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ASYMPTOTICS OF EXIT TIMES FOR MARKOV JUMP PROCESSES I

BY I. ISCOE[1] AND D. MCDONALD[2]

McMaster University and University of Ottawa

General conditions on general state space Markov jump processes are established for the asymptotic exponentiality of the distribution of the exit time into any small *forbidden* set. The error bounds for this exponential approximation tend to 0 as the size of the forbidden set tends to 0.

1. Introduction. Many problems relating to the reliability of large systems may be described by the first hitting time τ of some forbidden set F by a stationary Markov jump process with stationary probability measure π. Starting from an initial distribution $\hat{\pi}$, which is the stationary measure conditioned on starting inside $B \equiv F^c$, we consider the asymptotic behaviour of the distribution of the first hitting time as "$B \to S$" [which is shorthand for the convergence: $\pi(B) \to 1$ or $\pi(F) \to 0$]. We give general criteria for τ to be asymptotically exponentially distributed with a mean equal to the reciprocal of the smallest real eigenvalue $\Lambda(B)$ of an associated Dirichlet problem. Asymptotic exponentiality is known in some generality [see Keilson (1979)], so the novelty here is an explicit error bound for the exponential approximation.

The bounds are of the form (see Theorem 2.8):

$$\left|P_{\hat{\pi}}(\tau > t) - e^{-\Lambda(B)t}\right| \le \beta(B)e^{-\Lambda(B)t}, \tag{1.1}$$

where conditions are given which ensure $\beta(B) \to 0$ as $\pi(F) \to 0$. An immediate corollary is

$$\left|E_{\hat{\pi}}\tau - \frac{1}{\Lambda(B)}\right| \le \frac{\beta(B)}{\Lambda(B)} \tag{1.2}$$

(see Corollary 2.10). In the reversible case, a sharper upper bound is given [see (2.13)] in which $\beta(B)$ is proportional to $\Lambda(B)$ and which extends Theorem 3 in Aldous and Brown (1992).

In modern telecommunications networks, hitting the forbidden states may correspond to buffers or queues being filled beyond capacity, in which case messages are truncated or lost. These large deviations can be engineered to be highly improbable but not impossible. The distribution of and the mean time

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between such large deviations are important design features. For most practical purposes, the vital estimates from (1.1) and (1.2) are the upper bound on the probability of hitting the forbidden set within a fixed time interval and the lower bound on the expected value of this hitting time, since these bounds provide a degree of assurance against catastrophic failure.

In part II of this work, the estimates above are calculated explicitly for a Jackson queueing network where failure occurs if the number of customers in the queue at any one of the nodes of the network exceeds some specified level.

2. Setting; results on asymptotics. We begin by giving a precise description of the setting for our results. The proofs of technical details are deferred to the Appendix at the end of the paper, so as not to interrupt the exposition.

Let $(X_t;\, t \geq 0)$ be a continuous-time, Markov jump process on a measurable state space $(S, \mathscr{S})$. We denote the lifetime of the process by ζ; $(T_t;\, t \geq 0)$ is the usual semigroup of contraction operators on $\mathscr{B}(S) := \{f\colon S \to \mathbf{R} \mid f \text{ is bounded and measurable}\}$; $T_t f(x) = E_x[f(X_t);\, \zeta > t]$. The following theorem is taken from Dynkin (1965), Theorem 5.4, page 137.

THEOREM 2.1. *$(T_t;\, t \geq 0)$ is weakly continuous on all of $\mathscr{B}(S)$; its weak infinitesimal generator $-\mathscr{L}$ is given on its domain $\mathscr{D}_w(\mathscr{L}) \subset \mathscr{B}(S)$ by*

$$\mathscr{L}f(x) = \int_S J(x, dy)[f(x) - f(y)],$$

where $J(x, dy) = J(x)P(x, dy)$ and

$$J(x) = \frac{1}{E_x \tau^{(1)}}, \quad P(x, A) = P_x(X_{\tau^{(1)}} \in A),$$

where $\tau^{(1)}$ is the first jump time of the process.

We call $J(x)$ the *jump rate* and $J(x, dy)$ the *jump-rate kernel*.

We now assume that a stationary probability measure π is given on $(S, \mathscr{S})$. By the Cauchy–Schwarz inequality, T_t also acts as a contraction on $L^2(\pi) \equiv L^2(S, \mathscr{S}, \pi)$. Using the density of $\mathscr{D}_w(\mathscr{L})$ in $L^2(\pi)$, it is straightforward to show that $(T_t;\, t \geq 0)$ is strongly continuous on $L^2(\pi)$. (See Lemma A.1 in the Appendix.) As such, the infinitesimal generator $-L$ of $(T_t;\, t \geq 0)$ on $L^2(\pi)$ is a densely-defined, closed operator. [See, e.g., Dynkin (1965), pages 22 and 23.] A simple application of the bounded convergence theorem yields that $\mathscr{D}_w(\mathscr{L}) \subset \mathscr{D}(L)$, the domain of L, and $L|\mathscr{D}_w(\mathscr{L}) = \mathscr{L}$.

We make the standing assumption that our underlying process is *nonterminating*; that is, that $\zeta \equiv +\infty$.

We now fix a subset $B \in \mathscr{S}$ and define the stopping time

$$(2.1)\quad \tau^B := \inf\{t > 0\colon X_t \in B^c\} \qquad \left(\{\tau^B \leq t\} = \bigcup_{r \in \mathscr{D} \cap [0, t]} \{X_r \in B^c\}\right).$$

The "killed" semigroup [associated with $(X_t^B;\ t > 0)$, the process killed off B] is

$$T_t^B f(x) := E_x\big[\,f(X_t);\ \tau^B > t\big], \qquad x \in B,\ f \in L^2(B,\pi). \tag{2.2}$$

The "killed" generator $-L^B$ on $L^2(B,\pi)$ is formally defined by

$$\begin{aligned} L^B f(x) &:= \int_B J(x,dy)[\,f(x) - f(y)] \\ &\quad + K^B(x) f(x), \qquad x \in B,\ f \in L^2(B,\pi), \end{aligned} \tag{2.3}$$

where $K^B(x) := J(x, B^c)$, so that $K^B(x) = L^B\mathbf{1}$.

We now make the additional standing assumptions:

$$\int_S J(x)\pi(dx) < \infty, \qquad M(B) := \pi\text{-ess}\sup_{x\in B} J(x) < +\infty.$$

By Theorem 2.1, applied to $(X_t^B;\ t \geq 0)$, $\zeta = \tau^B$ on the state space $(B, \mathscr{S}\cap B)$, we have that the weak infinitesimal generator of $(T_t^B;\ t \geq 0)$ is a restriction of $-L^B$. Under the standing assumptions above we then have the following result which will be proved in the Appendix.

LEMMA 2.2. *$(T_t^B;\ t \geq 0)$ is uniformly continuous on $L^2(B,\pi)$ and its infinitesimal generator is $-L^B$, which is a bounded operator of norm less than or equal to $2M(B)$ on $L^2(B,\pi)$.*

Before stating our main results we must introduce some additional notation and definitions. To simplify many of the calculations in Sections 3 and 4, it will be convenient to normalize π restricted to B, which we denote by $\pi|B$, by setting [assuming that $\pi(B) > 0$, which we do, for the remainder of this article]

$$\hat{\pi} \equiv \hat{\pi}^B := [\pi(B)]^{-1}(\pi|B), \qquad L^2(\hat{\pi}) \equiv L^2(B;\hat{\pi}).$$

Using the following lemma, which will be proved in the Appendix, we may also define the resuscitation rate R^B to be the Radon–Nikodym derivative of the measure $\mu(dy) := \int_{B^c}\pi(dx)J(x,dy)$ with respect to $\pi|B$, π restricted to B.

LEMMA 2.3. *$\mu \ll \pi|B$ and $R^B := (d\mu/d\pi)|_B \leq M(B)$.*

There are three quantities related to the kernel J which will appear in the hypotheses and results of this and the following sections. They are the mean killing rate and standard deviations of the killing rate and resuscitation rate with respect to the probability $\hat{\pi}$ on B, namely,

$$\bar{\kappa} \equiv \bar{\kappa}^B := \int_B K^B(x)\hat{\pi}(dx), \qquad \kappa_1 := \|K^B - \bar{\kappa}\|_{\hat{\pi}}, \qquad \kappa_2 := \|R^B - \bar{\kappa}\|_{\hat{\pi}}. \tag{2.4}$$

We shall see below that the mean resuscitation rate coincides with $\bar{\kappa}$.

We denote the adjoint of an operator A on an L^2-space by A^*. As π is an invariant measure, $T_t^*\mathbf{1} = \mathbf{1}$ (= $T_t\mathbf{1}$, too) for all $t \geq 0$, and

$$\int T_t f(x)\pi(dx) = \int f(x)\pi(dx), \qquad f \in L^2(S, \pi).$$

Therefore, $L^*\mathbf{1} = 0$ and, of course, $L\mathbf{1} = 0$. Also, for all $f \in \mathscr{D}(L)$, $\int Lf\,d\pi = \int L^* f\,d\pi = 0$. The analogous results for L^B and $(L^B)^*$ are given in the next lemma, which reveals the utility of R^B. The proof is deferred to the Appendix.

Lemma 2.4. *For all $f \in L^2(\hat{\pi})$,*

$$\int_B L^B f\,d\hat{\pi} = \int_B f(y) R^B(y)\hat{\pi}(dy)$$

and

$$\int_B (L^B)^* f\,d\hat{\pi} = \int_B f(x) K^B(x)\hat{\pi}(dx).$$

In particular, $R^B = (L^B)^*\mathbf{1}$ and $\int_B R^B(y)\hat{\pi}(dy) = \int_B L^B\mathbf{1}\,d\hat{\pi} = \bar{\kappa}$.

We remark that the equality of the mean killing and resuscitation rates, namely, "$\int_B J(x, B^c)\pi(dx) = \int_{B^c} J(x, B)\pi(dx)$," just says that, in equilibrium, the rate of flow into and out of a set B coincides.

The following simple estimates for $\bar{\kappa}$, κ_1 and κ_2 may serve to make the hypotheses for our results more explicit and perhaps more practical, albeit less sharp. In particular, they show that $\bar{\kappa} \to 0$, and $\kappa_1, \kappa_2 \to 0$ as $B \to S$, in case $M(S) < \infty$.

Lemma 2.5.

(i) $$\bar{\kappa} \leq \int_{B^c} J(x)\pi(dx),$$

(ii) $$\kappa_1 \leq [M(B) \cdot \bar{\kappa}]^{1/2},$$

(iii) $$\kappa_2 \leq [M(B) \cdot \bar{\kappa}]^{1/2}.$$

Unless mentioned otherwise, if μ is a measure on B, a measurable subset of S, $L^2(B, \mu)$ will be considered as a real vector space of real-valued functions. One important exception will be when we discuss the spectral properties of operators related to T_t. In this case we, of course, must consider $L^2(B, \mu)$ as a complex vector space of complex-valued functions; we adopt the following convention:

$$(f, g)_\mu = \int \overline{f(x)} g(x)\mu(dx), \quad f, g \in L^2(B, \mu).$$

The nature of $L^2(B, \mu)$ will always be clear from the context, so we shall not adopt separate notations to distinguish the two possibilities. Note that T_t and all other related operators (such as L, T_t^B and L^B) have obvious, natural extensions from *real* $L^2(\mu)$ to *complex* $L^2(\mu)$. It should be noted that the norms of these extended operators, when bounded, do not change [see Lemma 7.5 in Davies (1980)].

The notion of a *spectral gap* is central to this article. We defer a detailed discussion of it until Section 4 but remark at this point that in the reversible case, in which T_t is self-adjoint on $L^2(S, \pi)$, the spectral gap (when positive) is the gap in $\sigma(L) \subset \mathbf{R}_+$ between the simple eigenvalue 0 and the rest of the spectrum.

DEFINITION 2.6. If $-L$ is the infinitesimal generator of $(T_t;\ t \geq 0)$ on $L^2(\pi)$,

$$\operatorname{Gap}(L) := \inf\{(f, Lf)_\pi\colon f \in \mathscr{D}(L), \|f\|_\pi = 1, (f, \mathbf{1})_\pi = 0\}.$$

The following will be standing assumptions throughout this article:

1. $(X_t;\ t \geq 0)$ is nonterminating.
2. π is an invariant probability.
3. $B \in \mathscr{S}$ with $0 < \pi(B) < 1$.
4. $\int J(x)\pi(dx) < \infty$ and $M(B) := \pi - \operatorname{ess\,sup}_{x \in B} J(x) < \infty$.
5. In the nonreversible case [i.e., when π is not a reversibility measure for $(X_t;\ t \geq 0)$], $\kappa_1, \kappa_2 \to 0$ *as* $B \to S$.
6. $\operatorname{Gap}(L) > 0$.

The following theorems are the main results of this article and the proofs are given in Sections 3 and 4. The symbol P_μ denotes the law of $(X_t;\ t \geq 0)$ initially (at $t = 0$) distributed according to the probability μ on $(S, \mathscr{S})$; E_μ denotes the corresponding expectation. Thus the process is stationary with respect to P_π.

THEOREM 2.7. *If the quantities* $\bar{\kappa}$, κ_1 *and* κ_2 [*defined at* (2.4)] *satisfy*

$$\bar{\kappa} < \operatorname{Gap}(L), \qquad 4\kappa_1\kappa_2 < [\operatorname{Gap}(L) - \bar{\kappa}]^2,$$

then L^B *and* $(L^B)^*$ *have a common positive, isolated eigenvalue* $\Lambda(B)$ *and associated real eigenfunctions* ϕ^B *and* ρ^B, *respectively, belonging to* $L^2(\hat{\pi})$ *and such that* $\int_B \rho^B \, d\hat{\pi} = 1$ *and* $\int_B \phi^B \rho^B \, d\hat{\pi} = 1$. *Moreover*, $0 < \Lambda(B) = \inf \operatorname{Re}\{\sigma(L^B)\}$, $\inf[\operatorname{Re}\{\sigma(L^B) \setminus \{\Lambda(B)\}\}] > 0$ *and*

(i) $$|\Lambda(B) - \bar{\kappa}| \leq 2\kappa_1\kappa_2/[\operatorname{Gap}(L) - \bar{\kappa}],$$

(ii) $$\|\rho^B - \mathbf{1}\|_{\hat{\pi}} \leq 2\kappa_2/[\operatorname{Gap}(L) - \bar{\kappa}],$$

(iii) $$\|\phi^B - \mathbf{1}\|_{\hat{\pi}} \leq \left\{ \frac{2\sqrt{(\operatorname{Gap}(L) - \bar{\kappa})^2 + 4\kappa_2^2}}{[\operatorname{Gap}(L) - \bar{\kappa}]^2 - 4\kappa_1\kappa_2} \right\} \kappa_1;$$

$$\left| \int_B \phi^B \, d\hat{\pi} - 1 \right| \leq \frac{4\kappa_1\kappa_2}{[\operatorname{Gap}(L) - \bar{\kappa}]^2 - 4\kappa_1\kappa_2}.$$

The asymptotic result (1.1) for a stationary starting measure then follows.

THEOREM 2.8. *If $\pi(B^c)$ is sufficiently small, then, for all $t \geq 0$,*

$$|P_{\hat{\pi}}(\tau > t) - e^{-\Lambda(B)t}| \leq \beta(B)e^{-\Lambda(B)t}, \tag{2.5}$$

where

$$\beta(B) := \frac{4}{(\text{Gap}(L) - \bar{\kappa})^2 - 4\kappa_1\kappa_2}\left[1 + \frac{\sqrt{(\text{Gap}(L) - \bar{\kappa})^2 + 4\kappa_2^2}}{\text{Gap}(L) - \bar{\kappa}}\right]\kappa_1\kappa_2. \tag{2.6}$$

In particular, $\pi(B^c)$ is sufficiently small if, in addition to the assumptions in Theorem 2.7,

$$\varepsilon(B) := \bar{\kappa} + \frac{2\kappa_1\kappa_2}{\text{Gap}(L) - \bar{\kappa}} + \varepsilon_0(B) < \text{Gap}(L), \tag{2.7}$$

where

$$\varepsilon_0(B) := 8(\text{Gap}(L) - \bar{\kappa})^{-1}\left[\text{Gap}(L) + 2M(B)\left(1 + \frac{2\kappa_2}{\text{Gap}(L) - \bar{\kappa}}\right)\right]\kappa_2. \tag{2.8}$$

We also have asymptotic results for a nonstationary starting measure $d\pi_0 = \rho_0\, d\hat{\pi}$, with $\rho_0 \in L^2(B; \hat{\pi})$.

THEOREM 2.9. *Suppose the hypotheses of Theorem 2.8 are satisfied and π_0 is a probability measure satisfying $d\pi_0 = \rho_0\, d\hat{\pi}$, with $\rho_0 \in L^2(B; \pi)$. Then*

$$|P_{\pi_0}(\tau > t) - e^{-\Lambda(B)t}| \leq \exp(-\Lambda(B)t)\beta(\rho_0, B),$$

where

$$\beta(\rho_0, B) := \left\{\frac{2\sqrt{(\text{Gap}(L) - \bar{\kappa})^2 + 4\kappa_2^2}}{[\text{Gap}(L) - \bar{\kappa}]^2 - 4\kappa_1\kappa_2}\right\} \times\left[\|\rho_0\|_{\hat{\pi}} + \|\rho_0 - \mathbf{1}\|_{\hat{\pi}} + \frac{2\kappa_2}{\text{Gap}(L) - \bar{\kappa}}\right]\kappa_1.$$

Theorem 2.8 will be proved in Section 4 by a spectral approach. There we use the main result of Stewart (1971) on the existence and approximation of invariant subspaces of operators to establish Theorem 2.7. As such, ϕ^B and $\rho^B d\hat{\pi}$ play the roles of the Perron–Frobenius eigenfunction and eigenmeasure for the positivity-preserving operators T_t^B. Heuristically, at least, as $B \to S$, L^B converges to L. Since $\Lambda(S) = 0$ and is an eigenvalue with corresponding constant eigenfunction $\mathbf{1}$, it is natural to expect that ϕ^B might converge to $\mathbf{1}$. This is indeed the case; moreover, the results of Stewart allow us to give the explicit estimates on the $L^2(\hat{\pi})$-distance between ϕ^B and $\mathbf{1}$. The connection between this and Theorem 2.8 is revealed by the following:

$$P_{\hat{\pi}}(\tau > t) = \int_B T_t^B \mathbf{1}\, d\hat{\pi} \quad \text{and} \quad T_t^B \phi^B = e^{-\Lambda(B)t}\phi^B.$$

The following corollary is an immediate consequence of (2.5) and, as the proof reveals, it depends solely on the form of the estimate (2.5) and not at all on the fact the τ arises as a hitting time of a Markov (jump) process.

COROLLARY 2.10. *Let $(B_n)_{n\in\mathbf{N}}$ be a sequence of sets in $\mathscr{S}$ satisfying the standing assumptions for all n and $\lim_{n\to\infty}\pi(B_n) = 1$. Set $\tau_n = \tau^{B_n}$. Then the law of $\Lambda(B_n)\tau_n$, with respect to $P_{\hat{\pi}_n}$ $[\hat{\pi}_n = [\pi(B_n)]^{-1}(\pi|B_n)]$, converges weakly, as $n \to \infty$, to that of a mean-one exponential random variable. Moreover,*

$$\lim_{n\to\infty} \Lambda(B_n) E_{\hat{\pi}}\tau_n = 1,$$

and, for any $B \in \mathscr{S}$ satisfying the assumptions of Theorem 2.7,

$$|E_{\hat{\pi}}\tau_B - \Lambda(B)^{-1}| \le \beta(B)/\Lambda(B),$$

where $\beta(B)$ is given by (2.6).

Using the estimate for $\Lambda(B)$ obtained from Theorem 2.7(i), we immediately get the following result.

COROLLARY 2.11. *If $\pi(B^c)$ is sufficiently small* (*as in Theorem* 2.8), *then*

$$P_{\hat{\pi}}(\tau \le t) \le t\Lambda(B) + \beta(B)$$

and

$$E_{\hat{\pi}}\tau^B \ge \left[\bar{\kappa} + 2\kappa_1\kappa_2/[\mathrm{Gap}(L) - \bar{\kappa}]\right]^{-1}(1 - \beta(B)),$$

where $\beta(B)$ is given by (2.6).

In the reversible case, in which π is in addition a reversibility measure for $(X_t;\, t \ge 0)$, Theorem 2.8 and Corollary 2.10 can be sharpened somewhat, with similar expressions corresponding to (2.6) and (2.7). Accordingly, we will treat the reversible case separately in Section 3, it being simpler to analyze from the standard spectral theory of self-adjoint operators. The following results will be proved there.

LEMMA 2.12. *Let $(X_t;\, t \ge 0)$ be reversible with respect to π. Set $\Lambda(B) := \inf \sigma(L^B)$. Then $\Lambda(B)$ is an isolated, simple eigenvalue of L^B. Moreover, $0 < \Lambda(B) \le \bar{\kappa} \to 0$, as $\pi(B^c) \to 0$, and* $\inf[\sigma(L^B) \setminus \{\Lambda(B)\}] \ge \mathrm{Gap}(L)$.

THEOREM 2.13. *If $(X_t;\, t \ge 0)$ is reversible with respect to π, then, for all $t \ge 0$,*

$$[1 - \beta_s(B)]e^{-\Lambda(B)t} \le P_{\hat{\pi}}(\tau > t) \le e^{-\Lambda(B)t},$$

where

$$\beta_s(B) = \min\left(\frac{\Lambda(B)}{\mathrm{Gap}(L)}, 1\right) \le \min\left(\frac{\bar{\kappa}}{\mathrm{Gap}(L)}, 1\right) \to 0 \tag{2.9}$$

as $\pi(B^c) \to 0$.

COROLLARY 2.14. *Suppose that* $(X_t;\ t \ge 0)$ *is reversible with respect to* π. *Let* $(B_n)_{n \in \mathbf{N}}$ *be a sequence of sets in* $\mathscr{S}$ *satisfying the standing assumptions for all n and* $\lim_{n\to\infty} \pi(B_n) = 1$. *Set* $\tau_n = \tau^{B_n}$. *Then the law of* $\Lambda(B_n)\tau_n$, *with respect to* $P_{\hat{\pi}_n}$ $[\hat{\pi}_n = [\pi(B_n)]^{-1}(\pi|B_n)]$, *converges weakly, as* $n \to \infty$, *to that of a mean-one exponential random variable. Moreover, for any* $B \in \mathscr{S}$ *satisfying the standing assumptions*

$$[1 - \beta_s(B)]\Lambda(B)^{-1} \le E_{\hat{\pi}}\tau \le \Lambda(B)^{-1},$$

where $\beta_s(B)$ *is described at* (2.9).

In closing this section, a few remarks are in order regarding the novelty and scope of the results above. Concerning Theorem 2.8, the technical assumptions on the size of $\pi(B^c)$ are imposed in order to apply the theorem of Stewart (1971); they are connected with the separation properties of certain spectra, which are not automatic. The conclusion of Stewart's theorem in our context is a result of Perron–Frobenius type. Now, such theorems are abundant in the literature—see Seneta (1981) for the matrix case (i.e., with countable state space) and Nummelin (1984) in the context of more general nonnegative kernels on a general measurable state space. However, these latter theorems will only yield results on the asymptotic behaviour as $t \to \infty$ of the tail of the distribution of τ^B for a *fixed* subset $B \subset S$.

Our principal interest is in the asymptotic behaviour as $B \to S$ of $P(\tau^B > t)$ for a *fixed* t. This behaviour was studied in a regenerative setting in Keilson (1979). In Theorem 8.2B there, the asymptotic exponentiality of $\tau/E_0\tau$ was shown without imposing the condition $\mathrm{Gap}(L) > 0$, where E_0 denotes the expectation associated with a chain started at a regenerative point denoted by 0. The only assumption is that $E_0\tau \to \infty$ as $B \to S$. The result of Stewart contains, in addition to statements of existence, quantitative estimates which suffice to derive our more precise result. Concerning the hypothesis $\mathrm{Gap}(L) >$ 0, we will give an example below to show that the conclusion of Theorem 2.8 can fail to hold in its absence.

Note that $\beta(B)$ can be of the same order as $\Lambda(B)$. Hence, for t fixed, the error between $\Lambda(B)t \sim 1 - e^{-\Lambda(B)t}$ and $1 - P_{\hat{\pi}}(\tau > t)$ would, in such a case, be of order $\Lambda(B)t$, which means the lower bound on $P_{\hat{\pi}}(\tau \le t)$ given by (2.5) may not be better than 0. However, it is the upper bound which is of practical interest.

Also regarding Corollary 2.10 and Theorem 2.8, it should be recalled that the function $u(x) := E_x\tau^B$ is the solution to the problem

$$L^B u = 1 \quad \text{on } B, \tag{2.10}$$

$$u = 0 \quad \text{on } B^c,$$

so that Corollary 2.10 gives the asymptotic behaviour of $\int_B u(x)\pi(dx)$, as $B \to S$. In the case of a countable state space S, if all states within B communicate (*within* B) and B is finite, then we may clump together all the states in B^c into one artificial state and treat (2.10) as a finite system of linear equations. However, the problem soon becomes numerically intractable as B becomes large. Thus the explicit and asymptotically sharp estimates provided by Corollary 2.10 and Theorem 2.8 may be more informative and efficient in practice than an attempt to solve (2.10) directly.

Finally, we present a simple counterexample to our main results when the hypothesis of the existence of a positive gap is not satisfied. The state space is $\mathbf{Z}$, the set of integers, and all jump rates equal 1. From 0 we jump to $n > 0$ with probability p_n. With probability $p_0 h_m$ we jump to state $-m$ where

$$\sum_{k=0}^{\infty} p_k = 1, \qquad \sum_{k=1}^{\infty} h_k = 1, \qquad \sum_{k=1}^{\infty} k h_k < \infty.$$

From $n > 0$ we jump directly back to 0. From $-m$ we jump to $-m + 1$, to $-m + 2$ and so on until we return to 0.

The stationary measure for this process is

$$\pi(n) \equiv \pi(\{n\}) = \begin{cases} \pi(0)p_n, & \text{if } n \geq 1, \\ \pi(0), & \text{if } n = 0, \\ \pi(0)p_0\hat{H}(-n), & \text{if } n < 0, \end{cases}$$

where, for $m = 1, 2, \dots,$

$$\hat{H}(m) = \sum_{k=m}^{\infty} h_k \quad \text{and} \quad \pi(0) = \left[1 + (1 - p_0) + p_0 \sum_{k=1}^{\infty} k h_k\right]^{-1}.$$

If $l > 0$, the time τ_l to exceed $l - 1$ satisfies, for $t \in \mathbf{N}$,

$$P_{\hat{\pi}}(\tau_l > t) \geq \pi(0)p_0 \sum_{m=t}^{\infty} h_m P_{\{-m\}}(\tau_l > t) \geq K\hat{H}(t),$$

where K is some constant, since starting at $-m$, τ_l is the sum of m exponentials. Now, picking $h_m = \mathscr{O}(1/m^{2+\varepsilon})$, we have $\hat{H}(t) = \mathscr{O}(1/t^{1+\varepsilon})$, so $P_{\hat{\pi}}(\tau_l > t)$ is certainly not exponentially small as $t \to \infty$. Consequently, the gap must be 0. Note that the point 0 is a recurrent point, so if E_0 denotes the associated expectation, then by Keilson (1979) $\tau_l / E_0 \tau_l$ tends to an exponential. From the estimate above, we must conclude $E_0 \tau_l / \Lambda(l)$ does not tend to 1.

3. The reversible case. In this section we suppose, in addition to the standing assumptions stated in the introduction just before Theorem 2.7, that π is a reversibility measure for the process $(X_t; t \geq 0)$; that is, T_t is self-adjoint on $L^2(S, \pi)$. In terms of the jump-rate kernel, $\int_{A_1} J(x, A_2)\pi(dx) = \int_{A_2} J(x, A_1)\pi(dx)$, for every $A_1, A_2 \in \mathscr{S}$. It follows that L is self-adjoint in $L^2(S, \pi)$ and that L^B and hence T_t^B are self-adjoint on $L^2(B, \hat{\pi})$. In order to handle the case where L is unbounded, we utilize the Dirichlet quadratic form [cf. Fukushima (1980)] associated with $-L$. [The essential technical difficulty in the unbounded case is that functions like $\mathbf{1}_B$, the indicator function of B, may not belong to $\mathscr{D}(L)$; however, see Lemma 3.1 below.]

The Dirichlet form $\mathscr{E}(\cdot,\cdot)$, associated with the infinitesimal generator $-L$ of $(T_t;\, t \geq 0)$, is defined by

$$\begin{aligned}\mathscr{E}(f,f) &:= (\sqrt{L}f, \sqrt{L}f)_\pi, \qquad f \in \mathscr{D}(\mathscr{E}) = \mathscr{D}(\sqrt{L})\\ &= (f, Lf)_\pi, \quad \text{if } f \in \mathscr{D}(L) \subset \mathscr{D}(\sqrt{L}),\end{aligned}$$

where $\sqrt{L}$ is defined as $\int_0^\infty \sqrt{\lambda}\, dE(\lambda)$, $(E(\lambda);\, \lambda \in \mathbf{R})$ being the spectral resolution of the identity for L. According to Lemma 1.3.4 of Fukushima (1980),

$$(3.1)\qquad \begin{aligned}& f \in \mathscr{D}(\mathscr{E}) \quad \text{whenever } \lim_{t \downarrow 0} \frac{1}{t}(f, f - T_t f)_\pi < +\infty \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(the limit always exists)},\\ &\mathscr{E}(f,f) = \lim_{t\downarrow 0}\frac{1}{t}(f, f - T_t f)_\pi \quad \text{(when finite)}.\end{aligned}$$

LEMMA 3.1. *Let $f \in L^2(B,\pi)$ and denote by $\bar f$ the zero-extension of f to all of S. Then $\bar f \in \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(\bar f, \bar f) = (f, L^B f)_{\pi|B}$.*

PROOF. For $f \in \mathscr{D}_w(\mathscr{L}) \subset \mathscr{D}(L) \cap \mathscr{B}(S)$,

$$(3.2)\qquad \begin{aligned}\mathscr{E}(f,f) = (f, Lf)_\pi &= \int J(x) f(x)^2 \pi(dx)\\ &\quad - \int f(x) \int J(x,dy) f(y)\pi(dx).\end{aligned}$$

Now, for $f \in \mathscr{B}(B)$, choose $f_n \in \mathscr{D}_w(\mathscr{L})$ such that $f_n \rightharpoonup \bar f$, as $n \to +\infty$ (i.e., pointwise and boundedly). Then, as $n \to \infty$, $f_n \to_{L^2} \bar f$ and by the dominated convergence theorem applied to the right-hand side of (3.2) [recall $\int_S J(x)\pi(dx) < \infty$],

$$\begin{aligned}\mathscr{E}(f_n, f_n) &\to \int_B J(x) f(x)^2 \pi(dx) - \int_B f(x) \int_B J(x,dy) f(y)\pi(dx)\\ &\equiv (f, L^B f)_{\pi|B}.\end{aligned}$$

Since $\mathscr{E}$ is a *closed* form, $\bar f \in \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(\bar f, \bar f) = (f, L^B f)_{\pi|B}$ in this case. Finally, for $f \in L^2(B,\pi)$, choose $f_n \in \mathscr{B}(B)$ with $f_n \to_{L^2} f$, as $n \to \infty$. Then, as $n \to \infty$, $\bar f_n \to_{L^2} \bar f$ and $\mathscr{E}(\bar f_n, \bar f_n) = (f_n, L^B f_n)_{\pi|B} \to (f, L^B f)_{\pi|B}$, since L^B is continuous by Lemma 6.3. Once again, by the closedness of $\mathscr{E}$, $\bar f \in \mathscr{D}(\mathscr{E})$ and $\mathscr{E}(\bar f, \bar f) = (f, L^B f)_{\pi|B}$. □

PROOF OF LEMMA 2.12. The lemma is an immediate consequence of the Rayleigh–Ritz and minimax variational characterizations [see, e.g., Reed and Simon (1978), Theorems 13.1 and 13.2] of $\Lambda(B)$, $\mathrm{Gap}(L)$ and the spectral gap of L^B. Indeed,

$$\Lambda(B) = \inf\left\{\frac{(f, L^B f)_{\hat{\pi}}}{\|f\|_{\hat{\pi}}^2} : 0 \neq f \in L^2(B, \hat{\pi})\right\}$$
$$\leq (\mathbf{1}_B, L^B \mathbf{1}_B)_{\hat{\pi}} / \|\mathbf{1}_B\|_{\hat{\pi}}$$
$$= \bar{\kappa} \to 0 \quad \text{as } \pi(B^c) \to 0$$

by Lemmas 2.4 and 2.5(i). Also by Lemma 3.1,

$$\sup_{0 \neq f \in L^2(\hat{\pi})} \inf_{0 \neq g \perp_{\hat{\pi}} f} \frac{(g, L^B g)_{\hat{\pi}}}{\|g\|_{\hat{\pi}}^2}$$
$$\geq \inf_{0 \neq g \perp_{\hat{\pi}} \mathbf{1}} \frac{(g, L^B g)_{\hat{\pi}}}{\|g\|_{\hat{\pi}}^2}$$
$$= \inf\left\{\frac{\mathscr{E}(g_1 \mathbf{1}_B, g_1 \mathbf{1}_B)}{\|g_1 \mathbf{1}_B\|_{\pi}^2} : 0 \neq g_1 \in L^2(S, \pi),\, g_1 \perp \mathbf{1}_B\right\}$$
$$\geq \inf\left\{\frac{E(g_2, g_2)}{\|g_2\|_{\pi}^2} : 0 \neq g_2 \in \mathscr{D}(\mathscr{E}),\, g_2 \perp \mathbf{1}\right\}$$
$$\equiv \mathrm{Gap}(L) > 0.$$

By the minimax principle, $\Lambda(B)$ is an isolated eigenvalue (of multiplicity 1) and $\inf[\sigma(L^B) \setminus \{\Lambda(B)\}] \geq \mathrm{Gap}(L)$.

Finally, if $\Lambda(B) = 0$, then if φ is an associated eigenfunction, normalized such that $\|\varphi\|_{\pi|B} = 1$, let $\bar{\varphi}$ denote its extension to S by $\bar{\varphi}|B^c = 0$ and set $\psi = \bar{\varphi} - (\bar{\varphi}, \mathbf{1})_{\pi}$. Then by the characterization of Gap(L) given above and Lemma 3.1

$$\|\psi\|_{\pi}\, \mathrm{Gap}(L) \leq \mathscr{E}(\bar{\psi}, \bar{\psi}) = \mathscr{E}(\bar{\varphi}, \bar{\varphi}) = (\varphi, L^B \varphi)_{\pi|B} = \Lambda(B) = 0.$$

Therefore $\bar{\varphi}$ is constant (π-a.e.). However $\bar{\varphi} = 0$ on B^c and $\pi(B^c) > 0$; so $\varphi \equiv 0$, a contradiction. □

PROOF OF THEOREM 2.13. For the upper bound, with $\mathbf{1}$ denoting the function with constant value 1,

$$P_{\hat{\pi}}(\tau > t) = \int_B T_t^B \mathbf{1}\, d\hat{\pi}$$
$$\leq \|T_t^B \mathbf{1}\|_{\hat{\pi}} \quad \text{by Cauchy–Schwarz}$$
$$\leq \|T_t^B\| = e^{-\Lambda(B)t}$$

by the spectral theorem.

For the lower bound we work in the (real) Hilbert space $L^2(\pi) \equiv L^2(S, \pi)$ and denote the inner product $(\cdot, \cdot)_{\pi}$ simply by $(\cdot, \cdot)$. Let $\mathscr{H}$ denote the (one-dimensional) subspace consisting of eigenfunctions of L^B, corresponding to the simple eigenvalue $\Lambda(B)$, extended to be 0 outside of B. Set

$$\mathbf{1} = \phi \oplus \psi, \quad \phi \in \mathscr{H}, \quad \psi \in \mathscr{H}^{\perp}. \tag{3.3}$$

Then

$$P_{\pi}(\tau > t) = \int_B \mathbf{1} \cdot T_t^B \mathbf{1}\, d\pi = \int_B [\phi \oplus \psi] \cdot T_t^B [\phi \oplus \psi]\, d\pi$$

$$= \|\phi\|^2 e^{-\Lambda(B)t} + \int_B \psi T_t^B \psi\, d\hat{\pi}$$

$$\geq \|\phi\|^2 e^{-\Lambda(B)t} = \left[1 - \|\psi\|^2\right] e^{-\Lambda(B)t}, \tag{3.4}$$

since T_t^B is a positive operator [in fact, $\sigma(T_t^B) = \exp(-\sigma(L^B)t)$], and, by (3.3),

$$\|\phi\|^2 = 1 - \|\psi\|^2. \tag{3.5}$$

Let $(E(\lambda);\, \lambda \in \mathbf{R})$ denote the spectral resolution of the identity associated with the self-adjoint operator L and set

$$d\mu(\lambda) := d(\psi, E(\lambda)\psi).$$

Now, $E(\lambda) = E(0)$ for all $\lambda \in [0, \mathrm{Gap}(L))$; so, for any $\lambda_* < 0 < \lambda^* < \mathrm{Gap}(L)$,

$$\mu((\lambda_*, \lambda^*]) = (\psi, [E(\lambda^*) - E(\lambda_*)]\psi)$$
$$= (\psi, [E(0) - 0]\psi)$$
$$= (\psi, \mathbf{1})^2 = \|\psi\|^4$$

by (3.3). Thus $\mu((-\infty, \mathrm{Gap}(L))) = \|\psi\|^4$. Therefore, in case L is bounded [i.e., if $M(S) < \infty$],

$$\|\psi\|^2 = \int_{-\infty}^{\infty} 1\, d\mu(\lambda) = \|\psi\|^4 + \int_{\mathrm{Gap}(L)}^{\infty} 1\, d\mu(\lambda)$$

$$\leq \|\psi\|^4 + [\mathrm{Gap}(L)]^{-1} \int_{\mathrm{Gap}(L)}^{\infty} \lambda\, d\mu(\lambda) \tag{3.6}$$

$$\leq \|\psi\|^4 + [\mathrm{Gap}(L)]^{-1} \cdot \int_0^{\infty} \lambda\, d(\psi, E(\lambda)\psi) \tag{3.7}$$

$$= \|\psi\|^4 + [\mathrm{Gap}(L)]^{-1} \cdot (\psi, L\psi) \tag{3.8}$$

$$= \|\psi\|^4 + [\mathrm{Gap}(L)]^{-1} \cdot (\phi, L\phi) \tag{3.9}$$

$$= \|\psi\|^4 + [\mathrm{Gap}(L)]^{-1} \cdot \Lambda(B)\|\phi\|^2$$

$$= \|\psi\|^4 + [\mathrm{Gap}(L)]^{-1} \cdot \Lambda(B)\left(1 - \|\psi\|^2\right) \tag{3.10}$$

by (3.5). The quadratic inequality between (3.6) and (3.10), for $\|\psi\|^2$, implies that

$$\|\psi\|^2 \leq \min\left(\frac{\Lambda(B)}{\mathrm{Gap}(L)}, 1\right). \tag{3.11}$$

The lower bound now follows from (3.4) and (3.11) since $\pi(B) < 1$.

In the general case where L is unbounded, we proceed as above with the following technical modifications. Rearranging the inequality $e^{-t} \le 1 - t + t^2/2$, for $t > 0$, we obtain

$$1 \le [1 - t/2]^{-1} \frac{[1 - e^{-t}]}{t} \le [1 - t/2]^{-1} \frac{[1 - \exp(-t\lambda/\mathrm{Gap}(L))]}{t},$$
$$\forall\, \lambda \ge \mathrm{Gap}(L), \forall\, t > 0.$$

Replacing t with $t \cdot \mathrm{Gap}(L)$ yields

$$1 \le \frac{[1 - t \cdot \mathrm{Gap}(L)/2]^{-1}}{\mathrm{Gap}(L)} \frac{[1 - e^{-t\lambda}]}{t}, \quad \forall\, \lambda \ge \mathrm{Gap}(L), \forall\, t > 0.$$

Substituting this into (3.6), we arrive at the following modifications of (3.7) and (3.8):

$$\begin{aligned}\|\psi\|^2 &\le \|\psi\|^4 + \frac{[1 - t \cdot \mathrm{Gap}(L)/2]^{-1}}{\mathrm{Gap}(L)} \int_0^\infty \frac{1 - e^{-t\lambda}}{t}\, d(\psi, E(\lambda)\psi) \\ &= \|\psi\|^4 + \frac{[1 - t \cdot \mathrm{Gap}(L)/2]^{-1}}{\mathrm{Gap}(L)} \left(\psi, \frac{\psi - T_t\psi}{t}\right).\end{aligned}$$

Letting $t \to 0^+$ and using (3.1), we obtain in place of (3.8) and (3.9):

$$\begin{aligned}\|\psi\|^2 &\le \|\psi\|^4 + [\mathrm{Gap}(L)]^{-1}\mathscr{E}(\psi, \psi) \\ &= \|\psi\|^4 + [\mathrm{Gap}(L)]^{-1}\mathscr{E}(\phi, \phi) \\ &= \|\psi\|^4 + [\mathrm{Gap}(L)]^{-1}(\phi|B, L^B[\phi|B])\end{aligned}$$

by Lemma 3.1. The rest of the proof remains unchanged. □

The proof of Corollary 2.14 is essentially the same as that of Corollary 2.10, to be given at the end of Section 4. In closing this section it should be noted that the derivations of the upper bound in Theorem 2.13, and hence the upper bound in Corollary 2.14, do not require that $\mathrm{Gap}(L) > 0$.

4. Gaps; proofs of general results.

4.1. *Spectral gaps*. We begin by recalling the notion of the numerical range of a bounded operator A on a complex Hilbert space $\mathscr{H}$ [on which we adopt the convention $(\lambda f, g) = \bar{\lambda}(f, g)$ for the inner product, $\lambda \in \mathbf{C}$]. The *numerical range*, $W(A)$, of A is defined by

$$W(A) := \{(f, Af): f \in \mathscr{H}, \|f\| = 1\}.$$

We recall the following well-known facts as summarized in Lawler and Sokal (1988):

1. $W(A)$ is convex.
2. $\sigma(A)$, the spectrum of A, is contained in $\overline{W(A)}$, the closure of $W(A)$.

3. If A is normal, then $\overline{W(A)}$ is the convex hull of $\sigma(A)$. However, in general, $\overline{W(A)}$ can be much larger than the convex hull of $\sigma(A)$, for example, if A is nilpotent.
4. If $\lambda \notin \overline{W(A)}$, then $\|(\lambda - A)^{-1}\| \le [\text{dist}(\lambda, W(A))]^{-1}$.

We state one more property of $W(A)$ as a proposition and indicate a proof, since it will play a fundamental role in our estimates. Set

$$w(A) := \inf \operatorname{Re} W(A); \tag{4.1}$$

note that $w(A)$ is finite since we are assuming that A is bounded: $|w(A)| \le \|A\|$.

Proposition 4.1. *If* $\operatorname{Re}\lambda < w(A)$, *then* $\|(\lambda - A)^{-1}\| \le [w(A) - \operatorname{Re}\lambda]^{-1}$.

Proof. Set $l = \operatorname{Re}\lambda$ and, for f of unit norm, let $g = (\lambda - A)^{-1}f$, so that $\lambda g = f + Ag$. Then, taking the real part of the inner product with g,

$$l\|g\|^2 = \operatorname{Re}(g, f) + \operatorname{Re}(g, Ag) \ge -\|g\| + w(A)\|g\|^2$$

by Cauchy–Schwarz. Dividing by $\|g\|$ and rearranging the inequality yields the result. □

Now consider a complex Hilbert space H, a (bounded) operator A on H and a fixed vector $\rho \in H$. Set $\mathscr{H} = \rho^{\perp} \equiv \{f \in H\colon (f, \rho) = 0\}$ and denote by Y the injection of $\mathscr{H}$ into H; Y^* is the adjoint projection. Finally, set $\tilde{A} = Y^*AY\colon \mathscr{H} \to \mathscr{H}$, the *compression* of A into $\mathscr{H}$.

Definition 4.2.

$$\Gamma_\rho \equiv \Gamma_\rho(A) := w(\tilde{A}) \equiv \inf \operatorname{Re}\{(f, Af)\colon \|f\| = 1, (f, \rho) = 0\}.$$

In particular, $\operatorname{Re}\{\sigma(\tilde{A})\} \subset [\Gamma_\rho, \infty)$; and in case ρ is an eigenvector of A with eigenvalue λ such that $\operatorname{Re}\lambda < \Gamma_\rho$, it is easy to check that $\sigma(A) - \{\lambda\}$ does not meet the strip $\{z \in \mathbf{C}\colon \operatorname{Re}\lambda < \operatorname{Re} z < \Gamma_\rho\}$. In this case it is appropriate to refer to $\Gamma_\rho - \operatorname{Re}\lambda$ as the *numerical gap* of A (with respect to ρ and λ). It may be smaller than the actual gap in the real part of the spectrum of A.

An immediate consequence of Proposition 4.1 and Definition 4.2 is the following corollary.

Corollary 4.3. *If* $\operatorname{Re}\lambda < \Gamma_\rho(A)$, *then* $\|(\lambda - \tilde{A})^{-1}\| \le [\Gamma_\rho(A) - \operatorname{Re}\lambda]^{-1}$.

The following proposition is a variant of a part of Theorem 2.3 of Liggett (1989); the proof is essentially the same.

Proposition 4.4. *Let* $(T_t;\ t \ge 0)$ *be a strongly continuous semigroup on a complex Hilbert space H with bounded infinitesimal generator* $-A$. *Then, for all* $f \in H$,

$$\|T_t f\| \ne e^{-w(A)t}\|f\|, \qquad t \ge 0.$$

In particular, if ρ is an eigenfunction of A^*, then, for all $f \in H$ such that $f \perp \rho$,

$$\|T_t f\| \le e^{-\Gamma_\rho(A)t}\|f\|, \qquad t \ge 0.$$

PROOF. Let $f \in H$ such that $\|f\| = 1$; w.l.o.g. assume $\|T_t f\| \neq 0$. Then

$$\frac{d}{dt}\|T_t f\|^2 = -2\operatorname{Re}(T_t f, AT_t f) \le -2w(A) \cdot \|T_t f\|^2.$$

Therefore, $\|T_t f\|^2 \le e^{-2w(A)\cdot t}\|f\|^2$. For the second part, simply apply the first part to $\mathscr{H} \equiv \rho^\perp$ and $\tilde{A} = A|\mathscr{H}$: $\mathscr{H} \to \mathscr{H}$ [since if $(f, \rho) = 0$, then $(Af, \rho) = (f, A^*\rho) = \lambda(f, \rho) = 0$, if $A^*\rho = \lambda\rho$; similarly $T_t(\mathscr{H}) \subset \mathscr{H}$ since $T_t^*\rho = e^{\lambda t}\rho$], recalling that $\Gamma_\rho = w(\tilde{A})$. □

REMARK 4.5. It is important to note that in case A (and hence T_t in the previous proposition) is a *real* operator on a *real* $L^2(\mu)$-space, then $w(A)$ and $\Gamma_\rho(A)$ [which by definition would be calculated on the complexification of $L^2(\mu)$] can be calculated as infimums using real functions f; that is, $w(A) = \inf_{\|f\|=1}(f, Af)$ and $\Gamma_\rho(A) = \inf\{(f, Af)\colon \|f\| = 1, \int f\rho\, d\mu = 0\}$ if ρ is real. Indeed, denoting these real counterparts temporarily by $w'(A)$ and $\Gamma_\rho'(A)$, clearly $w(A) \le w'(A)$. Conversely, if $f = g + ih$ with $g, h \in L^2(\mu)$ (and real) such that $1 = \|f\|^2 = \|g\|^2 + \|h\|^2$:

$$\operatorname{Re}(f, Af) = (g, Ag) + (h, Ah) \ge w'(A)\|g\|^2 + w'(A)\|h\|^2 = w'(A).$$

Taking the infimum over all such f then yields $w(A) \ge w'(A)$. Concerning $\Gamma_\rho(A)$ and $\Gamma_\rho'(A)$, the argument is the same, noting that if, as above, $f = g + ih$ and $f \perp \rho$, with ρ real, then $g \perp \rho$ and $h \perp \rho$. Thus Proposition 4.4 remains valid on a real Hilbert space.

Also, we remark that Proposition 4.4 remains valid (with the same proof) in the case that A is an unbounded operator, with the understanding that, in the definition of $W(A)$ and Definition 4.2, f is restricted to belong to $\mathscr{D}(A)$.

In particular, we can apply Proposition 4.4 to the case $H = L^2(S, \pi)$, $\rho = \mathbf{1}$, $T_t =$ "e^{-Lt}," and $A = L$. We write Gap(L) for $\Gamma_\mathbf{1}(L)$ and, as stated in Section 2, we assume that Gap(L) > 0. Since π was assumed to be an invariant probability, then 0 is an isolated eigenvalue of L^* and hence of L, and $\operatorname{Re}(\sigma(L) \setminus \{0\}) \subset [\operatorname{Gap}(L), \infty)$; and Proposition 4.4 describes a rate of relaxation to equilibrium [for a fuller discussion, see Liggett (1989), where, in particular, it is shown that Gap(L) ≥ 0]:

$$(4.2) \qquad \|T_t f - \textstyle\int f\, d\pi\|_\pi \le e^{-\operatorname{Gap}(L)t}\|f - \textstyle\int f\, d\pi\|_\pi, \qquad f \in L^2(S, \pi).$$

We now establish some simple relations between various "gaps." Recall that we are assuming $\pi(B) > 0$ and have set $\hat{\pi} = [\pi(B)]^{-1}\pi|B$.

DEFINITION 4.6.

$$\hat{\Gamma} \equiv \hat{\Gamma}^B := \Gamma_1(L^B) \equiv \inf\left\{(f, L^B f)_{\hat{\pi}}: \|f\|_{\hat{\pi}} = 1, \int_B f\,d\hat{\pi} = 0\right\},$$

$$\hat{\Gamma}^* := \Gamma_1([L^B]^*) \equiv \inf\left\{(f, [L^B]^* f)_{\hat{\pi}}: \|f\|_{\hat{\pi}} = 1, \int_B f\,d\hat{\pi} = 0\right\}.$$

Hence, we have the following result.

LEMMA 4.7. $\hat{\Gamma}^* = \hat{\Gamma}$.

REMARK. This is a special case of the more general result $w(A) = w(A^*)$ (since $\operatorname{Re} c = \operatorname{Re} \bar{c}$ for any $c \in \mathbf{C}$) applied to $A = \widetilde{L^B}$ (see Definition 4.2).

LEMMA 4.8. $\hat{\Gamma} \geq \operatorname{Gap}(L)$.

PROOF. For $g \in L^2(B; \hat{\pi})$ such that $\int g^2\,d\hat{\pi} = 1$ and $\int g\,d\hat{\pi} = 0$, extend g to all of S by setting $g = 0$ on B^c. For $n > 0$, set $g_n(x) = g(x)$ if $|g(x)| \leq n$ and 0 otherwise; and set $h_n = g_n - \int g_n\,d\pi$. For $\varepsilon > 0$, set

$$g_{n,\varepsilon} = \varepsilon^{-1}\int_0^\varepsilon T_t g_n\,dt, \qquad h_{n,\varepsilon} = \varepsilon^{-1}\int_0^\varepsilon T_t h_n\,dt.$$

Then $g_{n,\varepsilon}, h_{n,\varepsilon} \in \mathscr{D}_w(\mathscr{L}) \subset \mathscr{D}(L)$, $\int h_{n,\varepsilon}\,d\pi = 0$ and by Theorem 2.1, Lemma 2.2 and the dominated convergence theorem ($\int J(x)\pi(dx) < \infty$):

$$\begin{aligned}
(h_{n,\varepsilon}, \mathscr{L}h_{n,\varepsilon})_\pi &= (g_{n,\varepsilon}, \mathscr{L}g_{n,\varepsilon})_\pi \\
&= \int g_{n,\varepsilon}(x)\int [g_{n,\varepsilon}(x) - g_{n,\varepsilon}(y)] J(x, dy)\pi(dx) \\
&\to \int g_n(x)\int [g_n(x) - g_n(y)] J(x, dy)\pi(dx) \quad \text{as } \varepsilon \to 0, \\
&= (g_n, L^B g_n)_\pi \to (g, L^B g)_\pi \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore

$$\begin{aligned}
\operatorname{Gap}(L) &= \inf\left\{(f, Lf)_\pi: f \in \mathscr{D}(L), \|f\|_\pi = 1, \int f\,d\pi = 0\right\} \\
&\leq \inf\left\{(h, \mathscr{L}h)_\pi / \|h\|_\pi^2: h \in \mathscr{D}_w(\mathscr{L}), \int f\,d\pi = 0\right\} \\
&\leq \inf\left\{\lim_{n\to\infty}\lim_{\varepsilon\to 0}(h_{n,\varepsilon}, \mathscr{L}h_{n,\varepsilon})_\pi / \|h_{n,\varepsilon}\|_\pi^2: \|g\|_{\hat{\pi}} = 1, \int_B g\,d\hat{\pi} = 0\right\} \\
&= \inf\left\{(g, L^B g)_{\hat{\pi}}: \|g\|_{\hat{\pi}} = 1, \int_B g\,d\hat{\pi} = 0\right\} \\
&= \Gamma_1(L^B).
\end{aligned}$$

□

4.2. *Construction and approximation of Perron–Frobenius pairs.* In this subsection we prove Theorem 2.7. The tool we shall use is the following theorem, which is the restriction to bounded operators of the main theorem (Theorem 3.5) of Stewart (1971).

THEOREM 4.9. *Let A be a bounded linear operator on a separable Hilbert space $\mathscr{H}$. Let $\mathscr{X}$ be a subspace and denote its orthogonal complement by $\mathscr{Y}$. Also denote by X and Y the injections of $\mathscr{X}$ and $\mathscr{Y}$, respectively, into $\mathscr{H}$ and set*

$$B = X^*AX, \qquad H = X^*AY,$$
$$G = Y^*AX, \qquad C = Y^*AY,$$

where $*$ denotes the adjoint. Set $\gamma = \|G\|$, $\eta = \|H\|$ and $\delta = \|T^{-1}\|^{-1}$, where T is the linear operator on the space $\mathscr{B}(\mathscr{X}, \mathscr{Y})$ of bounded linear transformations from $\mathscr{X}$ to $\mathscr{Y}$ given by $TP = PB - CP$, and is assumed to be bijective. If $\gamma\eta/\delta^2 < 1/4$, then there is a $P \in \mathscr{B}(\mathscr{X}, \mathscr{Y})$ such that the range of $X + YP$ is an invariant subspace of A; $\|P\| \leq 2\gamma/\delta$. Moreover, the spectrum of A is the disjoint union

$$\sigma(A) = \sigma(B + HP) \cup \sigma(C - PH) \equiv \sigma_1 \cup \sigma_2,$$

and $\operatorname{dist}(\sigma_1, \sigma_2) > [\delta^2 - 4\gamma\eta]/\delta$.

Theorem 4.9 simplifies greatly in the case where $\dim \mathscr{X} = 1$. In this case B and $B + HP$ are scalar operators on $\mathscr{X}$, say $B = bI$ and $B + HP = \Lambda I$, and the operator T^{-1} can be identified explicitly. Since, for arbitrary $P \in \mathscr{B}(\mathscr{X}, \mathscr{Y})$, $TP = (b - C)P$, then $T^{-1}P = (b - C)^{-1}P$ [provided that $b \notin \sigma(C)$]. Thus $\delta \geq \|(b - C)^{-1}\|^{-1}$ and we obtain the following corollary.

COROLLARY 4.10. *Suppose, in addition to the hypotheses of Theorem* 4.9, $\dim \mathscr{X} = 1$, *with $B = bI$ and $b \notin \sigma(C)$ [in particular, if* $\operatorname{Re}(b) < w(C)$], *and* $\|(b - C)^{-1}\|^{-1} \geq \delta_0$, *for some $\delta_0 > 0$ such that $\gamma\eta/\delta_0^2 < 1/4$. Then, with $B + HP = \Lambda I$,*

(i) *Λ is an isolated eigenvalue of A; $|\Lambda - b| \leq 2\gamma\eta/\delta_0$:*

(ii) *if $0 \neq f \in \mathscr{X}$, then $\varphi := f \oplus YPf$ is an eigenvector associated with Λ and $\|\varphi - f\| \leq 2\gamma\delta_0^{-1}\|f\|$.*

We now apply this result to the operators L^B and $(L^B)^*$ on $L^2(\hat{\pi})$ with $\mathscr{X}$ being the subspace of constant functions. Note that both operators have obvious extensions to $L^2(\hat{\pi})$ considered as a complex vector space (for the spectral conclusions). It follows from the proof of Theorem 4.9 that P is a real operator; that is, P maps the real constants to real-valued functions in $\mathscr{Y}$.

PROOF OF THEOREM 2.7. The details of the proof of existence will only be given for ρ^B, those for ϕ^B being similar. We apply Corollary 4.10 to $A = (L^B)^*$

and $\mathscr{X}$= {constant functions}, and begin by identifying γ, η, b and δ_0. First,

$$\begin{aligned}\gamma &= \|Y^*[L^B]^* X\mathbf{1}\|_{\hat{\pi}} = \|[L^B]^*\mathbf{1} - (\mathbf{1},[L^B]^*\mathbf{1})_{\hat{\pi}}\mathbf{1}\|\\ &= \sqrt{\|R^B\|^2 - \bar{\kappa}^2} \equiv \kappa_2 \quad \text{by (2.4)},\\ \eta &= \sup\{\|X^*[L^B]^* f\|_{\hat{\pi}} : \|f\|_{\hat{\pi}} = 1, (\mathbf{1}, f)_{\hat{\pi}} = 0\}\\ &= \sup\{|(\mathbf{1},[L^B]^* f)_{\hat{\pi}}| : \|f\|_{\hat{\pi}} = 1, (\mathbf{1}, f)_{\hat{\pi}} = 0\}\\ &= \sup\{|(K^B - \bar{\kappa}\mathbf{1}, f)_{\hat{\pi}}| : \|f\|_{\hat{\pi}} = 1, (\mathbf{1}, f)_{\hat{\pi}} = 0\}\\ &= \|K^B - \bar{\kappa}\mathbf{1}\| \equiv \kappa_1,\end{aligned}$$

since $K^B - \bar{\kappa}$ is orthogonal to $\mathbf{1}$.

Now, with $B \equiv X^*(L^B)^*X = bI$,

$$b = (\mathbf{1}, X^*(L^B)^* X\mathbf{1})_{\hat{\pi}} = (L^B\mathbf{1}, \mathbf{1})_{\hat{\pi}} = \bar{\kappa}.$$

Also, by Definition 4.2 (and the ensuing remarks), Definition 4.6 and Lemmas 4.7 and 4.8 with $C = Y^*(L^B)^*Y$,

$$\inf \operatorname{Re}\{\sigma(C)\} \geq \hat{\Gamma}^* = \hat{\Gamma} \geq \operatorname{Gap}(L).$$

Therefore, since by hypothesis $\bar{\kappa} < \operatorname{Gap}(L)$, then $b \notin \sigma(C)$; moreover, by Corollary 4.3,

$$\|(b - C)^{-1}\| \leq [\hat{\Gamma}^* - b]^{-1} \leq [\operatorname{Gap}(L) - \bar{\kappa}]^{-1} =: \delta_0^{-1}.$$

With this choice of δ_0,

$$\gamma\eta/\delta_0^2 \leq \kappa_1\kappa_2/[\operatorname{Gap}(L) - \bar{\kappa}]^2 < 1/4$$

by hypothesis.

Thus we have the existence of $\Lambda(B)$ and $\rho^B := \mathbf{1} \oplus P_1\mathbf{1}$ from Corollary 4.10. (We subscript the transformations P, B, C and H from Theorem 4.9 to connect them with ρ^B; similarly, we will write P_2, B_2, C_2 and H_2 for the analogous transformations to be associated with ϕ^B below.) By construction $\int_B \rho^B \, d\hat{\pi} = 1$, and $\|\rho^B - \mathbf{1}\|_{\hat{\pi}} \leq 2\gamma\delta_0^{-1} \leq 2\kappa_2/[\operatorname{Gap}(L) - \bar{\kappa}]$. Also, from Corollary 4.10(i),

$$|\Lambda(B) - \bar{\kappa}| \equiv |\Lambda(B) - b| \leq 2\gamma\eta\delta_0^{-1} \leq 2\kappa_1\kappa_2/[\operatorname{Gap}(L) - \bar{\kappa}].$$

Similarly, we may apply Corollary 4.10 to L^B with the same $\mathscr{X}$ and $\mathscr{Y}$. Since $\hat{\Gamma} = \hat{\Gamma}^*$, $\sigma(L^B) = \overline{\sigma([L^B]^*)}$, $\sigma(C^*) = \overline{\sigma(C)}$ and the b and δ_0 associated with L^B coincide with those for $[L^B]^*$, we need only estimate $\gamma \equiv \gamma_2$ and $\eta \equiv \eta_2$:

$$\gamma = \|Y^*L^BX\mathbf{1}\| = \|Y^*K^B\| = \|K^B - \bar{\kappa}\mathbf{1}\| = \kappa_1$$

and

$$\begin{aligned}\eta &= \sup\left\{\|X^* L^B f\|_{\hat{\pi}} : \|f\|_{\hat{\pi}} = 1, (\mathbf{1}, f)_{\hat{\pi}} = 0\right\}\\ &= \sup\left\{\left|\left([L^B]^* \mathbf{1}, f\right)_{\hat{\pi}}\right| : \|f\|_{\hat{\pi}} = 1, (\mathbf{1}, f)_{\hat{\pi}} = 0\right\}\\ &= \sup\left\{\left|\left(R^B - \bar{\kappa}\mathbf{1}, f\right)\right| : \|f\|_{\hat{\pi}} = 1, (\mathbf{1}, f)_{\hat{\pi}} = 0\right\}\\ &= \|R^B - \bar{\kappa}\mathbf{1}\| \equiv \kappa_2.\end{aligned}$$

By hypothesis, $\gamma\eta\delta_0^{-2} < 1/4$. Therefore we have the existence of an eigenvector of L^B of the form $\psi = f \oplus P_2 f$ with $f = \alpha\mathbf{1}$, $\alpha \in \mathbf{R}$, for which

$$\int \varphi \rho^B \, d\hat{\pi} = (\mathbf{1} \oplus P_1\mathbf{1}, f \oplus P_2 f)_{\hat{\pi}} = \alpha + \alpha(P_1\mathbf{1}, P_2\mathbf{1})_{\hat{\pi}}.$$

Now

$$|(P_1\mathbf{1}, P_2\mathbf{1})_{\hat{\pi}}| \le \|P_1\| \cdot \|P_2\| \le \left[2\gamma_1\delta_0^{-1}\right]\left[2\gamma_2\delta_0^{-1}\right] \le \frac{4\kappa_1\kappa_2}{[\mathrm{Gap}(L) - \bar{\kappa}]^2} < 1$$

by hypothesis; so we may choose $\alpha = [1 + (P_1\mathbf{1}, P_2\mathbf{1})_{\hat{\pi}}]^{-1}$ and set ϕ^B to be the corresponding "φ," that is, $\phi^B = \alpha\mathbf{1} \oplus \mathrm{P}_2(\alpha\mathbf{1})$. Then

$$\begin{aligned}\left|\int_B \phi^B \, d\hat{\pi} - 1\right| &= |\alpha - 1| \le \frac{4\gamma_1\gamma_2\delta_0^{-2}}{1 - 4\gamma_1\gamma_2\delta_0^{-2}}\\ &\le \frac{4\kappa_1\kappa_2}{[\mathrm{Gap}(L) - \bar{\kappa}]^2 - 4\kappa_1\kappa_2}\end{aligned}$$

and

$$\begin{aligned}\|\phi^B - \mathbf{1}\|_{\hat{\pi}}^2 &= (\alpha - 1)^2 + \alpha^2\|P_2\mathbf{1}\|_{\hat{\pi}}^2\\ &\le (\alpha - 1)^2 + \alpha^2\|P_2\|^2\\ &\le \left(\frac{4\gamma_1\gamma_2\delta_0^{-2}}{1 - 4\gamma_1\gamma_2\delta_0^{-2}}\right)^2 + \left[1 - 4\gamma_1\gamma_2\delta_0^{-2}\right]^{-2} \cdot \left[2\gamma_2\delta_0^{-1}\right]^2\\ &= \left\{\frac{2\gamma_2\delta_0^{-1}}{1 - 4\gamma_1\gamma_2\delta_0^{-2}}\sqrt{1 + 4\gamma_1^2\delta_0^{-2}}\right\}^2\end{aligned}$$

and we conclude as with the previous estimate.

It remains to show that the eigenvalue Λ_2, say, associated with ϕ^B coincides with $\Lambda(B)$ and that $0 < \Lambda(B) = \inf \mathrm{Re}\{\sigma(L^B)\}$. First,

$$\begin{aligned}\Lambda(B) &= \Lambda(B) \cdot (\rho^B, \phi^B)_{\hat{\pi}} = ([L^B]^* \rho^B, \phi^B)_{\hat{\pi}}\\ &= (\rho^B, L^B\phi^B)_{\hat{\pi}} = (\rho^B, \Lambda_2\phi^B) = \Lambda_2.\end{aligned}$$

Finally, to check that $\Lambda(B) = \inf \mathrm{Re}\{\sigma(L^B)\}$, we need only show that the bottom of the numerical range, $w(C_1 - P_1 H_1)$, exceeds $\Lambda(B)$—see Theorem 4.9. For $f \in L^2(\hat{\pi})$ with $\|f\|_{\hat{\pi}} = 1$ and $(\mathbf{1}, f)_{\hat{\pi}} = 0$,

$$(f, [C_1 - P_1 H_1] f)_{\hat{\pi}} \geq \hat{\Gamma} - \|P_1\| \cdot \|H_1\| \geq \hat{\Gamma} - 2\gamma_1 \eta_1 \delta_0^{-1}.$$

Since $\Lambda(B) \leq \bar{\kappa} + 2\gamma_1\eta_1\delta_0^{-1}$, it suffices to ensure that $\bar{\kappa} + 4\gamma_1\eta_1\delta_0^{-1} < \hat{\Gamma}$, or, by Lemma 4.8, that $\bar{\kappa} + 4\kappa_1\kappa_2/[\mathrm{Gap}(L) - \bar{\kappa}] < \mathrm{Gap}(L)$, which was hypothesized.

Finally, to show that $\Lambda(B) > 0$, we use an argument similar to the one used in the proof of Lemma 2.12 combined with the truncation and smoothing used in Lemma 4.8. Extend ρ^B to be 0 off B and for $n > 0$, set $g_n(x) = \rho^B(x)$ if $|\rho^B(x)| \leq n$, $g_n(x) = 0$ otherwise. Then for $\varepsilon > 0$, define h_n, $g_{n,\varepsilon}$ and $h_{n,\varepsilon}$ as in the proof of Lemma 4.8. By the definition of Gap(L),

$$\|h_{n,\varepsilon}\|_\pi \,\mathrm{Gap}(L) \leq (h_{n,\varepsilon}, \mathscr{L} h_{n,\varepsilon})_\pi = (g_{n,\varepsilon}, \mathscr{L} g_{n,\varepsilon})_\pi.$$

Then by the dominated convergence theorem ($\int J(x)\pi(dx) < \infty$), we can let $\varepsilon \to 0$ to obtain

$$\|h_n\|_\pi \,\mathrm{Gap}(L) \leq (g_n, \mathscr{L} g_n)_\pi = (g_n, L^B g_n)_\pi.$$

Now, letting $n \to \infty$, we conclude that

$$\|\rho^B - \int \rho^B \, d\pi\|_\pi \mathrm{Gap}(L) \leq (\rho^B, L^B \rho^B)_\pi = ([L^B]^* \rho^B, \rho^B)_\pi$$
$$= \Lambda(B)\pi(B).$$

Therefore if $\Lambda(B) = 0$ then ρ^B would be constant (π-a.e.). However $\rho^B = 0$ on B^c and $\pi(B^c) > 0$; so the eigenfunction $\rho^B \equiv 0$, a contradiction. □

4.3. *Proofs of Theorems* 2.8 *and* 2.9 *and Corollary* 2.10. Before giving the proofs of Theorems 2.8 and 2.9 and Corollary 2.10, we first establish an underestimate of the gap $\Gamma_{\rho^B}(L^B)$.

LEMMA 4.11. *Under the hypothesis of Theorem* 2.7,

$$\Gamma_{\rho^B}(L^B) \geq \mathrm{Gap}(L) - \varepsilon_0(B),$$

where $\varepsilon_0(B)$ *was given at* (2.8).

PROOF. Given $f \in L^2(\hat{\pi})$ such that $\|f\|_{\hat{\pi}} = 1$ and $\int f \rho^B \, d\hat{\pi} = 0$, decompose

$$f \oplus \rho^B = \alpha \mathbf{1} \oplus g$$

for some $\alpha \in \mathbf{R}$ and $g \perp \mathbf{1}$ in $L^2(\hat{\pi})$, where $\mathbf{1}$ denotes the function on B with constant value 1. Note that

$$\alpha = \int (\alpha \mathbf{1} + g) \, d\hat{\pi} = \int (f + \rho^B) \, d\hat{\pi} = \int f d\hat{\pi} + 1 = \int f(\mathbf{1} - \rho^B) \, d\hat{\pi} + 1.$$

Therefore,

$$|\alpha - 1| \le \|f\|_{\hat\pi} \|\rho^B - \mathbf{1}\|_{\hat\pi} = \|\rho^B - \mathbf{1}\|_{\hat\pi}$$

and

$$\|f - g\|_{\hat\pi} \le |\alpha - 1| + \|\rho^B - \mathbf{1}\|_{\hat\pi} \le 2\|\rho^B - \mathbf{1}\|_{\hat\pi}.$$

Also

$$\|g\|_{\hat\pi} \le \|g - f\|_{\hat\pi} + \|f\|_{\hat\pi} \le 1 + 2\|\rho^B - \mathbf{1}\|_{\hat\pi},$$
$$\|g\|_{\hat\pi}^2 \ge (\|f\|_{\hat\pi} - \|f - g\|_{\hat\pi})^2 \ge 1 - 4\|\rho^B - \mathbf{1}\|_{\hat\pi}.$$

Therefore,

$$\begin{aligned} |(f, L^B f)_{\hat\pi} - (g, L^B g)_{\hat\pi}| &\le |(f - g, L^B f)_{\hat\pi} + (g, L^B[f - g])_{\hat\pi}| \\ &\le \|L^B\| \cdot \left[\|f - g\|_{\hat\pi} + \|g\|_{\hat\pi}\|f - g\|_{\hat\pi}\right] \\ &\le 2\|L^B\| \cdot \|\rho^B - \mathbf{1}\|_{\hat\pi} \cdot \left[1 + 1 + 2\|\rho^B - \mathbf{1}\|_{\hat\pi}\right] \\ &= 4\|L^B\| \cdot \|\rho^B - \mathbf{1}\|_{\hat\pi} \cdot \left[1 + \|\rho^B - \mathbf{1}\|_{\hat\pi}\right]. \end{aligned}$$

We can now compare the two gaps. Given $\varepsilon > 0$, if now f is also chosen to satisfy $(f, L^B f)_{\hat\pi} < \Gamma_{\rho^B} + \varepsilon$, then

$$\begin{aligned} \Gamma_{\rho^B} &> (f, L^B f)_{\hat\pi} - \varepsilon \ge (g, L^B g)_{\hat\pi} - 4\|L^B\| \cdot \|\rho^B - \mathbf{1}\|_{\hat\pi} \cdot \left[1 + \|\rho^B - \mathbf{1}\|_{\hat\pi}\right] - \varepsilon \\ &\ge \hat\Gamma \cdot \|g\|_{\hat\pi}^2 - 4\|L^B\| \cdot \|\rho^B - \mathbf{1}\|_{\hat\pi} \cdot \left[1 + \|\rho^B - \mathbf{1}\|_{\hat\pi}\right] - \varepsilon \\ &\ge \mathrm{Gap}(L) \cdot \left[1 - 4\|\rho^B - \mathbf{1}\|_{\hat\pi}\right] - 4\|L^B\| \cdot \|\rho^B - \mathbf{1}\|_{\hat\pi} \cdot \left[1 + \|\rho^B - \mathbf{1}\|_{\hat\pi}\right] - \varepsilon \end{aligned}$$

by Lemma 4.8. Since $\varepsilon > 0$ was arbitrary, the lemma then follows from the estimates:

$$\|\rho^B - \mathbf{1}\|_{\hat\pi} \le \frac{2\kappa_2}{\mathrm{Gap}(L) - \bar\kappa} \qquad \text{[cf. Theorem 2.7(ii)]}$$

and

$$\|L^B\| \le 2M(B). \qquad \square$$

PROOF OF THEOREM 2.9.

$$\begin{aligned} &|P_{\pi_0}(\tau > t) - e^{-\Lambda(B)t}| \\ &\quad = \left|\int_B T_t^B \mathbf{1}\, d\pi_0 - \int_B [T_t^B \phi^B] \rho^B\, d\hat\pi\right| \\ &\quad \le |(\rho_0 - \rho^B, T_t^B[\mathbf{1} - \phi^B])_{\hat\pi}| \\ &\qquad + |(\rho^B, T_t^B[\mathbf{1} - \phi^B])_{\hat\pi}| + |(T_t^B \phi^B, \rho_0 - \rho^B)_{\hat\pi}| \\ &\quad \le \left[\exp(-\Gamma_{\rho^B}(L^B)t)\right] \cdot \|\phi^B - \mathbf{1}\|_{\hat\pi} \cdot \|\rho_0 - \rho^B\|_{\hat\pi} \\ &\qquad + [\exp(-\Lambda(B)t)] \cdot \left|\int_B (\phi^B - 1)\rho_0\, d\hat\pi\right| \end{aligned} \tag{4.3}$$

by Cauchy–Schwarz and Proposition 4.4 (with $\rho \equiv \rho^B$ and $\mu \equiv \hat{\pi}$, $T_t \equiv T_t^B$ and $A \equiv L^B$) for the estimate of the first term. For the second term we have used the facts: $[T_t^B]^* \rho^B = e^{-\Lambda(B)t}\rho^B$ and $(\rho^B, \mathbf{1} - \phi^B)_{\hat{\pi}} = 1 - 1 = 0$ by Theorem 2.7. Then, by Lemma 4.11, the last estimate is less than or equal to $\beta(\rho_0, B, t)\exp(-\Lambda(B)t)$, where

$$\begin{aligned}\beta(\rho_0, B, t) &:= \|\phi^B - \mathbf{1}\|_{\hat{\pi}} \cdot \|\rho_0\|_{\hat{\pi}} \\ &\quad + \|\phi^B - \mathbf{1}\|_{\hat{\pi}} \cdot \|\rho_0 - \rho^B\|_{\hat{\pi}} \\ &\quad \times \exp([\Lambda(B) - (\mathrm{Gap}(L) - \varepsilon(B))]t) \\ &\le \beta(\rho_0, B)\end{aligned}$$

by Theorem 2.7(ii), provided $\Lambda(B) + \varepsilon_0(B) \le \mathrm{Gap}(L)$. This is guaranteed if (2.7) holds by Theorem 2.7(i). The term $\|\rho_0 - \rho^B\|_{\hat{\pi}} \le \|\rho_0 - \mathbf{1}\|_{\hat{\pi}} + \|\mathbf{1} - \rho^B\|_{\hat{\pi}}$ and the latter is estimated using Theorem 2.7 □

PROOF OF THEOREM 2.8. The proof follows by taking $\pi_0 = \hat{\pi}$ and $\rho_0 = \mathbf{1}$ in expression (4.3):

$$\begin{aligned}|P_{\hat{\pi}}(\tau > t) - e^{-\Lambda(B)t}| &= \left|\int_B T_t^B \mathbf{1}\, d\hat{\pi} - \int_B [T_t^B \phi^B]\rho^B\, d\hat{\pi}\right| \\ &\le \left[\exp\left(-\Gamma_{\rho^B}(L^B)t\right)\right] \cdot \|\phi^B - \mathbf{1}\|_{\hat{\pi}} \cdot \|\mathbf{1} - \rho^B\|_{\hat{\pi}} \\ &\quad + [\exp(-\Lambda(B)t)] \cdot \left|\int_B \phi^B\, d\hat{\pi} - 1\right|.\end{aligned}$$

Then, by Lemma 4.11 and Theorem 2.7(ii) and (iii), the last estimate is less than or equal to $\beta(B,t)\exp(-\Lambda(B)t)$, where $\beta(B,t)$ is given by

$$\begin{aligned}&\frac{4\kappa_1\kappa_2}{(\mathrm{Gap}(L) - \bar{\kappa})^2 - 4\kappa_1\kappa_2} \\ &\quad \times \left\{1 + \frac{\sqrt{(\mathrm{Gap}(L) - \bar{\kappa})^2 + 4\kappa_2^2}}{\mathrm{Gap}(L) - \bar{\kappa}} \exp([\Lambda(B) - (\mathrm{Gap}(L) - \varepsilon(B))]t)\right\} \\ &\le \beta(B),\end{aligned}$$

provided $\Lambda(B) + \varepsilon_0(B) \le \mathrm{Gap}(L)$. Again, this is guaranteed if (2.7) holds, by Theorem 2.7(i) □

PROOF OF COROLLARY 2.10. For $\theta \ge 0$,

$$\begin{aligned}&|E_{\hat{\pi}}[\exp(-\theta\Lambda(B)\tau)] - (1 + \theta)^{-1}| \\ &\quad = \left|\theta\int_0^\infty e^{-\theta t} P_{\hat{\pi}}(\Lambda(B)\tau \le t)\, dt - \theta\int_0^\infty e^{-\theta}[1 - e^{-t}]\, dt\right| \\ &\quad \le \theta \cdot \beta(B)\int_0^\infty e^{-(\theta+1)t}\, dt \quad \text{by Theorem 2.8} \\ &\quad = [\theta/(\theta + 1)]\beta(B) \to 0\end{aligned}$$

as $\pi(B^c) \to 0$. By the continuity theorem for Laplace transforms, we obtain the first part of the corollary. For the second part,

$$\begin{aligned} |E_{\hat{\pi}}\tau - \Lambda(B)^{-1}| &= \left| \int_0^\infty P_{\hat{\pi}}(\tau > t)\, dt - \int_0^\infty e^{-\Lambda(B)t}\, dt \right| \\ &\le \beta(B) \int_0^\infty e^{-\Lambda(B)t}\, dt, \quad \text{by Theorem 2.8} \\ &= \beta(B)/\Lambda(B). \end{aligned}$$

□

APPENDIX

Proofs of lemmas in Section 2. In this appendix we give the proofs of Lemmas 2.2, 2.3, 2.4 and 2.5. We begin with three preparatory lemmas. We assume the notation introduced in Section 2; in particular, we assume the standing hypotheses stated after Definition 2.6.

LEMMA A.1. *Let $(\mathcal{T}_t;\, t \ge 0)$ be a contraction semigroup on $L^2(\pi)$ such that, for all $t \ge 0$, $\mathcal{T}_t(\mathscr{B}(S)) \subset \mathscr{B}(S)$, and the restriction of $(\mathcal{T}_t;\, t \ge 0)$ to $\mathscr{B}(S)$ is weakly continuous (in the bounded-pointwise topology). Then $(\mathcal{T}_t;\, t \ge 0)$ is strongly continuous on $L^2(\pi)$. In particular, the semigroups $(T_t;\, t \ge 0)$ and $(T_t^B;\, t \ge 0)$, associated with the underlying and killed processes, are strongly continuous on $L^2(\pi)$ and $L^2(\hat{\pi})$, respectively.*

PROOF. Let $f \in \mathscr{B}(S)$. Then by hypothesis there exists a $\delta_f > 0$ such that $\forall\, x \in S$, $\lim_{t \to 0^+} \mathcal{T}_t f(x) = f(x)$ and $\sup_{0 \le t \le \delta_f} \|\mathcal{T}_t f - f\|_\infty < \infty$. Therefore, by the bounded convergence theorem

$$\|\mathcal{T}_t f - f\|_\pi^2 = \int_S |\mathcal{T}_t f(x) - f(x)|^2 \pi(dx) \to 0 \quad \text{as } t \to 0^+.$$

If now $f \in L^2(\pi)$, set $f_n = f\mathbf{1}_{\{|f| \le n\}} + n\mathbf{1}_{\{f > n\}} - n\mathbf{1}_{\{f < -n\}}$. Then $f \in \mathscr{B}(S)$ and $f_n \to_{L^2} f$, as $n \to \infty$, by the dominated convergence theorem. Then

$$\begin{aligned} \|\mathcal{T}_t f - f\|_\pi &\le \|\mathcal{T}_t [f - f_n]\|_\pi + \|\mathcal{T}_t f_n - f_n\|_\pi + \|f_n - f\|_\pi \\ &\le 2\|f_n - f\|_\pi + \|\mathcal{T}_t f_n - f_n\|_\pi, \end{aligned}$$

$$\limsup_{t \downarrow 0} \|\mathcal{T}_t f - f\|_\pi \le 2\|f_n - f\|_\pi \to 0 \quad \text{as } n \to \infty.$$

The application to $(T_t;\, t \ge 0)$ and $(T_t^B;\, t \ge 0)$ is justified by Theorem 2.1. Note that both semigroups do act on their respective L^2-spaces because π is an invariant probability. Indeed, with $A = S$ (respectively, B) and $\zeta = +\infty$ (respectively, τ^B), if $f, g\colon A \to \mathbf{R}$ are measurable, $f = g$ π-a.e., and $\int_A f^2\, d\pi < \infty$ then $T_t^A f = T_t^A g$ π-a.e. ($T_t^S \equiv T_t$); for

$$\int_A T_t^A\left[(f-g)^\pm\right] d\pi \le \int_S T_t^S\left[(\bar{f} - \bar{g})^\pm\right] d\pi = \int_S (\bar{f} - \bar{g})^\pm\, d\pi = 0$$

where $\bar{f}, \bar{g}$ denote the zero-extensions of f and g, respectively, to all of S.

The fact that $T_t^A f \in L^2(A, \pi|A)$, with $\|T_t^A f\|_\pi \le \|f\|_\pi$, follows from the Cauchy–Schwarz inequality. □

As noted after Theorem 2.1, the infinitesimal generator $-L$ of $(T_t; t \ge 0)$ on $L^2(\pi)$ is a densely defined, closed operator, and, moreover, $\mathscr{D}_w(\mathscr{L}) \subset \mathscr{D}(L)$, the domain of L, with $L|\mathscr{D}_w(\mathscr{L}) = \mathscr{L}$.

LEMMA A.2. *$\int\int J(x, dy) f(y)\pi(dx) = \int\int J(x, dy) f(x)\pi(dx) \equiv \int J(x) f(x)\pi(dx)$ for all measurable, nonnegative f.*

PROOF. If $f \in \mathscr{B}(S)$, then $\forall\, t \ge 0$, $\int T_t f(x)\pi(dx) = \int f(x)\pi(dx)$. If further $f \in \mathscr{D}_w(\mathscr{L})$, then $\int \mathscr{L} f(x)\pi(dx) = 0$ (by differentiating $d/dt|_{0^+}$ and using the bounded convergence theorem). By Theorem 2.1 and our integrabililty assumption on J, we can rearrange the last equality to yield the conclusion of the lemma in this case (with both sides being finite).

For general $f \in \mathscr{B}(S)$, $\exists\, f_n \in \mathscr{D}_w(\mathscr{L})$ with $f_n \rightharpoonup f$, as $n \to \infty$ (i.e., pointwise and boundedly). Applying the lemma to f_n and letting $n \to \infty$, using the bounded convergence theorem, yields the desired conclusion for f. Finally, if $f \ge 0$ and measurable, we can apply the previous result to $f \wedge n$ and let $n \to \infty$; by the monotone convergence theorem we obtain the lemma as stated. □

We have the following preliminary result for L^B, defined at (2.3).

LEMMA A.3. *L^B is a bounded operator, of norm $\le 2M(B)$, on $L^2(B, \pi)$.*

PROOF. Let $\bar{f}$ denote the zero extension of f to all of S. Then $L^B f(x) = J(x) f(x) - \int_S J(x, dy)\bar{f}(y)$. Clearly, $\|J \cdot f\|_\pi \le M(B) \cdot \|f\|_\pi$. By Cauchy–Schwarz, our boundedness assumption on J and Lemma A.2 (applied to $\bar{f}^2$),

$$\begin{aligned}\int_B \left[\int_S J(x, dy) \mathbf{1}_B(y) \bar{f}(y)\right]^2 \pi(dx) &\le M(B) \cdot \int_S \int_S J(x, dy)\, \bar{f}(y)^2 \pi(dx)\\ &= M(B) \cdot \int_S J(x)\, \bar{f}(x)^2 \pi(dx)\\ &\le M(B)^2 \cdot \int_B f(x)^2 \pi(dx).\end{aligned}$$

Note that L^B is well defined on equivalence classes. Indeed, if $h \ge 0$ and $h = 0$ π-a.e., then for π-a.e. $x \in B$:

$$L^B h(x) = \int_S \left[\bar{h}(x) - \bar{h}(y)\right] J(x, dy) = -\int_S \bar{h}(y) J(x, dy);$$

so by Lemma 6.2,

$$\int_B |L^B h(x)| \pi(dx) \le \int_S \int_S \bar{h}(y) J(x, dy) \pi(dx) = \int_S \bar{h}(x) J(x) \pi(dx) = 0.$$

Therefore $L^B h = 0$ π-a.e. Thus if $f = g$ π-a.e., we can apply the latter result to the positive and negative parts, $h = (f - g)^{\pm}$, to obtain $L^B f = L^B g$ π-a.e. □

PROOF OF LEMMA 2.2. Denote by $-\hat{\mathscr{L}}$ the weak infinitesimal generator of $(T_t^B;\, t \ge 0)$. By Theorem 2.1, $\hat{\mathscr{L}}$ is the restriction of L^B to $\mathscr{D}_w(\hat{\mathscr{L}}) \subset \mathscr{B}(B)$. By the bounded convergence theorem,

$$\int_B \left[\frac{T_t^B f(x) - f(x)}{t} + L^B f(x) \right]^2 \pi(dx) \to 0 \quad \text{as } t \to 0+.$$

Thus the infinitesimal generator, $-\hat{L}$ say, of $(T_t^B;\, t \ge 0)$ on $L^2(B, \pi)$ coincides with $-L^B$ on $\mathscr{D}_w(\hat{\mathscr{L}})$. Since $\hat{L}$ is closed, L^B is continuous (by Lemma A.3) and $\mathscr{D}_w(\hat{\mathscr{L}})$ is weakly dense in $\mathscr{B}(B)$ [see, e.g., Dynkin (1965), 1.15 B, page 40], and hence strongly dense in $L^2(B, \pi)$, then $\mathscr{D}(\hat{L}) = L^2(B, \pi)$ and $\hat{L} = L^B$. The bound on $\|L^B\|_\pi$ was derived in Lemma A.3. □

PROOF OF LEMMA 2.3. Let $A \in \mathscr{S} \cap B$. Then

$$\begin{aligned} \mu(A) &= \int_{B^c} \int_S \mathbf{1}_A(y) J(x, dy) \pi(dx) \\ &\le \int_S \int_S \mathbf{1}_A(y) J(x, dy) \pi(dx) \\ &= \int_S \int_S \mathbf{1}_A(x) J(x, dy) \pi(dx) \quad \text{by Lemma 6.2} \\ &\le M(B) \cdot \pi(A). \end{aligned}$$

□

PROOF OF LEMMA 2.4. For $f \in L^2(\hat{\pi})$, denote by $\bar{f}$ the zero extension of f off B. Then substituting $\bar{f}$ into Lemma 6.2 and rearranging yields

$$\int_B \int_B J(x, dy) f(x) \pi(dx) - \int_B \int_B J(x, dy) f(y) \pi(dx)$$
$$+ \int_B \int_{B^c} J(x, dy) f(x) \pi(dx) = \int_{B^c} \int_B J(x, dy) f(y) \pi(dx),$$

that is,

$$\int_B L^B f(x) \pi(dx) = \int_B f(y) R^B(y) \pi(dy).$$

Dividing by $\pi(B)$ yields the first result. The remaining results are evident from the facts $L^B \mathbf{1} = K^B$ and $(L^B)^* \mathbf{1} = R^B$. □

PROOF OF LEMMA 2.5. Substitute $f = \mathbf{1}_{B^c}$ into Lemma A.2. After some rearranging this yields

$$\int_B J(x, B^c)\pi(dx) = \int_{B^c} J(x, B)\pi(dx).$$

The estimate in (i) then follows. Estimate (ii) follows from

$$\kappa_1^2 = \int_B [K^B - \bar{\kappa}]^2 \, d\hat{\pi} = \int_B [K^B]^2 \, d\hat{\pi} - \bar{\kappa}^2$$

$$\leq M(B) \cdot \int_B K^B \, d\hat{\pi} \equiv M(B)\bar{\kappa}.$$

Estimate (iii) is similar. □

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DEPARTMENT OF MATHEMATICS AND STATISTICS
MCMASTER UNIVERSITY
HAMILTON, ONTARIO L8S 4K1
CANADA

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OTTAWA
585 KING EDWARD
OTTAWA, ONTARIO K1N 6N5
CANADA