof of (1.13) of Theorem 1.2.

THEOREM 2.3. *Let* $\{G(x), x \in R\}$ *be a mean zero Gaussian process with stationary increments and assume that* $\sigma^2(h)$ *is concave for* $h \in [0,\delta]$ *for some* $\delta > 0$ *and satisfies* $\lim_{h\to 0} \sigma(h)/h^{1/p} = \alpha$ *for some* $p \ge 2$ *and* $0 \le \alpha < \infty$. *Let* $\{\pi(n)\}_{n=1}^{\infty}$ *be partitions of* $\{[b_0(n), b_1(n)]\}_{n=1}^{\infty}$ *such that* $m(\pi(n)) = o(1/\log n)^{p/2}$, *and* $\lim_{n\to\infty} b_0(n) = b_0$ *and* $\lim_{n\to\infty} b_1(n) = b_1$. *Then*

$$\lim_{n\to\infty} \sum_{x_i\in\pi(n)} |G(x_i) - G(x_{i-1})|^p = E|\eta|^p \alpha^p (b_1 - b_0) \quad a.s., \tag{2.11}$$

where η *is a normal random variable with mean* 0 *and variance* 1.

PROOF. Let us consider Lemma 2.2 with $\{\pi(m)\}_{m=1}^{\infty}$ consisting of a single element $\pi(n)$. We will show that for all n sufficiently large,

$$\| \rho(\pi(n)) \| = o\left(\frac{1}{\log n}\right). \tag{2.12}$$

Assuming (2.12) for the moment, we see from (2.5) and the Borel–Cantelli lemma that

$$\lim_{n\to\infty} \left(||| G |||_{\pi(n),p} - \operatorname{med} ||| G |||_{\pi(n),p} \right) = 0 \quad \text{a.s.} \tag{2.13}$$

Set $\operatorname{med}(||| G |||_{\pi(n),p}) = M_n$. Then we have that

$$M_n \le 2E(||| G |||_{\pi(n),p}) \le 2\left(E ||| G |||^p_{\pi(n),p}\right)^{1/p}$$

$$= 2\left(E|\eta|^p\right)^{1/p} \left(\sum_{x_i\in\pi(n)} \sigma^p(x_i - x_{i-1}) \right)^{1/p}.$$

It follows from the hypotheses on σ^2 that

$$M_n \le C(E|\eta|^p)^{1/p}(b_1 - b_0)^{1/p} \qquad \forall\, n, \tag{2.14}$$

where C is an absolute constant. Choose some convergent subsequence $\{M_{n_i}\}_{i=1}^{\infty}$ of $\{M_n\}_{n=1}^{\infty}$ and suppose that

$$\lim_{i\to\infty} M_{n_i} = \overline{M}. \tag{2.15}$$

It then follows from (2.13) and (2.15) that

$$\lim_{i\to\infty} |||\, G \,|||_{\pi(n_i),p} = \overline{M} \quad \text{a.s.} \tag{2.16}$$

Let us also note that it follows from (2.5), (2.12) and (2.14) that for all $r > 0$, there exist finite constants $C(r)$ such that

$$E\, |||\, G \,|||^{r}_{\pi(n),p} \le C(r) \qquad \forall\, n \ge 1.$$

Thus, in particular, $\{|||\, G\,|||^{p}_{\pi(n),p};\ n = 1,\dots\}$ is uniformly integrable. This, together with (2.16), shows that

$$\lim_{i\to\infty} E\, |||\, G \,|||^{p}_{\pi(n_i),p} = \overline{M}^p. \tag{2.17}$$

Since it is obvious because of our assumption on σ^2 that

$$\lim_{n\to\infty} E\, |||\, G \,|||^{p}_{\pi(n),p} = (b_1 - b_0)\alpha^p E|\eta|^p,$$

we have that

$$\overline{M}^p = (b_1 - b_0)\alpha^p E|\eta|^p. \tag{2.18}$$

Thus the bounded set $\{M_n\}_{n=1}^{\infty}$ has a unique limit point $\overline{M}$. It now follows from (2.13) that

$$\lim_{n\to\infty} |||\, G \,|||^{p}_{\pi(n),p} = (b_1 - b_0)\alpha^p E|\eta|^p.$$

To complete the proof of (2.11) we need only establish (2.12). This will follow from Lemma 2.1 once we show that

$$\sum_j |\rho_{ij}(\pi(n))| \le 2\sigma^2(x_i - x_{i-1}) \le 2 \max_{x_i\in\pi(n)} \sigma^2(x_i - x_{i-1}). \tag{2.19}$$

To obtain (2.19) we first assume that $j > i$. We note from (2.3) that

$$\begin{aligned}\rho_{ij} &= -\tfrac{1}{2}\left[\sigma^2(x_{j-1} - x_{i-1}) - \sigma^2(x_{j-1} - x_i)\right]\\ &\quad + \tfrac{1}{2}\left[\sigma^2(x_j - x_{i-1}) - \sigma^2(x_j - x_i)\right]\\ &= -A_{j-1,i} + A_{j,i},\end{aligned}$$

where

$$A_{j,i} \equiv \tfrac{1}{2}\left[\sigma^2(x_j - x_{i-1}) - \sigma^2(x_j - x_i)\right].$$

Let us take n sufficiently large so that σ^2 is concave and monotonically increasing on $[0, \max_{x_i \in \pi(n)} |x_i - x_{i-1}|]$. This implies that $A_{j,i} \geq 0$ and also that $A_{j-1,i} \geq A_{j,i}$ for all $j \geq i$. Therefore

$$(2.20)\qquad \begin{aligned}\sum_{j=i+1}^{k} |\rho_{ij}| = \sum_{j=i+1}^{k} (A_{j-1,i} - A_{j,i}) = A_{i,i} - A_{k,i} \leq A_{i,i} \\ = \tfrac{1}{2}\sigma^2(x_i - x_{i-1}),\end{aligned}$$

where $k = k_{\pi(n)}$ is the number of partition points in $\pi(n)$.

Recall that $\sigma^2(h) = \sigma^2(-h)$ since the Gaussian processes have stationary increments. Therefore, similar to the above, for $j < i$, we set

$$\rho_{ij} = D_{j-1,i} - D_{j,i},$$

where now

$$D_{j,i} = \tfrac{1}{2}\left[\sigma^2(x_i - x_j) - \sigma^2(x_{i-1} - x_j)\right].$$

Using the monotonicity and concavity of σ^2 once more we see that $D_{j,i} \geq 0$, and also that $D_{j,i} \geq D_{j-1,i}$ for all $j < i$. Therefore,

$$(2.21)\qquad \begin{aligned}\sum_{j=1}^{i-1} |\rho_{ij}| &= \sum_{j=1}^{i-1} (D_{j,i} - D_{j-1,i}) \\ &= D_{i-1,i} - D_{0,i} \leq D_{i-1,i} = \tfrac{1}{2}\sigma^2(x_i - x_{i-1})\end{aligned}$$

and, of course,

$$(2.22)\qquad \rho_{i,i} = \sigma^2(x_i - x_{i-1}).$$

Using (2.20), (2.21) and (2.22) we obtain (2.19). This completes the proof of Theorem 2.3. □

Proof of Theorem 1.2. The proof of (1.12) follows immediately from Theorem 2.3. We simply take $b_0(n) = 0$ and $b_1(n) = a$ for all n. We proceed to the proof of (1.13). For clarity we write

$$\pi = \left[0 = x_0(\pi) < \cdots < x_{k_\pi}(\pi) = a\right].$$

We divide $[0, a]$ into m equal subintervals $I_{j,m}(a) \equiv [((j-1)/m)a, (j/m)a]$, $j = 1, \dots, m$. Using the partition points of π we define

$$(2.23)\qquad x_{k(j)}(\pi) = \sup_k \left\{x_k(\pi) \colon x_k(\pi) \leq \frac{j}{m}a\right\}, \qquad j = 0, \dots, m.$$

Consider the partitions (given by the increasing sequence of points)

$$(2.24)\qquad \begin{aligned}\pi(I_{j,m}(a)) = \left\{x_{k(j-1)}(\pi) < x_{k(j-1)+1}(\pi) < \cdots < x_{k(j)}(\pi)\right\}, \\ j = 1, \dots, m.\end{aligned}$$

For a partition π, we have

$$\begin{aligned}
&\sum_{x_i \in \pi} \left|G^2(x_i) - G^2(x_{i-1})\right|^p \\
(2.25)\quad &= \sum_{j=1}^{m} \sum_{x_i \in \pi(I_{j,m}(a))} \left|G^2(x_i) - G^2(x_{i-1})\right|^p \\
&\le 2^p \sum_{j=1}^{m} \sum_{x_i \in \pi(I_{j,m}(a))} \left|G(x_i) - G(x_{i-1})\right|^p \sup_{x_{k(j-1)}(\pi) \le x \le x_{k(j)}(\pi)} \left|G(x)\right|^p.
\end{aligned}$$

[The clarification of notation given prior to the statement of Theorem 1.1 is particularly relevant to the last two lines of (2.25) as well as to some similar statements involving subpartitions that are given later.] It is well known that under the hypothesis on σ^2 [see, e.g., Section IV, Theorem 1.3 in Jain and Marcus (1978)], the Gaussian process G has continuous sample paths almost surely. Using this fact and Theorem 2.3 we can take the limit, as n goes to infinity, of the terms to the right of the inequality in (2.25) to obtain

$$(2.26)\quad \begin{aligned}
&\limsup_{n\to\infty} \sum_{x_i \in \pi(n)} \left|G^2(x_i) - G^2(x_{i-1})\right|^p \\
&\le E|\eta|^p (2\alpha)^p \sum_{j=1}^{m} \frac{a}{m} \sup_{x \in I_{j,m}(a)} \left|G(x)\right|^p \quad \text{a.s.}
\end{aligned}$$

Similarly, we obtain

$$(2.27)\quad \begin{aligned}
&\liminf_{n\to\infty} \sum_{x_i \in \pi(n)} \left|G^2(x_i) - G^2(x_{i-1})\right|^p \\
&\ge E|\eta|^p (2\alpha)^p \sum_{j=1}^{m} \frac{a}{m} \inf_{x \in I_{j,m}(a)} \left|G(x)\right|^p \quad \text{a.s.}
\end{aligned}$$

Taking the limit of the right-hand sides of (2.26) and (2.27), as m goes to infinity, and using the definition of Riemann integration, we get (1.13). □

We will also use the following results in the proof of Theorem 1. The main part, (2.28), is due to Kawada and Kono.

THEOREM 2.4. *Let $\{G(x), x \in R\}$ be a mean zero Gaussian process with stationary increments. If $\sigma^2(h)$ is concave for $h \in [0, \delta]$ for some $\delta > 0$ and satisfies $\lim_{h\to 0} \sigma(h)/h^{1/p} = \alpha$ for some $p \ge 2$ and $0 \le \alpha < \infty$, then for $\varphi(x) = |x/\sqrt{2\log^+ \log 1/x}|^p$,*

$$(2.28)\quad \lim_{\delta\to 0} \sup_{\pi \in Q_a(\delta)} \sum_{x_i \in \pi} \varphi\left(\left|G(x_i) - G(x_{i-1})\right|\right) = \alpha^p a \quad a.s.$$

Also,

$$\lim_{\delta \to 0} \sup_{\pi \in Q_a(\delta)} \sum_{x_i \in \pi} \varphi(|G^2(x_i) - G^2(x_{i-1})|) = (2\alpha)^p \int_0^a |G(x)|^p \, dx \quad a.s. \tag{2.29}$$

PROOF. The statement in (2.28) follows from Theorems 3 and 4 in Kawada and Kono (1969). [It is easy to verify their conditions (ii)–(v) once one recognizes that they all follow from the concavity of $\sigma(h)$. To see this use Theorem 1.7.2b, page 39, of Bingham, Goldie and Teugels (1987).] Their results are for Gaussian processes with stationary increments defined on [0, 1]. If we apply their results to $H(ax) = G(x)$ with

$$E|H(a(x+h)) - H(ax)|^2 = \sigma^2(ah), \tag{2.30}$$

we get (2.28).

We now prove (2.29). Continuing the notation of the proof of Theorem 1.2, in addition to the subpartitions of π given by $\pi(I_{j,m}(a))$, $j = 1, \dots, m-1$ [see (2.24)], we define

$$\tilde{\pi}(I_{j,m}(a)) = \left\{ \frac{j-1}{m} a < x_{k(j-1)+1}(\pi) < \cdots < x_{k(j)}(\pi) \le \frac{j}{m} a \right\}, \quad j = 1, \dots, m. \tag{2.31}$$

[Note that $((j-1)/m)a$ and $(j/m)a$ are points in the partition given in (2.31).] We have

$$\begin{aligned} \sum_{x_i \in \pi} \varphi(|G^2(x_i) - G^2(x_{i-1})|) &= \sum_{j=1}^{m} \sum_{x_i \in \pi(I_{j,m}(a))} \varphi(|G^2(x_i) - G^2(x_{i-1})|) \\ &\le \sum_{j=1}^{m} \sum_{x_i \in \tilde{\pi}(I_{j,m}(a))} \varphi(|G^2(x_i) - G^2(x_{i-1})|) \\ &\quad + \sum_{j=1}^{m-1} \varphi\big(\big|G^2(x_{k(j)}(\pi)) - G^2(x_{k(j)+1}(\pi))\big|\big). \end{aligned} \tag{2.32}$$

To get (2.32) we added partition points at $\{((j-1)/m)a\}_{j=2}^{m}$. These points are included in the first term of (2.32). In the second term we have written the partitions that were present that bracketed the added points.

Since φ is regularly varying at zero, for any $0 < u < v < \infty$ and any $\varepsilon > 0$, if c is sufficiently small, we have $\varphi(cb) \le (1+\varepsilon)\varphi(c)|b|^p$ for all $b \in [u, v]$. However, for b sufficiently small, $\varphi(cb) \le \varphi(c)|b|^p$. Therefore for any $v < \infty$ and $\varepsilon > 0$, if c is sufficiently small, we have $\varphi(cb) \le (1+\varepsilon)\varphi(c)|b|^p$ for all $b \le v$. Now, since $G(x)$ is uniformly continuous almost surely on $[0, a]$, we can find a δ sufficiently small, depending on ε and ω, such that for all ω in a set of measure one,

$$
\begin{aligned}
&\sup_{\pi\in Q_a(\delta)}\sum_{x_i\in\pi}\varphi\big(\big|G^2(x_i)-G^2(x_{i-1})\big|\big)I\Big(\sup_{x\in[0,a]}|G(x)|\le v\Big)\\
&\qquad\le(1+\varepsilon)2^p\sum_{j=1}^{m}\sup_{\tilde\pi\in Q_a(\delta)}\sum_{x_i\in\tilde\pi(I_{j,m}(a))}\varphi\big(|G(x_i)-G(x_{i-1})|\big)\\
&\qquad\qquad\times\sup_{x\in I_{j,m}(a)}|2G(x)|^p\\
&\qquad\quad+m\sup_{\substack{|x-y|\le\delta\\ x,y\in[0,a]}}\varphi\Big(|G(x)-G(y)|\sup_{x\in[0,a]}|2G(x)|\Big),
\end{aligned}\tag{2.33}
$$

where $I(A)$ denotes the indicator function of the set A. It is well known (see, e.g, Section IV, Theorem 1.3, in Jain and Marcus (1978)] that

$$
\limsup_{\delta\to0}\sup_{|x-y|\le\delta}\frac{|G(x)-G(y)|}{\big(\delta^{2/p}(\log 1/\delta)\big)^{1/2}}\le C\quad\text{a.s.,}\tag{2.34}
$$

for some absolute constant C. Thus the last term in (2.33) is $o(\delta)$. Using this fact and taking the limit as δ goes to 0 in (2.33), we get by (2.28) that

$$
\begin{aligned}
&\lim_{\delta\to0}\sup_{\pi\in Q_a(\delta)}\sum_{x_i\in\pi}\varphi\big(\big|G^2(x_i)-G^2(x_{i-1})\big|\big)I\Big(\sup_{x\in[0,a]}|G(x)|\le v\Big)\\
&\qquad\le(1+\varepsilon)(2\alpha)^p\sum_{j=1}^{m}\frac{a}{m}\sup_{x\in I_{j,m}(a)}|G(x)|^p\quad\text{a.s.}
\end{aligned}\tag{2.35}
$$

Finally, taking the limit as m goes to infinity we get

$$
\begin{aligned}
&\lim_{\delta\to0}\sup_{\pi\in Q_a(\delta)}\sum_{x_i\in\pi}\varphi\big(\big|G^2(x_i)-G^2(x_{i-1})\big|\big)I\Big(\sup_{x\in[0,a]}|G(x)|\le v\Big)\\
&\qquad\le(1+\varepsilon)(2\alpha)^p\int_0^a|G(x)|^p\,dx\quad\text{a.s.,}
\end{aligned}\tag{2.36}
$$

and since this holds for all $\varepsilon>0$ and all v, we get (2.29) but with a less than or equal sign. To get the opposite inequality we note that

$$
\begin{aligned}
&\sup_{\pi\in Q_a(\delta)}\sum_{x_i\in\pi}\varphi\big(\big|G^2(x_i)-G^2(x_{i-1})\big|\big)\\
&\qquad\ge\sum_{j=1}^{m}\sup_{\tilde\pi\in Q_a(\delta)}\sum_{x_i\in\tilde\pi(I_{j,m}(a))}\varphi\Big(|G(x_i)-G(x_{i-1})|\inf_{x\in I_{j,m}(a)}|2G(x)|\Big).
\end{aligned}\tag{2.37}
$$

By an argument similar to the above we note that for any $\varepsilon>0$ and $u>0$, for c sufficiently small, we have $\varphi(cb)\ge(1-\varepsilon)\varphi(c)|b|^p$ for all $b\ge u$. Therefore for any $\varepsilon > 0$ we can find a $\delta = \delta(\varepsilon)$, sufficiently small, such that the last term in (2.37) is

$$\begin{aligned}&\ge (1-\varepsilon)2^p \sum_{j=1}^{m} \sup_{\pi \in Q_a(\delta)} \sum_{x_i \in \pi(I_{j,m}(a))} \varphi(|G(x_i) - G(x_{i-1})|) \\ &\quad \times \inf_{x \in I_{j,m}(a)} |G(x)|^p I\Big(\inf_{x \in I_{j,m}(a)} |G(x)| \ge u\Big).\end{aligned} \tag{2.38}$$

Taking the limit in (2.37) first as δ goes to zero and then as m goes to infinity, we get that the left-hand side of (2.29) is

$$\begin{aligned}&\ge (1-\varepsilon)(2\alpha)^p \int_0^a |G(x)|^p I(G(x) \ge u)\,dx \\ &\ge (1-\varepsilon)(2\alpha)^p \Big(\int_0^a |G(x)|^p\,dx - au^p\Big).\end{aligned}$$

Since this is true for all $\varepsilon > 0$ and all $u > 0$, we obtain (2.29) but with a greater than or equal sign. This completes the proof of Theorem 2.4. □

Our major concern with Gaussian processes is to show that (1.12), (1.13), (2.28) and (2.29) are satisfied by the stationary Gaussian processes with covariance given by (1.10). Let us denote these Gaussian processes by $\{G_\beta(x), x \in R\}$. The functions $\sigma^2(h)$ for these processes, as defined in (1.11), are

$$\sigma_\beta^2(h) = \frac{2}{\pi}\int_0^\infty \frac{1 - \cos\lambda h}{1 + \lambda^\beta}\,d\lambda. \tag{2.39}$$

THEOREM 2.5. *Let $\{G_\beta(x), x \in R\}$ be the mean zero Gaussian processes with covariance given by* (1.10) *and increments variance* [*see* (1.11)] *given by* (2.39). *These processes satisfy* (1.12) *and* (1.13) *for the sequences of partitions considered in Theorem* 1.2 *and* (2.28) *and* (2.29), *where $p = 2/(\beta - 1)$ and*

$$\alpha = \alpha_\beta = \left(\frac{2}{\pi}\int_0^\infty \frac{1 - \cos y}{y^\beta}\,dy\right)^{1/2}. \tag{2.40}$$

PROOF. We cannot use Theorems 1.2 and 2.4 immediately because we have not established that $\sigma_\beta^2(h)$ is concave in $[0, \delta]$. We could show this analytically and compute α but it is simpler and more interesting to give a probabilistic proof. We introduce two Gaussian processes with stationary increments $\{G_{\beta,i}(x), x \in I\}$, $i = 1, 2$, defined by

$$G_{\beta,i}(x) = \frac{1}{\sqrt{\pi}}\left(\int_0^\infty (1 - \cos\lambda x) f_i(\lambda)\,dB(\lambda) + \int_0^\infty (\sin\lambda x)\, f_i(\lambda)\,dB'(\lambda)\right),$$

where

$$f_1(\lambda) = \frac{1}{\lambda^{\beta/2}} \quad\text{and}\quad f_2(\lambda) = \frac{1}{(\lambda^\beta(1 + \lambda^\beta))^{1/2}}$$

and B and B' are independent Brownian motions. Let G_β, $G_{\beta,1}$ and $G_{\beta,2}$ be independent and note that $G_{\beta,1}(x)$ and $G_\beta(x) - G_\beta(0) + G_{\beta,2}(x)$ are equivalent Gaussian processes. (They have the same covariance.) Furthermore, we see by a change of variables that

$$(2.41)\qquad E\big(G_{\beta,1}(x+h) - G_{\beta,1}(x)\big)^2 = \frac{2}{\pi}\int_0^\infty \frac{1-\cos\lambda h}{\lambda^\beta}\,d\lambda = (\alpha_\beta)^2 h^{\beta-1}$$

and that

$$(2.42)\qquad E\big(G_{\beta,2}(x+h) - G_{\beta,2}(x)\big)^2 = O\big(h^{(2\beta-1)\wedge 2}\big)$$

as h goes to zero. This last relationship follows from simple estimates. We now see by (2.41) that (2.11) is satisfied by $G_{\beta,1}$ and that α is given by (2.40). Therefore (2.11) is also satisfied by $G_\beta(x) - G_\beta(0) + G_{\beta,2}(x)$. However, it follows from (2.42), as in (2.34), that

$$(2.43)\qquad \limsup_{\delta\to 0}\ \sup_{|x-y|\le\delta} \frac{|G_{\beta,2}(x) - G_{\beta,2}(y)|}{\big(\delta^{(2\beta-1)\wedge 2}(\log 1/\delta)\big)^{1/2}} \le C \quad \text{a.s.}$$

It is easy to see that this implies that $\lim_{n\to\infty} |||G_{\beta,2}|||_{\pi(n),p} = 0$ for the partitions $\pi(n)$ defined in Theorem 2.3. Therefore it follows from the triangle inequality, after we take the $1/p$th root of each side of (2.11), that $G_\beta(x) - G_\beta(0)$ satisfies (2.11). Since (2.11) is the same for $G_\beta(x)$ and $G_\beta(x) - G_\beta(0)$, we see that $G_\beta(x)$ also satisfies (2.11). This immediately implies that $G_\beta(x)$ satisfies (1.12). It also implies that it satisfies (1.13), since the proof of Theorem 1.2, (1.13) only requires that the Gaussian process is continuous and satisfies (2.11).

We now show that $G_\beta(x) - G_\beta(0)$ and hence $G_\beta(x)$ satisfies (2.28). It is clear that $G_{\beta,1}(x)$ satisfies (2.28) and therefore so does $G_\beta(x) - G_\beta(0) + G_{\beta,2}(x)$. For $x \in R$ define $\bar\varphi(x) = \varphi(|x|)$. Clearly $\bar\varphi(x)$ is convex for $x \in [-\delta,\delta]$ for some $\delta > 0$. Therefore for any $\varepsilon > 0$, for all $|a|$ and $|b|$ sufficiently small, depending on ε, we have

$$\bar\varphi\left(\frac{a}{1-\varepsilon}\right) \ge \frac{1}{1-\varepsilon}\bar\varphi(a+b) - \frac{\varepsilon}{1-\varepsilon}\bar\varphi\left(\frac{b}{\varepsilon}\right)$$

and

$$(2.44)\qquad \bar\varphi(a) \le (1-\varepsilon)\bar\varphi\left(\frac{a+b}{1-\varepsilon}\right) + \varepsilon\bar\varphi\left(\frac{-b}{\varepsilon}\right).$$

These inequalities follow from Jensen's inequality. We use them in (2.28) with φ replaced by $\bar\varphi$ and with $a = G_\beta(x_i) - G_\beta(x_{i-1})$ and $b = G_{\beta,2}(x_i) - G_{\beta,2}(x_{i-1})$. Since all these processes are uniformly continuous, there is no problem in taking the terms arbitrarily small. It should be clear now that in order to show that $G_\beta(x)$ satisfies (2.28), we need only show that for all $\varepsilon > 0$,

$$\lim_{\delta \to 0} \sup_{\pi \in Q_a(\delta)} \sum_{x_i \in \pi} \varphi\left(\frac{|G_{\beta,2}(x_i) - G_{\beta,2}(x_{i-1})|}{\varepsilon} \right) = 0 \quad \text{a.s.} \tag{2.45}$$

This follows immediately from (2.43) since $((2\beta - 1) \wedge 2)(p/2) > 1$. Thus we see that $G_\beta(x)$ satisfies (2.28). This also means that it satisfies (2.29) since in the proof of Theorem 2.4, we showed that any uniformly continuous process that satisfies (2.28) satisfies (2.29). This completes the proof of Theorem 2.5. □

We now have all the material about Gaussian processes that will be used in Section 3 to prove Theorem 1.1. In the rest of this section we show that the condition on the mesh size in Theorem 1.2 is close to "best possible." This will be used in Theorem 3.5 to obtain a similar result for the local times.

THEOREM 2.6. *For any $b > 0$ we can find a sequence of partitions $\{\pi_n\}_{n=1}^\infty$ with $m(\pi_n) \le b/\log n$ such that* (1.12) *and* (1.13) *of Theorem* 1.2 *are false for all of the Gaussian processes that satisfy the hypotheses of Theorem* 1.2 *with* $0 < \alpha < \infty$.

Theorem 2.6 shows that (1.12) and (1.13) are best possible for $p = 2$. There is a gap for $p > 2$. This theorem, for the case of (1.12), was proved by de la Vega (1974) when G is Brownian motion. Actually, he only states the result for $b \ge 3$. However a minor modification of his proof does give all $b > 0$. The proof of Theorem 2.6 closely follows the proof of de la Vega (1974).

PROOF. To simplify matters we take $\alpha = a = 1$. That the proof is valid for all $0 < \alpha < \infty$ and $0 < a < \infty$ should be obvious from the proof of this case. The set of partitions that we use to obtain the examples that establish Theorem 2.6 are the same as the ones used in de la Vega (1974). The description of these partitions that follows is taken, almost verbatim, from this reference.

Consider the sequence of partitions $\{\pi_n\}$ constructed as follows: π_0 is the partition consisting of the interval $[0, 1]$. For each integer $q \ge 1$, in turn, we add to the sequence, in an arbitrary order, all those partitions of $[0, 1]$ each of which contains for each integer k, $0 \le k \le 2^{q-1} - 1$, either the interval $J_q^k = [2k/2^q, 2k + 2/2^q]$ or both intervals $I_q^{2k} = [2k/2^q, 2k + 1/2^q]$ and $I_q^{2k+1} = [2k + 1/2^q, 2k + 2/2^q]$. Call Π_q the set containing these partitions.

There are $2^{2^{q-1}}$ partitions in Π_q. One of them has mesh 2^{-q}; all the others have mesh 2^{1-q}. Their ranks in the sequence $\{\pi_n\}$ are bounded above by

$$1 + \sum_{0 \le r \le q-1} 2^{2^r} < 2^{1+2^{q-1}}.$$

One can verify that $m(\pi_n) \le 3/\log n$ for all $n > 1$.

For the Gaussian processes considered in Theorem 1.2, we define for $0 \le k \le 2^{q-1} - 1$,

$$L(I_q^{2k}) = G\left(\frac{2k+1}{2^q}\right) - G\left(\frac{2k}{2^q}\right),$$

$$(2.46) \qquad L(I_q^{2k+1}) = G\left(\frac{2k+2}{2^q}\right) - G\left(\frac{2k+1}{2^q}\right),$$

$$L(J_q^k) = G\left(\frac{2k+2}{2^q}\right) - G\left(\frac{2k}{2^q}\right)$$

and

$$M_q^k = \max\left\{\left(L(I_q^{2k})\right)^p + \left(L(I_q^{2k+1})\right)^p, \left(L(J_q^k)\right)^p\right\}.$$

We have

$$(2.47) \qquad EM_q^k = \sigma^p(2^{-q}) E \max\{|\xi_q|^p + |\eta_q|^p, |\xi_q + \eta_q|^p\},$$

where ξ_q and η_q are normal random variables with mean zero variance 1 and

$$E\xi_q\eta_q = -\left(1 - \frac{\sigma^2(1/2^{q-1})}{2\sigma^2(1/2^q)}\right).$$

We now show that

$$(2.48) \qquad \lim_{q\to\infty} E \sup_{\pi\in\Pi_q} |||G|||_{\pi,p}^p = (1+c)E|\eta|^p$$

for some $c > 0$, where η is a normal random variable with mean zero and variance 1. To see this note that

$$(2.49) \qquad E \sup_{\pi\in\Pi_q} |||G|||_{\pi,p}^p = \sum_{k=1}^{2^{q-1}-1} E(M_q^k)$$

and so, by (2.47),

$$(2.50) \qquad \lim_{q\to\infty} E \sup_{\pi\in\Pi_q} |||G|||_{\pi,p}^p = \tfrac{1}{2} \lim_{q\to\infty} E \max\{|\xi_q|^p + |\eta_q|^p, |\xi_q + \eta_q|^p\}.$$

To evaluate the right-hand side of (2.50) we note that

$$(2.51) \qquad \lim_{q\to\infty} E\xi_q\eta_q = -(1 - 2^{(2/p)-1}).$$

This shows that for q sufficiently large, $f_q(\cdot,\cdot)$, the joint density of ξ_q and η_q, exists and is greater than zero on all of R^2. Therefore, $\lim_{q\to\infty} f_q(\cdot,\cdot) = f(\cdot,\cdot)$ is a strictly positive joint density of normal random variables with mean zero and variance 1 and with covariance given by the right-hand side of (2.51). Let

$g(x,y) = \max\{|x|^p + |y|^p, |x+y|^p\}$. Then

$$
\begin{aligned}
&E\max\{|\xi_q|^p + |\eta_p|^p, |\xi_q + \eta_q|^p\} \\
(2.52)\qquad &= \int_0^\infty \iint_{g(x,y)\ge\lambda} f_q(x,y)\,dx\,dy\,d\lambda \\
&= \int_0^\infty \lambda^{2/p} \iint_{g(u,v)\ge 1} f_q(\lambda^{1/p}u, \lambda^{1/p}v)\,du\,dv\,d\lambda.
\end{aligned}
$$

Taking the limit as q goes to infinity in (2.52) and using the dominated convergence theorem, we get

$$
\begin{aligned}
&\lim_{q\to\infty} E\max\{|\xi_q|^p + |\eta_q|^p, |\xi_q + \eta_q|^p\} \\
(2.53)\qquad &= \iint_{g(u,v)\ge 1}\int_0^\infty \lambda^{2/p} f(\lambda^{1/p}u, \lambda^{1/p}v)\,du\,dv\,d\lambda.
\end{aligned}
$$

By the same reasoning we have

$$
\begin{aligned}
2E|\eta|^p &= \lim_{q\to\infty} E\big(|\xi_q|^p + |\eta_q|^p\big) \\
(2.54)\qquad &= \iint_{|u|^p + |v|^p \ge 1}\int_0^\infty \lambda^{2/p} f(\lambda^{1/p}u, \lambda^{1/p}v)\,du\,dv\,d\lambda.
\end{aligned}
$$

Because of the different areas of integration in the (u,v)-plane we see that the right-hand side of (2.53) is equal to $(1+c)$ times the right-hand side of (2.54) for some $c > 0$. Using this in (2.50) we get (2.48).

We now show that

$$
(2.55)\qquad \lim_{q\to\infty}\sup_{\pi\in\Pi_q} |||G|||_{\pi,p}^p = (1+c)E|\eta|^p
$$

for c given in (2.48). To do this we use Lemma 2.2, exactly as it was used in the proof of Theorem 2.3, but with $|||G|||_{\pi(n),p}$ replaced by $\sup_{\pi\in\Pi_q}|||G|||_{\pi,p}$. Analagous to (2.13), we have

$$
(2.56)\qquad \lim_{q\to\infty}\left(\sup_{\pi\in\Pi_q}|||G|||_{\pi,p} - \operatorname{med}\left(\sup_{\pi\in\Pi_q}|||G|||_{\pi,p}\right)\right) = 0 \quad \text{a.s.},
$$

because, in this case, for q fixed, $\hat\sigma^2 \le 4^{1-q/p}$ for all q sufficiently large. Let $\tilde M_q = \operatorname{med}(\sup_{\pi\in\Pi_q}|||G|||_{\pi,p})$. By (2.47) and (2.49),

$$
(2.57)\qquad \tilde M_q \le 2E\left(\sup_{\pi\in\Pi_q}|||G|||_{\pi,p}\right) \le C
$$

for some constant $C < \infty$, independent of q. Using this in Lemma 2.2 [in particular, in (2.5) and (2.8)], we see that there exist finite constants $C(r)$ such that

$$
(2.58)\qquad E\sup_{\pi\in\Pi_q}|||G|||_{\pi,p}^r \le C(r) \qquad \forall\, q \ge 1.
$$

Following the approach of (2.15) and (2.16), we can show that

$$\lim_{q\to\infty} E \sup_{\pi\in\Pi_q} |||G|||^p_{\pi,p} = \lim_{q\to\infty} \left|\tilde{M}_q\right|^p. \tag{2.59}$$

Using (2.48), (2.59) and (2.56), we get (2.55).

We see from (2.55) that

$$\limsup_{n\to\infty} |||G|||^p_{\pi_n,p} = (1+c)E|\eta|^p. \tag{2.60}$$

This shows that (1.12) does not hold for a sequence of partitions $\{\pi_n\}$ for which $m(\pi_n) \le 3/\log n$. It is easy to improve this to get that (1.12) does not hold for sequences of partitions $\{\pi_n\}$ for which $m(\pi_n) \le b/\log n$ for all $b > 0$. For q fixed we create $2^{2^{q-1}} - 1$ different partitions on $[0, 2^{-j}]$ just as we did above on $[0,1]$. To each of these we add $2^q(2^j - 1)$ intervals of size $2^{-(j+q)}$ on $[2^{-j}, 1]$. We still have $2^{2^{q-1}} - 1$ partitions but now they are of length $2^{-(j+q)}$. Since the partitions on $[2^{-j}, 1]$ are all the same, the sum of the pth powers of the increments converge (as q goes to infinity) to $(1 - 2^{-j})E|\eta|^p$ but on $[0, 2^{-j}]$, as above, it converges to $2^{-j}(1+c)E|\eta|^p$. Thus we still have a counterexample to (1.12), but now $m(\pi_n) \le 2^{-j}(3/\log n)$. Since this holds for all $j \ge 0$ we obtain Theorem 2.5, as stated, for the case (1.12). □

As we stated above, the same proof works for all $0 < \alpha$, $a < \infty$. Thus if the partitions are imposed on $[0, a]$ and $\lim_{h\to\infty} \sigma(h)/h^{1/p} = \alpha$, we have

$$\lim_{q\to\infty} \sup_{\pi\in\Pi_q} |||G|||^p_{\pi,p} = (1+c)\alpha^p a E|\eta|^p. \tag{2.61}$$

We now show that this implies that

$$\lim_{q\to\infty} \sup_{\pi\in\Pi_q} |||G^2|||^p_{\pi,p} = (1+c)(2\alpha)^p E|\eta|^p \int_0^a |G(x)|^p\, dx \quad \text{a.s.} \tag{2.62}$$

The proof of (2.62) follows from (2.61) (in the case $\alpha = a = 1$), in the same way that (1.13) follows from (1.12) in (2.25)–(2.27). It is actually easier in this case because the sets of partitions Π_q are nested as q increases. Let q' and r be positive integers. Let $\{\Pi_{q',j}\}_{j=1}^{2^r}$ be a set of partitions of the form of $\Pi_{q'}$ on the interval $[(j-1)/2^r, j/2^r] \equiv I_{j,r}$ (rather than on $[0,1]$ as we did earlier in this proof). For $q = q' + r$, $\pi \in \Pi_q$ is a partition of $[0,1]$ formed by putting together one partition from each $\{\Pi_{q',j}\}_{j=1}^{2^r}$. We have

$$\begin{aligned}
\limsup_{q\to\infty} \sup_{\pi\in\Pi_q} |||G^2|||^p_{\pi,p} &= \limsup_{q\to\infty} \sum_{j=1}^{2^r} \sup_{\pi\in\Pi_{q',j}} |||G^2|||^p_{\pi,p} \\
&\le \sum_{j=1}^{2^r} \lim_{q'\to\infty} \sup_{\pi\in\Pi_{q',j}} |||G|||^p_{\pi,p} \sup_{x\in I_{j,r}} |2G(x)|^p \\
&= (1+c)2^p E|\eta|^p \sum_{j=1}^{2^r} \sup_{x\in I_{j,r}} |G(x)|^p \frac{1}{2^r}.
\end{aligned}$$

Since we can get the opposite inequality with the limit superior and supremum replaced by the limit inferior and infimum, we get (2.62) by the definition of Riemann integration. Also, as above, we can modify the partitions such that $m(\pi_n) < b/\log n$ for any $b > 0$.

REMARK 2.7. For later use let us note that (2.62), and the final comment in the above proof, imply that for any $b > 0$, we can find a sequence of partitions $\{\pi_n\}$ with $m(\pi_n) < b/\log n$ such that

$$(2.63)\qquad \limsup_{n\to\infty} |||\, G^2 \,|||^{\,p}_{\pi_n, p} = (1 + c)(2\alpha)^p E|\eta|^p \int_0^a |G(x)|^p \, dx \quad \text{a.s.},$$

for some $c > 0$. Also, by the proof of Theorem 2.5 and in particular (2.43), we see that (2.63) is also true with G and α replaced by G_β and α_β and $p = 2/(\beta - 1)$.

3. Local times. Our main tool is Lemma 4.3 in Marcus and Rosen (1992). The next result is an immediate corollary of this lemma.

LEMMA 3.1. *Let $\{L_t^x, (t, x) \in R^+ \times R\}$ be a jointly continuous local time of a real-valued symmetric stable process of index $1 < \beta \le 2$ and let $G = \{G(x), x \in R^+\}$ be the associated Gaussian process. Let (Ω_G, P_G) be the probability space of G. Let $B \in \mathscr{C}$, where $\mathscr{C}$ is the σ-algebra generated by the continuous functions on R, be such that $P_B(G_\cdot^2/2 \in B) = 1$. Then for almost all $\omega \in \Omega_G$ with respect to P_G,*

$$P^x\left(L_t^{\cdot} + \frac{G_\cdot^2(\omega)}{2} \in B \text{ for almost all } t \right) = 1,$$

where P^x is the probability measure corresponding to the symmetric stable process starting at x at time 0.

PROOF. This is a minor modification of Lemma 4.3 of Marcus and Rosen (1992). In that lemma the spatial variable is countable but since these processes are continuous on R we can think of them as defined on all of R. Also in that lemma, which is applicable in a more general setting than the one considered here, we are concerned with the lifetime of the Markov process, but this is infinite for Lévy processes. □

A good part of Theorem 1.1 can be obtained with no further consideration of local times.

PROOF OF THEOREM 1.1(ii), (iii) and (iv). We will first prove (ii). Let G_β be the Gaussian process associated with the symmetric stable process of index β. It follows from Theorem 2.5 that, under the condition on $m(\pi(n))$ given in (ii), we have

$$(3.1)\quad \lim_{n\to\infty} |||\, G_\beta^2/2 \,|||_{\pi(n),p} = \sqrt{2}\,\alpha(E|\eta|^p)^{1/p}\left(\int_0^a |G_\beta^2(x)/2|^{p/2}\,dx\right)^{1/p} \quad \text{a.s.},$$

where α is given in (2.40) and $p = 2/(\beta - 1)$. Therefore, by Lemma 3.1, for almost all $\omega \in \Omega_{G_\beta}$,

$$(3.2)\quad \begin{aligned} &\lim_{n\to\infty} \left|\left|\left| L_t + G_\beta^2(\omega)/2 \right|\right|\right|_{\pi(n),p} \\ &\quad = \sqrt{2}\,\alpha(E|\eta|^p)^{1/p}\left(\int_0^a |L_t^x + G_\beta^2(x,\omega)/2|^{p/2}\,dx\right)^{1/p} \end{aligned}$$

for almost all t a.s.

It follows that for almost all $\omega \in \Omega_{G_\beta}$,

$$(3.3)\quad \begin{aligned} &\limsup_{n\to\infty} |||\, L_t \,|||_{\pi(n),p} \\ &\le \sqrt{2}\,\alpha(E|\eta|^p)^{1/p}\left(\left(\int_0^a |L_t^x|^{p/2}\,dx\right)^{1/p} + \left(\int_0^a |G_\beta^2(x,\omega)/2|^{p/2}\,dx\right)^{1/p}\right) \\ &\quad + \limsup_{n\to\infty} \left|\left|\left| G_\beta^2(\omega)/2 \right|\right|\right|_{\pi(n),p} \quad \text{for almost all } t \text{ a.s.} \end{aligned}$$

Using (3.1) on the last term in (3.3) we see that for almost all $\omega \in \Omega_{G_\beta}$,

$$(3.4)\quad \begin{aligned} &\limsup_{n\to\infty} |||\, L_t \,|||_{\pi(n),p} \\ &\le \sqrt{2}\,\alpha(E|\eta|^p)^{1/p}\left(\left(\int_0^a |L_t^x|^{p/2}\,dx\right)^{1/p} + 2\left(\int_0^a |G_\beta^2(x,\omega)/2|^{p/2}\,dx\right)^{1/p}\right) \end{aligned}$$

for almost all t a.s.

Since the associated Gaussian processes have continuous sample paths, for all $\varepsilon > 0$,

$$(3.5)\quad P\left(\sup_{x\in[0,a]} |G_\beta(x)| < \varepsilon\right) > 0$$

[see, e.g., Theorem 2.6, Marcus and Rosen (1992)]. Therefore we can choose ω in (3.4) so that the integral involving the Gaussian process can be made arbitrarily small. Thus

$$(3.6)\quad \limsup_{n\to\infty} |||\, L_t \,|||_{\pi(n),p} \le \sqrt{2}\,\alpha(E|\eta|^p)^{1/p}\left(\int_0^a |L_t^x|^{p/2}\,dx\right)^{1/p}$$

for almost all t a.s.

By the same methods we can obtain the reverse of (3.6) for the limit inferior. Since $c(\beta) = (\sqrt{2}\,\alpha)^p E|\eta|^p$, we obtain Theorem 1.1(ii). We will postpone the verification of (1.4) until the end of this section.

To simplify the notation we denote, for real-valued functions $\{\tau(x), x \in R^+\}$ and $\{f(x), x \in [0, a]\}$,

$$\tilde{V}_{\tau, a}(f) = \lim_{\delta \to 0} \sup_{\pi \in Q_a(\delta)} \sum_{x_i \in \pi} \tau(|f(x_i) - f(x_{i-1})|). \tag{3.7}$$

The proof of (iv) is basically exactly the same as the proof of (ii). It follows from Theorem 2.5 that

$$\tilde{V}_{\varphi, a}(G_\beta^2/2) = (\sqrt{2}\alpha)^p \int_0^a |G_\beta^2(x)/2|^{p/2}\, dx \quad \text{a.s.}, \tag{3.8}$$

where φ is given in the statement of Theorem 2.5, α is given in (2.40) and $p = 2/(\beta - 1)$. Therefore, by Lemma 3.1, for almost all $\omega \in \Omega_{G_\beta}$,

$$\tilde{V}_{\varphi, a}(L_t + G_\beta^2(\cdot, \omega)/2) = (\sqrt{2}\alpha)^p \int_0^a |L_t^x + G_\beta^2(x, \omega)/2|^{p/2}\, dx \tag{3.9}$$

for almost all t a.s.

We note that for all $c > 0$, $\varphi(c|x|) \le (1 + \delta)c^p \varphi(|x|)$ for any $\delta > 0$, for all x sufficiently small. Using this and (2.44) we see that for almost all $\omega \in \Omega_{G_\beta}$ and $0 < \varepsilon \le 1/2$,

$$\begin{aligned} \tilde{V}_{\varphi, a}(L_t) &\le (1 + \delta)(1 - \varepsilon)^{(2-p)/2} \tilde{V}_{\varphi, a}(L_t + G_\beta^2(\cdot, \omega)/2) \\ &\quad + (1 + \delta)\varepsilon^{(2-p)/2} \tilde{V}_{\varphi, a}(G_\beta^2(\cdot, \omega)/2) \quad \text{for almost all } t \text{ a.s.} \end{aligned}$$

Therefore by (3.8) and (3.9) for almost all $\omega \in \Omega_{G_\beta}$,

$$\begin{aligned} \tilde{V}_{\varphi, a}(L_t) &\le (1 + \delta)(1 - \varepsilon)^{(2-p)/2} (\sqrt{2}\alpha)^p \left(\int_0^a |L_t^x + G_\beta^2(x, \omega)/2|^{p/2}\, dx \right) \\ &\quad + (1 + \delta)\varepsilon^{(2-p)/2} (\sqrt{2}\alpha)^p \int_0^a |G_\beta^2(x, \omega)/2|^{p/2}\, dx \end{aligned}$$

for almost all t a.s.

Using (3.5), we can choose an ω such that $\sup_{x \in [0, a]} |G_\beta(x, \omega)|$ can be made arbitrarily small. It follows that

$$\tilde{V}_{\varphi, a}(L_t) \le (\sqrt{2}\alpha)^p \int_0^a |L_t^x|^{p/2}\, dx \quad \text{for almost all } t \text{ a.s.} \tag{3.10}$$

A similar argument gives (3.10) with a greater than or equal sign. Thus we have

$$\tilde{V}_{\varphi, a}(L_t) = (\sqrt{2}\alpha)^p \int_0^a |L_t^x|^{p/2}\, dx \quad \text{for almost all } t \text{ a.s.} \tag{3.11}$$

We now show that (3.11) holds for all $t \in R^+$. Let Q be a countable dense subset of R^+. It follows from (3.11) that there exists a $t_0 \in R^+$ such that

$$\tilde{V}_{\varphi, b}(L_{t_0}) = (\sqrt{2}\alpha)^p \int_0^b |L_{t_0}^x|^{p/2}\, dx \qquad \forall\, b \in Q \quad \text{a.s.}, \tag{3.12}$$

and hence, since $\tilde{V}_{\varphi, b}(L_{t_0})$ is monotone in b, for all $b \in R^+$ almost surely.

Using (3.12) and the fact that φ is regularly varying at zero we see that for any $t \in R^+$,

$$
\begin{aligned}
&\tilde{V}_{\varphi, c(t_0/t)^{1/\beta}}\Big((t/t_0)^{1/\bar{\beta}} L_{t_0}\Big) \\
&\quad = (t/t_0)^{p/\bar{\beta}} \tilde{V}_{\varphi, c(t_0/t)^{1/\beta}}(L_{t_0}) \\
&\quad = (t/t_0)^{p/\bar{\beta}} (\sqrt{2}\,\alpha)^p \int_0^{c(t_0/t)^{1/\beta}} |L_{t_0}^x|^{p/2}\,dx \qquad (3.13)\\
&\quad = (\sqrt{2}\,\alpha)^p \int_0^c \Big|(t/t_0)^{1/\bar{\beta}} L_{t_0}^{y(t_0/t)^{1/\beta}}\Big|^{p/2}\,dy \quad \text{for all } c > 0 \text{ a.s.},
\end{aligned}
$$

where $1/\beta + 1/\bar{\beta} = 1$. As is well known, by rescaling the stable process, one sees that $(t/t_0)^{1/\bar{\beta}} L_{t_0}^{x(t_0/t)^{1/\beta}}$ is equal in distribution, as a function of x, to L_t^x. Therefore, by (3.13), we get that

$$
\tilde{V}_{\varphi, c}(L_t) = (\sqrt{2}\,\alpha)^p \int_0^c |L_t^x|^{p/2}\,dx \quad \text{for all } c > 0 \text{ a.s.} \tag{3.14}
$$

Thus we get (1.7) except for verification that $(\sqrt{2}\,\alpha)^p = c'(\beta)$. We will do this at the end of this section.

Finally we note that (iv) clearly implies (iii).

REMARK 3.2. We cannot use an argument similar to the one used to prove Theorem 1.1(iv) to show that Theorem 1.1(ii) holds for each $t \in R^+$ almost surely. This is because in (ii), as stated, the subsets of R^+ of measure zero for which (1.5) may not hold could depend on the particular sequence of partitions $\{\pi(n)\}$. Thus we cannot use scaling because we do not know if a sequence of partitions for which (1.5) holds for $L_{t_0}^{\cdot}$ will allow (1.5) to hold when they are rescaled, as we did above, to consider $L_t^{\cdot}$. Of course, what we can say is that for any $t \in R^+$, there are many sequences of partitions $\{\pi(n)\}$ with $m(\pi(n)) = o(1/\log n)^{1/(\beta-1)}$ for which (1.5) holds.

Before we can complete the proof of Theorem 1.1, we will need the following lemmas about the local times of symmetric stable processes of order $\beta > 1$. For a random variable, say Z, on the probability space of the Lévy process X, we denote by $\|Z\|_r$ the L^r norm of Z with respect to P^0, the probability measure of the process which is zero at time zero.

LEMMA 3.3. *Let $X = \{X(t),\ t \in R^+\}$ be a real-valued symmetric stable process of index $1 < \beta \le 2$ and let $\{L_t^x,\ (t, x) \in R^+ \times R\}$ be the local time of X. Then for all $x, y \in R$, $s, t \in R^+$ and integers $m \ge 1$,*

$$
\|L_t^x - L_t^y\|_{2m} \le C(\beta)((2m)!)^{1/(2m)} t^{(\beta-1)/(2\beta)} |x-y|^{(\beta-1)/2} \tag{3.15}
$$

and

$$
\|L_t^x - L_s^x\|_m \le C'(\beta)(m!)^{1/m} |t-s|^{(\beta-1)/\beta}, \tag{3.16}
$$

where $C(\beta)$ and $C'(\beta)$ are constants depending only on β.

Note that since (3.15) only depends on $|x - y|$, the inequality remains the same if we take the norm with respect to P^z for any $z \in R$. The inequality in (3.16) also remains unchanged if we take the norm with respect to P^z for any $z \in R$.

PROOF OF LEMMA 3.3. Let $p_t(u) \equiv p_t(x, x + u)$ denote the transition probability densities of X. By Lemma 1 of Rosen (1990b), we see that

$$\begin{aligned}\|L_t^x - L_t^y\|_{2m}^{2m} = (2m)! \int_{0 \le t_1 \le \cdots \le t_{2m} \le t} \cdots \int (p_{t_1}(x) + p_{t_1}(y)) \\ \times \prod_{i=2}^{2m} \left(p_{\Delta t_i}(0) - (-1)^{2m-i} p_{\Delta t_i}(x - y)\right) dt,\end{aligned} \tag{3.17}$$

where $\Delta t_i = t_i - t_{i-1}$. Writing $p_t(x)$ as the Fourier transform of its characteristic function, it is easy to see that $p_t(x) \le p_t(0)$. Thus we have that (3.17) is

$$\le (2m)! 2^m \left(\int_0^t p_s(0)\, ds\right)^m \left(\int_0^\infty (p_t(0) - p_t(x - y))\, dt\right)^m. \tag{3.18}$$

We obtain (3.15) from (3.18) by using (3.3) and (3.4) of Rosen (1990b).

To obtain (3.16) we note that

$$\|L_t^x - L_s^x\|_m = \|L_{t-s}^x \circ \theta_s\|_m = \left(E^0\{E^{X_s}(L_{t-s}^x)^m\}\right)^{1/m}, \tag{3.19}$$

where θ denotes the shift operator on the space of paths of X and without loss of generality, we assume that $t \ge s$. Note that for any z,

$$\begin{aligned}E^z(L_{t-s}^x)^m &= m! E^z\left(\int_{0 \le t_1 \le \cdots \le t_m \le t-s} \cdots \int \prod_{i=1}^m dL_{t_i}^x\right) \\ &= m! \int_{0 \le t_1 \le \cdots \le t_m \le t-s} \cdots \int p_{t_1}(x - z) \\ &\quad \times p_{t_2 - t_1}(0) \cdots p_{t_m - t_{m-1}}(0)\, dt_1 \cdots dt_m \\ &\le m! \left(\int_0^{t-s} p_r(0)\, dr\right)^m.\end{aligned} \tag{3.20}$$

Using (3.19) and (3.20) and (3.3) and (3.4) of Rosen (1990b), we get (3.16). □

LEMMA 3.4. *Let $X = \{X(t),\ t \in R^+\}$ be a real-valued symmetric stable process of index $1 < \beta \le 2$ and let $\{L_t^x, (t, x) \in R^+ \times R\}$ be the local time of X. Let $p = 2/(\beta - 1)$. Then for all partitions π of $[0, a]$, $s, t \in R^+$, with $s \le t$, and integers $m \ge 1$,*

$$\begin{aligned}&\left\| \,|||L_t|||_{\pi,p}^p - |||L_s|||_{\pi,p}^p \right\|_m \\ &\quad \le C(\beta, m, p) t^{(p-1)(\beta-1)/(2\beta)} |t - s|^{(\beta-1)/(2\beta)} a.\end{aligned} \tag{3.21}$$

In particular,

$$\| \,|||\, L_t \,|||\, _{\pi, p}^{p} \|_m^{1/p} \le C'(\beta, m, p) t^{(\beta-1)/(2\beta)} a^{1/p}, \tag{3.22}$$

where $C(\beta, m, p)$ and $C'(\beta, m, p)$ are constants depending only on β, m and p.

PROOF. For a partition π, we set

$$\Delta L_t^{x_i} = L_t^{x_i} - L_t^{x_{i-1}}, \qquad i = 1, \dots, k_\pi. \tag{3.23}$$

By the mean value theorem,

$$|u^p - v^p| \le p(u^{p-1} + v^{p-1})|u - v|.$$

Since $s \le t$, we see that

$$\begin{aligned} &\| \,|||\, L_t \,|||\, _{\pi, p}^{p} - \,|||\, L_s \,|||\, _{\pi, p}^{p} \|_m \\ &\quad \le \sum_{x_i \in \pi} \left\| |\Delta L_t^{x_i}|^p - |\Delta L_s^{x_i}|^p \right\|_m \\ &\quad \le \sum_{x_i \in \pi} p\left(\left\| |\Delta L_t^{x_i}|^{p-1} \right\|_{2m} + \left\| |\Delta L_s^{x_i}|^{p-1} \right\|_{2m} \right) \| \Delta L_t^{x_i} - \Delta L_s^{x_i} \|_{2m}. \end{aligned} \tag{3.24}$$

Let r be the smallest even integer greater than or equal to $2m(p-1)$. Then by Hölder's inequality and (3.15), we see that

$$\begin{aligned} \left\| |\Delta L_t^{x_i}|^{p-1} \right\|_{2m} &\le \| \Delta L_t^{x_i} \|_r^{p-1} \\ &\le D(\beta, m, p) t^{(p-1)(\beta-1)/(2\beta)} (x_i - x_{i-1})^{(p-1)(\beta-1)/2}, \end{aligned} \tag{3.25}$$

where $D(\beta, m, p) = (C(\beta)((2r)!)^{1/(2r)})^{p-1}$ and $C(\beta)$ is the constant in (3.16). We also have that

$$\| \Delta L_t^{x_i} - \Delta L_s^{x_i} \|_{2m} = \| \Delta L_{t-s}^{x_i} \circ \theta_s \|_{2m} = \left(E^0 \left\{ E^{X_s} (\Delta L_{t-s}^{x_i})^{2m} \right\} \right)^{1/2m}. \tag{3.26}$$

It follows from (3.15) and the remark immediately following the statement of Lemma 3.3 that for all $z \in R$,

$$\begin{aligned} \left(E^z (\Delta L_{t-s}^{x_i})^{2m} \right)^{1/2m} &= \| \Delta L_{t-s}^{x_i} \|_{2m} \\ &\le D'(\beta, m) |t - s|^{(\beta-1)/(2\beta)} |x_i - x_{i-1}|^{(\beta-1)/2}, \end{aligned} \tag{3.27}$$

where $D'(\beta, m) = C(\beta)((2m)!)^{1/(2m)}$. Combining (3.24)–(3.27) we see that

$$\begin{aligned} &\| \,|||\, L_t \,|||\, _{\pi, p}^{p} - \,|||\, L_s \,|||\, _{\pi, p}^{p} \|_m \\ &\quad \le 2pD(\beta, m, p) D'(\beta, m) t^{(p-1)(\beta-1)/(2\beta)} |t - s|^{(\beta-1)/(2\beta)} \\ &\qquad \times \sum_{x_i \in \pi} (x_i - x_{i-1})^{(\beta-1)/2} (x_i - x_{i-1})^{(p-1)(\beta-1)/2}. \end{aligned} \tag{3.28}$$

This gives (3.21) since the sum in (3.28) is equal to a. The statement in (3.22) follows from (3.21) by setting $s = 0$. □

PROOF OF THEOREM 1.1 (i). Although in (i) we are dealing with a weaker form of convergence than in (ii), the only way that we know to prove (i) is through (ii). The main point is to show that for any sequence of partitions $\{\pi(n)\}$ of $[0, a]$ with $\lim_{n\to\infty} m(\pi(n)) = 0$, $|||L_t|||_{\pi(n),p}$ converges in probability for each $t \in R^+$, where $p = 2/(\beta - 1)$. Suppose that this is not the case. Then there exists a subsequence $\{\pi(n(k))\}$ of $\{\pi(n)\}$ with $m(\pi(n(k))) = o(1/\log k)^{1/\beta - 1}$ with further subsequences $\{\pi(n(k_j))\}_{j=1}^{\infty}$ and $\{\pi(n(k'_j))\}_{j=1}^{\infty}$ for which

$$(3.29)\qquad P\Big(\big|\, |||L_t|||_{\pi(n(k_j)),p} - |||L_t|||_{\pi(n(k'_j)),p} \big| > \varepsilon\Big) > \varepsilon \qquad \forall\, j \ge 1.$$

It follows from (3.21) that we can find a $\delta > 0$ such that for all $s \in [t, t+\delta]$,

$$P\Big(\big|\, |||L_s|||_{\pi(n(k_j)),p} - |||L_t|||_{\pi(n(k_j)),p} \big| > \varepsilon/4\Big) \le \varepsilon/4 \qquad \forall\, j \ge 1,$$

and similarly with k_j replaced by k'_j. Therefore we have, for all $s \in [t, t+\delta]$ and all j greater than or equal to 1, that

$$\begin{aligned} &P\Big(\big|\, |||L_s|||_{\pi(n(k_j)),p} - |||L_s|||_{\pi(n(k'_j)),p} \big| > \varepsilon/2\Big) \\ &\quad\ge P\Big(\big|\, |||L_t|||_{\pi(n(k_j)),p} - |||L_t|||_{\pi(n(k'_j)),p} \big| > \varepsilon\Big) \\ &\qquad - P\Big(\big|\, |||L_s|||_{\pi(n(k_j)),p} - |||L_t|||_{\pi(n(k_j)),p} \big| > \varepsilon/4\Big) \\ &\qquad - P\Big(\big|\, |||L_s|||_{\pi(n(k'_j)),p} - |||L_t|||_{\pi(n(k'_j)),p} \big| > \varepsilon/4\Big) \ge \varepsilon/2. \end{aligned}$$

This implies that $|||L_s|||_{\pi(n(k)),p}$ does not converge in probability for any of the values of $s \in [t, t+\delta]$. This is not possible since the fact that $m(\pi(n(k))) = o(1/\log k)^{1/(\beta-1)}$ implies by (ii) that $|||L_s|||_{\pi(n(k)),p}$ converges, as k goes to infinity, for almost all $s \in R^+$, almost surely. Therefore we see that

$$(3.30)\qquad |||L_t|||_{\pi(n),p} \quad \text{converges in probability for each } t \in R^+.$$

Also, by (3.22), for fixed t there exists a constant $C(m)$, depending only on m, such that

$$\big\|\, |||L_t|||_{\pi(n),p} \big\|_m \le C(m) \qquad \forall\, n \ge 1.$$

Therefore $\{|||L_t|||_{\pi(n),p}\}_{n=1}^{\infty}$ is uniformly integrable. This fact and (3.30) give (1.3) for each $t \in R^+$. Uniformity in t on bounded intervals of R^+ follows easily from (3.21). All that remains in (i) is to evaluate the constant, which we will do below. □

THEOREM 3.5. *For all $b > 0$, we can find a sequence of partitions $\{\pi_n\}_{n=1}^{\infty}$ with $m(\pi_n) \le b/\log n$ such that, almost surely, the sequence*

$$\left\{ \sum_{x_i \in \pi(n)} \left| L_t^{x_i} - L_t^{x_{i-1}} \right|^{2/(\beta-1)} \right\}_{n=1}^{\infty}$$

does not converge whatever the value of $1 < \beta \le 2$.

PROOF. By Remark 2.7 and Lemma 3.1, we see that for all $b > 0$, we can find a sequence of partitions $\{\pi_n\}_{n=1}^{\infty}$ with $m(\pi_n) \le b/\log n$ such that for almost all $\omega \in \Omega_{G_\beta}$,

$$\begin{aligned}\limsup_{n\to\infty} & \left|\left|\left| L_t + G_\beta^2(\omega)/2 \right|\right|\right|_{\pi_n, p} \\ (3.31) \qquad &= \sqrt{2}\,\alpha(1+c)^{1/2}(E|\eta|^p)^{1/p}\left(\int_0^a \left| L_t^x + G_\beta^2(x,\omega)/2\right|^{p/2} dx\right)^{1/p} \\ &\qquad \text{for almost all } t \text{ a.s.,}\end{aligned}$$

where $c > 0$. $L_t^{\cdot}$ is as given in (1.5), G_β is the associated Gaussian process and $p = 2/(\beta - 1)$. Therefore for almost all $\omega \in \Omega_{G_\beta}$,

$$\begin{aligned}\limsup_{n\to\infty} & |||\, L_t \,|||_{\pi_n, p} \\ (3.32) \qquad &\ge \sqrt{2}\,\alpha((1+c)E|\eta|^p)^{1/p}\left(\int_0^a \left| L_t^x + G_\beta^2(x,\omega)/2\right|^{p/2} dx\right)^{1/p} \\ &\qquad - \limsup_{n\to\infty} \left|\left|\left| G_\beta^2(\omega)/2 \right|\right|\right|_{\pi_n, p} \quad \text{for almost all } t \text{ a.s.}\end{aligned}$$

By (2.63) we see that the final term in (3.32) is also equal to an integral with respect to $G_\beta(x, \omega)$ and, as we have already seen many times, we can take $\sup_{x\in[0,a]}|G_\beta(x,\omega)|$ arbitrarily small. [For example, see the proof of Theorem 1.1(ii).] Thus we get

$$\begin{aligned}\limsup_{n\to\infty} & |||\, L_t \,|||_{\pi_n, p}^p \\ (3.33) \qquad &\ge (1+c)(\sqrt{2}\,\alpha)^p E|\eta|^p \int_0^a |L_t^x|^{p/2}\, dx \quad \text{for almost all } t \text{ a.s.}\end{aligned}$$

We will show below that $(\sqrt{2}\,\alpha)^p E|\eta|^p = c(\beta)$. Accepting this we see by (1.5) that the limit of the left-hand side of (3.33) exists almost surely along some subsequence of $\{\pi_n\}$ and is not the same as the right-hand side of (3.33). Thus we have established Theorem 3.5. □

It is interesting to note and easy to see that we actually have equality in (3.33).

PROOF OF THEOREM 1.1 (Constants). In order to complete the proof of Theorem 1.1, we need to show that

$$(3.34) \qquad c(\beta) = \left(\sqrt{2}\,\alpha_\beta\right)^p E|\eta|^p \quad \text{and} \quad c'(\beta) = \left(\sqrt{2}\,\alpha_\beta\right)^p,$$

where α_β is given in (2.40), η is a normal random variable with mean 0 and variance 1 and $p = 2/(\beta - 1)$.

We will first consider the case $\beta = 2$ and show that $\alpha_2 = 1$. One sees from (2.39) and (2.40), by a change of variables, that

$$\alpha_2^2 = \lim_{h \downarrow 0} \frac{2(u^1(0) - u^1(h))}{h} \equiv -2\frac{d^+}{dx}u^1(x)|_{x=0}, \tag{3.35}$$

where $u^1(x)$, the 1-potential density of the Markov process defined in (1.1), is the covariance of the associated Gaussian process. [See (1.10) and (1.11), where $u^1(x) = g(0, x)$.] By definition,

$$u^1(x) = \int_0^\infty e^{-t}p_t(x)\,dt,$$

where $p_t(\cdot)$ is the probability transition density of the process defined in (1.1). Now, note that in the definition of standard Brownian motion, that is, normalized so that $EB(1)^2 = 1$, t is replaced by $t/2$ in (1.1). Let $\tilde{p}_t$ and $\tilde{u}^\alpha$ be the transition probability density function and α-potential density of standard Brownian motion. Then, clearly $p_t(x) = \tilde{p}_{2t}(x)$ and

$$u^1(x) = \tfrac{1}{2}\int_0^\infty e^{-(t/2)}\tilde{p}_t(x)\,dt = \tfrac{1}{2}\tilde{u}^{1/2}(x). \tag{3.36}$$

By Itô and McKean [(1965), (31), page 17],

$$\tilde{u}^\alpha(x) = \frac{\exp(-\sqrt{2\alpha}\,x)}{\sqrt{2\alpha}}$$

and so

$$\frac{d^+}{dx}\tilde{u}^\alpha(x)|_{x=0} = -1 \qquad \forall\, \alpha > 0. \tag{3.37}$$

It follows from (3.35)–(3.37) that $\alpha_2^2 = 1$ which gives us $c(2) = c'(2) = 2$ in agreement with the values given in Theorem 1.1.

Referring to (1.13) and the proof of Theorem 1.1, we see that (1.3) holds, in general, with the constant $c(2)$ replaced by $(\sqrt{2}\,\alpha)^2$, for α as given in the statement of Theorem 1.2 for the associated Gaussian process. We have just shown that for standard Brownian motion,

$$\frac{d^+}{dx}\tilde{u}^1(x)|_{x=0} = -1,$$

so by (3.35) $\alpha^2 = 2$ in this case. Thus, if in (1.3) we considered the local time of standard Brownian motion, in place of $c(2) = 2$, we would have the constant 4. This is the result mentioned in (1.9).

We now consider the case $1 < \beta < 2$. By Ibragimov and Linnik [(1971), 2.6.32, page 88],

$$(\alpha_\beta)^2 = -\frac{2}{\pi}\cos\left(\frac{\pi}{2}(\beta - 1)\right)\Gamma(1 - \beta).$$

Since

$$\Gamma(\beta)\Gamma(1-\beta) = \frac{\pi}{\sin \pi\beta}$$

[see,. e.g., (30), page 198, Ahlfors (1966)] and

$$\cos\left(\frac{\pi}{2}(\beta - 1)\right) = \sin\left(\frac{\pi}{2}\beta\right),$$

we see that

$$\left(\sqrt{2}\,\alpha_\beta\right)^p = \left(\frac{2}{\Gamma(\beta)\sin((\pi/2)(\beta-1))}\right)^{1/(\beta-1)},$$

which is the value given for $c'(\beta)$ in Theorem 1.1. Also, since

$$E|\eta|^p = \frac{2^{p/2}}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right),$$

we get the expression for $c(\beta)$ given in (1.4). This completes the proof of Theorem 1.1. □

REMARK 3.6. We noted in the Introduction that Theorem 1.1, with convergence in L^2, is given in Rosen (1990) in the case $\beta = 1 + (1/k)$, where k is an integer greater than or equal to 1. In Rosen (1990) the constant $c(1 + (1/k))$ is actually written as

$$\tilde{c}_k = (2k-1)(2k-3)\cdots 3\cdot 1(4\rho)^k = E|\eta|^{2k}(4\rho)^k,$$

where

$$\rho = \int_0^\infty (p_t(0) - p_t(1))\,dt = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{1-e^{i\lambda}}{\lambda^\beta}\,d\lambda = \frac{1}{\pi}\int_0^\infty \frac{1-\cos\lambda}{\lambda^\beta}\,d\lambda = \frac{\alpha_\beta^2}{2}.$$

Thus

$$(4\rho)^k = \left(\sqrt{2}\,\alpha_\beta\right)^{2k}$$

and consequently,

$$\tilde{c}_k = E|\eta|^{2k}\left(\sqrt{2}\,\alpha_\beta\right)^{2k} = c\left(1 + \frac{1}{k}\right).$$

This follows from (3.34) because when $\beta = 1 + (1/k)$, $p = 2k$.

Lastly let us note that the discrepancy between (1.3), in which $c(\beta) = 2$ and (1.9), is due to the fact that Brownian motion is defined with a different scaling in (1.1) than usual. This is explained just above in the portion of the proof of Theorem 1.1 that deals with evaluation of the constants $c(\beta)$ and $c(\beta')$.

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