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FRANK SPITZER'S WORK ON RANDOM WALK AND BROWNIAN MOTION

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Sums of independent identically distributed (i.i.d.) random variables were among the first subjects to be studied in probability theory. The sequence of partial sums $S_n := \sum_1^n X_i$, $n \geq 1$, for i.i.d. random variables is called a *random walk*. Around 1650, Fermat, Pascal and Huygens already solved a number of absorption problems for very special one- and two-dimensional random walks which arose in gambling and de Moivre obtained his local central limit theorem for sums of binomial random variables in 1733 [see Hald (1990) for the early history of probability and statistics]. Because random walks have been studied so long, our knowledge of their properties is very detailed. Nevertheless, random walks continue to be fascinating because elegant new properties are still being discovered.

Many early investigations dealt with limit theorems for S_n and, not surprisingly, these made strong assumptions on the common distribution of the X_i's. One of the directions of random walk theory has been to generalize limit laws such as the central limit theorem and the law of the iterated logarithm to settings with nonidentically distributed variables, or to finding higher order terms in the convergence to limit laws [e.g., the Berry–Esseen theorem, or expansions in the central limit theorem and various other topics which can be found in Petrov (1975)], or to prove refined invariance principles, which give information about the distribution of functionals of the whole sample path $\{S_k,\ k \leq n\}$, rather than about the distribution of S_n only [e.g., Donsker's theorem and Strassen's law of the iterated logarithm; see Billingsley (1968) and Bingham (1986), respectively]. Generally speaking, this type of result gives detailed and sometimes rather technical information about the random walk under rather strong assumptions on the distribution of F. It is often required that F have a second moment or regularly varying tails.

Frank Spitzer's interest was more in the direction of finding relationships which made *no a priori assumptions whatsoever* on the underlying distribution. This type of result relies solely on the fact that the X_i are independent and identically distributed. A classical example of this kind of approach is the determination by Lévy and Khinchine of all possible limit laws for $b_n^{-1}(S_n - a_n)$ for suitable constants a_n and b_n, and of necessary and sufficient conditions on the common distribution of the X_i for convergence of $b_n^{-1}(S_n - a_n)$ to any of the possible limit laws. [Gnedenko and Kolmogorov (1954) or Feller (1971, Chapter 17) are standard references for this general theory.]

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This point of view of Frank's, as well as questions and suggestions by Frank, has been the inspiration of several of my own articles. Also, a number of papers were written jointly with Frank. I look back with pleasure and gratitude on the time of my interaction with Frank. It was a stimulating time and Frank has had a major influence on my work, for which I am thankful.

Most of Frank's contributions for random walk have been incorporated in his beautiful book *Principles of Random Walk* [S18]. ([Sx] refers to reference number x in the bibliography of Spitzer's, elsewhere in this issue.) They fall mainly in the following two categories:

1. combinatorial results on fluctuation theory;
2. potential theory for random walks.

Below we shall outline some of Frank's principal results in these categories. We shall also include a brief discussion of Frank's work on

3. recurrence criterion for random walk,
4. random walk in random environment

and on

5. Brownian motion.

Section 3 still belongs to his random walk results but is worth singling out. Even though Brownian motion is a separate category, it has many analogies with random walk, and some of these also show up in Frank's work, as we shall see.

Throughout, $X, X_1, X_2, \dots$ are i.i.d. random variables with common distribution F, and $S_n = \Sigma_1^n X_i$. Only at the end of Section 2 and in Section 4 will we allow a more general interpretation of S_n.

1. Combinatorial results. Erdös and Kac (1947) proved that

$$(1)\qquad P\left\{\frac{1}{n}N_n \le x\right\} \to \frac{1}{\pi}\int_0^x \frac{dt}{\sqrt{t(1-t)}} = \frac{2}{\pi}\arcsin\sqrt{x}, \qquad 0 \le x \le 1,$$

where

$$N_n = \text{number of positive terms among } S_1, \dots, S_n.$$

This is a special case of the arcsine limit law. The only conditions for (1) in Erdös and Kac were that F have mean zero and a finite second moment. (In fact they even have a limit theorem for N_n when the X_i are not identically distributed.) In Kac (1951) it is shown that

$$\int_0^1 I_{[0,\infty)}(S_t)\,dt$$

even has an arcsine distribution when $\{S_t\}_{t \ge 0}$ is any symmetric stable process (which does not have to have second, or even first, moments). This surprising general validity of the arcsine law was explained by Sparre Andersen (1949). He showed that the distribution of N_n (for fixed n) is the same for all continuous symmetric distributions F. This proof is based on a combinatorial lemma which shows that under a linear independence condition on a set of numbers $\{x_1, \dots, x_n\}$, the number of choices for $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ and permutations σ of $\{1, \dots, n\}$ for which $\sum_1^n \varepsilon_i x_{\sigma_k} > 0$, is independent of $\{x_1, \dots, x_n\}$. In Sparre Andersen (1953) and (1954) this was generalized by proving, among other results, that the distribution of N_n and such quantities as $L_n :=$ first index at which $\max(0, S_1, \dots, S_n)$ is taken on, depends only on the numbers $a_k = P\{S_k > 0\}$. Expressions for the generating functions of N_n and L_n in terms of the $\{a_k\}$, and arcsine limit laws for $n^{-1}N_n$ and $n^{-1}L_n$ are also given.

Another related combinatorial result from about that time appeared in Kac (1954). With some assistance from Chung and Hunt he showed that if $x_1, \dots, x_n$ are any real numbers, σ a permutation of $\{1, \dots, n\}$ and $\nu(\sigma) =$ number of positive terms in the sequence $S_i(\sigma x) := \sum_{j=1}^i x_{\sigma_j}$, $i = 1, \dots, n$, then

$$\sum_\sigma \max\{0, S_1(\sigma x), \dots, S_n(\sigma x)\} = \sum_\sigma \nu(\sigma) x_{\sigma_1}, \tag{2}$$

where the sum over σ runs over $\mathfrak{S}_n$, the group of all permutations of n elements. Actually the identity (2) was discovered when proving a formula for

$$E[\max\{0, S_1, \dots, S_{n-1}\} I\{S_n = 0\}]$$

when F is a symmetric distribution on $\mathbb{Z}$, which Kac had deduced from a purely analytical theorem of Szegö on determinants of Toeplitz matrices.

Spitzer [S3] realized that all these results had a common combinatorial origin. Because $X_1, \dots, X_n$ are i.i.d., one has

$$Ef(X_1, \dots, X_n) = Ef(X_{\sigma_1}, \dots, X_{\sigma_n})$$

for any permutation σ of $\{1, \dots, n\}$ and any bounded function f. Therefore, also for any subgroup $\mathfrak{E}_n$ of $\mathfrak{S}_n$ with cardinality $|\mathfrak{E}_n|$,

$$Ef(X_1, \dots, X_n) = E\frac{1}{|\mathfrak{E}_n|} \sum_{\sigma \in \mathfrak{E}_n} f(X_{\sigma_1}, \dots, X_{\sigma_n}). \tag{3}$$

For some f and for $\mathfrak{E}_n = \mathfrak{S}_n$ or $\mathfrak{E}_n$ equal to the subgroup of cyclical permutations one can express

$$\frac{1}{|\mathfrak{E}_n|} \sum_{\sigma \in \mathfrak{E}_n} f(x_{\sigma_1}, \dots, x_{\sigma_n})$$

in a different form, for any real numbers $x_1, \dots, x_n$. This, combined with (3), may yield an expectation which is simpler to evaluate than the original $Ef(X_1, \dots, X_n)$, or at least involves other interesting functions of $X_1, \dots, X_n$. As an example we mention the following (Theorem 2.2 in [S3]): For any permutation τ of $\{1, \dots, n\}$, let $\alpha_1(\tau), \dots, \alpha_{k(\tau)}(\tau)$ be its decomposition into cycles (i.e., $\alpha_1(\tau), \dots, \alpha_{k(\tau)}(\tau)$ form a partition of $\{1, \dots, n\}$ and the integers in any $\alpha_j(\tau)$ are moved around cyclically by τ). Define

$$T(\tau x) = \sum_{i=1}^{k(\tau)} \Bigg(\sum_{j \in \alpha_i(\tau)} x_j \Bigg)^{+}.$$

Then the sets

$$\{S(\sigma x)\}_{\sigma \in \mathfrak{S}_n} \quad \text{and} \quad \{T(\tau x)\}_{\tau \in \mathfrak{S}_n}$$

are identical. After a little work Spitzer derived from this the following representation of the generating function of $\max\{0, S_1, \dots, S_n\}$. For $|t| < 1$ and $|z| \le 1$,

$$\sum_{n=0}^{\infty} t^n E\{z^{\max\{0, S_1, \dots, S_n\}}\} = \exp\Bigg[\sum_{n=1}^{\infty} \frac{t^n}{n} E\{z^{S_n^+}\} \Bigg]. \tag{4}$$

In fact, he even gives the (somewhat more complicated) formula for the joint generating function of S_n and $\max\{0, S_1, \dots, S_n\}$.

To be sure, the right-hand side of (4) usually can not be evaluated explicitly. However, the right-hand side of (4) equals $[c(t) f_i(t; z) f_e(t; 1)]^{-1}$ where

$$f_i(t; z) := \exp\Bigg(- \sum_1^{\infty} \frac{t^n}{n} E\{z^{S_n}; S_n > 0\} \Bigg),$$

$$f_e(t; z) := \exp\Bigg(- \sum_1^{\infty} \frac{t^n}{n} E\{z^{S_n}; S_n < 0\} \Bigg),$$

$$c(t) := \exp\Bigg(- \sum_1^{\infty} \frac{t^n}{n} P\{S_n = 0\} \Bigg).$$

As observed by Baxter (1961) and Kemperman (1961) (see also [S18], Section 17), f_i and f_e are inner and outer functions, respectively, that is, they are analytic functions of z on the interior and exterior of the unit circle, respectively. Moreover, one has the simple factorization

$$1 - tE\{z^X\} = c(t) f_i(t; z) f_e(t; z)$$

and one can therefore use Wiener–Hopf theory to "determine" and to obtain information about f_i and f_e [see, for instance, Kemperman (1961, Section 13) and (1963), Prabhu (1965, Section 4.6) and Feller (1971, Chapter 12)]. The basic relations of fluctuation theory have by now been derived in many ways. Wendel (1958) and Baxter (1961) follow an analytic approach, Feller (1971, Chapter 12) a largely probabilistic approach, while Rota (1969) and Rota and Smith (1972) treat the problem purely algebraically. Also Pollaczek (1957) [see equation (7.16)] derived a formula which is essentially the same as (4), by using contour integrations. Other order statistics of $S_0, \dots, S_n$ besides $\max\{0, S_1, \dots, S_n\}$ are treated in Pollaczek (1952), Wendel (1960) and Port (1963).

In [S4], [S9] and [S8] Frank used the relationship between the Wiener–Hopf equation and $\max\{0, S_1, \dots, S_n\}$ in the opposite direction; he used probability theory to obtain results for the Wiener–Hopf equation.

The random variable $\max\{0, S_1, \dots, S_n\}$ appears frequently in queueing theory. When X has the distribution of the difference of a service and an interarrival time, then the waiting time of the nth customer has the same distribution as $\max\{0, S_1, \dots, S_n\}$. In fact this was the motivation of Pollaczek (1957). Spitzer's results can therefore be used to derive general facts about a single server queue [see Prabhu (1965), Chapter 4]. Equation (4) has also been used fruitfully to obtain asymptotic information about the distribution of $\max\{S_1, \dots, S_n\}$ (see, for instance, Spitzer [S3], Section 4, Darling (1956), Feller (1971), Section 12.5).

Spitzer also noted that

$$\{\max\{0, S_1, \dots, S_n\} = 0\} = \{S_k \le 0, 1 \le k \le n\} = \{T > n\},$$

where T is the first upward ladder index, that is,

$$T = \inf\{k: S_k > 0\}.$$

Thus (4) for $z = 0$ yields

$$\sum_0^{f} t^n P\{T \ll n\} = \exp\left[\sum_1^{f} \frac{t^n}{n} P\{S_n \le 0\}\right],$$

and one can therefore also "find" the distribution of T, and one can even find the joint distribution of T and S_T [see Baxter (1961), Example 2.2, and [S18], Proposition 17.5]. This is very useful for renewal theory; it was also used by Frank to establish the existence of a certain limit of hitting probabilities and the existence of the potential kernel for a one-dimensional random walk with mean zero and finite variance [see Section 2].

To conclude this section we mention that analogously to (4) one can derive a formula for

$$\sum_{n=0}^{\infty} t^n E\{z^{N_n}\}$$

in terms of the numbers $a_k = P\{S_k > 0\}$ (see [S3], Section 5). As mentioned before, this result is originally due to Sparre Andersen (1954). In [S3], Section 7, it is used to derive the general arcsine limit law for $n^{-1}N_n$ (under somewhat weaker conditions than known before) which was the original motivation for the developments of this section.

2. Potential theory for random walks. The relationship between classical potential theory and Brownian motion has been well known since the probabilistic treatment of the Dirichlet problem and boundary values of harmonic functions by Kakutani (1944) and Doob (1954). This classical potential theory is the potential theory associated with the fundamental solution of

$$\Delta u(x) = -\kappa_d \delta_y(x), \tag{5}$$

where $\kappa_d = 2\pi$ in dimension $d = 2$ and $\kappa_d = (d-2)$ times the surface area of the unit sphere S^{d-1} in dimension $d \geq 3$. The solution of (5) is given by Green's kernel

$$u_y(x) = \|x - y\|^{2-d}$$

in dimension $d \geq 3$ and by

$$u_y(x) = -\log \|x - y\|$$

in dimension two. Much of the classical potential theory deals with properties of "potentials," that is, of functions of the form $\int u_y(x) f(y)\, dy$, or more generally $\int u_y(x)\mu(dy)$ for a signed measure μ. By abstracting the properties of potentials axiomatic treatments have been given of potential theory [see Brelot (1952) for a survey]. It was known that potentials with respect to other kernels than $u_y(x)$ satisfy many of these axioms. In a series of influential papers Hunt (1957–1958) showed that there is a close parallel between the potential theory for general kernels and the theory of transient Markov processes (corresponding to the case $d \geq 3$ above). In particular, every transient Markov process $\{X_t\}_{t\geq 0}$ has associated with it the kernel $\int P\{X_t \in dy | X_0 = x\}\, dt$ which satisfies most of the axioms for potential kernels. In the case of a transient Markov chain $\{Y_n\}_{n\geq 0}$ on a countable state space, the kernel is represented by

$$\begin{aligned} G(x,y) &:= \sum_{n=0}^{\infty} P^n(x,y) \\ &= E\left\{ \sum_{n=0}^{\infty} I[Y_n = y] \,\middle|\, Y_0 = x \right\}, \end{aligned}$$

where P is the matrix of transition probabilities for $\{Y_n\}$. Note that for fixed y, $u_y(x) = u(x,y)$ satisfies

$$(P - I)u_y(x) = -\delta(x,y), \tag{6}$$

where $\delta(x,y) = 1$ or 0 according as $x = y$ or $x \neq y$. (6) is an analogue of (5), as becomes apparent by looking at the special case when $\{Y_n\}$ is a simple random walk on $\mathbb{Z}^d$. In this case the left-hand side of (6) is

$$\frac{1}{2d} \sum_{i=1}^{d} \left[u_y(x + e_i) + u_y(x - e_i) - 2u_y(x) \right]$$

($e_i = i$th coordinate vector); this is the discrete analogue of the Laplacian. For a general random walk on $\mathbb{Z}^d$ as considered here, the matrix P of transition probabilities is given by $P(x,y) = P\{X = y - x\}$.

Kemeny and Snell (1961, 1963) and Spitzer in [S14] and [S16] set themselves the task of finding the proper analogue of a potential kernel for recurrent Markov chains. Frank was in particular interested in the special Markov chain given by a recurrent random walk on $\mathbb{Z}^d$, $d = 1$ or 2. For a recurrent (irreducible) Markov chain the series for $G(x,y)$ above is identically equal to ∞, so this definition cannot be used without change for a recurrent chain. A natural attempt to circumvent this difficulty is to replace $G(x,y)$ by $G(x,y)$ "minus an infinite constant"; that is one tries to define

$$A(x,y) := \sum_{n=0}^{\infty} (a_n - P^n(x,y)) \tag{7}$$

for a sequence of constants a_n, as a replacement for $-G(x,y)$. Formally $-A(x,y)$ is also a solution of (6). [The change in sign introduced here is insignificant; it will lead to a positive kernel which has a certain probability interpretation ([S14], Section 4).] One now tries to choose the a_n such that the series in (7) converges. Once one has such a_n's it still needs to be shown that $-A$ solves (6). A good choice for a_n seems to be $a_n = P^n(z,z)$ for any fixed z. Kemeny and Snell (1961, 1963) also discuss the convergence for this choice of a_n for a general Markov chain. However, in this generality (7) may fail to converge. Frank showed that, for *any* random walk on $\mathbb{Z}^d$,

$$A(x,y) = \sum_{n=0}^{\infty} (P^n(0,0) - P^n(x,y)) \tag{8}$$

converges and $-A$ is a positive solution of (6). For $d = 1$ this result is very delicate, as indicated by the fact that, as far as the author knows, it is still not known whether the series in (8) always converges absolutely.

For the remainder of this section it is convenient to allow for a starting point of the random walk which is different from the origin. We therefore take $S_n = S_0 + \Sigma_1^n X_i$, where S_0 may be nonzero, in contrast to the preceding. As before, the X_i are i.i.d. and take values in the integer lattice $\mathbb{Z}^d$. Once one has the above results about A, all kinds of relations for hitting probabilities for finite sets follow. For instance, Frank shows that

$$\begin{aligned} &P\{S_n \text{ visits } y \text{ before returning to } x | S_0 = x\} \\ &\quad = [A(x-y) + A(y-x)]^{-1}, \end{aligned} \tag{9}$$

and shows how in general the transition probabilities of the imbedded random walk on a finite set F (i.e., S_n studied only at the times when $S_n \in F$) can be expressed in terms of the restriction of $A(x,y)$ to $F \times F$. He also showed that

$$\lim_{|x| \to \infty} P\{S \text{ first enters } F \text{ at } y | S_0 = x\}$$

has a limit, which can be expressed in terms of A. (For $d = 1$ one must take the limit as $x \to +\infty$ and as $x \to -\infty$ separately; these limits may differ.) Frank also found the most general positive solution f to

$$(P - I)f = g$$

for a positive g with finite support. Again the solutions can be simply expressed by means of the kernel A.

In a series of papers by Frank, Ornstein, and the author [S15, S17] and Kesten (1963) these results were used to prove a number of very general ratio theorems for "taboo probabilities," that is, for transition probabilities for random walk limited to the complement of a set. Let $B \subset \mathbb{Z}^d$ be a *finite* set and

$$T_B := \min\{n \geq 1\colon S_n \in B\} \qquad (= \infty \text{ if no such } n \text{ exists})$$

its entry time. With only aperiodicity assumptions,

$$\lim_{n\to\infty} \frac{P\{T_B \geq n | S_0 = x\}}{P\{T_B \geq n | S_0 = 0\}} = \delta(x,0) + A(x,0), \tag{10}$$

whether S_n is persistent or not [note that the definition (8) for A makes sense for a transient random walk as well, for then the right-hand side of (8) converges absolutely]. This is a partial analogue for general random walk of a theorem of Hunt (1956) for the rate of approach (as time $\to \infty$) of a solution of the heat equation on $\mathbb{R}^2 \setminus B$ to the solution of a Dirichlet problem on $\mathbb{R}^2 \setminus B$, for compact B. [Hunt is able to replace the denominator of (10) by the explicit function $\log n$, because he works with a specific process, namely, two-dimensional Brownian motion.]

For recurrent random walk in dimension ≥ 2, or in dimension 1, but with $EX^2 = \infty$, (10) can be refined to

$$\lim_{n\to\infty} \frac{P\{S_n = y, T_B \geq n | S_0 = x\}}{\sum_{u,v \in B} P\{S_n = v, T_B \geq n | S_0 = u\}} = \tilde{g}_B(x) \cdot \tilde{g}_{-B}(-y) \tag{11}$$

for a certain function $\tilde{g}$ which can be defined by means of Frank's results on limits of hitting probabilities [cf. Kesten (1963)]. Equation (11) was originally conjectured by Frank as an analogue of the well-known result that high powers of a finite matrix whose largest eigenvalue is simple look more and more like a matrix of rank one. By the Perron–Frobenius theory, this is always the case for a positive irreducible finite matrix [see for instance the Appendix of Karlin (1966)]. Equation (11) is an analogue of this for the infinite matrix Q which is the restriction of P to $B^c \times B^c$. In fact for $x, y \notin B$, the numerator of (11) is precisely $Q^n(x,y)$; note also that the right-hand side of (11) is a *product* of a function of x and a function of y, that is, a matrix of rank one.

Frank also discussed the meaning of the capacity and Robin's constant of a finite set for recurrent random walk (see [S14] and [S18], Section 14). For transient random walk he identifies the equilibrium charge of a finite set with the escape probability ([S18], Section 25).

Since the appearance of Frank's book [S18] many extensions of his potential theoretic work have been published. We mention only a few of these. Frank and the author in 1965 showed how to carry over much of the theory to a random walk S_n taking values in a countable abelian group $\mathscr{G}$. Ornstein (1969) and Port and Stone (1969) showed how to define the recurrent potential kernel for a random walk taking values in $\mathbb{R}^d$, or even a locally compact abelian group rather than one with a countable state space. Finally, Brunel and Revuz (1974–1977) extended the theory to random walks on arbitrary (not necessarily abelian) locally compact groups. These extensions to groups raised the question for what groups $\mathscr{G}$ does there exist a recurrent random walk which does not live on a subgroup of $\mathscr{G}$ (i.e., for which $\mathscr{G} =$ closure of $\bigcup_1^\infty \operatorname{supp}(S_n)$).

Even though there were many intermediate contributions [see in particular Baldi, Lohoué and Peyrière (1977)] a satisfactory answer to this difficult question was only given by Varopoulos in 1986 [see Varopoulos (1990) for a summary and references].

3. Recurrence criterion. In connection with the question of recurrence of random walks we wish to mention a rather specific, but quite elegant result of Frank's. To put this in context, we remind the reader that Polya (1921) proved that simple random walk on $\mathbb{Z}^d$ is recurrent if $d = 1$ or 2 and transient if $d \geq 3$. Chung and Fuchs (1951) proved the following necessary and sufficient condition for recurrence of a general random walk: If

$$\phi(\theta) = Ee^{i\theta X},$$

then S_n is (interval) recurrent if and only if for some $\varepsilon > 0$ (and then for all $\varepsilon > 0$)

$$\lim_{t \uparrow 1} \int_{|\theta| \leq \varepsilon} \operatorname{Re} \frac{1}{1 - t\phi(\theta)} \, d\theta = \infty. \tag{12}$$

Here X may take values in $\mathbb{R}^d$, in which case θ is also to be taken in $\mathbb{R}^d$ and θX is to be replaced by $\langle \theta, X \rangle$, the inner product of θ and X; $|\theta|$ is then the Euclidean norm of θ, and the integral in (12) is a d-dimensional integral. Even though this is a very appealing and useful criterion, it still involves the unpleasant limit as $t \uparrow 1$. Can one take the limit inside the integral, that is, can (12) be replaced by

$$\int_{|\theta| \leq \varepsilon} \operatorname{Re} \frac{1}{1 - \phi(\theta)} \, d\theta = \infty\,? \tag{13}$$

Note that this makes sense because $\operatorname{Re}[1 - \phi(\theta)]^{-1} \geq 0$, as one easily sees. Frank showed that (13) is indeed necessary and sufficient for interval recurrence of a random walk on $\mathbb{Z}^d$ (see [S18], Theorem 8.2). The extensions of this criterion to $\mathbb{R}^d$ and abelian group valued random walks can be found in [S20], Ornstein (1969) and Port and Stone (1969). Perhaps somewhat surprisingly, it is by no means easy to prove that (13) is equivalent to (12); in fact as far as the author knows it has not been decided whether the left-hand sides of (12) and (13) are always equal. Frank's proof rests on the whole potential theoretical apparatus. In particular, it uses (9) plus the nontrivial fact that $A(x, y) \to \infty$ as $x \to \infty$ or $y \to \infty$.

4. Random walk in random environment. Frank was one of the first people to consider random walk in random environment. He thought up the model described below by himself, but on a trip to the Soviet Union found out that several Russian mathematicians were working on the same model which had arisen in some applied situations [see Chernov (1967) and Temkin (1972)]. Eventually this trip by Frank to the Soviet Union resulted in the joint paper [S41] with Kesten and Kozlov. We note that S_n in this model is *not* the sum of i.i.d. random variables, so that the term "random walk" here is used in a different sense than in the preceding sections. Consider an i.i.d. family $\{p(v, \cdot\,)$: $v \in \mathbb{Z}^d\}$ of probability vectors on $\mathbb{Z}^d$. Thus, for each v, $p(v, w) \geq 0$ and $\sum_{w \in \mathbb{Z}^d} p(v, w) = 1$. These $p(v, \cdot\,)$'s form the environment. Once it is chosen it stays fixed forever. *Conditionally on the environment* one now takes S_n as a Markov chain with transition probabilities

$$P\{S_{n+1} = y | S_n = x\} = p(x, y - x).$$

What makes this process difficult and interesting is that $\{S_n\}$ *by itself is not Markovian*; every time when S takes a step away from x we obtain some more information about the environment at x, that is, about $p(x, \cdot\,)$. This model was the subject of a number of Ph.D. dissertations at Cornell, for students of Frank as well as of the author. [See Solomon (1975), Ritter (1976), Kalikow (1981) and Key (1984).] Criteria for recurrence were given and limit laws were investigated. In the one-dimensional case under the further restriction that S_n can move only one step to the left or one step to the right and that $S_n \to \infty$ w.p.1 [or equivalently, $E(\log(p(0,1)/p(0,-1)) > 0)$] the possible limit laws for S_n (under mild conditions) were fully determined in [S41]. The model has become very popular with statistical physicists because typically in dimension 1, S_n should not be normalized by $\sqrt{n}$ to obtain a limit law. The literature on this model is far too extensive to cite. We content ourselves with the mention of two fundamental papers: (a) Sinai (1982) which studies the so-called critical case in one dimension, and proves that $(\log n)^{-2} S_n$ has a limit law in this case; and (b) Bricmont and Kupiainen (1991) which shows that in certain cases in dimension greater than or equal to 3, S_n has diffusive behavior, that is, $\{n^{-1/2} S_{[nt]}\}_{t \geq 0}$ converges weakly to a d-dimensional Brownian motion.

5. Brownian motion. The processes with stationary independent increments, also called Lévy processes, are continuous time analogues of random walks and share many of their properties. The simplest and best known among these is Brownian motion, which corresponds to a random walk with finite variance. Frank's contributions have mainly been to two-dimensional Brownian motion. This process visits w.p.1 each open subset of $\mathbb{R}^2$ infinitely often (i.e., for arbitrarily large t) but it does not hit points; in other words, if $B(t)$ is a two-dimensional Brownian motion, then for each x, $T < \infty$ and $\varepsilon > 0$,

$$P\{|B(t) - x| < \varepsilon \text{ for some } t > T\} = 1,$$

but

$$P\{B(t) = x \text{ for some } t > 0\} = 0.$$

We say that this process is interval recurrent but not point recurrent. Already in his 1953 Ph.D. dissertation (see [S5]) Frank investigated how close $B(t)$ comes to a fixed point x. First he finds the asymptotic behavior of the hitting probability of small discs. For fixed $0 < t_1 < t_2$,

$$\lim_{r \downarrow 0} \log\left(\frac{1}{r}\right) P\left\{\min_{t_1 \le t \le t_2} |B(t) - B(0)| < r\right\} = \frac{1}{2} \log \frac{t_2}{t_1}. \tag{14}$$

He then uses this to prove that for a given positive, decreasing function g, one has

$$P\{|B(t) - x| \ge g(t)\sqrt{t} \text{ eventually}\} = \begin{cases} 0, \\ 1, \end{cases} \tag{15}$$

according as

$$\sum_{k=1}^{\infty} \frac{1}{k|\log g(k)|} \begin{cases} \text{diverges}, \\ \text{converges}. \end{cases} \tag{16}$$

An analogous result for the rate of escape to infinity of a higher dimensional random walk is due to Dvoretzky and Erdös (1951). By changing $B(t)$ to the equivalent process $tB(1/t)$, Frank also obtains an analogous result for the behavior of $B(t) - B(0)$ for small t, rather than large t. An analogue of the results (14)–(16) for random walks in the domain of attraction of a symmetric stable law is in [S2].

The preceding result is close in flavor to some of Frank's later work on random walk. Rather different though is Frank's result (still from his thesis) on the winding number of two-dimensional Brownian motion. He shows that if $\theta(t)$, $t \ge 1$, is a continuous choice of the argument of $B(t)$, then

$$\frac{1}{\log t}\theta(t) \text{ has a Cauchy limit distribution as } t \to \infty. \tag{17}$$

Both of these results have led to many further investigations. For a recent paper on the fine structure of two-dimensional Brownian motion which uses Frank's estimate (14), see LeGall (1991) in the Festschrift in honor of Frank. This Festschrift also contains the article Yor (1991) with a considerable generalization of Frank's result (17) on the winding number. Other papers have studied further aspects of the entanglement of planar Brownian motion and related questions for Brownian motion on manifolds [e.g., Lyons and McKean (1984)].

In [S19] Frank deals with a rather different aspect of Brownian motion. This paper dates from the same time as the potential theoretical papers discussed in Section 2, and deals with a continuous time potential theoretical question. It gives for a Brownian motion $B(\cdot)$ in dimensions 2 and 3 an asymptotic expansion as $t \to \infty$ for the expected volume of the set

$$W_A(t) = \bigcup_{s \le t} (B(s) + A). \tag{18}$$

This is the set swept out by moving A along the Brownian path up till time t, $\{B(s)\colon s \le t\}$. In the special case when A is the unit ball this set is called the Wiener sausage. Its analogue for a discrete time random walk which takes values in $\mathbb{Z}^d$ is the range of the random walk up till time n, that is, the set

$$R_n = \{S_k : 0 \le k \le n\}.$$

In the discussion to Kingman (1973) Frank gave us a nice application of Kingman's subadditive ergodic theorem, the existence w.p.1 of the limits

$$\begin{aligned} &\frac{1}{t}\,(\text{volume of } W_A(t)) \quad \text{as } t \to \infty, \\ &\frac{1}{n}\,(\text{cardinality of } R_n) \quad \text{as} \quad n \to \infty. \end{aligned} \tag{19}$$

The first limit is the capacity of A in dimension 3, and the second limit is

$$P\{S_n \neq 0 \text{ for all } n \ge 1 | S_0 = 0\}.$$

The volume of $W_A(t)$ minus the volume of A is also interpreted in [S19] as the total heat flow in time t from A to A^c.

More precise results about the distribution of the Wiener sausage, going beyond just the strong law (19), can be found in LeGall (1988); large deviation estimates for this volume have been considered in Donsker and Varadhan (1975). Limit laws for the range of a random walk have been considered by Jain and Pruitt (1972) (and its references). Finally, Port (1991) in the Festschrift for Frank considers analogous problems for the "Lévy sausage," the volume swept out by a ball moving along the path of a Lévy process.

Frank Spitzer has had a major influence on probability theory in the last 30–40 years, both through his own work, and through the work of his students and collaborators. His enthusiasm was infectious. He was always searching for elegant new phenomena. In fact he had neither the inclination, nor much patience for extending known results if this did not lead to some surprises. Extending the validity of some theorems, by which most of us make a living, held little appeal to him. I still remember that Frank once complained to me after a visitor filled the blackboard in his room with lots of formulae which Frank thought boring: "What did he expect me to do? Eat a bunch of formulae?" Frank's work has given major impetus to what one can call "the general theory of random walk": results which hold for *all* random walks, such as the existence of the potential kernel, the existence of limits of hitting distributions and ratio limit theorems. His work also helped explain the combinatorial nature of results for the maxima of random walks and various other quantities studied in fluctuation theory. Frank's work on Brownian motion has led to deep studies of the winding number of Brownian motion and asymptotics of the Wiener sausage. After random walk and Brownian motion Frank turned to models which involve more dependence. He was one of the creators of what is nowadays called interacting particle systems. This field is so important and huge that a separate survey by David Griffeath in this issue is devoted to it.

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