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TAIL PROBABILITIES OF THE MAXIMA OF GAUSSIAN RANDOM FIELDS[1]

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This paper gives a general two-term approximation for the tail probability of the maxima of a class of differentiable Gaussian random fields and illustrates its potential statistical applications.

1. Introduction. Let $\{Z(t),\ t \in I\}$ be a nonsingular Gaussian random field with zero mean and unit variance where I is a d-dimensional indexing set. Various statistical problems can be reduced to calculating the tail probability

$$P\Big\{\max_{t\in I} Z(t) \geq z\Big\} \tag{1}$$

as $z \to \infty$. Typical examples can be found in projection pursuit [cf. Sun (1991)] and change point problems [cf. Hogan and Siegmund (1986)].

According to Adler (1990), page 5, when $d = 1$ there are only six special cases (i.e., six kinds of covariance functions) for which exact formulas for the tail probability (1) are known. For other cases, we can only obtain approximate results. A *one*-term approximation formula for (1) can be found in Adler (1981), page 160, if the Gaussian random field $Z(t)$ is *homogeneous*. However, this one-term approximation formula is often not good enough for the following two reasons. First, the one-term approximation is frequently inaccurate, especially when d is large. Sun (1991), for example, discusses exploratory projection pursuit, which is concerned with finding the "nonlinear" structure of high-dimensional data. In this application, we observe data points with dimension p and use a projection index with J terms to enumerate the amount of nonlinear structure in one-dimensional projections. Sun (1991) then expresses the maximum of the projection index over all projections as the maximum of a d-dimensional nonsingular differentiable Gaussian random field where $d = p + J - 2$. Evaluating the tail probability (1) helps to decide whether the nonlinear structure is real or just random variation in the data. Typically, we are interested in $p \geq 4$ and $J \geq 3$, and hence $d \geq 5$. In this case the one-term approximation performs very poorly and additional terms are needed to get sufficient accuracy. Second, the $Z(t)$'s derived from many

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statistical problems are *not* homogeneous. Therefore, it is an interesting and important question whether we can get a *higher*-order approximation formula for a *wider* class of random fields.

The ideas from Adler (1981) and Chapter 12 of Leadbetter, Lindgren and Rootzen (1983) can be used to obtain a general one-term approximation formula for a wider class of random fields [cf. Hogan and Siegmund (1986) and Bickel and Rosenblatt (1973)]. Unfortunately, these ideas do not seem to be helpful in obtaining a higher-order approximation to (1).

In this paper we use a method which we call the "tube method" to obtain a *higher*-order approximation formula to (1) for a *wider* class of random fields. The intuitive idea is as follows.

Let $\tilde{d} = d$ if d is even and $\tilde{d} = d - 1$ if d is odd. If $Z(t)$ is differentiable, there is a neat and simple connection between $Z(t)$ and some manifold through a Karhunen–Loève expansion of $Z(t)$. (See Section 2.) When the Karhunen–Loève expansion of $Z(t)$ is finite, the $\tilde{d}$-term expansion of (1) in Theorem 3.1 is an easy consequence of Weyl's formula (1939) for the volume of a tube around a manifold. When the Karhunen–Loève expansion of $Z(t)$ is not finite, the lengthy proofs of a general two-term approximation to (1) introduced in Theorems 3.2 and 3.3 involve approximating $Z(t)$ by some $Z_k(t)$ which has a finite Karhunen–Loève expansion. Difficulty arises because our approximation is in the tail of the distribution: (a) For a fixed z and appropriate $Z_k(t)$'s, $P\{\max Z_k(t) \geq z\} \to P\{\max Z(t) \geq z\}$ as $k \to \infty$. (b) For a fixed k, there is a higher-order approximation to $P\{\max Z_k(t) \geq z\}$ as $z \to \infty$ (see Theorem 3.1). (c) Our purpose is to approximate $P\{\max Z(t) \geq z\}$ as $z \to \infty$ by building bridges between (a) and (b). Therefore, it is a very delicate situation where we need to choose a k based on z and the convergence rate of the Karhunen–Loève expansion of $Z(t)$.

To study the convergence rate, we generalize the definitions of standard Fourier series and standard Karhunen–Loève expansion and make a connection between them. (See Section 6.) To take care of the remainder term $P\{\max[Z(t) - Z_k(t)] \geq z\}$ as $z \to \infty$, we use Borell's (1975) inequality.

The first term of our general two-term approximation (Theorem 3.3) is the same as the one-term approximation given by Adler (1981), page 159, when $Z(t)$ is homogeneous and differentiable. Since our final formulas for the two coefficients depend only on the covariance function of the random field, familiarity with elementary differential geometry is not needed to apply the two-term approximation. In fact, in the context of projection pursuit, Sun (1991) constructed tables for those constants.

The organization of the paper is as follows. Some preliminaries and the connection between $Z(t)$ and some manifold are presented in Section 2. Our main results, the $\tilde{d}$ and two-term approximate formulas, are given in Section 3. The geometrical meanings of the two coefficients in the two-term approximation and their applicability in practice are discussed in Section 4. The theorems in Sections 3 and 4 are proved in Sections 5 and 6. Some related known results and definitions in differential geometry are briefly introduced in the Appendix.

Notation. Throughout this paper, $\exists$ means there exists, s.t. means such that, i.i.d. means independent, identically distributed, a.s. means almost surely, r.v. means random variable, PP means projection pursuit, $\|A\|$ is the determinant of a matrix A, $\|u\| = (\Sigma u_l^2)^{1/2}$ if u is a vector $(u_1, u_2, \dots)^\tau$, $\tilde{d} = d$ if d is even and $\tilde{d} = d - 1$ if d is odd. $f \in \mathscr{C}^n(I)$ denotes that the (vector) function f has lth-order bounded mixed continuous partial derivatives for $l = 1, 2, \dots, n$ on its domain I. A random field $Z(t)$ is said to be differentiable in I if it has mean square differentiability: $\exists$ random fields $Z_1(t), \dots, Z_d(t)$ s.t. for all $t \in I$ and $i = 1, \dots, d$,

$$\lim_{h \to 0} E\left\{\left[\frac{Z(t + h\delta_i) - Z(t)}{h} - Z_i(t)\right]^2\right\} = 0.$$

2. Connections and preliminaries. As early as the 1930s, there were statistical questions which called for the calculation of the volume of a tube around a manifold [cf. Hotelling (1939)]. For recent applications, see Knowles and Siegmund (1988). In this section we give a connection between a differentiable Gaussian random field and a differentiable manifold, introduce definitions related to the tube of a manifold embedded in a unit sphere and present Weyl's (1939) formula for the volume of the tube.

Connection. Assume $\{Z(t),\ t \in I\}$ is a d-dimensional differentiable Gaussian random field with $\mathscr{E}[Z(t)] = 0$, $\mathscr{E}[(Z(t))^2] = 1$. $Z(t)$ is said to have a uniformly convergent *Karhunen–Loève expansion* in $t \in I$, if there exist i.i.d. $\mathscr{N}(0,1)$ r.v.'s $X_1, X_2, \dots$ and a sequence of twice continuously differentiable functions $u_1(t), u_2(t), \dots,$ such that as $k \to \infty$,

$$\sum_{l=1}^{k} u_l(t) X_l \to Z(t) \quad \text{uniformly in } t \in I, \text{ a.s.} \tag{2}$$

We shall write this expansion as $Z(t) = \sum_{l=1}^{\infty} u_l(t) X_l$. Note that the Karhunen–Loève expansion defined here does *not* require the orthogonality

$$\int_I u_i(t) u_j(t)\, dt = 0 \quad \text{for } i \neq j \tag{3}$$

as the *standard Karhunen–Loève expansion* does [cf. Adler (1990) and the Appendix]. Set

$$\tilde{Z}_k(t) = \sum_{l=1}^{k} u_l(t) X_l, \qquad Z_k(t) = \sum_{l=1}^{k} v_l(t) X_l \tag{4}$$

for $k = 1, 2, \dots,$ where $v_l(t) = u_l(t)/\sigma_k(t)$ and $\sigma_k^2(t) = \sum_{l=1}^{k} (u_l(t))^2$. Then

$$\mathscr{V}^k = \{v^k(t) : t \in I,\ v^k(t) = (v_1(t), \dots, v_k(t))\}$$

is a d-dimensional *manifold* embedded in S^{k-1}, the unit sphere in $\mathscr{R}^k$, and

$$\mathscr{U} = \{u(t) : t \in I,\ u(t) = (u_1(t), u_2(t), \dots)\}$$

is a d-dimensional *manifold* embedded in S^∞, the unit sphere in $\mathscr{R}^\infty$. Here $S^\infty = \{x: x = (x_1, x_2, \dots) \in \mathscr{R}^\infty, \sum_{i=1}^\infty x_i^2 = 1\}$. Therefore, there is a connection between a differentiable manifold embedded in the unit sphere and a differentiable Gaussian random field, through a uniformly convergent Karhunen–Loève expansion of the random field.

Existence of a Karhunen–Loève expansion. Let $r(s,t)$ be the covariance function of a nonsingular differentiable Gaussian random field $Z(t)$ on a d-dimensional rectangle I with mean 0 and variance 1. In general, by Mercer's theorem, $r(s,t)$ has an absolutely and uniformly convergent eigenvalue (λ_l)–eigenfunction $(\Lambda_l(t))$ expansion

$$r(s,t) = \sum_{l=1}^{\infty} \lambda_l \Lambda_l(s) \Lambda_l(t), \tag{5}$$

where the λ_l's are in decreasing order $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq 0$ and

$$\int_I r(s,t)\Lambda_l(t)\,dt = \lambda_l \Lambda_l(s), \qquad \int_I \Lambda_l(t)\Lambda_m(t)\,dt = \delta_{lm}.$$

This expansion (5) is called the *Mercer expansion* [cf. Courant and Hilbert (1953), pages 138–140]. Under an additional mild condition on $r(s,t)$ [cf. Lemma A.1 by Garsia (1972)], there exist i.i.d. $\mathscr{N}(0,1)$ r.v.'s $X_1, X_2, \dots$, such that

$$\sum_{l=1}^{k} u_l(t) X_l \to Z(t) \quad \text{uniformly in } t \in I, \text{ a.s.}$$

as $k \to \infty$, where $u_l(t) = \sqrt{\lambda_l}\,\Lambda_l(t)$ for $l = 1, 2, \dots$. Here the "mild" condition is tantamount to the sample path continuity of $Z(t)$ [cf. Theorem 3.8 etc. in Adler (1990)]. Therefore, there exists a Karhunen–Loève expansion to a Gaussian random field in most cases. (See also a generalization of the existence lemma, Lemma A.2 in the Appendix.)

Definitions. The following definitions are intuitive if one thinks about them geometrically. Note that the value of k can be infinite.

DEFINITION 2.1 (Tube). The *tube* with radius r of a manifold $\mathscr{U}^k = \{u^k(t): u^k(t) = (u_1(t), \dots, u_k(t)), t \in I\}$ embedded in S^{k-1} is

$$\mathscr{T}_k(r) = \left\{y: y \in S^{k-1}, \inf_{t \in I} \|y - u^k(t)\| \leq r\right\}. \tag{6}$$

Define the *cross section* of $u^k(t)$ in $\mathscr{T}_k(r)$ by

$$C(u^k(t)) = \{y: y \in \mathscr{T}_k(r), y - u^k(t) \perp \dot{u}^k(t)\}$$

[the dots denote partial differentiation with respect to t_i, $i = 1, \dots, d$; see Johansen and Johnstone (1990)]. A tube $\mathscr{T}_k(r)$ has no *self-overlap* if any two different cross sections of the tube do not overlap: $C(u^k(t_1)) \cap C(u^k(t_2)) = \varnothing$ for all $t_1 \neq t_2 \in I$.

DEFINITION 2.2 (Critical). The *critical radius* of the tube $\mathscr{T}_k(r)$ is

$$r_{kc} = \inf\{r: r \geq 0,\ \mathscr{T}_k(r) \text{ has self-overlap}\}.$$

The *critical point* of the tube is $d_k = r_{kc}^2/(2 - r_{kc}^2)$. The *semicritical radius* of the tube $\mathscr{T}_k(r)$ is

$$\tilde{r}_{kc} = \inf\left\{r: r \geq 0, \exists\, y(t,\xi) = u^k(t) + \sum_{i=1}^{k-d-1} \xi_i n_i^k(t),\ \frac{y(t,\xi)}{\|y(t,\xi)\|} \in \mathscr{T}_k(r), \text{s.t. } J(y) = 0\right\}.$$

Here $k < \infty$, $n_i^k(t)$, $i = 1, 2, \ldots, k - d - 1$, are mutually orthogonal unit normal vectors of $\mathscr{U}^k$ at t and are orthogonal to $u^k(t)$. $J(y)$ is the volume element (Jacobian) at $y = y(t,\xi) = (y_1(t,\xi), \ldots, y_k(t,\xi))^T$:

$$J(y) \equiv \left\| y, \frac{\partial y}{\partial t_1}, \ldots, \frac{\partial y}{\partial t_d}, n_1^k(t), n_2^k(t), \ldots, n_{k-d-1}^k(t) \right\|.$$

The *semicritical point* of the tube is $\tilde{d}_k = \tilde{r}_{kc}^2/(2 - \tilde{r}_{kc}^2)$.

According to the multivariate inverse function theorem, for $k < \infty$ and $r > 0$,

$$\begin{aligned}
&\left\{\exists\, y(t,\xi) = u^k(t) + \sum_{i=1}^{k-d-1} \xi_i n_i^k(t),\ \frac{y(t,\xi)}{\|y(t,\xi)\|} \in \mathscr{T}_k(r), \text{s.t. } J(y) = 0\right\} \\
&\quad \subset \Bigg\{\exists\, y(t,\xi) = u^k(t) + \sum_{i=1}^{k-d-1} \xi_i n_i^k(t),\ \frac{y(t,\xi)}{\|y(t,\xi)\|} \in \mathscr{T}_k(r), \\
&\qquad \text{s.t. } \exists\, t' \neq t,\ t' \in \text{a small neighborhood of } t \subset \mathscr{T}_k(r), \\
&\qquad \frac{y(t,\xi)}{\|y(t,\xi)\|} = \frac{y(t',\xi')}{\|y(t',\xi')\|},\ y(t',\xi') = u^k(t') + \sum_{i=1}^{k-d-1} \xi_i' n_i^k(t')\Bigg\} \\
&\quad \subset \{\mathscr{T}_k(r) \text{ has self-overlap}\},
\end{aligned}$$

hence $r_{kc} \leq \tilde{r}_{kc}$.

DEFINITION 2.3 (Full rank). A real vector function $f = (f_1, \ldots, f_n)$ is *full rank* if there is no Borel measurable function F such that

$$f_i = F\big(f_1, \ldots, \check{f}_i, \ldots, f_n\big)$$

for any $i = 1, \ldots, n$, where the $\check{}$ denotes missing.

Weyl's formula (1939). Suppose $\mathscr{U}^k$ is a smooth manifold (without boundary) embedded in S^{k-1} for a finite k. Let $\theta = \arccos(1 - r^2/2)$ be the spherical radius related to a radius r. Then the volume of the tube $\mathscr{T}_k(r)$ is

$$(7) \qquad V(r) \begin{cases} = \dfrac{2\pi^{m/2}}{\Gamma(m/2)} \displaystyle\sum_{e=0,\text{ even}}^{d} \kappa_e J_e(\theta), & \text{if } r \le r_{kc}, \\ \le \dfrac{2\pi^{m/2}}{\Gamma(m/2)} \displaystyle\sum_{e=0,\text{ even}}^{d} \kappa_e J_e(\theta), & \text{if } r \le \tilde{r}_{kc}. \end{cases}$$

Here $m = k - d - 1$, κ_0 is the volume of $\mathscr{U}^k$, κ_e's for $e = 0, 2, \dots$ are other integral invariants of $\mathscr{U}^k$, $J_e(\theta)$'s are some incomplete beta functions

$$J_0(\theta) = \int_0^\theta \sin^{m-1}(x)\cos^d(x)\,dx,$$

$$m(m+2)\dots(m+e-2)J_e(\theta) = \int_0^\theta \sin^{m+e-1}(x)\cos^{d-e}(x)\,dx,$$

$$e = 2, 4, \dots, \le d,$$

r_{kc} and $\tilde{r}_{kc}$ are the critical and semicritical radius.

In the simple case where $d = 1$, $k = 3$, the first term in Weyl's formula (7) indicates that if r is smaller than the critical radius, the dotted area in unit sphere S^2 (see Figure 1) is equal to $2\kappa_0 \sin(\theta)$ if the curve is closed. Here κ_0 is the length of the line.

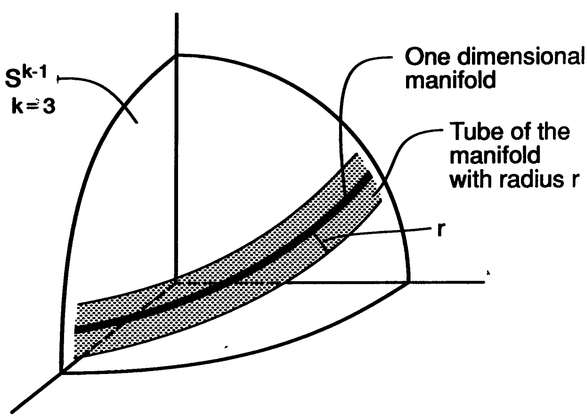

FIG. 1. *Tube of the one-dimensional manifold in S^2.*

3. Approximation for the tail probabilities. In this section Theorem 3.1 gives a $\tilde{d}$-term approximation to (1) when the corresponding Karhunen–Loève expansion of $Z(t)$ is finite. Theorems 3.2 and 3.3 give respectively a two-term upper and a two-term approximation formula to (1) when the corresponding Karhunen–Loève expansion is infinite.

$\tilde{d}$-term approximation.

LEMMA 3.1. *Suppose that $Z(t)$ is a d-dimensional nonsingular three times differentiable Gaussian random field on I, a compact subset of Euclidean space $\mathscr{R}^d$, with mean 0 and a finite Karhunen–Loève expansion*

$$Z(t) = \sum_{l=1}^{k} u_l(t) X_l,$$

where $\|u^k(t)\| = 1$ for $u^k(t) = (u_1(t), \dots, u_k(t))$, and $X_1, X_2, \dots, X_k$ are i.i.d. $\mathscr{N}(0,1)$ r.v.'s. Then the critical radius r_{kc} of the manifold $\mathscr{U}^k = \{u^k(t): t \in I\}$ is positive.

The idea of proving Lemma 3.1 is to assume $r_{kc} = 0$ and then find a contradiction.

THEOREM 3.1. *Let $Z(t)$ be a nonsingular Gaussian random field on a d-dimensional Borel measurable set $I \subset \mathscr{R}^d$ with mean 0, variance 1 and covariance function $r(s,t)$. If $r(s,t) \in C^3$ has finite expansion*

$$r(s,t) = \sum_{l=1}^{k} u_l(s) u_l(t), \qquad k < \infty,$$

and the manifold $\mathscr{U}^k = \{u^k(t): t \in I\}$ has no boundary, then as $z \to \infty$,

$$P\Big\{\max_{t \in I} Z(t) \geq z\Big\} = \kappa_0 \psi_0(z) + \kappa_2 \psi_2(z) + \cdots + \kappa_{\tilde{d}} \psi_{\tilde{d}}(z) + o(\psi_{\tilde{d}}(z)). \tag{8}$$

Here $\kappa_0, \kappa_2, \dots, \kappa_{\tilde{d}}$ are the same constants as those in Weyl's formula for the manifold $\mathscr{U}^k$ and

$$\psi_e(z) = \frac{1}{2^{1+e/2} \pi^{(d+1)/2}} \int_{z^2/2}^{\infty} u^{(d+1-e)/2-1} \exp\{-u\}\, du, \qquad e = 0, 2, 4, \dots, \tilde{d}. \tag{9}$$

Discussion of conditions in Theorem 3.1. A key assumption in Theorem 3.1 is that the manifold $\mathscr{U}^k$ formed from the expansion of $r(s,t)$ has *no boundary*. It is easy to check in the following two situations. The first case is when $\mathscr{U}^k$ or its image is known explicitly, for example, in the case of the PP regression index suggested by Johansen and Johnstone (1990). The second case is when we can parametrize the domain of Z in terms of α in $S^d = \{\alpha: \alpha = (\alpha_1, \dots, \alpha_{d+1}), \|\alpha\| = 1\}$ or there is a one-to-one continuous mapping from the interior of the parameter space I to S^d and onto from I to S^d. For example, this is the case of Friedman's (1987) PP index and Johansen and Johnstone's (1990) PP regression index. When the corresponding manifold does have a boundary, some corrections to the approximation are needed. We do not discuss this case here; interested readers are referred to Sun and Loader (1991).

Another key assumption in Theorem 3.1 is that $r(s,t)$ has a *finite* term expansion, or there exists a $\tilde{Z}(t)$ identically distributed as $Z(t)$ which has a finite Karhunen–Loève expansion. The Gaussian random field $Z(t)$ corresponding to the PP regression index, suggested by Johansen and Johnstone (1988), has a finite Karhunen–Loève expansion. However, in most cases, it is hard to prove that $r(s,t)$ has a finite term expansion. If $Z(t)$ is the Gaussian random field corresponding to Friedman's (1987) PP index, for example, we cannot prove and doubt that $Z(t)$ has a finite Karhunen–Loève expansion. Therefore, it is helpful if we have a result which holds even if the Karhunen–Loève expansion is infinite.

Two-term approximation. If the Karhunen–Loève expansion of $Z(t)$ is infinite, it is a much harder problem to calculate the tail probability of the maximum of $Z(t)$ in (1). The reason is that Weyl's formula works only for finite cases ($k < \infty$).

THEOREM 3.2. *Suppose $Z(t)$ is a d-dimensional nonsingular Gaussian random field on a bounded d-dimensional rectangle I with mean* 0, *variance* 1 *and covariance function $r(s,t)$ which satisfies the regularity conditions* R.1 (*for $m = 6$*), R.2 *and* R.3 *described below. Then as $z \to \infty$,*

$$P\Big\{\max_{t\in I} Z(t) \geq z\Big\} \leq \kappa_0\psi_0(z) + \kappa_2\psi_2(z) + o(\psi_2(z)), \tag{10}$$

where $\psi_e(z)$'s are given in (9), *κ_0 and κ_2 are the same constants as those in Weyl's formula for a manifold. The constants depend only on the double mixed derivatives of $r(s,t)$.*

THEOREM 3.3. *Suppose $Z(t)$ satisfies the conditions in Theorem* 3.2 *and* R.4 *described below. Then as $z \to \infty$,*

$$P\Big\{\max_{t\in I} Z(t) \geq z\Big\} = \kappa_0\psi_0(z) + \kappa_2\psi_2(z) + o(\psi_2(z)), \tag{11}$$

where κ_0, $\kappa_2\psi_0(z)$ and $\psi_2(z)$ are the same as those in Theorem 3.2.

Note that Theorems 3.2 and 3.4 can be generalized to the case where I is a bounded convex set in $\mathscr{R}^d$ since Lemma 6.1 remains valid.

The four *regularity conditions* R.1–R.4 on $r(s,t)$ are certain differentiability conditions on $r(s,t)$.

R.1. *One of the following is true for some positive integer* m.
(1) *There exist functions* f *and* g, *such that*

$$r(s,t) = g(f(s) - f(t)),$$

where $f, g \in \mathscr{C}^{md^2}(I)$, g *is an even real function in each of its coordinate(s),* f *is a real vector function with full rank.*
(2) *There exist integers* $d_3 < \infty$, $d_1 < d$ *and functions* f, h_{ij}, h_i *for* $i, j = 1, \dots, d_3$, *such that*

$$r(s,t) = \sum_{i,j=1}^{d_3} h_i(s^{(1)}) h_j(t^{(1)}) h_{ij}(f(s^{(2)}) - f(t^{(2)})),$$

where $f, h_{ij} \in \mathscr{C}^{md_2(d_2+1)}(I)$, $d_2 = d - d_1$, $h_i \in \mathscr{C}^4$, h_{ij} *are even functions in each of their coordinates,* f *is a real vector function with full rank,* $s^{(1)} = (s_1, \dots, s_{d_1})$, *and* $s^{(2)} = (s_{d_1+1}, \dots, s_d)$.

R.2. *The* $d \times d$ *matrix* $R(t) = (\partial^2 r(s,t)/\partial s_i \, \partial t_j|_{s=t})_{d\times d}$ *is nonsingular on* I.

R.3. *The manifolds* $\mathscr{V}^k = \{v^k(t)\colon t \in I,\ v^k(t) = (v_1(t), \dots, v_k(t))\}$, *derived from an expansion of* $r(s,t) = \sum_{l=1}^{\infty} u_l(s) u_l(t)$, *have no boundary for all* $k > d$. *Here* $v_l(t)$ *is defined in* (4).

R.4. *For some* $c_0 > 0$, *the critical radius* r_{kc} *of the tube* $\mathscr{T}_k(r)$ *of the manifolds* $\mathscr{V}^k$ *satisfies* $r_{kc} \ge c_0$.

Discussion of conditions R.1–R.4 *in Theorems* 3.2 *and* 3.3. The two representations of $r(s,t)$ in R.1 can be examined easily. The homogeneous random fields have covariance functions like (1) in R.1. The non-homogeneous random fields similar to the one derived from Friedman's PP index have covariance functions like (2) in R.1. The differentiability required in R.1 implies that the condition of Corollary A.1 in the Appendix holds with $\alpha = 2$. Hence a uniformly convergent Karhunen–Loève expansion of $Z(t)$ exists.

Conditions R.1 (for $m = 6$) and R.2 together ensure the following two properties of the related manifolds derived from an expansion of $r(s,t)$. The properties will be applied to show the two-term (upper) approximation formulas.

PROPERTY 1. There exists $c_0 > 0$ such that the semicritical radius $\tilde{r}_{kc} \ge c_0 > 0$ (cf. Proposition 5.1).

PROPERTY 2. There is a uniformly convergent Karhunen–Loève expansion of $Z(t)$ satisfying some regularity conditions for the rate of the convergence of

$$\sum_{l=k+1}^{\infty} (u_l(t))^2 \to 0 \quad \text{as } k \to \infty$$

(cf. Proposition 5.2).

Condition R.3 is the first key assumption in Theorem 3.1. See the discussion there.

Condition R.4 is confirmed by some numerical examples about r_{kc}'s (and hence d_k's) in Johansen and Johnstone (1990). In most applications, the λ_l's decrease rapidly as $l \to \infty$, viz. the first few terms of the Karhunen–Loève expansion usually dominate the variability of $Z(t)$. For example, this behavior appears in the application to image processing [cf. Yaglom (1987a) and Yaglom (1987b), note 118]. Hence R.4 and Property 2 are reasonable. The drawback of condition R.4 is that it is relatively hard to check. We also believe that there are similar conditions to R.1 (for $m = 6$) and R.2 which are sufficient for $r_{kc} \geq c_0 > 0$, condition R.4. (See the following conjecture.) Without R.4, we still have a two-term upper approximation formula in Theorem 3.2.

In summary, R.1–R.4 are all reasonable, where R.1–R.3 can be examined easily.

Conjecture. The conditions in Theorem 3.2 are sufficient for (11), that is, R.1, R.2 and R.3 are sufficient for R.4, or for the lower bound to hold.

REMARK. The condition $f, g \in \mathscr{C}^{md^2}$ $(m = 6)$ requires a lot of differentiability from the covariance function. In two dimensions, $md^2 = 24$. However, a careful reader of the proofs of Theorems 3.2 and 3.3 will see that these regularity conditions, especially condition R.1 (for $m = 6$), *can be weakened.*

4. Geometric meanings for κ_0 and κ_2. The two coefficients κ_0, κ_2 in Weyl's formula are two geometric constants of the d-dimensional differentiable manifold $\mathscr{U} = \{u(t)\colon u(t) = (u_1(t), \dots, u_k(t))\}$. Here k is finite or infinite. A d-dimensional differentiable manifold is determined by its *metric tensor*, which has the components defined by the following inner product:

$$g_{ij}(t) = \left\langle \frac{\partial u(t)}{\partial t_i}, \frac{\partial u(t)}{\partial t_j} \right\rangle = \sum_{l=1}^{k} \frac{\partial u_l(t)}{\partial t_i} \cdot \frac{\partial u_l(t)}{\partial t_j} \tag{12}$$

for $i, j = 1, \dots, d$, $t = (t_1, \dots, t_d) \in I$. More specifically, in principle everything we need to know about the manifold $\mathscr{U}$, like the *volume* and some total *scalar curvature* (cf. the Appendix) or κ_0 and κ_2, can be represented as some functions of $R(t)$, $t \in I$. Here $R(t)$ is a $d \times d$ symmetric matrix formed from the functions $g_{ij}(t)$. $R(t)$ is called the *metric tensor matrix* in this paper.

If we know the Karhunen–Loève expansion of $Z(t)$ explicitly, $R(t)$ is known explicitly. In general, however, we only know the existence of the expansion. For example, in the case of the Friedman's (1987) PP index, we do not know the explicit Karhunen–Loève expansion of the related $Z(t)$. Lemma 4.1 builds a connection between $R(t)$ and the covariance function of a Gaussian random field. This connection enables one to calculate the metric tensor matrix without knowing the Karhunen–Loève expansion explicitly.

LEMMA 4.1. *Let $r(s,t)$ be the covariance function of a nonsingular differentiable Gaussian random field $Z(t)$ on a d-dimensional compact set I which satisfies* R.1 *for $m = 6$. Then there exists a $\tilde{Z}(t)$ identically distributed as $Z(t)$ which has a uniformly convergent Karhunen–Loève expansion*

$$\tilde{Z}(t) = \sum_{l=1}^{k} u_l(t) X_l$$

for a finite or infinite k. In particular, the metric tensor of $\mathscr{U} = (u_1(t), \dots, u_k(t))$ is

$$g_{ij}(t) = \sum_{l=1}^{k} \frac{\partial u_l(t)}{\partial t_i} \frac{\partial u_l(t)}{\partial t_j} = \left. \frac{\partial^2 r(s,t)}{\partial s_i \, \partial t_j} \right|_{s=t}. \tag{13}$$

The proof of this lemma is given in Section 6. Note that when $Z(t)$ is homogeneous, $g_{ij}(t)$ is the corresponding second-order spectral moment of $Z(t)$ which does not depend on t.

The first coefficient κ_0 in the approximation formula (8) is the *volume* (or area) of the manifold [Weyl (1939)]. The volume of a manifold is relatively easy to calculate compared to other geometric terms. Weyl (1939) did not explain the explicit geometric meaning of the second coefficient κ_2. He vaguely described it as "certain integral invariant." Theorem 4.1 shows the explicit geometric meaning of κ_2 and a formula for it based on $R(t)$.

THEOREM 4.1. *Let κ_0 and κ_2 be the two coefficients (constants) in Weyl's formula for Theorems* 3.1–3.3. *Then κ_0 is the volume of a manifold and κ_2 is some total scalar curvature. Further, if condition* R.1 *holds for $m = 6$, we have*

$$\begin{aligned} \kappa_0 &= \int_I \|R(t)\|^{1/2} \, dt_1 \dots dt_d, \\ \kappa_2 &= \int_I \left(-\frac{S}{2} - \frac{d(d-1)}{2} \right) \|R(t)\|^{1/2} \, dt_1 \dots dt_d. \end{aligned} \tag{14}$$

Here S is the intrinsic scalar curvature of the manifold which has $R(t) = (g_{ij}(t))_{d\times d}$, $g_{ij}(t) = \partial^2 r(s,t)/\partial s_i \, \partial t_j|_{s=t}$, as its matrix of metric tensor [cf. Kreyszig (1968), *page* 310, *or the Appendix*]. *When $Z(t)$ is homogeneous, the scalar curvature $S = 0$.*

PROOF. Lemma 4.1 says that $R(t) = (\partial^2 r(s,t)/\partial s_i \, \partial t_j|_{s=t})_{d\times d}$. The expression for κ_0 is then automatic. The calculation involved to obtain the final geometric meaning of κ_2 or expression (14) is enormous. The key idea of the proof is to manipulate patiently the complicated expression for κ_2 in Weyl's paper and apply the definition of *scalar curvature* as in Definition A.5.

There is a less computational approach to the proof of (14) by using slightly more advanced techniques. We omit the details.

When $Z(t)$ is homogeneous, the $g_{ij}(t)$'s are independent of t and hence $S = 0$. □

REMARK. In the case of Friedman's projection pursuit index, the corresponding metric tensor matrix is diagonal, and hence κ_0 and κ_2 are easy to obtain. It is also helpful to express a multiple integral as the expectation of a function of some simple random variables. In this way, we can use the Monte Carlo method to estimate κ_0 and/or κ_2 quickly by a computer. For examples, see Sun (1991).

5. Related theorems and proofs for Section 3.

PROOF OF LEMMA 3.1. Notice that I is compact and $u^k(t)$ is three times differentiable on I. If $r_{kc} = 0$, there exist $t_1, t_2 \in I$, $t_1 \neq t_2$, such that $u^k(t_1) = u^k(t_2)$, which is equivalent to $Z(t_1) = Z(t_2)$ a.s., by the assumed finite representation for $t \in I$.

However, $Z(t_1) = Z(t_2)$ contradicts the assumed nonsingularity of Z. Hence $r_{kc} > 0$. □

PROOF OF THEOREM 3.1. First, define $\tilde{Z}(t) = \sum_{l=1}^k u_l(t) X_l$, where $X_1, X_2, \dots, X_k$ are i.i.d. $\mathscr{N}(0,1)$ r.v.'s. Then $\tilde{Z}(t)$ and $Z(t)$ are identically distributed, which implies

$$P\Big\{\max_{t\in I} Z(t) \geq z\Big\} = P\Big\{\max_{t\in I} \tilde{Z}(t) \geq z\Big\}.$$

Second, by Lemma 3.1, the critical radius r_{kc}, and hence the critical point $d_k = r_{kc}^2/(2 - r_{kc}^2)$, of the manifold $\mathscr{U}^k = \{u^k(t) = (u_1(t), \dots, u_k(t)):\ t \in I\}$, are positive.

Third, since $U \equiv (X_1/\|X\|, X_2/\|X\|, \dots, X_k/\|X\|)$ is uniformly distributed on the unit sphere S^{k-1} and is independent of $\|X\|$, we can rewrite the tail probability as follows:

$$\begin{aligned} P\Big\{\max_{t\in I} Z(t) \geq z\Big\} &= P\Big\{\max_{t\in I} \tilde{Z}(t) \geq z\Big\} \\ &= \int_z^\infty P\Big\{\max_{t\in I}\langle u^k(t), U\rangle \geq \frac{z}{x}\Big\} P\{\|X\| \in dx\} \qquad (15) \\ &= \int_z^{(1+d_k)z} + \int_{(1+d_k)z}^\infty \equiv A + B. \end{aligned}$$

Fourth, we calculate A by applying Weyl's formula and bound B by some elementary probability inequalities. Let $r_{kz} \equiv [2(1 - z/x)]^{1/2}$ and $\theta \equiv \arccos(1 - r_{kz}^2/2) = \arccos(z/x)$. If $z < x < (1 + d_k)z$, then

$$0 < r_{kz} < \sqrt{2d_k/(1 + d_k)} = r_{kc}.$$

Applying Weyl's formula in Lemma 2.1, we have that for $x \in (z, (1 + d_k)z)$,

$$\begin{aligned} P\left\{\max_{t \in I}\langle u^k(t), U\rangle \geq \frac{z}{x}\right\} &= P\left\{U\colon \inf_{t \in I}\|u^k(t) - U\| \leq \sqrt{2\left(1 - \frac{z}{x}\right)}\right\} \\ &= \frac{1}{\omega_{k-1}} \frac{2\pi^{m/2}}{\Gamma(m/2)} \sum_{e=0,\text{ even}}^{d} \kappa_e J_e(\theta) \qquad (16) \\ &= \frac{\Gamma(k/2)}{\Gamma(m/2)\pi^{(d+1)/2}} \sum_{e=0,\text{ even}}^{d} \kappa_e J_e(\theta). \end{aligned}$$

Here $m = k - d - 1$ and $\omega_{k-1} = 2\pi^{k/2}/\Gamma(k/2)$ is the surface area of S^{k-1}.

Note $\|X\|$ is the square root of a χ^2 r.v. with k degrees of freedom which has the density function $f_k(x) = x^{k-1}\exp\{-x^2/2\}/\{2^{(k-2)/2}\Gamma(k/2)\}$. We have from (16) that

$$\begin{aligned} A &= \int_z^{(1+d_k)z} P\left\{\max_{t \in I}\langle u^k(t), U\rangle \geq \frac{z}{x}\right\} P\{\|X\| \in dx\} \\ &= \frac{\Gamma(k/2)}{\Gamma(m/2)\pi^{(d+1)/2}} \sum_{e=0,\text{ even}}^{d} \kappa_e \int_z^{\infty} f_k(x) J_e(\theta)\, dx \\ &\quad - \frac{\Gamma(k/2)}{\Gamma(m/2)\pi^{(d+1)/2}} \sum_{e=0,\text{ even}}^{d} \kappa_e \int_{(1+d_k)z}^{\infty} f_k(x) J_e(\theta)\, dx \\ &\equiv A_1 + A_2. \end{aligned}$$

Straightforward calculation shows

$$\begin{aligned} A_1 &= \frac{\Gamma(k/2)}{\Gamma(m/2)\pi^{(d+1)/2}} \sum_{e=0,\text{ even}}^{d} \kappa_e \int_z^{\infty} f_k(x) J_e(\theta)\, dx \\ &= \kappa_0 \psi_0(z) + \kappa_2 \psi_2(z) + \cdots + \kappa_{\tilde{d}} \psi_{\tilde{d}}(z), \\ A_2 &= -\frac{\Gamma(k/2)}{\Gamma(m/2)\pi^{(d+1)/2}} \sum_{e=0,\text{ even}}^{d} \kappa_e \int_{(1+d_k)z}^{\infty} f_k(x) J_e(\theta)\, dx \\ &= o(\psi_{\tilde{d}}(z)), \\ B &= \int_{(1+d_k)z}^{\infty} P\left\{\max_{t \in I}\langle u^k(t), U\rangle \geq \frac{z}{x}\right\} P\{\|X\| \in dx\} \\ &\leq \int_{z(1+d_k)}^{\infty} P\{\|X\| \in dx\} = o(\psi_{\tilde{d}}(z)). \end{aligned}$$

The summation of A_1, A_2 and B here leads to (8). □

PROOF OF THEOREMS 3.2 AND 3.3. Suppose $Z(t)$, $t \in I$, is a d-dimensional nonsingular Gaussian random field with mean 0 and variance 1. Assume $\tilde{Z}(t)$ is identically distributed as $Z(t)$ and has a uniformly convergent Karhunen–Loève expansion in $t \in I$: $\tilde{Z}(t) = \sum_{l=1}^{\infty} u_l(t) X_l$, where $X_1, X_2, \dots$ are i.i.d. $\mathscr{N}(0,1)$ r.v.'s. Note that $\|u(t)\|^2 = \sum_{l=1}^{\infty} u_l^2(t) = 1$ because of $\operatorname{cov}\{Z(t)\} = 1$. Let $\tilde{Z}_k(t)$ and $Z_k(t)$ be the partial sums of $\tilde{Z}(t)$ defined in (4). Define

$$a_k^2 = \max_{t \in I} \sum_{l=k+1}^{\infty} u_l^2(t), \qquad b_k^2 = \min_{t \in I} \sum_{l=k+1}^{\infty} u_l^2(t). \tag{17}$$

Write κ_e^k as the integral invariants in Weyl's formula when the manifold is

$$\mathscr{V}^k = \{v^k(t) : v^k(t) = (v_1(t), \dots, v_k(t))\}$$

with $v_l(t) = u_l(t)/\sigma_k(t)$ and $\sigma_k^2(t) = \sum_{l=1}^{k} u_l^2(t)$. Denote κ_e as κ_e^k when $k = \infty$.

In the following, we shall give Propositions 5.1 and 5.2, then use them to prove Theorems 3.2 and 3.3. The two properties given in the discussion of the regularity conditions R.1–R.4 are the immediate consequences of Propositions 5.1 and 5.2.

PROPOSITION 5.1. *Suppose $Z(t)$ is a nonsingular differentiable Gaussian random field on a d-dimensional compact space I with mean* 0, *variance* 1 *and covariance function $r(s,t)$ which satisfies the regularity conditions* R.1 (*for* $m = 6$) *and* R.2. *Then there exists a uniformly convergent expansion* $r(s,t) = \sum_{l=1}^{\infty} u_l(s) u_l(t)$ *and a constant* $c_0 > 0$ *such that for all* k,

$$\tilde{r}_{kc} \ge c_0 > 0. \tag{18}$$

Here $\tilde{r}_{kc}$ is the semicritical radius of the manifold $\mathscr{V}^k = \{v^k(t)$: $v^k(t) = (u_1(t)/\sigma_k(t), u_2(t)/\sigma_k(t), \dots, u_k(t)/\sigma_k(t)), t \in I\}$, $\sigma_k^2(t) = \sum_{l=1}^{k} u_l^2(t)$.

PROOF. Let the uniformly convergent expansion $r(s,t) = \sum_{l=1}^{\infty} u_l(s) u_l(t)$ be the one given in Lemma 6.1. By the assumptions, the manifolds $\mathscr{V}^k$ and $\mathscr{U} = \{u(t)$: $u(t) = (u_1(t), u_2(t), \dots), t \in I\}$ are three times differentiable. Assume there is no such c_0 for (18). Then for some ε_k which decrease to 0 as $k \to \infty$, there are $n_k \to \infty$ such that

$$\tilde{r}_{n_k c} < \varepsilon_k, \tag{19}$$

that is, $\lim_{k \to \infty} \tilde{r}_{n_k c} = 0$. Without loss of generality, we denote n_k by k.

According to Definition 2.2,

$$\tilde{r}_{kc} = \inf\left\{ r : r \ge 0, \exists\, y(t,\xi) = v^k(t) + \sum_{i=1}^{k-d-1} \xi_i n_i^k(t), \right.$$

$$\left. \frac{y(t,\xi)}{\|y(t,\xi)\|} \in \mathscr{T}(r), \text{ s.t. } J(y) = 0 \right\},$$

where $n_i^k(t)$, $i = 1, 2, \ldots, k-d-1$, are mutually orthogonal unit normal vectors of $\mathscr{V}^k$ at $v^k(t)$ and are orthogonal to $v^k(t)$. For the tube of $\mathscr{V}^k$ with radius $r_k = 2\tilde{r}_{kc}$, there is

$$y = y(t,\xi) = v^k(t) + \sum_{i=1}^{k-d-1} \xi_i n_i^k(t), \qquad y(t,\xi)/\|y(t,\xi)\| \in \mathscr{T}(r_k),$$

such that

$$(20) \qquad J(y) \equiv \left\| y, \frac{\partial y}{\partial t_1}, \ldots, \frac{\partial y}{\partial t_d}, n_1^k(t), n_2^k(t), \ldots, n_{k-d-1}^k(t) \right\| = 0.$$

In the following, we show that (19) and (20) produce a contradiction and hence (18) holds.

Expressing $\partial n_l^k(t)/\partial t_i$ by the Weingarten equation (cf. Definition A.5), we have

$$(21) \qquad \frac{\partial n_l^k(t)}{\partial t_i} = -\sum_j L_i^j(l) \frac{\partial v^k(t)}{\partial t_j} + \cdots,$$

where $-L_i^j(l)$ is the coefficient of $\partial n_l^k(t)/\partial t_i$ in the direction $\partial v^k(t)/\partial t_j$, and "$+ \cdots$" are components orthogonal to the tangent space spanned by $\partial v^k(t)/\partial t_i$, $i = 1, \ldots, d$, and hence are some linear combinations of $n_1^k(t), n_2^k(t), \ldots, n_{k-d-1}^k(t), v^k(t)$. Therefore,

$$\frac{\partial y}{\partial t_i} = \frac{\partial\left(v^k(t) + \sum_{l=1}^{k-d-1} \xi_l^k n_l^k(t)\right)}{\partial t_i} = \frac{\partial v^k(t)}{\partial t_i} + \sum_{l=1}^{k-d-1} \xi_l^k \frac{\partial n_l^k(t)}{\partial t_i}$$

$$= \frac{\partial v^k(t)}{\partial t_i} + \sum_{l=1}^{k-d-1} \xi_l^k \left(-\sum_j L_i^j(l) \frac{\partial v^k(t)}{\partial t_j} + \cdots \right).$$

Let

$$\frac{\partial \tilde{y}}{\partial t_i} = \frac{\partial v^k(t)}{\partial t_i} + \sum_{l=1}^{k-d-1} \xi_l^k \left(-\sum_j L_i^j(l) \frac{\partial v^k(t)}{\partial t_j} \right).$$

Then $\partial y/\partial t_i = \partial \tilde{y}/\partial t_i + \cdots$ and

$$(22) \qquad \left(\frac{\partial \tilde{y}}{\partial t_1}, \ldots, \frac{\partial \tilde{y}}{\partial t_d} \right) = \left(\frac{\partial v^k(t)}{\partial t_1}, \ldots, \frac{\partial v^k(t)}{\partial t_1} \right) \left(I_{d\times d} - \sum_{l=1}^{k-d-1} \xi_l^k \tilde{L}(l) \right),$$

where $\tilde{L}(l)$ is the $d \times d$ matrix $\tilde{L}(l) = (L_i^j(l))_{d\times d}$. "$+ \cdots$" are some linear combinations of $n_1^k(t), n_2^k(t), \ldots, n_{k-d-1}^k(t), v^k(t)$. Therefore,

$$
\begin{aligned}
(J(y))^2 &= \left\| y, \frac{\partial y}{\partial t_1}, \ldots, \frac{\partial y}{\partial t_d}, n_1^k(t), n_2^k(t), \ldots, n_{k-d-1}^k(t) \right\|^2 \\
&= \left\| v^k(t), \frac{\partial \tilde{y}}{\partial t_1}, \ldots, \frac{\partial \tilde{y}}{\partial t_d}, n_1^k(t), n_2^k(t), \ldots, n_{k-d-1}^k(t) \right\|^2 \\
&= \left\| \left(\frac{\partial \tilde{y}}{\partial t_1}, \ldots, \frac{\partial \tilde{y}}{\partial t_d}, n_1^k(t), n_2^k(t), \ldots, n_{k-d-1}^k(t), v^k(t) \right)^\tau \right. \\
&\qquad \left. \times \left(\frac{\partial \tilde{y}}{\partial t_1}, \ldots, \frac{\partial \tilde{y}}{\partial t_d}, n_1^k(t), n_2^k(t), \ldots, n_{k-d-1}^k(t), v^k(t) \right) \right\| \\
&= \left\| \left(\frac{\partial \tilde{y}}{\partial t_1}, \ldots, \frac{\partial \tilde{y}}{\partial t_d} \right)^\tau \left(\frac{\partial \tilde{y}}{\partial t_1}, \ldots, \frac{\partial \tilde{y}}{\partial t_d} \right) \right\| \\
&= \left\| \left(\frac{\partial v^k(t)}{\partial t_1}, \ldots, \frac{\partial v^k(t)}{\partial t_d} \right)^\tau \left(\frac{\partial v^k(t)}{\partial t_1}, \ldots, \frac{\partial v^k(t)}{\partial t_d} \right) \right\| \\
&\qquad \times \left\| I_{d\times d} - \sum_{l=1}^{k-d-1} \xi_l^k \tilde{L}(l) \right\|^2 \quad \text{[by (22)]} \\
&= \left\| \left(g_{ij}^k(t) \right) \right\|_{d\times d} \left\| I_{d\times d} - \sum_{l=1}^{k-d-1} \xi_l^k \tilde{L}(l) \right\|^2 .
\end{aligned} \tag{23}
$$

Here $(g_{ij}^k(t))$ is the $d \times d$ matrix with elements

$$
g_{ij}^k(t) = \frac{\partial v^k(t)^\tau}{\partial t_i} \frac{\partial v^k(t)}{\partial t_j} = \sum_{l=1}^{k} \frac{\partial v_l(t)}{\partial t_i} \frac{\partial v_l(t)}{\partial t_j} .
$$

Let $g_{ij}(t) = \sum_{l=1}^{\infty} \partial u_l(t)/\partial t_i \, \partial u_l(t)/\partial t_j$; we shall prove under (19) that as $k \to \infty$,

$$
\left\| \left(g_{ij}^k(t) \right)_{d\times d} \right\| \to \left\| \left(g_{ij}(t) \right)_{d\times d} \right\| \geq c_1, \tag{24}
$$

$$
\left\| I_{d\times d} - \sum_{l=1}^{k-d-1} \xi_l^k \tilde{L}(l) \right\| \to c_2 \tag{25}
$$

for some $c_1, c_2 > 0$. (23), (24) and (25) together contradict (20), and therefore (18) holds.

By Lemma 6.1, for some constant $c > 0$, and any $t, s \in I$,

$$
|u_k(t) u_k(s)| \leq \frac{c}{k^6}, \qquad |u_k(t)| \leq \frac{c}{k^3}, \qquad \left| \frac{\partial u_k(t)}{\partial t_i} \right| \leq \frac{c}{k^2}, \qquad \left| \frac{\partial^2 u_k(t)}{\partial t_i \partial t_j} \right| \leq \frac{c}{k} .
$$

This implies that as $k \to \infty$, uniformly in s, t,

$$\sum_{l=1}^{k} u_l(s) u_l(t) \to r(s,t), \qquad \sum_{l=1}^{k} u_l(s) \frac{\partial u_l(t)}{\partial t_i} \to \frac{\partial r(s,t)}{\partial t_i},$$

$$(26) \quad \sum_{l=1}^{k} u_l(s) \frac{\partial^2 u_l(t)}{\partial t_i \, \partial t_j} \to \frac{\partial^2 r(s,t)}{\partial t_i \, \partial t_j}, \qquad \sum_{l=1}^{k} \frac{\partial u_l(s)}{\partial s_i} \frac{\partial u_l(t)}{\partial t_j} \to \frac{\partial^2 r(s,t)}{\partial s_i \, \partial t_j},$$

$$\sum_{l=1}^{k} \frac{\partial^2 u_l(s)}{\partial s_i \, \partial t_j} \frac{\partial u_l^2(t)}{\partial t_{i'} \, \partial t_{j'}} \to \frac{\partial^4 r(s,t)}{\partial s_i \, \partial s_j \, \partial t_{i'} \, \partial t_{j'}}.$$

Now as $r(t,t) = 1$ and $r(s,t) = \sum_{l=1}^{\infty} u_l(s) u_l(t)$ uniformly in $s, t \in I$, we have that

$$(27) \qquad \sigma_k^2(t) = \sum_{l=1}^{k} u_l(t) u_l(t) \to 1 \quad \text{uniformly as } k \to \infty.$$

That $r(s,t)$ is maximized at $s = t$ gives $\partial r(s,t)/\partial s_i|_{s=t} = 0$. Thus we have by R.2 that as $k \to \infty$, uniformly in t,

$$\frac{\partial \sigma_k}{\partial t_i} = \frac{\partial}{\partial t_i} \left\{ \sum_{l=1}^{k} u_l(t)^2 \right\}^{1/2} = \frac{1}{\sigma_k} \sum_{l=1}^{k} \frac{\partial u_l(t)}{\partial t_i} u_l(t)$$

$$\to \sum_{l=1}^{\infty} \frac{\partial u_l(t)}{\partial t_i} u_l(t) = \left. \frac{\partial r(s,t)}{\partial s_i} \right|_{s=t} = 0,$$

$$(28) \quad g_{ij}^k(t) = \left\langle \frac{\partial v^k}{\partial t_i}, \frac{\partial v^k}{\partial t_j} \right\rangle = \frac{1}{\sigma_k^2} \sum_{l=1}^{k} \left(\frac{\partial u_l(t)}{\partial t_i} \frac{\partial u_l(t)}{\partial t_j} \right) - \frac{\partial \sigma_k}{\partial t_i} \frac{\partial \sigma_k}{\partial t_j} \frac{1}{\sigma_k^2}$$

$$\to \sum_{l=1}^{\infty} \frac{\partial u_l(t)}{\partial t_i} \frac{\partial u_l(t)}{\partial t_j} = g_{ij}(t).$$

By Lemma 4.1, (26) gives

$$g(t) \equiv \left(g_{ij}(t) \right)_{d \times d} = \left(\left. \frac{\partial^2 r(s,t)}{\partial s_i \, \partial t_j} \right|_{s=t} \right)_{d \times d}.$$

Hence (28), the nonsingularity and continuity of $(\partial^2 r(s,t)/\partial s_i \, \partial t_j|_{s=t})_{d \times d}$ on the compact set I give the following two results: (i) (24) holds, and (ii) the inverse matrices $g^{-1,k}(t) = (g^{k,ij}(t))_{d \times d}$ of $g^k(t)$ and $g^{-1}(t) = (g^{ij}(t))_{d \times d}$ of $g(t)$ exist for k greater than some positive integer k_0. The elements of these inverse matrices have a uniform upper bound and $\|g^{-1}(t)\|$ has a positive lower bound, that is, for some $M, M' > 0$,

$$(29) \qquad g^{k,ij}(t) \le M, \qquad g^{ij}(t) \le M, \qquad \|g^{-1}(t)\| \ge M'.$$

Let $L(l) \equiv (L_{ij})_{d \times d} = g^k \tilde{L}(l)$, $g^k = (g_{ij}^k)_{d \times d}$, a $d \times d$ symmetrical matrix, $u^k(t) = (u_1(t), \dots, u_k(t))^\tau$. By the Gauss formula given in Definition A.5, R.1

(for $m = 6$) and (26), we have

$$\sum_l L_{ij}^2(l) \leq \sum_l \left\langle \frac{\partial^2 v^k}{\partial t_i \partial t_j}, n_l^k \right\rangle \leq \left\| \frac{\partial^2 v^k}{\partial t_i \partial t_j} \right\|^2$$

$$= \left\| \frac{1}{\sigma_k(t)} \frac{\partial^2 u^k(t)}{\partial t_i \partial t_j} - \frac{1}{\sigma_k^2(t)} \left(\frac{\partial u^k(t)}{\partial t_i} \frac{\partial \sigma_k(t)}{\partial t_j} + \frac{\partial u^k(t)}{\partial t_j} \frac{\partial \sigma_k(t)}{\partial t_i} \right) \right.$$

$$\left. + \frac{u^k(t)}{2\sigma_k^3(t)} \frac{\partial \sigma_k(t)}{\partial t_i} \frac{\partial \sigma_k(t)}{\partial t_j} - \frac{u^k(t)}{\sigma_k^2(t)} \frac{\partial^2 \sigma_k(t)}{\partial t_i \partial t_j} \right\|^2 \tag{30}$$

$$\leq 16 \left(\left\| \frac{\partial^2 u^k(t)}{\partial t_i \partial t_j} \right\|^2 + \left\| \frac{\partial u^k(t)}{\partial t_i} \right\|^2 \left(\frac{\partial \sigma_k(t)}{\partial t_j} \right)^2 + \left\| \frac{\partial u^k(t)}{\partial t_j} \right\|^2 \left(\frac{\partial \sigma_k(t)}{\partial t_i} \right)^2 \right.$$

$$\left. + \frac{1}{4} \left(\frac{\partial \sigma_k(t)}{\partial t_i} \frac{\partial \sigma_k(t)}{\partial t_j} \right)^2 + \left(\frac{u^k(t)}{\sigma_k^2(t)} \frac{\partial^2 \sigma_k(t)}{\partial t_i \partial t_j} \right)^2 \right)$$

$$\leq M''$$

for some $M'' > 0$. Equations (29) and (30) imply

$$\sum_l \left(L_i^j(l) \right)^2 \leq M''' \tag{31}$$

for some $M''' > 0$. On the other hand, as $y(t, \xi)/\|y(t, \xi)\| \in \mathscr{T}(r_k)$, we have $\Sigma_l (\xi_l^k)^2/(1 + \Sigma_l (\xi_l^k)^2) \leq r_k^2$, which implies by (19) that $\Sigma_l (\xi_l^k)^2 \leq r_k^2/(1 - r_k^2) \to 0$ as $k \to \infty$. Consequently, from (31) we have that

$$\left\| I_{d \times d} - \sum_{l=1}^{k-d-1} \xi_l^k \tilde{L}(l) \right\| \to 1 > 0$$

as $k \to \infty$. This gives (25). Therefore, (24) and (25) hold. Hence (18) is proved. □

PROPOSITION 5.2. *Suppose $r(s,t)$ is the nonnegative definite covariance function of a Gaussian random field $Z(t)$ on a d-dimensional rectangle I. Assume the regularity condition* R.1 *holds for $m = 6$. Then there is a uniformly convergent Karhunen–Loève expansion $\Sigma_{l=1}^{\infty} u_l(t) X_l$ for $\tilde{Z}(t)$ which is distributed identically as $Z(t)$ and the corresponding κ_0^k, κ_0 and a_k in* (17) *satisfy the following requirement*:

$\exists\ \varepsilon_0, \varepsilon_1 > 0$, *and $K, c > 0$ such that*

$$a_k^2 \leq \frac{c}{k^{4+\varepsilon_1}} \quad \text{for } k > K, \tag{32}$$

and consequently

$$\kappa_0^k - \kappa_0 = o\left(\frac{1}{k^{1+\varepsilon_0}}\right). \tag{33}$$

We need Lemma 5.1 in our proof of Proposition 5.2.

LEMMA 5.1. *Suppose $Z(t)$ is a d-dimensional Gaussian random field on a bounded compact set I in $\mathscr{R}^d$ with mean 0, variance 1 and covariance function $r(s, t)$, which has uniformly convergent expansion $r(s, t) = \sum_{l=1}^{\infty} u_l(s) u_l(t)$ in $t \in I$.*

Further, assume $\partial^4 r(s,t)/\partial^2 s_i\, \partial^2 t_j$ is uniformly bounded, the $d \times d$ matrix $(\partial^2 r(s, t)/\partial s_i\, \partial t_j|_{s=t})_{d\times d}$ is continuous and nonsingular on I, $\sum_{l=1}^{k} \partial u_l(t)/\partial t_i\, \partial u_l(t)/\partial t_j$ and $\sum_{l=1}^{k} \partial u_l(t)/\partial t_i\, u_l(t)$ converge uniformly in I as $k \to \infty$, and, for some $\varepsilon_0, \varepsilon_1 > 0$,

$$\sum_{l=k+1}^{\infty} \frac{\partial u_l(t)}{\partial t_i} u_l(t) = o\left(\frac{1}{k^{1/2+\varepsilon_0}}\right),$$

$$\left|\sum_{l=k+1}^{\infty} \frac{\partial u_l(t)}{\partial t_i} \frac{\partial u_l(t)}{\partial t_j}\right| = o\left(\frac{1}{k^{1+\varepsilon_1}}\right)$$

as $k \to \infty$. Then "$a_k^2 \le c/(k^{1+\varepsilon_2})$ for some $\varepsilon_2 > 0$" gives

$$\kappa_0^k - \kappa_0 = o\left(\frac{1}{k^{1+\varepsilon_3}}\right)$$

for some $\varepsilon_3 > 0$, that is, (32) implies (33). Here κ_0^k and κ_0 correspond to the manifold formed from the $u_l(t)$'s.

PROOF. Suppose $R(t) = (g_{ij}(t))_{d\times d}$ is the matrix formed by the metric tensor with components $g_{ij}(t)$ of the manifold $\mathscr{U}$ of $Z(t)$, $R^k(t) = (g_{ij}^k(t))_{d\times d}$ is the matrix formed by the metric tensor of the manifold $\mathscr{V}^k$ of $Z(t)$ (cf. Definition 4.1).

In the following, we shall prove that

$$\begin{aligned} \kappa_0^k - \kappa_0 &= \int_{t\in I} \sqrt{\|R^k(t)\|}\, dt - \int_{t\in I} \sqrt{\|R(t)\|}\, dt \\ &\le c \max_{i,j=1,\dots,d} |g_{ij}(t) - g_{ij}^k(t)| \end{aligned} \tag{34}$$

for some positive constant c, and as $k \to \infty$,

$$\max_{i,j=1,\dots,d} |g_{ij}(t) - g_{ij}^k(t)| = o\left(\frac{1}{k^{1+\varepsilon_3}}\right) \tag{35}$$

for some $\varepsilon_3 > 0$. The lemma is an immediate result of (34) and (35).

Similarly to the proof of Proposition 5.1, we have $g_{ij}^k(t) \to g_{ij}(t)$ as $k \to \infty$, uniformly in t. Further,

$$
\begin{aligned}
&|g_{ij}^k(t) - g_{ij}(t)| \\
(36)\quad &= \left| \frac{1}{\sigma_k^2} \sum_{l=1}^{k} \frac{\partial u_l(t)}{\partial t_i} \frac{\partial u_l(t)}{\partial t_j} - \frac{\partial \sigma_k}{\partial t_i} \frac{\partial \sigma_k}{\partial t_j} \frac{1}{\sigma_k^2} - \sum_{l=1}^{\infty} \frac{\partial u_l(t)}{\partial t_i} \frac{\partial u_l(t)}{\partial t_j} \right| \\
&\le \left| \left(\frac{1}{\sigma_k^2} - 1 \right) \sum_{l=1}^{k} \frac{\partial u_l(t)}{\partial t_i} \frac{\partial u_l(t)}{\partial t_j} \right| + \left| \sum_{l=k+1}^{\infty} \frac{\partial u_l(t)}{\partial t_i} \frac{\partial u_l(t)}{\partial t_j} \right| + \left| \frac{\partial \sigma_k}{\partial t_i} \frac{\partial \sigma_k}{\partial t_j} \frac{1}{\sigma_k^2} \right|.
\end{aligned}
$$

Now (27) and the assumptions give that as $k \to \infty$,

$$
\begin{aligned}
&\left| \left(\frac{1}{\sigma_k^2} - 1 \right) \sum_{l=1}^{k} \frac{\partial u_l(t)}{\partial t_i} \frac{\partial u_l(t)}{\partial t_j} \right| + \left| \sum_{l=k+1}^{\infty} \frac{\partial u_l(t)}{\partial t_i} \frac{\partial u_l(t)}{\partial t_j} \right| \\
&\quad \le \frac{a_k^2}{\sigma_k^2} \left| \sum_{l=1}^{k} \frac{\partial u_l(t)}{\partial t_i} \frac{\partial u_l(t)}{\partial t_j} \right| + \left| \sum_{l=k+1}^{\infty} \frac{\partial u_l(t)}{\partial t_i} \frac{\partial u_l(t)}{\partial t_j} \right| \\
&\quad \le \frac{a_k^2}{\sigma_k^2} \left| \left. \frac{\partial^2 r(s,t)}{\partial s_i \partial t_j} \right|_{s=t} \right| + \left(1 + \frac{a_k^2}{\sigma_k^2} \right) \left| \sum_{l=k+1}^{\infty} \frac{\partial u_l(t)}{\partial t_i} \frac{\partial u_l(t)}{\partial t_j} \right| \\
&\quad = o\left(\frac{1}{k^{1+\min\{\varepsilon_1, \varepsilon_2\}}} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
\left| \frac{\partial \sigma_k}{\partial t_i} \frac{\partial \sigma_k}{\partial t_j} \frac{1}{\sigma_k^2} \right| &= \left| \left(\sum_{l=1}^{k} \frac{\partial u_l(t)}{\partial t_i} u_l(t) \right) \left(\sum_{m=1}^{k} \frac{\partial u_m(t)}{\partial t_j} u_m(t) \right) \frac{1}{\sigma_k^4} \right| \\
&\le \left| \sum_{l=k+1}^{\infty} \frac{\partial u_l(t)}{\partial t_i} u_l(t) \right| \left| \sum_{m=k+1}^{\infty} \frac{\partial u_m(t)}{\partial t_j} u_m(t) \right| = o\left(\frac{1}{k^{1+2\varepsilon_0}} \right).
\end{aligned}
$$

Hence from (36), we have as $k \to \infty$,

$$
|g_{ij}(t) - g_{ij}^k(t)| = o\left(\frac{1}{k^{1+\varepsilon_3}} \right)
$$

uniformly in $t \in I$, where $\varepsilon_3 = \min\{\varepsilon_1, \varepsilon_2, 2\varepsilon_0\}$. (35) is proved.

The following representation,

$$
R(t) = \left(\left. \frac{\partial^2 r(s,t)}{\partial s_i \partial t_j} \right|_{s=t} \right)_{d \times d}
$$

and the assumption on $r(s,t)$ imply that the determinant $\|R(t)\|$ has a positive lower bound, say l_b. Therefore, (35) gives $\|R^k(t)\| = \|(g_{ij}^k(t))\| > l_b/2$ for all $k > K$, some positive constant.

Considering $\|\cdot\|$ as a multivariate function of d^2 variables g_{ij}, we obtain the following inequality by the definitions of κ_0 and κ_0^k:

$$\kappa_0^k - \kappa_0 = \int_{t\in I}\sqrt{\|R^k(t)\|}\,dt - \int_{t\in I}\sqrt{\|R(t)\|}\,dt$$
$$\le c(d^2, l_b)\max_{i,j=1,\dots,d}|g_{ij}(t) - g_{ij}^k(t)|.$$

Here the constant $c(d^2, l_b)$ only depends on d^2 and the lower bound l_b of $\|R(t)\|$. (34) is proved. Hence Lemma 5.1 is true. □

PROOF OF PROPOSITION 5.2. By Lemma 6.1, there exists a uniformly convergent expansion for $r(s,t) = \sum_{l=1}^{\infty}u_l(s)u_l(t)$, such that for some constant $c > 0$ and any $t, s \in I$,

$$|u_k(t)u_k(s)| \le \frac{c}{k^6}, \qquad \left|\frac{\partial u_k(t)}{\partial t_i}\right| \le \frac{c}{k^4}.$$

Hence for some constant c' we have

$$\sum_{l=k+1}^{\infty}\frac{\partial u_l(t)}{\partial t_i}u_l(t) \le \frac{c'}{k^4}, \qquad \sum_{l=k+1}^{\infty}\frac{\partial u_l(t)}{\partial t_i}\frac{\partial u_l(t)}{\partial t_j} \sim \frac{c'}{k^3},$$

$$b_k^2 \le a_k^2 = \max_{t\in I}\sum_{l=k+1}^{\infty}u_l^2(t) \le \frac{c'}{k^5},$$

and the following two series converge uniformly in t as $k \to \infty$:

$$\sum_{l=1}^{k}\frac{\partial u_l(t)}{\partial t_i}u_l(t), \qquad \sum_{l=1}^{k}\frac{\partial u_l(t)}{\partial t_i}\frac{\partial u_l(t)}{\partial t_j}.$$

Define $\tilde{Z}(t) = \sum_{l=1}^{\infty}u_l(t)X_l$, where $X_1, X_2, \dots$ are i.i.d. $\mathscr{N}(0,1)$ r.v.'s. Then $\tilde{Z}(t)$ and $Z(t)$ are identically distributed and all the conditions in Lemma 5.1 hold. Hence (33) also holds by Lemma 5.1. The proposition is proved. □

Condition 5.1 and Condition 5.2 for the rate of the convergence of

$$\sum_{l=k+1}^{\infty}(u_l(t))^2 \to 0 \quad \text{as } k \to \infty$$

are implied by (32), which is a consequence of the regularity conditions R.1 and R.2, by Proposition 5.2. We shall use these two conditions to prove Theorems 3.2 and 3.3. Therefore, we leave room for readers to see that R.1–R.4 can be weakened in Theorems 3.2 and 3.3.

CONDITION 5.1. *Let $Z(t)$ be a d-dimensional Gaussian random field with a uniformly convergent Karhunen–Loève expansion. There are $z_0 > 1$ and $\varepsilon_0 > 0$*

such that for all $z > z_0$, $\exists$ a, $k \le z^{2-\varepsilon_0}$ *and* $\varepsilon_{kz} > 0$ *with*

$$a_k^2 \le \frac{\varepsilon_{kz}^2}{(1+\delta)}.$$

Here $\delta = \delta(z)$ *satisfies that as* $z \to \infty$, $\delta \cdot z^2/2 - 2\log z \to \infty$, *and* $\varepsilon_{kz} = o(1/z^2)$ *for* $k \le z^{2-\varepsilon_0}$, *where* a_k^2 *is defined in* (17).

CONDITION 5.2. *For the same random field* $Z(t)$, k *and* ε_{kz} *in Condition* 5.1,

$$\kappa_0^k \frac{\psi_0(z')}{\psi_0(z)} - \kappa_0 = o\left(\frac{1}{z^2}\right), \qquad \kappa_0^k \frac{\psi_0(z'')}{\psi_0(z)} - \kappa_0 = o\left(\frac{1}{z^2}\right)$$

as $z \to \infty$, *where* $\psi_0(z)$ *is defined in* (9),

$$z' = z\frac{1-\varepsilon_{kz}}{\left(1-b_k^2\right)^{1/2}}, \qquad z'' = z\frac{1-\varepsilon_{kz}}{\left(1-a_k^2\right)^{1/2}}.$$

Here a_k^2, b_k^2 *are defined in* (17) *and* κ_0 *and* κ_0^k *are introduced after* (17).

Condition 5.2 can be simplified. For $e = 0, 2, \dots, \le d$, as $z, k \to \infty$, $k \le z^{2-\varepsilon_0}$,

$$\begin{aligned}\frac{\psi_e(z')}{\psi_e(z)} &= \frac{\int_{z'^2/2}^{\infty} u^{(d+1-e)/2-1}\exp\{-u\}\,du}{\int_{z^2/2}^{\infty} u^{(d+1-e)/2-1}\exp\{-u\}\,du}\\ &\sim \frac{z'^{d-e-1}}{z^{d-e-1}}\exp\left\{-\frac{z'^2}{2}+\frac{z^2}{2}\right\}\\ &= 1 - \frac{z^2}{2}\left(\varepsilon_{kz}^2 - 2\varepsilon_{kz} + b_k^2\right)(1+o(1)).\end{aligned} \tag{37}$$

Similarly, as $z, k \to \infty$, $k \le z^{2-\varepsilon_0}$,

$$\frac{\psi_0(z'')}{\psi_0(z)} = 1 - \frac{z^2}{2}\left(\varepsilon_{kz}^2 - 2\varepsilon_{kz} + a_k^2\right)(1+o(1)).$$

Since

$$\kappa_0^k \frac{\psi_0(z')}{\psi_0(z)} - \kappa_0 = \left(\kappa_0^k - \kappa_0\right)\frac{\psi_0(z')}{\psi_0(z)} - \kappa_0\left(\frac{\psi_0(z')}{\psi_0(z)} - 1\right),$$

a sufficient condition for Condition 5.2 is the following:

$$\begin{aligned}&\kappa_0^k - \kappa_0 = o\left(\frac{1}{z^2}\right),\\ &\left(b_k^2 - 2\varepsilon_{kz} + \varepsilon_{kz}^2\right)z^4 = o(1),\\ &\left(a_k^2 - 2\varepsilon_{kz} + \varepsilon_{kz}^2\right)z^4 = o(1).\end{aligned} \tag{38}$$

LEMMA 5.2. *Condition* 5.1, (38) *and hence Condition* 5.2 *are implied by* (32).

PROOF. Without loss of generality, assume $\varepsilon_0, \varepsilon_1 < 1$.
Choose $k = z^{2-\varepsilon_0\varepsilon_1/4}$, $z_0 = 1.1$, $\delta = 1.1^{\varepsilon_1 - \varepsilon_2\varepsilon_1^2/8} - 1$, $\varepsilon_{kz}^2 = c/k^{4+\varepsilon_1/2}$. Then

$$b_k^2 \le a_k^2 \le \frac{c}{k^{4+\varepsilon_1}} = \frac{\varepsilon_{kz}^2}{k^{\varepsilon_1/2}} \sim \frac{\varepsilon_{kz}^2}{z^{\varepsilon_1 - \varepsilon_0\varepsilon_1^2/8}} \le \frac{\varepsilon_{kz}^2}{1+\delta}$$

for $z > z_0$, and $\varepsilon_{kz} \sim k^{-2-\varepsilon_1/4} \sim o(z^{-4})$. Hence Condition 5.1 holds.
On the other hand,

$$b_k^2 \le a_k^2 \le \frac{\varepsilon_{kz}^2}{1+\delta} \sim o\left(\frac{1}{z^8}\right).$$

By (33), $\kappa_0^k - \kappa_0 = o(k^{-1-\varepsilon_0}) = o(z^{-(2-\varepsilon_0\varepsilon_1/4)(1+\varepsilon_0)}) = o(z^{(-2+2\varepsilon_0-\varepsilon_0\varepsilon_1/4-\varepsilon_1\varepsilon_0^2/4)})$. Therefore, (38) and hence Condition 5.2 hold. □

PROOF OF THEOREM 3.3. By Proposition 5.2 and Lemma 5.2, there exists a $\tilde{Z}(t)$ identically distributed as $Z(t)$ which has a uniformly convergent Karhunen–Loève expansion in $t \in I$:

$$\tilde{Z}(t) = \sum u_l(t) X_l$$

and satisfies Conditions 5.1 and 5.2 with ε_{kz}^2 as in the proof of Lemma 5.2. Hence

$$P\left\{\max_{t\in I} Z(t) \ge z\right\} = P\left\{\max_{t\in I} \tilde{Z}(t) \ge z\right\}.$$

To prove (11), we need to show that as $z \to \infty$,

$$\frac{P\{\max_{t\in I} \tilde{Z}(t) \ge z\} - \kappa_0\psi_0(z)}{\kappa_2\psi_2(z)} = 1 + o(1). \tag{39}$$

Assume $\tilde{Z}^k(t)$and $Z^k(t)$ are the finite k partial sums defined in (4). On the one hand, since

$$\begin{aligned} P\left\{\max_{t\in I}\tilde{Z}(t) \ge z\right\} &= P\left\{\max_{t\in I}\left(\tilde{Z}(t) - \tilde{Z}_k(t) + \tilde{Z}_k(t)\right) \ge z\right\} \\ &\le P\left\{\max_{t\in I}\tilde{Z}_k(t) \ge z(1-\varepsilon_{kz})\right\} \\ &\quad + P\left\{\max_{t\in I}\left(\tilde{Z}(t) - \tilde{Z}_k(t)\right) \ge z\varepsilon_{kz}\right\}, \end{aligned}$$

we have

$$\frac{P\{\max_{t\in I}\tilde{Z}(t)\geq z\}-\kappa_0\psi_0(z)}{\kappa_2\psi_2(z)} \tag{40}$$
$$\leq\frac{P\{\max_{t\in I}\tilde{Z}_k(t)\geq z(1-\varepsilon_{kz})\}-\kappa_0\psi_0(z)}{\kappa_2\psi_2(z)}$$
$$+\frac{P\{\max_{t\in I}(\tilde{Z}(t)-\tilde{Z}_k(t))\geq z\varepsilon_{kz}\}}{\kappa_2\psi_2(z)}\equiv A+B.$$

On the other hand, as

$$P\Big\{\max_{t\in I}\tilde{Z}(t)\geq z\Big\}=P\Big\{\max_{t\in I}\big(\tilde{Z}(t)-\tilde{Z}_k(t)+\tilde{Z}_k(t)\big)\geq z\Big\}$$
$$\geq P\Big\{\max_{t\in I}\tilde{Z}_k(t)-\max_{t\in I}|\tilde{Z}(t)-\tilde{Z}_k(t)|\geq z,$$
$$\max_{t\in I}|\tilde{Z}(t)-\tilde{Z}_k(t)|<z\varepsilon_{kz}\Big\}$$
$$\geq P\Big\{\max_{t\in I}\tilde{Z}_k(t)\geq z(1+\varepsilon_{kz})\Big\}$$
$$-P\Big\{\max_{t\in I}|\tilde{Z}(t)-\tilde{Z}_k(t)|\geq z\varepsilon_{kz}\Big\},$$

we see that

$$\frac{P\{\max_{t\in I}\tilde{Z}(t)\geq z\}-\kappa_0\psi_0(z)}{\kappa_2\psi_2(z)} \tag{41}$$
$$\geq\frac{P\{\max_{t\in I}\tilde{Z}_k(t)\geq z(1+\varepsilon_{kz})\}-\kappa_0\psi_0(z)}{\kappa_2\psi_2(z)}$$
$$-\frac{P\{\max_{t\in I}|\tilde{Z}(t)-\tilde{Z}_k(t)|\geq z\varepsilon_{kz}\}}{\kappa_2\psi_2(z)}\equiv A'+B'.$$

In the following, it is enough to choose k as some function of z s.t. as $z\to\infty$:

$$A\leq 1+o(1),\qquad B=o(1), \tag{42}$$

$$A'\geq 1+o(1),\qquad B'=o(1). \tag{43}$$

We choose the same k as that in Lemma 5.2. Without loss of generality, $\tilde{Z}(t)$ will be denoted by $Z(t)$.

PROOF OF "$B=o(1)$." Condition 5.1 implies

$$\frac{\varepsilon_{kz}^2}{a_k^2}\geq 1+\delta \tag{44}$$

for some δ which satisfies $\delta \cdot z^2/2 - \log z^2 \to \infty$ as $z \to \infty$, and therefore,

$$\frac{z\varepsilon_{kz}}{a_k} \to \infty \quad \text{as } z \to \infty. \tag{45}$$

Let $\bar{\sigma}_k^2(t) = \sum_{l=k+1}^{\infty} u_l^2(t)$, $\bar{Z}(t) = (Z(t) - \tilde{Z}_k(t))/\bar{\sigma}_k(t)$. Let m denote the median of the distribution of $\sup_{t\in I}|\bar{Z}(t)|$. Using Borell's (1975) inequality [cf. also Adler (1990)] we have that for $y > 0$,

$$P\Big\{\max_{t\in I}|\bar{Z}(t)| \geq y\Big\} \leq 1 - \Phi(y - m).$$

Hence as $z \to \infty$,

$$\begin{aligned}
B &= \frac{P\{\max_{t\in I}\bar{Z}(t)\bar{\sigma}_k(t) \geq z\varepsilon_{kz}\}}{\kappa_2\psi_2(z)} \\
&\leq \frac{P\{\max_{t\in I}\bar{Z}(t) \geq z\varepsilon_{kz}/a_k\}}{\kappa_2\psi_2(z)} \\
&\leq \frac{1 - \Phi(z\varepsilon_{kz}/a_k - m)}{\kappa_2\psi_2(z)} \\
&\sim \frac{\left\{\sqrt{2\pi}\left(\dfrac{z\varepsilon_{kz}}{a_k} - m\right)\right\}^{-1} \exp\left\{-\dfrac{1}{2}\left(\dfrac{z\varepsilon_{kz}}{a_k} - m\right)^2\right\}}{\kappa_2 \dfrac{1}{4\pi^{(d+1)/2}} \int_{z^2/2}^{\infty} u^{(d-3)/2}\exp\{-u\}\,du} \qquad \text{[by (45)]} \\
&= o(1) \quad \text{[by (44)]}.
\end{aligned}$$

Here c is some positive constant.

PROOF OF "$A \leq 1 + o(1)$."

$$\begin{aligned}
A &= \frac{P\{\max_{t\in I}Z_k(t)\sigma_k(t) \geq z(1 - \varepsilon_{kz})\} - \kappa_0\psi_0(z)}{\kappa_2\psi_2(z)} \\
&\leq \frac{P\{\max_{t\in I}Z_k(t) \geq z'\} - \kappa_0\psi_0(z)}{\kappa_2\psi_2(z)},
\end{aligned} \tag{46}$$

where $z' = z(1 - \varepsilon_{kz})/(1 - b_k^2)^{1/2}$ and $b_k^2 = \min_{t\in I}\sum_{l=k+1}^{\infty}u_l^2(t)$ is defined in (17).

Let $v^k(t) = (u_1(t)/\sigma_k(t), u_2(t)/\sigma_k(t), \dots, u_k(t)/\sigma_k(t))$ and $d_k = r_{kc}^2/(2 - r_{kc}^2)$. Under R.3, a similar proof to that of Theorem 3.1 gives that

$$\begin{aligned}
P\Big\{\max_{t\in I}Z_k(t) \geq z'\Big\} &= \int_{z'}^{\infty} P\Big\{\max_{t\in I}\langle u^k(t), U\rangle \geq \frac{z'}{x}\Big\} P\{\|X\| \in dx\} \\
&= \int_{z'}^{(1+d_k)z'} + \int_{(1+d_k)z'}^{\infty} \\
&= \kappa_0^k\psi_0(z') + \kappa_2^k\psi_2(z') + R_{kz'},
\end{aligned} \tag{47}$$

where

$$
\begin{aligned}
R_{kz'} &= -\frac{\Gamma(k/2)}{\Gamma(m/2)\pi^{(d+1)/2}} \sum_{e=0,2} \kappa_e^k \int_{(1+d_k)z'}^{\infty} f_k(x) J_e(\theta)\, dx \\
&\quad + \frac{\Gamma(k/2)}{\Gamma(m/2)\pi^{(d+1)/2}} \sum_{e=4,\text{ even}}^{d} \kappa_e \int_{z'}^{(1+d_k)z'} f_k(x) J_e(\theta)\, dx \\
&\quad + \int_{z'(1+d_k)}^{\infty} P\left\{ \max_{t\in I} \langle u^k(t), U\rangle \geq \frac{z'}{x} \right\} P\{\|X\| \in dx\} \\
&\equiv R_{1kz'} + R_{2kz'} + R_{3kz'}.
\end{aligned} \tag{48}
$$

Here $J_e(\cdot)$ is defined in Lemma 2.2 and $f_k(x)$ is the density function of $\|X\|$. Therefore,

$$
\begin{aligned}
A &\leq \frac{\kappa_0^k \psi_0(z') - \kappa_0 \psi_0(z)}{\kappa_2 \psi_2(z)} + \frac{\kappa_2^k \psi_2(z')}{\kappa_2 \psi_2(z)} - \frac{R_{kz'}}{\kappa_2 \psi_2(z)} \\
&\equiv A_1 + A_2 + A_3,
\end{aligned}
$$

and it is enough to show that as $z \to \infty$,

$$
A_1 = \frac{\kappa_0^k \psi_0(z') - \kappa_0 \psi_0(z)}{\kappa_2 \psi_2(z)} \to 0, \tag{49}
$$

$$
A_2 = \frac{\kappa_2^k \psi_2(z')}{\kappa_2 \psi_2(z)} \to 1, \tag{50}
$$

$$
A_3 = -\frac{R_{kz'}}{\kappa_2 \psi_2(z)} \to 0. \tag{51}
$$

Condition 5.2 implies (49).

It is obvious that $\kappa_2^k \to \kappa_2$ as $k \to \infty$ by (28). Under Condition 5.1, $b_k^2 \leq a_k^2 \leq \varepsilon_{kz}^2/(1+\delta) = o(z^{-2})$. This gives $b_k^2 - 2\varepsilon_{kz} + \varepsilon_{kz}^2 = o(z^{-2})$. Hence, by the same derivation as for (37), we have that $\psi_2(z')/\psi_2(z) \to 1$ as $z \to \infty$. Thus (50) is valid.

Condition 5.1 gives $R_{2kz'} = o(\psi_2(z')) = o(\psi_2(z))$ as $z \to \infty$. It is also easy to see that for some $c > 0$,

$$
\begin{aligned}
R_{3kz'} &\leq \int_{(1+d_k)z'}^{\infty} P\{\|X\| \in dy\}, \\
R_{1kz'} &\leq c \frac{\Gamma((d+1)/2)}{2\pi^{(d+1)/2}} \int_{(1+d_k)z'}^{\infty} P\{\|X\| \in dy\}.
\end{aligned}
$$

From R.4, $d_k \equiv r_{kc}^2/(2 - r_{kc}^2) \geq r_{kc}^2/2 \geq c_0^2/2 > 0$. Thus a simple calculation gives

$$
\int_{(1+d_k)z'}^{\infty} P\{\|X\| \in dx\} \leq \int_{(1+c_0^2/2)z'}^{\infty} f_k(x)\, dx \leq R_1 \int_{z'^2/2}^{\infty} u^{(d-1)/2-1} \exp\{-u\}\, du
$$

for $z'^2/2 \geq (k-d+1)/\{2(c_0^2 + c_0^4/4)\}$ and

$$R_1 = \frac{(1+c_0^2/2)^k}{\Gamma(k/2)}\left(\frac{z'^2}{2}\right)^{(k-d+1)/2} \exp\left\{-c_0^2\frac{z'^2}{2} - \frac{c_0^4}{4}\frac{z'^2}{2}\right\} \to 0.$$

By Stirling's formula for $\Gamma(k/2)$, we have for $k \leq cz^{2-\varepsilon_0}$, $R_1 \to 0$ as $z \to \infty$. Hence (51) is valid and consequently "$A \leq 1 + o(1)$" is proved.

"$A' \geq 1 + o(1)$" and "$B' = o(1)$" can be proved similarly.

As described in Section 4, κ_0, κ_2 of a d-dimensional differentiable manifold $\mathscr{U} = \{u(t): u(t) = (u_1(t), u_2(t), \dots, u_k(t))\}$ are determined by the *metric tensor* g_{ij} of $\mathscr{U}$. By Lemma 4.1, $g_{ij}(t) = \{\partial^2 r(s,t)/\partial s_i\, \partial t_j\}_{s=t}$, which implies that κ_0 and κ_2 depend only on the double mixed derivatives of $r(s,t)$. □

PROOF OF THEOREM 3.2. The proof is almost the same as that of Theorem 3.3, only with some minor changes.

By Proposition 5.1,

$$\tilde{r}_{kc} \geq c_0 > 0 \tag{52}$$

for some $c_0 > 0$. As (52) is similar to the regularity condition R.4, we call it R.4′. In other words, R.1–R.3 and R.4′ are satisfied in Theorem 3.2.

The minor changes include:

1. R.4 in the proof of Theorem 3.3 is replaced by R.4′.
2. The critical point d_k is replaced by the semicritical point $\tilde{d}_k$.
3. (47) is changed into

$$P\left\{\max_{t \in I} Z_k(t) \geq z'\right\} \leq \kappa_0^k \psi_0(z') + \kappa_2^k \psi_2(z') + R_{kz'}.$$

4. All the arguments about (43) are deleted.

Therefore, "$A \leq 1 + o(1)$" and "$B = o(1)$" still hold under the conditions of Theorem 3.2, which implies (10). The theorem is proved. □

6. Related theorems and proofs for Section 4. This section makes a connection between a Fourier series and a Karhunen–Loève expansion. In this way, we can control the convergence rate of the Karhunen–Loève expansion and prove Lemma 4.1. We first give our definition of a Fourier series and discuss some relevant properties.

DEFINITION 6.1 (Fourier kernel). A $\{n(1), \dots, n(d)\}$ Fourier kernel $f^i_{n(1),\dots,n(d)}(t)$ on a d-dimensional rectangle I is one of the following $d(d+1)/2$ combinations of sine and cosine functions:

$$f^1_{n(1),\dots,n(d)}(t) = \sin(b_{n(1)}t_1)\sin(b_{n(2)}t_2)\cdots\sin(b_{n(d)}t_d),$$
$$f^2_{n(1),\dots,n(d)}(t) = \cos(b_{n(1)}t_1)\sin(b_{n(2)}t_2)\cdots\sin(b_{n(d)}t_d),$$
$$f^3_{n(1),\dots,n(d)}(t) = \sin(b_{n(1)}t_1)\cos(b_{n(2)}t_2)\cdots\sin(b_{n(d)}t_d),$$
$$\vdots$$
$$f^{d(d+1)/2}_{n(1),\dots,n(d)} = \cos(b_{n(1)}t_1)\cos(b_{n(2)}t_2)\cdots\cos(b_{n(d)}t_d),$$

which are mutually orthogonal when certain relations among the b_n and the lengths of sides of I are satisfied:

$$\int_I f^i_{n(1),\dots,n(d)}(t)\, f^{i'}_{n'(1),\dots,n'(d)}(t)\, dt$$

$$(53)\qquad = \begin{cases} a^i_{n(1),\dots,n(d)} > 0, & \text{if } i = i', \{n(1),\dots,n(d)\} \\ & \qquad = \{n'(1),\dots,n'(d)\}, \\ 0, & \text{otherwise,} \end{cases}$$

for $i, i' = 1,\dots,(d+1)d/2$, $n(i), n'(i) = 1, 2, \dots,$ where $b_{n(i)}/n(i) \ge b_{i0} > 0$ for $n(i) \ge n(i_0)$. Here b_{i0} and $n(i_0)$ are positive numbers.

Assume that the orthogonality condition (53) holds. For $i = 1,\dots,(d+1)d/2$, $n(i) = 1, 2, \dots,$ let

$$c^i_{n(1),\dots,n(d)} = \frac{1}{a^i_{n(1),\dots,n(d)}} \int_I f(t)\, f^i_{n(1),\dots,n(d)}(t)\, dt.$$

The following series with real coefficients $c^i_{n(1),\dots,n(d)}$ and the Fourier kernels $f^i_{n(1),\dots,n(d)}(t)$ is called a *Fourier series*:

$$\sum_{n(1)=0,\dots,n(d)=0}^{\infty} \sum_{i=1}^{d(d+1)/2} c^i_{n(1),\dots,n(d)} f^i_{n(1),\dots,n(d)}(t).$$

REMARK 6.1. A routine generalization of the standard theorems in Walker (1988), pages 185–216, Carslaw (1930), page 225, Sneddon (1961), pages 34 and 40, and Tolstov (1962), pages 125–180, shows that all the properties about the coefficients of the above Fourier series, its convergence, integration and differentiation are the same as those for the *regular* Fourier series where the coefficients $b_{n(i)} = cn(i)$ for some positive c, $i = 1,\dots,d(d+1)/2$.

For a regular Fourier series of a function, where $b_{n(i)} = cn(i)$, $i = 1,\dots,d(d+1)/2$, that the function takes the same value at all its end points is a necessary condition for its Fourier series to converge to itself in its entire domain I. Under our definition of a (semi) Fourier series, it is not necessary for a function to have the same value at the end points for such convergence. Here is one example.

Let $r(s,t)$ be the covariance function of a Brownian motion $B(t)$ on $(0,1)$. Then $r(s,t) = \min\{s,t\}$ for $0 \le s,\ t \le 1$, and $r(0,0) \ne r(1,1)$. However, the following Fourier series convergence uniformly to $r(s,t)$:

$$r(s,t) = 2 \sum_{n=1}^{\infty} \left(\frac{\sin(n - 1/2)\pi t}{(n - 1/2)\pi} \right) \left(\frac{\sin(n - 1/2)\pi s}{(n - 1/2)\pi} \right).$$

Here $d = 1$, $b_{n(1)} = (n(1) - 1/2)\pi$. This expansion also gives a uniformly convergent Karhunen–Loève expansion of $B(t)$:

$$B(t) = \sqrt{2} \sum_{n=1}^{\infty} \frac{\sin(n - 1/2)\pi t}{(n - 1/2)\pi} X_n,$$

where $X_1, X_2, \dots$ are some i.i.d. $\mathscr{N}(0,1)$ r.v.'s [cf. Yaglom (1987), pages 448–450].

In fact, let $f(t)$ be a function defined on a rectangular region $I \equiv I_1 \times \cdots \times I_d$. If it has the same value at its end points and its jth order mixed partial derivative is either continuous or satisfies the Dirichlet condition (see Definition A.6) for $j = 1, 2, \dots, d$, there is a Fourier series for f "which converges uniformly to f in any interval which contains neither in its interior nor at an end any point of the discontinuity point of the function" [cf. Carslaw (1930), page 275, and Walker (1988), (3.6), page 185]. Here $I_i = [a_i, b_i]$ for $i = 1, \dots, d$.

If $f(t)$ on I does not have the same value at its end points, we can always transform $f(t)$, as indicated below, into a new function [cf. (54)] which has the same value at all end points. This new function has jth-order bounded continuous mixed partial derivatives or its jth-order mixed partial derivatives satisfy the Dirichlet condition only if $f(t)$ does so. Without loss of generality, assume that $d = 2$, that is, $I = [a_1, b_1] \times [a_2, b_2]$. Then the new function at $t = (t_1, t_2)$,

$$\begin{aligned} f_{\text{new}}(t) &= f(t_1, t_2) - \frac{t_1 - a_1}{b_1 - a_1}(f(b_1, a_2) - f(a_1, a_2)) \\ &\quad - \frac{t_2 - a_2}{b_2 - a_2}(f(a_1, b_2) - f(a_1, a_2)) \\ &\quad - \frac{(t_1 - a_1)(t_2 - a_2)}{(b_1 - a_1)(b_2 - a_2)}(f(b_1, b_2) - f(b_1, a_2) \\ &\qquad - f(a_1, b_2) + f(a_1, a_2)) \end{aligned} \tag{54}$$

has the same value $f(a_1, a_2)$ at the four end points (a_1, a_2), (a_1, b_2), (b_1, a_2) and (b_1, b_2), and the difference between the old and new functions, viz. the quadratic function

$$\begin{aligned} f_{\text{diff}}(t) &= \frac{t_1 - a_1}{b_1 - a_1}(f(b_1, a_2) - f(a_1, a_2)) + \frac{t_2 - a_2}{b_2 - a_2}(f(a_1, b_2) - f(a_1, a_2)) \\ &\quad + \frac{(t_1 - a_1)(t_2 - a_2)}{(b_1 - a_1)(b_2 - a_2)}(f(b_1, b_2) - f(b_1, a_2) \\ &\qquad - f(a_1, b_2) + f(a_1, a_2)) \end{aligned}$$

has a uniformly convergent Fourier series by a similar trick used for $r(s,t)$ of $B(t)$ [cf. Yaglom (1987a), pages 448–450]. Therefore, if $f(t)$ has jth-order mixed continuous derivatives for $j = 1, \dots, d$, f_{new} does also and hence has a

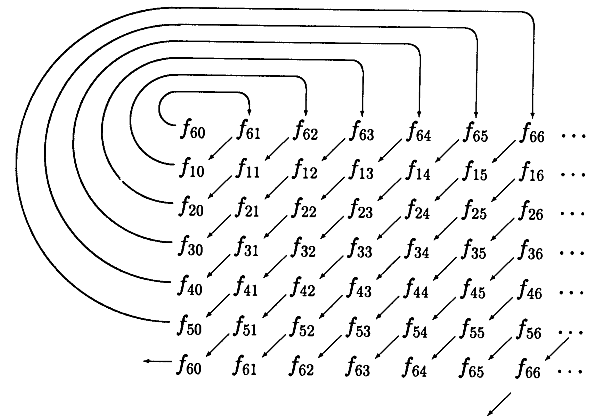

FIG. 2. *The ordering diagram in the bivariate case.* f_{ij} *represents* $d(d+1)/2$ *components with the ordering*

$$f_{ij}^1 \to f_{ij}^2 \to \cdots \to f_{ij}^{d(d+1)/2}.$$

uniformly convergent Fourier expansion. This implies that $f(t)$ has a uniformly convergent trigonometric series. This trigonometric series is a summation of two Fourier series. We call the summation of a finite number of Fourier series a *semi-Fourier series*. The properties of the coefficients of a semi-Fourier series are similar to those of a Fourier series.

In the following, we can assume any function under consideration has the same value at all the end points.

DEFINITION 6.2 (Regular ordering). A *regular ordering* of a multidimensional indexing Fourier series to a univariate indexing Fourier series is $\sum_{l=1}^{\infty} c_l f_l(t)$, where c_l is the coefficient of the lth Fourier kernel in the dictionary ordering. (See Figure 2 for the ordering diagram in the bivariate case.)

Under this ordering, $f_{n(1),\dots,n(d)}^i(t)$ is ordered as f_l for some $l \le d(d+1)/2\,(\max_{i=1,\dots,d} n(i))^d$, where $i = 1,\dots,d(d+1)/2$.

We shall use the following Lemma 6.1 to prove Lemma 4.1.

LEMMA 6.1. *Suppose that a nonnegative definite covariance function* $r(s,t)$ *of a* d*-dimensional random field* $Z(t)$ *on a rectangle* I *satisfies the regularity condition* R.1 *for* $m = k$. *Then there is a uniformly convergent expansion for* $r(s,t) = \sum_{l=1}^{\infty} u_l(s)u_l(t)$, *where* $|u_l(s)u_l(t)| \le c/l^k$ *for all* $s,t \in I$, *for some positive constant* c.

PROOF. *Case* 1: *Part* (1) *of* R.1 *holds.* Without loss of generality, assume $f(s) \equiv s$, $I = [-\pi/2, \pi/2] \times \cdots \times [-\pi/2, \pi/2]$. Then $r(s,t) = r(s-t) =$

$r(w_1, w_2, \dots, w_d)$ is symmetric in terms of each $w_i = s_i - t_i$, $i = 1, \dots, d$, on $[-\pi, \pi] \times \cdots \times [-\pi, \pi]$ and is bounded, even and has up to kd^2 mixed bounded continuous partial derivatives. Hence r has a uniformly convergent Fourier series [cf. (3.6) in Walker (1988)]. Further, without loss of generality again, assume $r(\cdot)$ in terms of the w_i's has the same value at its end points.

In the following, we expand $r(w_1, w_2, \dots, w_d)$ into a Fourier series according to its coordinates, one by one.

Expanding $r(w_1, w_2, \dots, w_d)$ into a Fourier series in terms of w_1, we see that this series is a cosine series as $r(\cdot)$ is symmetric in w_1:

$$r(w_1, w_2, \dots, w_d) = \sum_{n=0}^{\infty} c_n^1(w_2, \dots, w_d)\cos(nw_1), \tag{55}$$

where the coefficients $c_n^1(w_2, \dots, w_d)$ are symmetric in terms of $w_2, \dots, w_d$ by virtue of the same symmetry in $r(\cdot)$.

Similarly to (55), for $m = 0, 1, 2, \dots$ we expand $c_m^1(w_2, \dots, w_d)$ into a Fourier series in terms of w_2, which is still a cosine series in w_2:

$$c_m^1(w_2, \dots, w_d) = \sum_{n=0}^{\infty} c_{n,m}^2(w_3, \dots, w_d)\cos(nw_2).$$

The new coefficients $c_{n,m}^2(w_2, \dots, w_d)$ are still symmetric in terms of $w_3, \dots, w_d$.

Repeating the above procedures for $w_3, \dots, w_d$, we have

$$\begin{aligned} r(w_1, w_2, \dots, w_d) &= \sum_{n(1), \dots, n(d)=0}^{\infty} c_{n(1), \dots, n(d)} \cos(n(1)w_1) \cdots \cos(n(d)w_d) \\ &= \sum_{n(1), \dots, n(d)=0}^{\infty} c_{n(1), \dots, n(d)}\big[\cos(n(1)s_1)\cos(n(1)t_1) \\ &\qquad + \sin(n(1)s_1)\sin(n(1)t_1)\big] \\ &\qquad \dots \big[\cos(n(d)s_d)\cos(n(d)t_d) + \sin(n(d)s_d)\sin(n(d)t_d)\big] \\ &= \sum_{n(1), \dots, n(d)=0}^{\infty} c_{n(1), \dots, n(d)} \sum_{i=1}^{d(d+1)/2} \Lambda^i_{n(1), \dots, n(d)}(t)\Lambda^i_{n(1), \dots, n(d)}(s), \end{aligned} \tag{56}$$

where $\Lambda^i_{n(1), \dots, n(d)}(t)$ is the ith configuration of the following $(p-1)p/2$ combination of sines and cosines:

$$\begin{gathered} \sin(n(1)t_1)\sin(n(2)t_2) \cdots \sin(n(d)t_d) \\ \cos(n(1)t_1)\sin(n(2)t_2) \cdots \sin(n(d)t_d) \\ \sin(n(1)t_1)\cos(n(2)t_2) \cdots \sin(n(d)t_d) \\ \vdots \\ \cos(n(1)t_1)\cos(n(2)t_2) \cdots \cos(n(d)t_d) \end{gathered}.$$

Since $r(\cdot)$ is nonnegative definite, $c_{n(1),\dots,n(d)}$ is nonnegative. Similar to a theorem in Walker (1988), page 199, we have

$$|c_{n(1),\dots,n(d)}| \le \frac{c}{n(1)^{kd} \cdots n(d)^{kd}}$$

for some positive constant c. Therefore, using the regular ordering in Definition 6.2 and the remark following the definition, the reordered Fourier series

$$\sum_{l=1}^{\infty} c_l \sum_{i=1}^{d(d+1)/2} \Lambda_l^i(t)\Lambda_l^i(s)$$

has the property $c_l \le c/l^k$ for some $c > 0$. This Fourier series uniformly converges to $r(s,t)$ for $k > 2$. Therefore, the conclusion of Lemma 6.1 is true in Case 1.

Case 2: *Part* (2) *of* R.1 *holds*. Similar to Case 1, without loss of generality, assume the function under consideration has the same value at its end points and $f(s) = s$.

From (56), we have

$$\begin{aligned} r(s,t) &= \sum_{i,j=1}^{d_3} h_i(s^{(1)})h_j(t^{(1)}) \sum_{n(1),\dots,n(d_2)=0}^{\infty} c^{ij}_{n(1),\dots,n(d_2)} \\ &\quad\times \sum_{i=1}^{d_2(d_2+1)/2} \Lambda^i_{n(1),\dots,n(d_2)}(t)\Lambda^i_{n(1),\dots,n(d_2)}(s) \\ &= \sum_{n(1),\dots,n(d_2)=0}^{\infty} \sum_{i,j=1}^{d_3} h_i(s^{(1)})h_j(t^{(1)})c^{ij}_{n(1),\dots,n(d_2)} \\ &\quad\times \sum_{i=1}^{d_2(d_2+1)/2} \Lambda^i_{n(1),\dots,n(d_2)}(t)\Lambda^i_{n(1),\dots,n(d_2)}(s) \\ &= \sum_{n(1),\dots,n(d_2)=0}^{\infty} c_{n(1),\dots,n(d_2)}(s^{(1)},t^{(1)})\Lambda^i_{n(1),\dots,n(d_2)}(t)\Lambda^i_{n(1),\dots,n(d_2)}(s), \end{aligned} \tag{57}$$

where

$$c_{n(1),\dots,n(d_2)}(s^{(1)},t^{(1)}) = \sum_{i,j=1}^{d_1} h_i(s^{(1)})h_j(t^{(1)})c^{ij}_{n(1),\dots,n(d)}.$$

Integrating by parts, we see that there exists a constant $C_{i,j;n(1),\dots,n(d_2)}$ such that

$$c^{ij}_{n(1),\dots,n(d_2)} = \frac{C_{i,j;n(1),\dots,n(d_2)}}{n(1)^{k(d_2+1)} \cdots n(d_2)^{k(d_2+1)}},$$

where $C_{i,j;n(1),\dots,n(d_2)}$ depends only on $i, j; n(1),\dots,n(d_2)$ and lth mixed order partial derivatives of h_{ij} for $l = 1, 2, \dots, 6d_2(d_2 + 1)$. Now define

$$\tilde{c}_{n(1),\dots,n(d)}(s^{(1)}, t^{(1)}) = n(1)^{kd_2+1} \cdots n(d)^{kd_2+1} c_{n(1),\dots,n(d)}(s^{(1)}, t^{(1)}).$$

It is easy to see that $\tilde{c}_{n(1),\dots,n(d)}(s^{(1)}, t^{(1)})$ is bounded uniformly in $n(1), \dots, n(d_2)$. It is a continuous symmetric kernel in $s^{(1)}, t^{(1)}$ which has up to fourth mixed order partial derivatives. Hence $\tilde{c}_{n(1),\dots,n(d_2)}(s^{(1)}, t^{(1)})$ has a uniformly convergent Mercer expansion,

$$\sum_{l=1}^{\infty} u_l^{n(1),\dots,n(d_2)}(s^{(1)}) u_l^{n(1),\dots,n(d_2)}(t^{(1)}),$$

which implies that

$$\begin{aligned} r(s,t) = & \sum_{n(1),\dots,n(d_2)=0}^{\infty} \frac{1}{n(1)^{kd_2+1} \cdots n(d)^{kd_2+1}} \\ & \times \left(\sum_{l=1}^{\infty} u_l^{n(1),\dots,n(d_2)}(s^{(1)}) u_l^{n(1),\dots,n(d_2)}(t^{(1)}) \right) \\ & \times \Lambda^i_{n(1),\dots,n(d_2)}(t) \Lambda^i_{n(1),\dots,n(d_2)}(s). \end{aligned}$$

Therefore, by the regular ordering from multiple index $\{n(1), \dots, n(d_2), l\}$ to univariate index, the conclusion of Lemma 6.1 is true in Case 2. □

PROOF OF LEMMA 4.1. By Lemma 6.1, for $m = 6$ there exists a uniformly convergent expansion of $r(s,t)$, such that

$$r(s,t) = \sum_{l=1}^{\infty} u_l(s) u_l(t), \tag{58}$$

where $u_l(s)u_l(t) \le c/l^m$ for some $c > 0$. Therefore, as $k \to \infty$, $\sum_{l=1}^{k} \partial u_l(s)/\partial s_i \, \partial u_l(t)/\partial t_j$ converges uniformly in I. We can exchange the summation and differentiation as follows [cf. Walker (1988), Theorem 5.5, page 26]:

$$\begin{aligned} \frac{\partial^2 r(s,t)}{\partial s_i \, \partial t_j} &= \frac{\partial^2}{\partial s_i \, \partial t_j} \sum_{l=1}^{k} u_l(s) u_l(t) \quad \text{[by (58)]} \\ &= \sum_{l=1}^{k} \frac{\partial^2}{\partial s_i \, \partial t_j} u_l(s) u_l(t) = \left\langle \frac{\partial u}{\partial s_i}, \frac{\partial u}{\partial s_j} \right\rangle = g_{ij}(t) \quad \text{by definition.} \end{aligned}$$

Therefore, Lemma 4.1 is valid. □

Connection. If $Z(t)$ is a d-dimensional nonsingular Gaussian random field satisfying R.1 with mean 0 and variance 1, there is a uniformly convergent (semi) Fourier expansion for $r(s,t)$ (Lemma 6.1). Under the same conditions, there is a Karhunen–Loève expansion for $\tilde{Z}(t)$ based on this (semi) Fourier expansion (Proposition 5.2), where $\tilde{Z}(t)$ and $Z(t)$ are identically distributed. This gives a connection between a (semi) Fourier series and Karhunen–Loève expansion.

APPENDIX

More on the existence of a Karhunen–Loève expansion. Let $r(s,t)$ be the covariance function $\mathrm{Cov}(Z(s), Z(t))$ of a random field $Z(t)$ on I, a d-dimensional compact space. Define $\Delta r(s,t) = r(s,s) + r(t,t) - 2r(s,t)$ and

$$p(u) = \max_{|s-t| \le u\sqrt{d}} \sqrt{\Delta r(s,t)}\,. \tag{59}$$

LEMMA A.1 [Garsia (1972); a sufficient condition]. *Suppose $Z(t)$ is a d-dimensional nonsingular random field with mean* 0 *and covariance function $r(s,t)$.*

If

$$\int_0^1 \sqrt{-\log u}\, dp(u) < \infty, \tag{60}$$

then there exists a partial sum formed from the first k eigenvalue–eigenfunctions of $r(s,t)$:

$$\tilde{Z}_k(t) = \sum_{l=1}^{k} \sqrt{\lambda_l}\,\Lambda_l(t) X_l,$$

which converges to $Z(t)$ uniformly in t on I as $k \to \infty$, with probability 1. *Here $X_1, X_2, \dots$ are i.i.d. $\mathcal{N}(0,1)$ r.v.'s, λ_l is the lth eigenvalue of $r(s,t)$, $\Lambda_l(t)$ is the corresponding eigenfunction of λ_l* [*cf.* (5)], *and $p(u)$ is as defined in* (59).

The existence of an orthogonal eigenvalue–eigenfunction expansion (*Mercer expansion*) of the covariance function $r(s,t)$ of a random field $Z(t)$ is given by Mercer [cf. Courant and Hilbert (1953), pages 138–140, or (5)]. The corresponding series $\sum_{l=1}^{\infty} u_l(t) X_l$ as in Lemma A.1 is called the uniformly convergent *standard Karhunen–Loève expansion* of $Z(t)$, where $u_l(t) = \sqrt{\lambda_l}\,\Lambda_l(t)$ for $l = 1, 2, \dots$.

REMARK A.1. If $\Delta r(s,t)$ satisfies the *Lipschitz-α* condition:

$$\Delta r(s,t) \le c\|s-t\|^{\alpha} \quad \text{for all } s,t \in I, \tag{61}$$

for some positive constants c and α, then (60) in Lemma A.1 is valid. This immediately gives Corollary A.1.

COROLLARY A.1. *Suppose $Z(t)$ is a nonsingular Gaussian random field on a bounded d-dimensional closed space I, with mean* 0, *variance* 1 *and* *covariance function* $r(s,t)$ *which satisfies, for some* $\alpha \in (0,2]$,

$$r(s,t) = 1 - \sum_{l=1}^{d} a_l(s,t)|s_l - t_l|^{\alpha} + o(|s-t|^{\alpha})$$

as $|s-t| \to 0$, *where the* $a_l(s,t)$'s *are bounded and nonnegative on* I. *Then* $Z(t)$ *has a uniformly convergent standard Karhunen–Loève expansion in* $t \in I$ [*cf*. (2)].

REMARK A.2. Suppose $Z(t)$ is a d-dimensional homogeneous random field and the covariance function $r(s,t) = r(s-t)$ of $Z(t)$ has two continuous derivatives with respect to w $(= s - t)$ on its bounded domain. Then $r(s,t)$ satisfies a Lipschitz-2 condition, that is, (61) holds with $\alpha = 2$. Hence $Z(t)$ has a uniformly convergent standard Karhunen–Loève expansion on its bounded domain I.

LEMMA A.2 (Extension to nonorthogonal series). *Suppose* $Z(t)$ *is a* d-*dimensional nonsingular random field on a bounded domain* I, *with mean* 0 *and covariance function* $r(s,t)$, *and*

$$\int_0^1 \sqrt{-\log u}\, dp(u) < \infty.$$

If there is an expansion of $r(s,t) = \sum_{l=1}^{\infty} \lambda_l \Lambda_l(s) \Lambda_l(t)$, *such that* $\sum_{l=1}^{k} |\lambda_l \Lambda_l(t) \Lambda_l(s)|$ *is dominated by* $\sum_{l=1}^{k} c|\lambda_l|$ *for some* $c > 0$, *which converges as* $k \to \infty$, *then the random field* $\tilde{Z}(t)$:

$$\tilde{Z}(t) = \sum_{l=1}^{\infty} \sqrt{\lambda_l}\, \Lambda_l(t) X_l,$$

is identically distributed as $Z(t)$ *and has a uniformly convergent Karhunen–Loève expansion on its bounded domain* I, *where* $X_1, X_2, \ldots$ *are i.i.d.* $\mathscr{N}(0,1)$ *r.v.'s.*

PROOF. Just follow the proof in Adler (1981) from pages 52 to 57 and notice the following three things. First, our Λ_i is ϕ_i and X_i is θ_i in Adler's proof. Second, the inequality signs in Adler's proof need to be switched at three places (misprints): line 4 from the bottom on page 52, line 10 from the bottom on page 55 and line 7 from the bottom on page 55. Third, there is a small change to match our assumption on the expansion in (A.5), where the Λ_l's do not have to form an orthogonal system. The small change is to replace four lines (lines 11 to 14) on page 56 with

$$E\left\{ \sum_{k=1}^{\infty} \lambda_k \theta_k^2(\omega) \right\} \le c \sum_{k=1}^{\infty} |\lambda_k| < \infty. \qquad \square$$

REMARK A.3. Lemma A.2 says that under certain conditions for the covariance function of a random field $Z(t)$, there exists a random field $\tilde{Z}(t)$ which is identically distributed as $Z(t)$ and has a uniformly convergent Karhunen–Loève expansion on its bounded domain I. Lemma A.1 says that if the expansion of $r(s,t)$ is orthogonal, there is a Karhunen–Loève expansion which converges to $Z(t)$ itself on its bounded domain I, viz. $\tilde{Z}(t) \equiv Z(t)$. These two lemmas on the existence of a Karhunen–Loève expansion play the same role in finding an approximation formula for $P\{\max_{t\in I} Z(t) \geq z\}$ as $z \to \infty$, since

$$P\Big\{\max_{t\in I} Z(t) \geq z\Big\} = P\Big\{\max_{t\in I} \tilde{Z}(t) \geq z\Big\}$$

and $\tilde{Z}(t)$ has a uniformly convergent Karhunen–Loève expansion to itself as $Z(t)$ in Lemma A.1 does.

Some basic concepts and formulas in differential geometry and Fourier series. Suppose $\mathscr{U} = \{u(t): u(t) = (u_1(t), u_2(t), \dots, u_k(t)) \in S^{k-1},\ t \in I\}$ is a d-dimensional differentiable manifold on metric space I, $k \leq \infty$. Its *metric tensor* is the inner product

$$g_{ij}(t) = \left\langle \frac{\partial u(t)}{\partial t_i}, \frac{\partial u(t)}{\partial t_j} \right\rangle = \sum_{l=1}^{k} \frac{\partial u_l(t)}{\partial t_i} \cdot \frac{\partial u_l(t)}{\partial t_j}$$

of the partial derivatives of u. Here $i, j = 1, 2, \dots, d$, $t = (t_1, t_2, \dots, t_d) \in I$. The inverse matrix $R^{-1}(t)$ of the *metric tensor matrix* $R(t) = (g_{ij}(t))$ is written as

$$R^{-1}(t) = (g^{ij}(t))_{d\times d}. \tag{62}$$

Definition A.1 (Volume). The volume, or area of a subset $M \subset \mathscr{U}$ is

$$V(M) = \int_{t\in u^{-1}(M)} \sqrt{\|R(t)\|}\, dt,$$

where $\|R(t)\|$ is the determinant of $R(t)$ [cf. Kreyszig (1968)]. Hence the volume of $\mathscr{U}$ is $V(\mathscr{U}) = \int_{t\in I} \sqrt{\|R(t)\|}\, dt$.

Definition A.2 (Riemannian curvature tensor). The Riemannian curvature tensor of $\mathscr{U}$ is the tensor with components

$$R^l_{ijk}(t) = \frac{\partial \Gamma^l_{ik}(t)}{\partial t_j} - \frac{\partial \Gamma^l_{ij}(t)}{\partial t_k} + \sum_{p=1}^{d} \big(\Gamma^p_{ik}(t)\Gamma^l_{pj}(t) - \Gamma^p_{ij}(t)\Gamma^l_{pk}(t)\big),$$

where the Γ^k_{ij} are called the *Christoffel symbols* defined on $\mathscr{U}$:

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^{d} g^{lk}(t) \left(\frac{\partial g_{lj}(t)}{\partial t_i} - \frac{\partial g_{ij}(t)}{\partial t_l} + \frac{\partial g_{il}(t)}{\partial t_j} \right), \tag{63}$$

with g^i_j as in (62) [cf. Kreyszig (1968), (41.4), page 134].

DEFINITION A.3 (Ricci curvature tensor). The Ricci curvature tensor of $\mathscr{U}$ is the tensor with components

$$R_{ij}(t) = \sum_k R^k_{ijk}(t)$$

[cf. Kreyszig (1968), page 309].

DEFINITION A.4 (Scalar curvature). The scalar curvature of $\mathscr{U}$ is

$$S(t) = \sum_{i=1, j=1}^{d} g^{ij}(t) R_{ij}(t)$$

[cf. Kreyszig (1968), page 310].

DEFINITION A.5 (Formulas of Gauss and Weingarten). Let n_l, $l = 1, 2, \ldots,$ be the normal vectors orthogonal to the tangent space spanned by $\partial u(t)/\partial t_l$, $l = 1, \ldots, d$.

The *Gauss formula* of $\mathscr{U}$ gives a linear expansion of $\partial^2 u(t)/\partial t_i \, \partial t_j$ in terms of $\partial u(t)/\partial t_k$, $k = 1, \ldots, d$, and n_l, $l = 1, 2, \ldots$:

$$\frac{\partial^2 u(t)}{\partial t_i \, \partial t_j} = \sum_l \Gamma^l_{ij} \frac{\partial u(t)}{\partial t_l} + \sum_l L_{ij}(l) n_l,$$

where $L_{ij}(l) = \langle \partial^2 u(t)/\partial t_i \, \partial t_j, n_l \rangle = \sum_m g_{im} L^m_j(l)$, Γ^l_{ij} is the Christoffel symbol defined as in Definition A.2.

The *Weingarten equation* of $\mathscr{U}$ gives linear expansion of $\partial n_l(t)/\partial t_i$ in terms of $\partial u(t)/\partial t_k$, $k = 1, \ldots, d$, and n_l, $l = 1, 2, \ldots$:

$$\frac{\partial n^k_l(t)}{\partial t_i} = -\sum_j L^j_i(l) \frac{\partial u^k(t)}{\partial t_j} + \cdots,$$

where $-L^j_i(l)$ is the coefficient of $\partial n^k_l(t)/\partial t_i$ in the direction $\partial u^k(t)/\partial t_j$, and "$+ \cdots$" are components orthogonal to the tangent space spanned by $\partial u^k(t)/\partial t_j$, $i = 1, \ldots, d$ [cf. Kreyszig (1968)].

DEFINITION A.6 (Dirichlet condition). A function $f(x_1, \ldots, x_d)$ defined on a rectangular region $I = [a_1, b_1] \times \cdots \times [a_d, b_d]$ satisfies the *Dirichlet condition*, if one of the following two conditions holds:

1. f is bounded on I. For any fixed x_j, $j \neq i$ of $j \in (1, \ldots, d)$, the interval (a_i, b_i) can be broken up into a finite number of open partial intervals, in each of which $f(x_1, \ldots, x_d)$, as a function of x_i, is monotonic for $i = 1, \ldots, d$.
2. $f(x_1, \ldots, x_d)$ has a finite number of points of infinite discontinuity in I. When arbitrary small neighborhoods of these points are excluded, f is bounded on I, and the remainder of (a_i, b_i) can be broken up into a finite number of open partial intervals (when x_j is fixed for $j \neq i$), in each of

which f is monotonic in terms of x_i, for $i = 1, \dots, d$. Further, the infinite integral $\int_I f(x_1, \dots, x_d)\, dx_1 \dots dx_d$ is absolutely convergent [cf. Carslaw (1930), page 226].

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