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FRANK SPITZER'S PIONEERING WORK ON INTERACTING PARTICLE SYSTEMS

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In the late 1960's Frank Spitzer's research underwent a phase shift—he began to study spatially distributed interacting stochastic systems, a subject that was still in its infancy. Later, in the preface to his 1976 second edition of *Principles of Random Walk*, he would write the following:

> New types of random walk problems are now in the stage of pioneering work. This came about because the simple model of a single particle, performing a random walk with given transition probabilities, may be regarded as a crude approximation to more elaborate random walk models. ... In other models one considers the simultaneous random walk of a finite or even infinite system of particles, with certain types of interaction between the particles. But this is an entirely different story.

Although the roots of interacting systems can be traced to earlier modeling efforts in applied fields such as statistical physics, computer science and population genetics [cf. Glauber (1963), Kimura and Weiss (1964) and von Neumann (1966)], Spitzer and his Russian counterpart R. L. Dobrushin are widely credited as co-founders of a mathematical theory that has now evolved into one of the richest and most vital areas of probability. This legacy is best documented in the excellent books by Liggett (1985) and Durrett (1988), which not only detail much of the work described below, but also consolidate two decades of research by Spitzer's colleagues and students.

The early 1970's were a doubly fortunate time for me to be a graduate student in stochastic processes at Cornell University. First, I could learn the subject from three masters: Itô, Kesten and Spitzer. However, I was also lucky to find myself among an unusual concentration of eager disciples, several of whom have become my close friends and collaborators. We responded to interacting systems as a promising new paradigm for understanding the organizational principles that underlie many fundamental "real world" phenomena. All of us were equally inspired by Spitzer's keen sense of aesthetics, which was guided by the beauty of mathematics as much as any physical motivation. Although he admired exceptional technical ability, Frank seemed to favor elegance above all. He set high standards for himself and his students while communicating genuine enthusiasm for good work.

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What follows is a rather subjective annotated bibliography of Spitzer's writings on interacting particle systems (IPS). I include a sampling of papers by other authors, either because Frank admired them, or because I feel they illustrate his lasting impact on contemporary research. The bibliography is divided into six groupings, more or less in chronological order. References of the form [S*n*] refer to the publication list that begins on page 622 in this issue of the *Annals*.

Not surprisingly, Spitzer's first couple of papers about interacting systems recall his earlier work on random walks and Brownian motion.

[S27] Uniform motion with elastic collision of an infinite particle system. *J. Math. Mech.* **18** (1969) 973–989.

[S28] Random processes defined through the interaction of an infinite particle system. *Probability and Information Theory* (Proc. International Symposium, McMaster University, Hamilton, Ontario, 1968). *Lecture Notes in Math.* **89** (1969) 201–223. Springer, Berlin.

Frank was intrigued by Harris (1965), in which an infinite system of reflecting Brownian particles on the line was analyzed. Starting from a Poisson distribution, Harris proved that the displacement of a tagged particle is of order $t^{1/4}$ at time t, due to the congestion caused by other particles. At the end of the paper Harris considered a related model in which the individual particles execute *deterministic* motions with independent mean-0 random initial velocities and elastic collisions ("1-d billiards"). In this case he showed that the displacement of a tagged particle is of order $t^{1/2}$ at time t, and conjectured that the trajectory, suitably scaled, should converge (weakly) to a Brownian motion. Spitzer proved this conjecture in [S27] by establishing convergence of finite dimensional distributions and tightness. Then in [S28] he discussed a more general class of IPS on $\mathbb{R}$ including some variants on the Harris billiards. Most notable was the result of his student R. Holley [Cornell dissertation, Holley (1969)]. In a masterpiece of technical virtuosity, Holley proved that a massive 1-d particle, buffeted by infinitely many small ones, executes an Ornstein–Uhlenbeck process under a suitable limiting scheme. Frank found such results satisfying for the precise way they demonstrate that basic diffusions arise as large-scale limits of deterministic particle dynamics. In retrospect, it is worth noting that his first models were *deterministic* systems with random initial states.

The most elementary interactions in particle systems that preclude multiple occupancy are *exclusion* (= reflection or elastic collision), *coalescence* (two particles merge into one) and *annihilation* (two particles cancel one another). When crossed with the simplest particle motions, deterministic (constant velocity) or stochastic (completely random motion), one gets six basic classes of one-dimensional model. Within each class, system behavior is essentially the same whether time and space are discrete or continuous. Linear ordering of the particles facilitates various exact and asymptotic calculations that are much more difficult in higher dimensions. Table 1 lists some of the many papers in this $d = 1$ setting that may be viewed as descendants of the early

TABLE 1

Motion	Interaction: Exclusion	Coalescence	Annihilation
Deterministic	Harris (1965) Spitzer [S27]	Fisch (1993)	Fisch (1992)
Stochastic	Harris (1965) Spitzer [S29] Arratia (1983)	Bramson and Griffeath (1980b) Arratia (1979)	Erdös and Ney (1974) Bramson and Griffeath (1980b) Arratia (1979)

work of Harris and Spitzer. Highlights include: (i) the construction in Arratia (1979) of coalescing Brownian motions starting with *every* point of the real line occupied; (ii) Arratia's (1983) solution of the tagged particle problem for Spitzer's symmetric exclusion process on $\mathbb{Z}$, following the method of Harris (1965); and (iii) exact asymptotics for annihilating deterministic particles on $\mathbb{Z}$ having Bernoulli ± 1-valued initial velocities, derived in Fisch (1992) by means of an interesting exact connection with simple random walk.

Spitzer's focus turned next to a general framework for interacting particle systems η_t as Markov processes on a configuration space. He settled on $\mathbb{Z}^d$ as the set of possible locations for particles, presumably out of affection for the discrete setting and to avoid additional technical difficulties associated with the continuum alternative. He chose continuous time so that at most one local transition occurs at any instant of time and the generator Ω for η_t takes the relatively simple form [see Liggett (1985)]

$$\Omega f(\eta) = \sum_T \int c_T(\eta, d\zeta)\left[f(\eta^\zeta) - f(\eta)\right], \tag{1}$$

$c_T(\eta, d\zeta)$ being the exponential rate of change from configuration η to the modification η^ζ that has a new configuration ζ on the finite set $T \subset \mathbb{Z}^d$. In this context he wrote what I consider to be his best two papers on particle systems:

[S29] Interaction of Markov processes. *Adv. in Math.* **5** (1970) 246–290.
[S34] Recurrent random walk of an infinite particle system. *Trans. Amer. Math. Soc.* **198** (1974) 191–199.

In [S29], Spitzer introduced several classes of models of type (1) for which one can explicitly identify invariant measures. Many of these models are motivated by statistical mechanics and formulated in terms of an interaction potential $U(x, y)$. At that time Frank was inspired by the seminal work of Dobrushin (1968) and Ruelle (1969) on Gibbs random fields. He wanted to find a collection of simple random evolutions that admit such Gibbs distributions as steady states. Part of the grand design was a hope that one could gain deeper understanding of the equilibria by exploiting the dynamics.

Spitzer discovered a gold mine in [S29]—the fact that certain basic IPS have tractable invariant measures, even for $d > 1$, was a promising development. It suggested that such systems might be surprisingly amenable to rigorous mathematical analysis. Frank was especially interested in proving *convergence theorems*, that is, determining the ergodic theory of his models. On *finite* lattices ($\mathbb{Z}^d$ mod L with wrap-around, say) many of his systems can be viewed as irreducible denumerable Markov chains, so the Markov chain ergodic theorem applies. But he was more intrigued by the *uncountable* configuration spaces $\{0,1\}^{\mathbb{Z}^d}$ (at most one particle per site), or $\{0,1,\dots\}^{\mathbb{Z}^d}$ (any number of indistinguishable particles per site), for which a whole new theory was needed. In particular these *infinite* particle systems entailed infinitely many changes of state in any time interval no matter how small. In the introduction to [S29] Frank openly admitted that his favorite systems were not known to exist. He simply called his results "conjectures" and stressed "the need for further work." Fortunately, Liggett (1972, 1985) resolved this uncertain state of affairs by proving existence and uniqueness of Markov processes with generators of type (1) (for f that depend on finitely many coordinates). Liggett's theorem imposes only mild and natural assumptions on the "jump rates" c, and so applies to virtually any reasonable local interaction with uniformly bounded transition intensities.

Of all the models Spitzer discusses in [S29], surely the most important is *simple symmetric exclusion*. Frank was very fond of a result from the 1950's, due to Doob and Derman, about systems of independent particle motions. For either Brownian motions on $\mathbb{R}^d$ or simple random walks on $\mathbb{Z}^d$, any (homogeneous) Poisson point process is invariant. He considered a corresponding continuous-time lattice model η_t with at most one particle per site and the simplest imaginable interaction: whenever a particle tries to jump to a site that is already occupied that jump is suppressed. Frank observed that η_t has a one-parameter family of invariant measures that is also the simplest imaginable: The Bernoulli product measures μ_α; $0 \le \alpha \le 1$. Curiously, in spite of the exclusive interaction, an initially uncorrelated homogeneous configuration remains uncorrelated at all times. Because of this fact, several additional striking structural features, and its special significance as a prototype for diffusion of a lattice gas, simple symmetric exclusion is widely revered as one of the most beautiful models of contemporary probability theory.

The basic ergodic theory of exclusion is laid out in [S34] and the companion papers Liggett (1973) and Liggett (1974). The starting point is Spitzer's *duality equation* from [S29]:

$$P^\eta(\eta_t \equiv 1 \text{ on } A) = P^A(\eta = 1 \text{ on } A_t), \tag{2}$$

where A is any *finite* subset of $\mathbb{Z}^d$ and A_t is the exclusion process starting from A. This reduces distributional characteristics of the infinite system to quantities involving the countable Markov chain A_t. Various *couplings*, constructions of two or more systems on a common probability space, also play a key role in the Spitzer–Liggett analysis. For instance, one can couple η_t to an independent particle system of Doob–Derman type, or to the *stirring* modification in which particles swap places when one attempts to jump to the position occupied by the other. Or two copies of η_t with different initial configurations can be coupled in various ways. The independent and stirring systems are useful for comparisons because their individual particles execute bona fide simple random walks to which classical theory applies. Over the past twenty years duality and coupling have turned out to be the two most versatile and productive tools for rigorous analysis of interacting systems.

Using duality, coupling, and other clever techniques, Spitzer and Liggett were able to show that the only extreme translation invariant equilibrium states for simple symmetric exclusion on $\mathbb{Z}^d$ are the Bernoulli product measures μ_α, and to identify the domains of attraction of the μ_α. Separate arguments are needed for "recurrent" ($d = 1, 2$) and "transient" ($d \geq 3$) models; the theory of discrete harmonic functions plays a key role. Chapter VIII, Section 1 of Liggett (1985) presents this elegant analysis in complete detail.

Liggett went on to write several papers about more general exclusion models, some of them proposed by Spitzer in [S29]. His extensive investigation of variants without symmetry or nearest neighbor structure constitutes a beautiful chapter in stochastic process theory, full of power and ingenuity. Another outstanding paper in this area is Kipnis and Varadhan (1985), where the position of a tagged particle in the d-dimensional simple symmetric exclusion model η_t is shown to obey a central limit theorem with the usual $\sqrt{t}$ scaling for any $d > 1$. Thus the Harris–Arratia identification of $t^{1/4}$ scaling for $d = 1$ represents the only case in which particle fluctuations are "subdiffusive." Most recently, exclusion processes have become the favorite toy models of *stochastic hydrodynamics*: the study of density profile evolutions for IPS. Over the past decade, since the seminal paper of Rost (1981), dozens of articles on particle hydrodynamics have appeared. Andjel, Bramson and Liggett (1988), for example, analyze the connection between asymmetric exclusion and *shocks* in an associated pde known as Burger's equation. See also the survey of De Masi, Ianiro, Pellegrinotti and Presutti (1984).

[S30] Markov random fields and Gibbs ensembles. *Amer. Math. Monthly* **78** (1971) 142–154.

[S31] Random fields and interacting particle systems. Notes on lectures given at the 1971 MAA Summer Session, Williams College, Williamstown, MA, Mathematical Association of America, Washington, D.C.

[S35] Introduction aux processus de Markov à paramètre dans Z_ν. *Ecole d'Eté de Probabilités de Saint-Flour III–1973. Lecture Notes in Math.* **390** (1974) 114–189. Springer, Berlin.

[S37] Markov random fields on an infinite tree. *Ann. Probab.* **3** (1975) 387–398.

[S40] Phase transition in one-dimensional nearest neighbor systems. *J. Funct. Anal.* **20** (1975) 240–255.

Spitzer was greatly impressed by the work of Dobrushin (1968) and Ruelle (1969) on the mathematical foundations of statistical mechanics. Although dynamics remained the focus of his research, Frank became intrigued with the theory of (discrete) *random fields*, lattice distributions that play the role of invariant measures for IPS. Simultaneously with others he discovered a beautiful equivalence [S30] of *Markov random fields* and *Gibbs ensembles*. Roughly speaking, a Markov field η has the property that the conditional distribution of $\eta(x)$ given η off x agrees with the distribution of $\eta(x)$ given $\eta(\partial x)$, where ∂x denotes the *nearest neighbors* of x. A Gibbs ensemble is a random field with cylinder probabilities expressed in terms of the exponential of a nearest neighbor potential. Frank's rather arduous proof that these two recipes coincide was later reduced to a simple application of the Möbius inversion formula by Grimmett (1973). A general and abstract framework for this circle of ideas was developed subsequently by Dobrushin, C. Preston, H. Föllmer, E. B. Dynkin and others.

My first introduction to Spitzer's work was the "Williamstown Notes" [S31] that he prepared in conjunction with an MAA summer school in 1971. While an undergraduate at Dartmouth College, I remember poring over a mimeographed preprint of these notes (which I have kept to this day). They begin with a gentle introduction to Markov chains, adapted to suit the IPS framework. Simple exclusion is introduced, but thereafter the notes focus on models of statistical mechanics: Gibbs random fields and their dynamical counterparts, *stochastic Ising models*. After discussing selected topics from [S29] and [S30], Frank's lectures culminate with a demonstration of phase transition in the basic Ising model on $\mathbb{Z}^2$. I still remember thinking that the Peierls argument he presents in Chapter 7 of [S31] was about the most beautiful proof I had ever seen.

The subsequent Saint-Flour notes [S35] reflect a more mature theory, following two more years of fundamental work by Spitzer, his co-workers and students. Liggett had proved his existence and uniqueness theorem, while Holley had established basic properties of *attractive* (monotone) systems and obtained some elegant results for stochastic Ising models. Spitzer's students R. Thompson and K. Logan had written Cornell dissertations motivated by statistical mechanics. At that time Frank was optimistic that the dynamics of Glauber-type *spin systems* might yield additional insights into the deep secrets of their equilibria: the Gibbs states. More than anyone else, Holley took this calling to heart. Over the past twenty years, often in collaboration with D. Stroock, he has written a large collection of masterful papers that explore deep and important aspects of the relationship between reversible Ising dynamics and their corresponding Gibbs fields. For references, see Liggett (1985) and Holley's contribution to the Spitzer *Festschrift* [Durrett and Kesten (1991)].

In his articles [S37] and [S40], Spitzer makes elegant excursions into the theory of (translation-invariant nearest-neighbor) Markov random fields on tractable graphs. Paper [S37], based in part on work of Preston, studies the structure of Markov fields on homogeneous N-ary trees. The absence of loops enables a prescription of cylinder probabilities in terms of a Markov chain. This setting provides explicit criteria for uniqueness/nonuniqueness of Markov fields with prescribed local characteristics (conditional probabilities). Paper [S40] studies Markov fields on $\mathbb{N}^{\mathbb{Z}}$—the lattice is the one-dimensional integers, but the type space is denumerably infinite. In contrast to the case of finite type space, there can be more than one Gibbs field with given conditional probabilities, that is, *phase transition* can occur. A representation theorem for the class of Gibbs fields is given, an interesting example of phase transition is worked out in detail, and then the discussion concludes with a list of conjectures and problems. As was often the case, Frank's investigations quickly inspired a flurry of activity by colleagues and students. While I was a graduate student at Cornell, three "in-house" research projects grew out of [S40], resolving most of the open problems mentioned there. Frank's student J. T. Cox wrote a very nice dissertation on *entrance laws*, a one-sided variant of the Markov field theory that allows denumerable Markov chains started at time $t = -\infty$ to "come in from ∞ (in the state space)." Another Cornell student, S. Kalikow, found a surprising example of a nonstationary entrance law that first coincides with a stationary distribution at a deterministic time ($t = 0$, say). And, even before Frank's paper appeared in print, Kesten (1976) proved that the class of *translation invariant* Gibbs states must either be empty or consist of a single stationary Markov chain. A survey of random fields including these and related results, and many additional references, may be found in Griffeath (1976).

[S36] Random time evolution of infinite particle systems. *Proc. Internat. Congress Math.* (Vancouver 1974) **2** 169–171.

[S39] Random time evolution of infinite particle systems. *Adv. in Math.* **16** (1975) 139–143; Also appeared in *Surveys in Applied Mathematics* (N. Metropolis, S. Orszag and G.-C. Rota, eds.). Academic, New York, 1976.

Articles [S36] and [S39] are two short overviews of IPS theory circa 1974 with the same title. The first is a very brief sketch of his talk to the 1974 International Congress in Vancouver; the second contains identical references but elaborates somewhat on the outline of the first. These accounts continue to focus on connections with statistical mechanics, recapitulating the Saint Flour Notes for the most part. Frank singles out the seminal work of Dobrushin (1971), acknowledging his independent discovery of particle system dynamics that have prescribed Markov or Gibbs fields as equilibria. Dobrushin's extraordinary 1971 paper presented a unified approach to existence, uniqueness and ergodicity of infinite spin systems. Such a process has two possible states per site, ± 1 say. Each site x flips its state at an exponential rate $c_x(\eta)$ that depends only on the configuration of η in a local neighborhood of x. Roughly, the interaction is *weak* if $|c_x(\eta) - c_x(\eta')| < \varepsilon$ whenever $\eta(x) = \eta'(x)$, that is, if the flip rates at different sites are nearly independent. A system η_t is said to be *ergodic* if it converges (in distribution) to a unique equilibrium μ starting from any initial distribution π. Dobrushin used a combination of coupling and duality to prove ergodicity in the case of sufficiently weak interaction (small ε). But his exposition was rather difficult to penetrate, so

Spitzer, Harris, Holley, Liggett and others dedicated their collective energies to understanding the essential techniques.

To my knowledge, [S39] includes the first reference in print to the "major open problem" that has come to be known as the *positive rates conjecture*. Based largely on the fact that no finite range, homogeneous Gibbs potential in one dimension can admit phase transition, and the closely-related ergodicity of corresponding stochastic Ising models, a "yes" answer was suspected to the question:

(3) Is any homogeneous, finite-range, one-dimensional spin system with positive flip rates ergodic?

Work on this problem has been a major impetus to the theory ever since.

At about the same time, Harris (1974) established a phase transition for the *basic contact process* ξ_t in one dimension, a spin system prototype for contagion in which infected sites (1's) recover at exponential rate δ, while healthy sites (0's) are infected at rate β by each *neighboring* infective. Since the "all 0's" configuration is a trap for the process ξ_t, its rates are not uniformly positive and ergodicity amounts to global recovery (in distribution) starting from any initial configuration. Harris proved existence of a *critical value* $\lambda_c \in (0, \infty)$ such that ξ_t is ergodic if $\beta/\delta < \lambda_c$ whereas the infection persists starting from "all 1's" if $\beta/\delta > \lambda_c$.

Another triumph of the mid-1970's was the detailed analysis of the *basic voter model* ζ_t on $\mathbb{Z}^d$ by Holley and Liggett (1975). In this system, folks of two competing opinions continuously reevaluate their own views in light of their neighbors'. Specifically, the individual at x changes opinion at a rate proportional to the number of neighbors who disagree. Again, this system does not have uniformly positive flip rates; in this case both of the consensus configurations "all 0's" and "all 1's" are traps, so the system cannot possibly be ergodic. Holley and Liggett showed that the asymptotic behavior of ζ_t is *dimension-dependent*, reflecting the Polya dichotomy for recurrence/transience of simple random walk. If $d \geq 3$, the model has a one-parameter family of nondegenerate extreme invariant equilibria ν_α, and settles down to one of these starting from any nice initial measure. If $d = 1$ or 2, on the other hand, ζ_t *clusters* starting from any nontrivial product measure μ_θ, meaning that arbitrarily large connected components of common opinion arise as $t \to \infty$. This latter scenario constitutes one of the simplest instances of self-organization starting from randomness.

Spitzer mentioned both the contact process and the voter model in [S36] and [S39], describing the abovementioned results as "deep" and "most surprising." These nonreversible systems do not really fit within the framework of traditional statistical mechanics. Rather, they arise as mathematically tractable models from a much broader terrain of spatial interactions. It is interesting to note that Frank dedicated [S39] to Ulam, who along with von Neumann is considered the founder of the theory of *cellular automata* [cf. Ulam (1952) and von Neumann (1966)]. Cellular automata (CA) are deterministic, local, homogeneous, discrete-time dynamics that update in parallel; see Toffoli and Margolus (1987) for an overview of the field including a brief history. Nowadays, the discrete-time synchronous analogs of Spitzer's particle systems are often called *random cellular automata* (RCA), meaning that stochastic ingredients enter into their local update rules. Despite the popularity of Conway's *Game of Life* in the 1970's, and the obvious similarity of the IPS and CA paradigms, only in the past few years have researchers from the two areas begun to collaborate. Increasingly, IPS and CA models are being investigated by mathematical physicists, chemists, biologists and computer scientists.

In fact, variants on both the contact process and the voter model had been considered previously, outside the context of statistical mechanics. An RCA version of the contact process, sometimes called the *Russian lamps*, was studied by Toom (1968), Vasershtein (1969) and Vasersthein and Leontovich (1970) within the context of reliable computation in neural networks. These remarkable papers established an exact connection between the lamps and the *oriented percolation* model of Hammersley (1959), developed the discrete-time counterpart of Dobrushin's ergodic theorem via coupling, and formulated both the sample path and analytical versions of discrete-time duality theory. The voter model, on the other hand, arose naturally within the context of biology and population genetics as a prototype for spatially-distributed selectively-neutral genetic drift. Its qualitative dimension dependence was noted by Clifford and Sudbury (1973). A multitype generalization known as the *stepping-stone model* had been analyzed a decade earlier by Kimura and Weiss (1964), based on ideas that date back to Wright (1943).

[S43] Stochastic time evolution of one dimensional infinite particle systems. *Bull. Amer. Math Soc.* **83** (1977) 880–890.

Article [S43] is based on an invited address that Spitzer delivered at the January 1976 meeting of the AMS in San Antonio. There he introduced a new class of one-dimensional interactions known as *nearest particle systems*. Just as for spin systems, each site of $\mathbb{Z}$ is either occupied by a particle or vacant, and transitions are governed by flip rates. But now these birth and death rates at x are of the form $\beta(l,r)$ and $\delta(l,r)$, where l and r are the distances from x to the nearest particle to the left of x and to the right of x, respectively. [One can obtain finite nearest particle systems by suitable choice of rates $\beta(\infty,r)$, $\beta(l,\infty)$, $\delta(\infty,r)$ and $\delta(l,\infty)$ to dictate transitions of the extremal occupied sites.] Frank discovered that for certain choices of the flip rates, for example,

$$(4) \qquad \beta(l,r) = c\left(\frac{1}{l} + \frac{1}{r}\right)^p, \qquad \delta(l,r) \equiv 1, \qquad p > 1, c > c_p,$$

where c_p is easily computable, the corresponding infinite nearest particle system has an invariant *renewal measure* μ, and is reversible with respect to μ. [In example (4) the interparticle spacings of μ are iid with density $\beta(n) = n^{-p}$.] Thus he exhibited another rich class of interactions with equilibria from classical probability theory. The connection is limited to $d = 1$, but has given rise to a rich and detailed theory, including the evaluation for various *critical exponents*, much of which is developed in Chapter VII of Liggett (1985). Again, at the time of Frank's lecture his calculations were largely formal: Since his new systems were not local, Liggett's existence theorem failed to apply to basic cases such as (4). By the time [S43] appeared the following year, his student L. Gray had written a dissertation on "controlled spin systems" that established the existence and uniqueness of nearest particle systems under natural assumptions.

In one of my very favorite pieces of mathematics, Holley and Liggett (1978) found it fruitful to think of the basic one-dimensional contact process ξ_t as a nonreversible nearest particle system with $\beta(1,1) = 2\lambda$, $\beta(1,r) = \beta(l,1) = \lambda$ for $r, l > 1$, $\beta(l,r) = 0$ otherwise, and $\delta \equiv 1$. Motivated by Spitzer's reversible examples, they sought an initial renewal measure μ_λ with the property that ξ_t stays *above* μ_λ in an appropriate sense at all times $t > 0$. Amazingly, they were able to carry out this survival strategy for $\lambda \geq 2$, thereby obtaining far and away the best rigorous upper bound on the critical value λ_c (even to this day). What better indication of the vision and scope of Spitzer's model-building abilities?

Section 7 of Chapter VII in Liggett (1985) lists 18 challenging problems about finite and infinite nearest particle systems that were unsolved when the book was written. One of the most intriguing dealt with the (nonreversible) *uniform nearest particle process* with rates

$$\beta(l,r) = \frac{b}{l+r-1}, \qquad \delta(l,r) \equiv 1, \qquad c > 0, p > 1.$$

In words, a particle is born within each unoccupied interval at rate b, the position of the particle being uniformly distributed over that interval; deaths occur at rate 1. It is not hard to see that such a process dies out if $b \leq 1$; Liggett asked the critical value b_c for survival. The recent and impressive solution is due to Mountford (1992). Improving on some partial results of Bramson, he proves that $b_c = 1$.

[S46] Infinite systems with locally interacting components. *Ann. Probab.* **9** (1981) 349–364.

[S47] (with T. M. Liggett). Ergodic theorems for coupled random walks and other systems with locally interacting components. *Z. Wahrsch. Verw. Gebiete* **56** (1981) 443–468.

Spitzer's final two papers on particle systems introduced yet another new collection of interacting systems, these with spins in $\mathbb{N}$ or $\mathbb{R}$ rather than a two-point set. Paper [S46] is based on his 1979 Wald Lectures to the IMS; the subsequent collaboration [S47] with Liggett provides the technical underpinnings. His last models are called *smoothing* and *potlatch* processes and *coupled random walk*. They are interrelated by an intricate web of duality equations and martingales that also makes connections with previously studied systems such as the voter model and coalescing random walks. An elegant treatment of these so-called *linear* dynamics may be found in Chapter IX of

Liggett (1985), which also describes the "second-moment method" for proving phase transitions, developed subsequently by Holley and Liggett.

In the early 1980's there was a methodological war underway within the so-called "particle mafia." Duality equations like (2) had become the dominant tool of the day, but there were two competing approaches to duality: one based on semigroup theory, martingale problems and differential equations, the other relying on graphical representation, percolation theory and sample path methods. A fundamental paper by Holley and Stroock (1976) championed the former perspective, which Frank seemed to favor. In [S46] he mentions several essentially analytical results, for example, exact asymptotics for coalescing random walks, derived by Bramson and Griffeath (1980a) based on stepping stone results of Sawyer (1979). The competing approach to duality, advocated by Harris (1976, 1978), won a major battle when Gray (1982) proved that any one-dimensional nearest-neighbor translation invariant attractive spin system with positive flip rates is ergodic. Gray's affirmative answer to an important special case of (3) seems to *require* a proof based on graphical analysis. The phenomenal development of *percolation theory* throughout the 1980's [cf. Kesten (1982) and Grimmett (1989)] introduced powerful new sample-path techniques that apply to particle systems as well. This connection culminated in the proof of Bezuidenhout and Grimmett (1990) that the basic contact process dies out at the critical value $\lambda = \lambda_c$ in any dimension d.

Now that some dust has cleared, the future of rigorous results concerning interacting systems seems certain to entail an exciting interplay of analysis, combinatorics and topology. Durrett (1988) gives an excellent overview of many topics not contained in Liggett (1985), including a friendly introduction to stochastic shape theory. For a popular account of some very recent developments, see Durrett (1992). Not surprisingly, computer technology and theoretical computer science are having an increasing impact on the subject. For instance Gacs (1985) uses Turing machine ideas to argue for the existence of nonergodic $d = 1$ RCA rules with uniformly positive transition probabilities. From his perspective this is not so much a counterexample to the (discrete-time) positive rates conjecture, but rather a proof that reliable one-dimensional parallel computation is possible at low noise levels.

In closing, I would like to make a few somewhat more personal remarks about Frank Spitzer. Above all I want this article to document his tremendous generosity and support in sharing his ideas and enthusiasms with his students and co-workers. As one of my colleagues once remarked, Frank had more beautiful results hidden away in his office drawer than most of us publish in our lifetime. During the 1980's, as his health was failing, he would often dig into that drawer and produce lovely computations that inspired substantial research projects. For instance, my work with Ted Cox in the mid-1980's on the large deviations of particle system occupation times grew out of Spitzer's evaluation of the cumulant generating function for the Doob–Derman model in three or more dimensions [cf. Cox and Griffeath (1984)]. Another beautiful unpublished formula, concerning random walk on a torus, played a key role in

Cox (1989), which derives exact asymptotics for the time until consensus in the voter model on a large finite box in $\mathbb{Z}^d$.

As already mentioned in Harry Kesten's companion article, Frank was continually searching for "new phenomena" and exhorting his students and co-workers to do the same. He sometimes talked about "skimming the cream" off a discovery, implicitly leaving the curds and whey to others. In later years our conversations focused more and more on *self-organization*: The ability of locally interacting systems, initially disordered, to evolve toward coherent, large-scale spatial structures. His interest in this theme was already evident in [S27], which concludes by analyzing a one-dimensional point process that tends toward equal spacing. Now that interactive computer experimentation and visualization facilitate the empirical study of a vast menagerie of nonlinear spatially-distributed dynamical systems, self-organization is rapidly becoming a major theme of scientific investigation. Especially in this arena, where probability theory is sure to play a major role, I feel certain that Frank Spitzer's visionary ideas will impact future generations even beyond the spheres of mathematics and statistical physics.

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