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MULTIDIMENSIONAL REACTION–DIFFUSION EQUATIONS WITH WHITE NOISE BOUNDARY PERTURBATIONS[1]

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In this paper we study multidimensional stochastic reaction–diffusion equations (SRDE's) with white noise boundary data. More precisely, we consider a general SRDE with Robin data to be a white noise field. Because this boundary data is very irregular, we formulate a set of conditions that a random field must satisfy to solve the SRDE. We show that a unique solution exists, and we study the boundary-layer behavior of the solution. This boundary-layer analysis reveals some natural restrictions on the reaction term of the SRDE that ensure that the reaction term does not qualitatively affect the boundary layer. The boundary-layer analysis also leads to the definition of some functional Banach spaces into which are encoded the boundary-layer degeneracies and that would be the natural settings for other analyses of the SRDE of this paper (e.g., large deviations and central limit theorems, approximation theorems).

Introduction. In this paper we study the stochastic reaction–diffusion equation (SRDE)

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u + (b, \nabla u) + cu + f(x,u), \\ u(0,\cdot) &= u_0, \\ (\nu, \nabla u) + \beta(x)u|_{\mathbb{R}_+ \times \partial M} &= \sigma(x)\zeta, \end{aligned} \tag{I.1}$$

where the space variable takes values on some n-dimensional ($n \geq 2$) Riemannian manifold M with boundary and where ζ is a space–time white noise field on $\mathbb{R}_+ \times \partial M$. We use ν to denote the inward-pointing normal vector field on ∂M. Our aim in studying (I.1) is twofold. First, we want to find a natural way to interpret the SRDE (I.1) and show that there is a unique solution for this interpretation. Second, we are interested in the structure of the solutions, particularly near the boundary. Because ζ is a generalized function, the solution of (I.1) is degenerate near the boundary, and we would like to have a basic understanding of these degeneracies.

This work is essentially an extension of the efforts of Freidlin and Wentzell, who in [10] considered the problem for $n = 1$ (see also [9]). We shall see that

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in the multidimensional case considered here, the boundary behavior is more intricate than in [10]. Another difference is that our study of the boundary layer will stem almost entirely from a statistical analysis, whereas in the one-dimensional case of [10], there was a natural integration by parts that $\mathbb{P}$-a.s. transformed the white noise boundary data into continuous functions. Related efforts that lead in a slightly different direction are in [7].

The organization of this paper is as follows. Our assumptions and notation will be given in the next section. Then in Section 2 we shall see exactly how we may understand (I.1). We shall also see that locally ζ is a generalized function that may be thought of as the n-fold derivative of a continuous but nondifferentiable function. Thus, the understanding of (I.1) must define the notion of a weak solution that may have boundary-layer degeneracies. Section 3 is devoted to briefly recalling some facts about the fundamental solution of (I.1). This leads to the efforts of Sections 4 and 5, where we study the simpler linear case of (I.1) when $f \equiv 0$. In Section 4 we show that this simplified version of (I.1) has a unique solution that may be described by a natural integration of Section 3's fundamental solution against the boundary data $\sigma\zeta$. In Section 5 we study the boundary layer of this solution. We finally return in Section 6 to the fully nonlinear SRDE (I.1) and bring everything together by describing the solution of (I.1) as a nonlinear transformation of the solution of the linear SRDE of Sections 4 and 5. The interaction of this functional transformation with the boundary layer is studied by encoding the boundary-layer degeneracies into certain functional Banach spaces and then interpreting this functional transformation as a mapping between such Banach spaces.

1. Assumptions and notation. In this paper, M is an n-dimensional, compact, connected, Riemannian C^∞ manifold with smooth boundary. We denote by ∂M the boundary of M and $M^\circ := M \sim \partial M$ is the interior of M. We denote the Riemannian metric tensor by $(\cdot,\cdot)$, and the associated fiber metric on TM by $\|\cdot\|$. Also associated with $(\cdot,\cdot)$ are the Riemannian volume element, denoted by α_0, and the corresponding volume measure α on $(M, \mathcal{B}(M))$. The gradient operator defined by $(\cdot,\cdot)$ is denoted by ∇ and the adjoint of $-\nabla$ with respect to α is the divergence operator, div, and $\Delta := \operatorname{div} \nabla$ is the Laplace–Beltrami operator (our sign conventions follow [8]). The inward-pointing unit normal vector field on ∂M is denoted by ν, as we already mentioned (note that in [8], ν denotes the unit normal *outward*-pointing vector). In (I.1), b is a C^∞ vector field on M, c and β are some C^∞ functions on M and ∂M, respectively, and u_0 is some continuous function on M. We will discuss the nonlinearity f later in Section 6. The distance function on M defined by $(\cdot,\cdot)$ will be denoted by $d(\cdot,\cdot)$ and for any subset S of M and any point x in M, $\operatorname{dist}(x,S) := \inf_{y\in S} d(x,y)$ as usual. We will also let $\operatorname{rad}(M)$ denote the radius of M.

Although the results of this paper will be phrased in a geometric-invariant language, some of our intuition and some of our more delicate calculations will come from looking at, respectively, $\mathbb{R}^n$ or the n-dimensional Euclidean half-space, which are by definition models for open neighborhoods, which are, respectively, in M° or near ∂M. We denote the half-space as

$$H_n := \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

and, consistent with our previous notation, we let ∂H_n be its boundary and H_n° be its interior. We always assume that H_n is endowed with the standard Euclidean metric. The Laplace–Beltrami operator on H_n is

$$\widehat{\Delta} := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}. \tag{1.1}$$

We shall denote by $\|\cdot\|_{\mathbb{R}^n}$ and $\|\cdot\|_{\mathbb{R}^{n-1}}$ the Euclidean norms on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively. Whereas the canonical local diffeomorphisms between neighborhoods in M° and $\mathbb{R}^n$ are given by normal coordinates, the canonical local diffeomorphisms between open neighborhoods of M near ∂M and H_n are *Fermi* coordinates (see [14]).

We will also use some of the geometry of ∂M. The volume element α_0 on M and the volume measure α on $(M, \mathcal{B}(M))$ naturally induce a volume element $\overline{\alpha}_0$ on ∂M and a volume measure $\overline{\alpha}$ on $(\partial M, \mathcal{B}(\partial M))$. The Riemannian metric $(\cdot,\cdot)$ also induces a Riemannian metric tensor $(\cdot,\cdot)^-$ on ∂M, under which ∂M is itself a Riemannian manifold. Unfortunately, the fact that ∂M is not necessarily connected (consider an annulus) means that we cannot immediately define a distance function on ∂M from $(\cdot,\cdot)^-$. We must first break ∂M into its connected components, which we shall denote by $\{\partial_i M: i = 1, 2, \dots, \bar{i}\}$ (because M is compact, $\bar{i}$ is finite) and then use $(\cdot,\cdot)^-$ to separately define a distance function $\overline{d}_i$ on each $\partial_i M$. Each $\partial_i M$ has its own injectivity radius, $\overline{\varepsilon_{injr,i}}$, which must be positive because $\partial_i M$ is compact, and its own $\overline{d}_i$-radius, $\overline{\mathrm{rad}}_i(\partial_i M)$, which must similarly be finite. We may then define

$$\overline{\varepsilon_{injr}} := \min_{1 \leq i \leq \bar{i}} \overline{\varepsilon_{injr,i}} \quad \text{and} \quad \overline{\mathrm{rad}}(\partial M) := \max_{1 \leq i \leq \bar{i}} \overline{\mathrm{rad}}_i(\partial_i M).$$

Our choice of the notation $\overline{\alpha}_0$, $\overline{\alpha}$, $(\cdot,\cdot)^-$ and so forth reflects our convention that objects associated with an $n-1$-dimensional manifold, be it $\mathbb{R}^{n-1}$ or ∂M, have overbars, whereas objects associated with n-dimensional manifolds, such as $\mathbb{R}^n$ or M, do not have overbars.

As a final requirement, we assume that there is an underlying probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ on which all random variables are defined.

2. The meaning of the SRDE. We now develop the meaning of the SRDE (I.1). We begin by using the volume measure $\overline{\alpha}$ and Lebesgue measure on $\mathbb{R}_+$ to give a "natural" meaning of the space–time white noise field ζ. We show that $\mathbb{P}$-a.s., ζ may be understood as a classical generalized function on $\mathbb{R}_+ \times \partial M$. Thus the solution of (I.1) should be degenerate near the boundary, so we must give an appropriate weak formulation of (I.1). We finish the section by listing the resulting conditions that a random field must satisfy to "solve" (I.1).

To start, we define ζ in a fairly standard way (see [3, 16]) as a Gaussian additive set function; this is appropriate if we view ζ as the effect of a large number of independent random particles hitting the boundary. We need to fix some sigma-finite measure μ on $(\mathbb{R}_+ \times \partial M, \mathcal{B}(\mathbb{R}_+ \times \partial M))$ and then take ζ to be a random set function on $\mathcal{B}(\mathbb{R}_+ \times \partial M)$ such that:

wn. For any finite number of sets $A_1, A_2, \dots, A_m$ in $\mathcal{B}(\mathbb{R}_+ \times \partial M)$ of finite μ-measure, $\zeta(A_1), \zeta(A_2), \dots, \zeta(A_m)$ are jointly Gaussian random variables with means zero and covariances $\mathbb{E}[\zeta(A_i)\zeta(A_j)] = \mu(A_i \cap A_j)$ for all $1 \le i, j \le m$.

By standard results [3], such a random set function ζ does in fact exist. A natural choice of the measure μ in our case is the product measure $\mu = \mathrm{Leb} \times \overline{\alpha}$, where Leb is the Lebesgue measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. Thus we define ζ:

DEFINITION 2.1. The *white noise field* ζ is a random set function on $\mathcal{B}(\mathbb{R}_+ \times \partial M)$ satisfying *wn* with $\mu = \mathrm{Leb} \times \overline{\alpha}$.

Although our decision to use the measure $\overline{\alpha}$ on $(\partial M, \mathcal{B}(\partial M))$ may appear somewhat arbitrary, this only slightly limits the generality of the SRDE (I.1). If we want to use $\mu = \mathrm{Leb} \times \tilde{\alpha}$, where $\tilde{\alpha}$ is any other finite measure on $(\partial M, \mathcal{B}(\partial M))$ that is absolutely continuous with respect to $\overline{\alpha}$, we may simply use ζ as in Definition 2.1 and replace σ in (I.1) by $\sigma\sqrt{d\tilde{\alpha}/d\overline{\alpha}}$ (we should also then require that $\sqrt{d\tilde{\alpha}/d\overline{\alpha}}$ be smooth).

Some quick estimates show that ζ, as given by Definition 2.1, may also be locally treated as the signed variation of a continuous, but nondifferentiable function (see also [16]). If we fix a $T > 0$ and take any oriented chart (φ, U) on ∂M such that $\varphi(U) = \{\overline{x} \in \mathbb{R}^{n-1}: \|\overline{x}\|_{\mathbb{R}^{n-1}} \le 1\}$, then we may define

$$\widehat{W}(t, \overline{x}) := \zeta\Big([0,t] \times \varphi^{-1}([0, \overline{x}])\Big) \tag{2.1}$$

for all $0 \le t \le T$ and all $\overline{x}$ in $\mathbb{R}^{n-1}$ such that $\|\overline{x}\|_{\mathbb{R}^{n-1}} \le 1$. Here $[0, \overline{x}]$ is the rectangle in $\mathbb{R}^{n-1}$ having 0 and $\overline{x}$ as opposite vertices. For any $0 \le s \le t \le T$ and such $\overline{x} = (x_1, x_2, \dots, x_{n-1})$ and $\overline{y} = (y_1, y_2, \dots, y_{n-1})$,

$$\begin{aligned}
&\mathbb{E}\big[|\widehat{W}(t,\overline{x}) - \widehat{W}(s,\overline{y})|^2\big] \\
&\quad = (\mathrm{Leb} \times \overline{\alpha})\Big(\Big([0,t] \times \varphi^{-1}([0,\overline{x}])\Big) \triangle \Big([0,s] \times \varphi^{-1}([0,\overline{y}])\Big)\Big) \\
&\quad \le \mathrm{Leb}\big([0,t] \triangle [0,s]\big)\overline{\alpha}(\partial M) + T\overline{\alpha}\Big(\varphi^{-1}([0,\overline{x}]) \triangle \varphi^{-1}([0,\overline{y}])\Big) \\
&\quad \le \overline{\alpha}(\partial M)|t-s| + T\overline{K}\left(\sum_{i=1}^{n-1} |x_i - y_i|\right),
\end{aligned}$$

where $\triangle$ denotes the symmetric difference set operation and

$$\overline{K} := \sup\big\{\|\overline{\alpha}_0(\overline{x})\|_{\Lambda^{n-1}(\partial M)}: \overline{x} \in U,\ \|\varphi(\overline{x})\|_{\mathbb{R}^{n-1}} \le 1\big\},$$

where $\|\cdot\|_{\Lambda^{n-1}(\partial M)}$ is the standard fiber metric on $\Lambda^{n-1}(\partial M)$, the vector bundle of $n-1$-degree exterior forms on ∂M (see [1], Proposition 6.2.11). By the celebrated Garsia, Rodemich and Rumsey bounds [2, 12, 16], we can then show that for every $\eta > 0$, there is a version of $\widehat{W}$ that is Hölder continuous of exponent $\frac{1}{2} - \eta$ jointly in all of its arguments. By using the additivity of ζ and (2.1), we can then see that for any rectangle in $\{\bar{x} \in \mathbb{R}^{n-1}: \|\bar{x}\|_{\mathbb{R}^{n-1}} \leq \frac{1}{2}\}$ and any $0 \leq s \leq t \leq T$, $\zeta([s,t] \times \varphi^{-1}(A))$ is equal to the variation of $\widehat{W}$ over $[s,t] \times A$, which heuristically may be written as

$$\zeta\big([s,t] \times \varphi^{-1}(A)\big) = \int_{[s,t] \times A} \frac{\partial^n \widehat{W}}{\partial t\, \partial x_1\, \partial x_2 \cdots \partial x_{n-1}} dx_1\, dx_2 \cdots dx_{n-1}.$$

To see that this expression is only heuristic and cannot be made rigorous, we may calculate that the variation of $\widehat{W}$ along any axis is like that of a Wiener process. For any fixed $\bar{x}$ in $\mathbb{R}^{n-1}$ such that $\|\bar{x}\|_{\mathbb{R}^{n-1}} \leq \frac{1}{2}$, $s \mapsto \widehat{W}(s,\bar{x})$ is a Wiener process on $[0,T]$, as a simple covariance analysis reveals. Second, for any fixed $0 < t \leq T$, any fixed $(x_1, x_2, \ldots, x_{n-2})$ in $\mathbb{R}^{n-2}$ such that $\|(x_1, x_2, \ldots, x_{n-2})\|_{\mathbb{R}^{n-2}} \leq \frac{1}{2}$ and any fixed i in $\{1, 2, \ldots, n-1\}$, a simple time-change argument shows that

$$\widehat{W}\big(t, (x_1, x_2, \ldots, x_{i-1}, y, x_i, \ldots, x_{n-2})\big) = \widetilde{W}\big(\tau(y)\big)$$

for y small enough, where $\widetilde{W}$ is some Wiener process and τ is some increasing and perhaps random mapping from $\mathbb{R}_+$ to itself that satisfies $\underline{K} y \leq \tau(y) \leq \overline{K} y$ for all y small enough. Here $\underline{K}$ is some positive constant that reflects the fact that $\overline{\alpha}_0$ is nondegenerate.

Returning now to (I.1), we see from the foregoing analysis that the boundary data ζ are very irregular, which forces us to formulate (I.1) in a weak sense. To simplify the notation, we define the second-order operator

$$\mathcal{L}\varphi := \tfrac{1}{2}\Delta\varphi + (b, \nabla\varphi) + c\varphi, \qquad \varphi \in C^\infty(M), \tag{2.2}$$

and its adjoint (with respect to α)

$$\mathcal{L}^*\varphi := \tfrac{1}{2}\Delta\varphi - (b, \nabla\varphi) + (c - \operatorname{div} b)\varphi, \qquad \varphi \in C^\infty(M).$$

We similarly define two first-order operators on functions that are differentiable at the boundary:

$$\begin{aligned} \overline{\mathcal{L}}\varphi &:= (\nu, \nabla\varphi) + \beta\varphi, \\ \overline{\mathcal{L}}^*\varphi &:= (\nu, \nabla\varphi) + \big(\beta - 2(b,\nu)\big)\varphi, \qquad \varphi \in C^\infty(M). \end{aligned} \tag{2.3}$$

The relevant Green's formula is thus

$$\begin{aligned} &\int_{x \in M} g(x)(\mathcal{L}h)(x)\alpha(dx) + \tfrac{1}{2}\int_{\bar{x} \in \partial M} g(\bar{x})(\overline{\mathcal{L}}h)(\bar{x})\overline{\alpha}(d\bar{x}) \\ &\quad = \int_{x \in M} h(x)(\mathcal{L}^*g)(x)\alpha(dx) + \tfrac{1}{2}\int_{\bar{x} \in \partial M} h(\bar{x})(\overline{\mathcal{L}}^*g)(\bar{x})\overline{\alpha}(d\bar{x}), \end{aligned}$$

which holds for all g and h in $C^\infty(M)$. We will also use the collection of test functions

$$C_c^\infty(\mathbb{R}_+ \times M) := \{\varphi \in C^\infty(\mathbb{R}_+ \times M)\text{: supp } \varphi \text{ is compact}\}$$

with supp φ denoting the support of φ. The weak formulation of (I.1) is then given by an integration by parts of u against any such test function. This would imply that at least formally,

$$\begin{aligned}
&\int_{t=0}^{\infty}\int_{x\in M^\circ} u(t,x)\left(\frac{\partial\varphi}{\partial t} + \mathcal{L}^*\varphi\right)(t,x)\alpha(dx)\,dt \\
&\qquad + \int_{t=0}^{\infty}\int_{x\in M^\circ} f\big(x,u(t,x)\big)\varphi(t,x)\alpha(dx)\,dt \\
&\qquad = -\int_{x\in M} u_0(x)\varphi(0,x)\alpha(dx) - \frac{1}{2}\int_{t=0}^{\infty}\int_{\bar{x}\in\partial M} u(t,\bar{x})(\overline{\mathcal{L}}^*\varphi)(t,\bar{x})\overline{\alpha}(d\bar{x})\,dt \\
&\qquad + \frac{1}{2}\int_{t=0}^{\infty}\int_{\bar{x}\in\partial M}\varphi(t,\bar{x})\sigma(\bar{x})\zeta(dt,d\bar{x})
\end{aligned} \tag{2.4}$$

for any φ in $C_c^\infty(\mathbb{R}_+ \times \partial M)$. To make this rigorous, we need to define the last integral and we need to define the first two integrals so that they allow for the anticipated boundary-layer degeneracies of u.

The definition of the last integral in (2.4) is the easiest of these two problems. For any set A in $\mathcal{B}(\mathbb{R}_+ \times \partial M)$ of finite Leb $\times\ \overline{\alpha}$ measure, we define the *integral* of χ_A by

$$\int_{t=0}^{\infty}\int_{\bar{x}\in\partial M}\chi_A(t,\bar{x})\zeta(dt,d\bar{x}) := \zeta(A).$$

We may then define, in the obvious way, integrals of linear combinations of such simple functions and then complete the definition, again in the obvious way (see [16]) by using the natural isometric mapping of $L^2(\mathbb{R}_+ \times \partial M)$ into $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Slightly more difficult is the question of what to do with the first, second and fourth integrals in (2.4). The biggest problem is the fourth integral, that is, the $\overline{\alpha}(dx)\,dt$ integral, because u might not even exist on the boundary. Because u will exist, however, inside M°, the next best thing might be to "shift" the boundary in some orderly manner into M°. The presence of a Riemannian distance function on M suggests that for each $\varepsilon > 0$ we shrink M to

$$M_\varepsilon := \{x \in M\text{: dist}(x, \partial M) > \varepsilon\}, \qquad \varepsilon > 0,$$

and then somehow replace M by M_ε and ∂M by ∂M_ε in the first three terms of (2.4). We then wish to define these integrals as limits, if they exist, of these ε-integrals. Clearly the first two limits will exist if and only if the following two limits exist for any arbitrarily chosen φ in $C_c^\infty(\mathbb{R}_+ \times \partial M)$:

$$\begin{aligned}(2.5)\qquad &\lim_{\varepsilon \to 0} \int_{t=0}^{\infty} \int_{x \in M_\varepsilon} u(t,x)\varphi(t,x)\alpha(dx)\,dt,\\ &\lim_{\varepsilon \to 0} \int_{t=0}^{\infty} \int_{x \in M_\varepsilon} f\big(x, u(t,x)\big)\varphi(t,x)\alpha(dx)\,dt, \qquad \mathbb{P}\text{-a.s.}\end{aligned}$$

Taking up next the question of the fourth integral of (2.4), we would like to change the region of integration to $\mathbb{R}_+ \times \partial M_\varepsilon$. Unfortunately, this is not directly possible because $(\overline{\mathcal{L}}^*\varphi)(t,x)$ is defined only for x in ∂M. We must instead look at $(\overline{\mathcal{L}}^*\varphi)(t,\overline{x})$ for $\overline{x}$ in ∂M but $u(t,\overline{y})$ for $\overline{y}$ in ∂M_ε, where $\overline{x}$ and $\overline{y}$ are related by some shift operation (or more precisely, a "tubular neighborhood" [6]). Once again, the Riemannian metric on M gives us a natural tool, this time in the form of the exponential map, which we shall denote by E. By standard results ([6] or [14], Chapter 2.3)], there is an $\varepsilon_{\text{tub}} > 0$ such that the mapping

$$\theta_\varepsilon(\overline{x}) := E\big(\varepsilon\nu(\overline{x})\big), \qquad \overline{x} \in \partial M,$$

is a diffeomorphism from ∂M to ∂M_ε for each $0 \le \varepsilon < \varepsilon_{\text{tub}}$. Furthermore, θ_0 is the identity map on ∂M. For convenience, we shall fix for the rest of this paper some $\varepsilon'_{\text{tub}}$ such that

$$0 < \varepsilon'_{\text{tub}} < \varepsilon_{\text{tub}}$$

and consider only ε in $[0, \varepsilon'_{\text{tub}}]$. This will avoid any problems stemming from degeneracies of the mapping θ_ε for ε near ε_{tub}. We thus see that an appropriate integrand for the modified fourth term of (2.4) should contain $(\overline{\mathcal{L}}^*\varphi)(t,\overline{x})$ and $u(t,\overline{y})$, where $\overline{x}$ and $\overline{y}$ are related by $\overline{y} = \theta_\varepsilon(\overline{x})$. Thus, we approximate the fourth term of (2.4) by

$$(2.6)\qquad \tfrac{1}{2} \int_{t=0}^{\infty} \int_{\overline{x} \in \partial M} u\big(t, \theta_\varepsilon(\overline{x})\big)(\overline{\mathcal{L}}^*\varphi)(t,\overline{x})\overline{\alpha}(d\overline{x})\,dt$$

for ε small. As we did with the first two terms of (2.4), we will define the fourth term of (2.4) by the limit, (if it exists) of (2.6) as ε tends to zero. Clearly this limit will exist if and only if

$$(2.7)\qquad \lim_{\varepsilon \to 0} \int_{t=0}^{\infty} \int_{\overline{x} \in \partial M} u\big(t, \theta_\varepsilon(\overline{x})\big)\varphi(t,\overline{x})\overline{\alpha}(d\overline{x})\,dt, \qquad \mathbb{P}\text{-a.s.},$$

exists for any arbitrarily chosen φ in $C_c^\infty(\mathbb{R}_+ \times \partial M)$.

We finally can give some rigorous conditions that a random field u on $\mathbb{R}_+ \times M^\circ$ must satisfy to be a weak solution of (I.1). We want some continuity:

a.1. The random field u is $\mathbb{P}$-a.s. continuous in $\mathbb{R}_+ \times M^\circ$.

We also need the limits (2.5) and (2.7) to exist:

a.2. The limits of (2.5) and (2.7) exist for each φ in $C_c^\infty(\mathbb{R}_+ \times M^\circ)$.

Third, we need (2.4) to be true if we replace the first, second and fourth integrals by the appropriate approximate integrals, namely, that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{t=0}^{\infty}\int_{x\in M_\varepsilon} u(t,x)\left(\frac{\partial\varphi}{\partial t}+\mathcal{L}^*\varphi\right)(t,x)\alpha(dx)\,dt\\
&\quad+\lim_{\varepsilon\to 0}\int_{t=0}^{\infty}\int_{x\in M_\varepsilon} f\big(x,u(t,x)\big)\varphi(t,x)\alpha(dx)\,dt\\
&\quad=-\int_{x\in M} u_0(x)\varphi(0,x)\alpha(dx)\\
&\quad-\lim_{\varepsilon\to 0}\frac{1}{2}\int_{t=0}^{\infty}\int_{\bar{x}\in\partial M} u\big(t,\theta_\varepsilon(\overline{x})\big)(\overline{\mathcal{L}}^*\varphi)(t,\overline{x})\overline{\alpha}(d\overline{x})\,dt\\
&\quad+\frac{1}{2}\int_{t=0}^{\infty}\int_{\bar{x}\in\partial M}\varphi(t,\overline{x})\sigma(\overline{x})\zeta(dt,d\overline{x}).
\end{aligned}\tag{2.8}
$$

This is the third condition:

a.3. For each φ in $C_c^\infty(\mathbb{R}_+\times M)$, (2.8) holds.

We give the following definition.

DEFINITION 2.2. We say that a random field u solves the problem (I.1) for the nonlinearity f, the initial condition u_0 and the boundary data ζ if conditions *a*.1–*a*.3 hold.

3. The Robin kernel. The natural place to start searching for solutions of (I.1) is the deterministic PDE,

$$
\frac{\partial v}{\partial t}=\mathcal{L}v+\psi,\qquad v(0,\cdot)=v_0,\qquad \overline{\mathcal{L}}v|_{\mathbb{R}_+\times\partial M}=\overline{\psi},\tag{3.1}
$$

for some functions ψ in $C^\infty(\mathbb{R}_+\times M)$, $\overline{\psi}$ in $C^\infty(\mathbb{R}_+\times\partial M)$ and v_o in $C^\infty(M)$. This corresponds to replacing $(t,x)\mapsto f(x,u(t,x))$ in (I.1) with ψ some smooth function of t and x, but which does not depend upon u, and smoothing off the random field $\sigma\zeta$. The solution of (3.1) can be explicitly given in terms of the *Robin kernel* p^N; for each y in M, p_y^R is a C^∞ function of (t,x) in $\mathbb{R}_+\times M^\circ\sim\{(0,y)\}$ that satisfies

$$
\frac{\partial p_y^R}{\partial t}=\mathcal{L}p_y^R,\qquad \lim_{t\to 0}p_y^R(t,\cdot)=\delta_y,\qquad \overline{\mathcal{L}}p_y^R|_{\mathbb{R}_+\times\partial M}=0,\tag{3.2}
$$

where all of the relevant derivatives exist and p_y^R and these derivatives are continuous on $\mathbb{R}_+\times M\sim\{(0,y)\}$. Here the middle equality is understood in the sense of distributions with δ_y being the Dirac measure on $(M,\mathcal{B}(M))$ concentrated at y, and the last equality is interpreted as $\overline{\mathcal{L}}p_y^R(t,x):=\lim_{x\to\bar{x},x\in M^\circ}(\widetilde{\nu}(x),\nabla p_y^R(t,x))+\widetilde{\beta}(x)p_y^R(t,x)$ for each $(t,\overline{x})$ in $\mathbb{R}_+\times\partial M$, where $\widetilde{\nu}$ and $\widetilde{\beta}$ are any extensions of ν and β into M°. The next result tells us that the solution of (3.2) exists and that it is regular when $x\neq y$:

PROPOSITION 3.1. *The Robin kernel p^R exists uniquely; that is, there is a unique solution in $C^\infty(\mathbb{R}_+ \times M^\circ \sim \{(0,y)\})$ for each y in M. Furthermore, the mapping $(t,x,y) \mapsto p_y^R(t,x)$ is C^∞ on $\mathbb{R}_+ \times M^\circ \times M \sim \{(0,x,x): x \in M\}$.*

We can use the Robin kernel and the next several results to prove that the unique solution of (3.1) is given by

$$\begin{aligned} v(t,x) = &\int_{s=0}^{t} \int_{y \in M^\circ} p_y^R(t-s,x)\psi(s,y)\alpha(dy)\,ds + \int_{y \in M^\circ} p_y^R(t,x)v_\circ(y)\alpha(dy) \\ &-\tfrac{1}{2}\int_{s=0}^{t}\int_{\bar{y} \in \partial M} p_{\bar{y}}^R(t-s,x)\overline{\psi}(s,\bar{y})\overline{\alpha}(d\bar{y})\,ds \end{aligned} \tag{3.3}$$

for all $t \geq 0$ and x in M.

To get a better idea of the behavior of p^R, which by (3.3) should play some role in our study of (I.1), we may consider a canonical model for (3.2):

$$\frac{\partial \widehat{p}^N_{(\bar{y},y_n)}}{\partial t} = \frac{1}{2}\widehat{\Delta}\widehat{p}^N_{(\bar{y},y_n)}, \qquad \lim_{t \to 0} \widehat{p}^N_{(\bar{y},y_n)}(t,\cdot) = \delta_{(\bar{y},y_n)}, \qquad \frac{\partial \widehat{p}^N_{(\bar{y},y_n)}}{\partial x_n}\big(t,(\bar{x},0)\big) = 0,$$

the solution of which (subject to the correct conditions on growth for x large) is

$$\begin{aligned} \widehat{p}^N_{(\bar{y},y_n)}\big(t,(\bar{x},x_n)\big) := &\exp\left[-\frac{\|\bar{x}-\bar{y}\|^2_{\mathbb{R}^{n-1}} + |x_n - y_n|^2}{2t}\right](2\pi t)^{-n/2} \\ &+ \exp\left[-\frac{\|\bar{x}-\bar{y}\|^2_{\mathbb{R}^{n-1}} + |x_n + y_n|^2}{2t}\right](2\pi t)^{-n/2} \end{aligned} \tag{3.4}$$

for all $t > 0$, $(\bar{x},x_n)$ and $(\bar{y},y_n)$ in H_n. Some of the theory showing that p^R, the solution of (3.2), may indeed be thought of as a perturbation of $\widehat{p}^N$ is given in Appendix A. In particular, we have the following proposition.

PROPOSITION 3.2. *Let K be any compact subset of M°. Then for all x and y in K such that $d(x,y) < \operatorname{dist}(x,\partial M)$,*

$$p_y^R(t,x) = \exp\left[-d^2(x,y)/(2t)\right](2\pi t)^{-n/2}\big(1+o(1)\big), \qquad t \to 0,$$

uniformly over all such x and y.

We also have the following results on the boundary behavior of p^R:

PROPOSITION 3.3. *Let $T > 0$ be some fixed time horizon. The Robin kernel p^R has the following behavior on $[0,T]$:*

(a) *There is a positive constant $\kappa_1(T) \leq 1$ such that*

$$\left|p_y^R(t,x)\right| \leq \kappa_1^{-1}(T)\exp\left[-\kappa_1(T)\frac{d^2(x,y)}{t}\right]t^{-n/2} \tag{3.5}$$

for all $0 < t \le T$, *x in* M° *and y in* M.

(b) *For every* $\lambda > 0$, *there is a positive constant* $\kappa_2(T,\lambda) \le 1$ *such that*

$$\begin{aligned}&\left\|\nabla p^R_{\bar y}(t,x)\right\| \\ &\quad \le \kappa_2^{-1}(T,\lambda)\exp\left[-\kappa_2(T,\lambda)\frac{d^2(x,\bar y)}{t}\right]t^{-n/2+\lambda/2-1/2}\big(\mathrm{dist}(x,\partial M)\big)^{-\lambda}\end{aligned} \tag{3.6}$$

for all $0 < t \le T$, *x in* M° *and $\bar y$ in* ∂M.

(c) *For every* $\lambda \ge 0$, *there is a positive constant* $\kappa_3(T,\lambda) \le 1$ *such that*

$$\begin{aligned}\left|\frac{\partial p^R_{\bar y}}{\partial t}(t,x)\right| &\le \kappa_3^{-1}(T,\lambda)\exp\left[-\kappa_3(T,\lambda)\frac{d^2(x,\bar y)}{t}\right] \\ &\quad \times t^{-n/2+\lambda-1}\big(\mathrm{dist}(x,\partial M)\big)^{-\lambda}\end{aligned} \tag{3.7}$$

for all $0 < t \le T$, *x in* M° *and $\bar y$ in* ∂M.

Analogous bounds for $\widehat{p}^N$ are easy to prove. Although one could take $\lambda = 0$ in claim (b) in these analogous bounds for $\widehat{p}^N$, our method of proof in Appendix A is not refined enough to do so for p^R. Also, although the results of Proposition 3.3 give the natural generalization of the behavior of $\widehat{p}^N$, the formulae (3.5)–(3.7) are not the easiest to deal with, because the dependences on t and x are so closely intertwined. A more useful collection of standard bounds may be found by recalling that

$$C(\lambda) := \sup_{x \ge 0} x^{\lambda/2}e^{-x} \tag{3.8}$$

is finite for every $\lambda > 0$. We then have (see [8], Chapter 7) the following proposition.

PROPOSITION 3.4. *Let $T > 0$ be some fixed time horizon. The Robin kernel p^R has the following behavior on* $[0,T]$:

(a) *For every pair of nonnegative constants* $\lambda_1 \le n$ *and* $\lambda_2 \ge 0$, *there is a positive constant* $\kappa_1'(T,\lambda_1,\lambda_2)$ *such that*

$$\left|p^R_{\bar y}(t,x)\right| \le \kappa_1'(T,\lambda_1,\lambda_2)t^{(\lambda_2-\lambda_1)/2}d^{-n+\lambda_1}(x,\bar y)\big(\mathrm{dist}(x,\partial M)\big)^{-\lambda_2} \tag{3.9}$$

for all $0 < t \le T$, *x in* M°, *and $\bar y$ in* ∂M.

(b) *For every pair of nonnegative constants* $\lambda_1 \le n$ *and* $\lambda_2 > 0$, *there is a positive constant* $\kappa_2(T,\lambda_1,\lambda_2)$ *such that*

$$\begin{aligned}\left\|\nabla p^R_{\bar y}(t,x)\right\| &\le \kappa_2'(T,\lambda_1,\lambda_2)t^{(\lambda_2-\lambda_1-1)/2}d^{-n+\lambda_1}(x,\bar y) \\ &\quad \times \big(\mathrm{dist}(x,\partial M)\big)^{-\lambda_2}\end{aligned} \tag{3.10}$$

for all $0 < t \le T$, *x in* M° *and $\bar y$ in* ∂M.

(c) *For every pair of constants $\lambda_1 \le n$ and $\lambda_2 \ge 0$, there is a positive constant $\kappa_3(T, \lambda_1, \lambda_2)$ such that*

$$\left|\frac{\partial p_{\bar{y}}^R}{\partial t}(t,x)\right| \le \kappa_3'(T,\lambda_1,\lambda_2)t^{(\lambda_1-\lambda_2-2)/2}d^{-n+\lambda_1}(x,\bar{y}) \times \left(\operatorname{dist}(x,\partial M)\right)^{-\lambda_2} \tag{3.11}$$

for all $0 < t \le T$, x in M° and $\bar{y}$ in ∂M.

PROOF. The proofs are simple. To show (3.9), use (3.5) and the obvious fact that $d(x,\bar{y}) \ge \operatorname{dist}(x, \partial M)$. Then calculate that

$$\begin{aligned}
&\exp\left[-\kappa_1(T)\frac{d^2(x,\bar{y})}{t}\right] \\
&\quad\le \left(\frac{t}{\kappa_1(T)d^2(x,\bar{y})}\right)^{(n-\lambda_1+\lambda_2)/2} \\
&\qquad\times\left(\frac{\kappa_1(T)d^2(x,\bar{y})}{t}\right)^{(n-\lambda_1+\lambda_2)/2}\exp\left[-\kappa_1(T)\frac{d^2(x,\bar{y})}{t}\right] \\
&\quad\le \kappa_1^{(\lambda_1-n-\lambda_2)/2}(T)C(n-\lambda_1+\lambda_2)t^{(n-\lambda_1+\lambda_2)/2}d^{-n+\lambda_1}(x,\bar{y})\left(\operatorname{dist}(x,\partial M)\right)^{-\lambda_2}.
\end{aligned}$$

Similarly, for (3.10) and (3.11), we can take $\lambda = \lambda_2$ in parts (b) and (c) of Proposition 3.3. This gives us the $\operatorname{dist}(x, \partial M)^{-\lambda_2}$ term. We then calculate that for $i = 2$ or $i = 3$ and any $\lambda_1 \le n$,

$$\begin{aligned}
&\exp\left[-\kappa_i(T,\lambda_2)\frac{d^2(x,\bar{y})}{t}\right] \\
&\quad\le \left(\frac{t}{\kappa_i(T,\lambda_2)d^2(x,\bar{y})}\right)^{(n-\lambda_1)/2} \\
&\qquad\times\left(\frac{\kappa_i(T,\lambda_2)d^2(x,\bar{y})}{t}\right)^{(n-\lambda_1)/2}\exp\left[-\kappa_i(T,\lambda_2)\frac{d^2(x,\bar{y})}{t}\right] \\
&\quad\le \kappa_i^{(\lambda_1-n)/2}(T,\lambda_2)C(n-\lambda_1)t^{n/2-\lambda_1/2}d^{-n+\lambda_1}(x,\bar{y}).
\end{aligned}$$

The proof is complete. □

These results will be used in conjunction with the following result, which gives the bounds of (3.9)–(3.11) some of the same integrability properties as the bounds (3.5)–(3.7):

LEMMA 3.5. *For every positive constant λ, the quantity*

$$\Xi_\lambda := \sup_{\substack{x\in M\\ 0\le\varepsilon\le\varepsilon'_{\text{tub}}}} \int_{y\in M} d^{-(n-1)+\lambda}\left(x,\theta_\varepsilon(\bar{y})\right)\bar{\alpha}(d\bar{y})$$

is finite.

PROOF. Use normal coordinates on ∂M. See also [8], Chapter 7. □

In addition to using (3.2), we shall also use the *adjoint equation* for p^R. For each y in M, we may define

$$p_y^{R,*}(t,x) := p_x^R(t,y), \qquad t > 0,\ x \in M. \tag{3.12}$$

Then we have the following classical result.

PROPOSITION 3.6. *For each y in M, $p_y^{R,*}$ is in $C^\infty(\mathbb{R}_+ \times M^\circ \sim \{(0,y)\})$ and satisfies the PDE*

$$\frac{\partial p_y^{R,*}}{\partial t} = \mathcal{L}^* p_y^{R,*}, \qquad \lim_{t \to 0} p_y^{R,*}(t,\cdot) = \delta_y, \qquad \overline{\mathcal{L}}^* p_y^{R,*}\big|_{\mathbb{R}_+ \times \partial M} = 0, \tag{3.13}$$

where the second and third of these equalities are interpreted in the same way as the second and third equalities of (3.2).

4. Existence and uniqueness for a linear problem. In this section we use the Robin kernel of the last section to study a simple linear version of (I.1), namely,

$$\frac{\partial u_l}{\partial t} = \mathcal{L} u_l, \qquad u_l(0,\cdot) \equiv 0, \qquad \overline{\mathcal{L}} u_l|_{\mathbb{R}_+ \times \partial M} = \sigma(x)\zeta. \tag{4.1}$$

This is of course of the same form as (I.1) except that we have set $f \equiv 0$ and $u_0 \equiv 0$. This *linear* stochastic PDE will be important in our study of (I.1) because a *nonlinear* transformation of it will give a solution of (I.1).

Representation (3.3) suggests a natural guess for the solution of (4.1):

$$u_l(t,x) := -\tfrac{1}{2} \int_{s=0}^{t} \int_{\bar{y} \in \partial M} p_{\bar{y}}^R(t-s,x)\sigma(\bar{y})\zeta(ds,d\bar{y}), \qquad t \geq 0,\ x \in M^\circ. \tag{4.2}$$

For every $t \geq 0$ and x in M°, this stochastic integral is well defined because

$$\int_{s=0}^{\infty} \int_{\bar{y} \in \partial M} \left|p_{\bar{y}}^R(t-s,x)\right|^2 |\sigma(\bar{y})|^2 \overline{\alpha}(d\bar{y})\, ds < \infty,$$

as is clear from the smoothness result of Proposition 3.1. A more extensive use of the properties of p^R shows that the guess (4.2) is correct:

PROPOSITION 4.1. *The random field u_l defined by* (4.2) *solves* (4.1) *in the sense of Definition* 2.2. *It is also $\mathbb{P}$-a.s. in $C^\infty(\mathbb{R}_+ \times M^\circ)$.*

PROOF. Consider first the regularity of u_l. The smoothness result of Proposition 3.1 implies that for all integers k and l,

$$\int_{s=0}^{t} \int_{\bar{y} \in \partial M} \left\| \left(D^k \frac{\partial^l}{\partial t^l} p_{\bar{y}}^R \right)(t-s,x) \right\|_{T^{0,k}M}^2 |\sigma(\bar{y})|^2 \overline{\alpha}(d\bar{y})\, ds$$

is uniformly bounded as t and x range over any compact subset of $\mathbb{R}_+ \times M^\circ$. Here D^k is the kth order tensor derivative operator and $\|\cdot\|_{T^{0,k}M}$ is the standard fiber metric on the vector bundle $T^{0,k}M$ (see [11]). Thus by standard results, not only is condition $a.1$ true, but moreover, u_l has a C^∞ version.

Next, we verify condition $a.2$. Select any test function φ in $C_c^\infty(\mathbb{R}_+ \times M)$. Take any $T > 0$ such that $\operatorname{supp}\varphi \subset [0,T] \times M$; this will allow us to reduce the integrals in (2.5) and (2.7) to finite horizons. Consider (2.5), which, because here $f \equiv 0$, is equivalent to the existence of the first limit of (2.5). For simplicity, set

$$I_1^\varphi(\varepsilon) := \int_{t=0}^\infty \int_{x \in M_\varepsilon} u(t,x)\varphi(t,x)\alpha(dx)\,dt, \qquad 0 < \varepsilon \le \varepsilon'_{\text{tub}}. \tag{4.3}$$

Using the stochastic Fubini theorem, we can rewrite this as

$$I_1^\varphi(\varepsilon) = -\tfrac{1}{2} \int_{s=0}^\infty \int_{\bar{y} \in \partial M} \left(\int_{t=s}^\infty \int_{x \in M_\varepsilon} p_{\bar{y}}^R(t-s,x)\varphi(t,x)\,dt\ \alpha(dx) \right) \sigma(\bar{y})\zeta(ds,d\bar{y})$$

for all $0 < \varepsilon < \varepsilon'_{\text{tub}}$. The natural limit point of I_1^φ for ε tending to zero is

$$\begin{aligned} I_1^\varphi(0) := -\tfrac{1}{2} \int_{s=0}^\infty \int_{\bar{y} \in \partial M} &\left(\int_{t=s}^\infty \int_{x \in M^\circ} p_{\bar{y}}^R(t-s,x)\varphi(t,x)\alpha(dx)\,dt \right) \\ &\times \sigma(\bar{y})\zeta(ds,d\bar{y}), \end{aligned} \tag{4.4}$$

and we shall verify that $\mathbb{P}$-a.s. $\lim_{\varepsilon \to 0} I_1^\varphi(\varepsilon) = I_1^\varphi(0)$ by using Kolmogorov's continuity theorem and the fact that I_1^φ is a Gaussian field. For any $0 \le \varepsilon_1 < \varepsilon_2 \le \varepsilon'_{\text{tub}}$,

$$\begin{aligned} &\mathbb{E}\big[|I_1^\varphi(\varepsilon_1) - I_1^\varphi(\varepsilon_2)|^2\big] \\ &\quad = \tfrac{1}{4} \int_{s=0}^T \int_{\bar{y} \in \partial M} \left(\int_{t=s}^T \int_{x \in M_{\varepsilon_1} \sim M_{\varepsilon_2}} p_{\bar{y}}^R(t-s,x)\varphi(t,x)\alpha(dx)\,dt \right)^2 \\ &\qquad\quad \times |\sigma(\bar{y})|^2 \overline{\alpha}(d\bar{y})\,ds \\ &\quad \le \tfrac{1}{4} \sup_{\substack{t' \ge 0 \\ x \in M}} |\varphi(t',x)|^2 \sup_{\bar{z} \in \partial M} |\sigma(\bar{z})|^2 \\ &\qquad\quad \times \int_{s=0}^T \int_{\bar{y} \in \partial M} \left(\int_{t=0}^s \int_{x \in M_{\varepsilon_1} \sim M_{\varepsilon_2}} |p_{\bar{y}}^R(t,x)|\alpha(dx)\,dt \right)^2 \overline{\alpha}(d\bar{y})\,ds. \end{aligned} \tag{4.5}$$

We shall use Proposition 3.4 to bound the term in parentheses. Note that by expanding the volume form α in Fermi coordinates (see [14], Theorem 9.2.2), one can see that there exists a positive constant $\hbar$ such that

$$\int_{x \in M \sim M_{\varepsilon'_{\text{tub}}}} \psi(x)\alpha(dx) \le \hbar \int_{\eta=0}^{\varepsilon'_{\text{tub}}} \int_{\bar{x} \in \partial M} \psi\big(\theta_\varepsilon(\bar{x})\big)\, \overline{\alpha}\,(d\bar{x})\,d\eta$$

for all $\mathcal{B}(M)$-measurable mappings ψ from M to $[0,\infty)$. To proceed with our analysis of (4.5), fix any $0 < \varsigma < 1$; then use Proposition 3.4 with $\lambda_1 = 1 + \varsigma$ and

$\lambda_2 = \varsigma$. We can thus calculate that

$$
\begin{aligned}
&\int_{x \in M_{\varepsilon_1} \sim M_{\varepsilon_2}} \left|p_{\bar{y}}^R(s,x)\right| \alpha(dx) \\
&\qquad \le K_1'(T, 1+\varsigma, \varsigma) s^{-1/2} \hbar \int_{\eta=\varepsilon_1}^{\varepsilon_2} \int_{\bar{x} \in \partial M} d^{-(n-1)+\varsigma}\big(\theta_\eta(\bar{x}), \bar{y}\big) \eta^{-\varsigma} \overline{\alpha}(d\bar{x})\, d\eta \\
&\qquad \le K_1'(T, 1+\varsigma, \varsigma) s^{-1/2} \hbar \int_{\eta=\varepsilon_1}^{\varepsilon_2} \eta^{-\varsigma} \int_{\bar{x} \in \partial M} d^{-(n-1)+\varsigma}\big(\theta_\eta(\bar{x}), \bar{y}\big) \overline{\alpha}(d\bar{x})\, d\eta \\
&\qquad \le K_1'(T, 1+\varsigma, \varsigma) s^{-1/2} \hbar \Xi_\varsigma \left\{ \int_{\eta=\varepsilon_1}^{\varepsilon_2} \eta^{-\varsigma}\, d\eta \right\}
\end{aligned} \tag{4.6}
$$

for all $\bar{y}$ in ∂M, $0 < s \le T$, and $0 \le \varepsilon_1 < \varepsilon_2 \le \varepsilon'_{\text{tub}}$. Because $\eta \mapsto \eta^{-\varsigma}$ is decreasing on $(0, \infty)$,

$$
\int_{\eta=\varepsilon_1}^{\varepsilon_2} \eta^{-\varsigma}\, d\eta \le \int_{\eta=0}^{\varepsilon_2-\varepsilon_1} \eta^{-\varsigma}\, d\eta = (1-\varsigma)^{-1} |\varepsilon_2 - \varepsilon_1|^{1-\varsigma}. \tag{4.7}
$$

Inserting this into (4.6) and thence into (4.5), we can see that for all $0 \le \varepsilon_1 < \varepsilon_2 \le \varepsilon'_{\text{tub}}$,

$$
\begin{aligned}
\mathbb{E}\left[\left|I_1^\varphi(\varepsilon_1) - I_1^\varphi(\varepsilon_2)\right|^2\right] &= \tfrac{1}{4} \sup_{\substack{t' \ge 0 \\ x \in M}} |\varphi(t,x')|^2 \sup_{\bar{z} \in \partial M} |\sigma(\bar{z})|^2 \\
&\quad \times \left\{K_1'(T, 1+\varsigma, \varsigma) \hbar \Xi_\varsigma\right\}^2 \overline{\alpha}(\partial M)(2T^2)(1-\varsigma)^{-2} \\
&\quad \times |\varepsilon_2 - \varepsilon_1|^{2(1-\varsigma)}.
\end{aligned}
$$

Then (2.5) follows from this by applying Kolmogorov's continuity theorem and using the fact that I_1^φ is Gaussian.

To verify (2.7), we may define in a way analogous to (4.3),

$$
I_2^\varphi(\varepsilon) := \int_{t=0}^{\infty} \int_{\bar{x} \in \partial M} u\big(t, \theta_\varepsilon(\bar{x})\big) \varphi(t, \bar{x})\, \overline{\alpha}(d\bar{x})\, dt, \tag{4.8}
$$

with alternate representation

$$
I_2^\varphi(\varepsilon) = -\tfrac{1}{2} \int_{s=0}^{\infty} \int_{\bar{y} \in \partial M} \left(\int_{t=s}^{\infty} \int_{\bar{x} \in \partial M} p_{\bar{y}}^R\big(t-s, \theta_\varepsilon(\bar{x})\big) \varphi(t, \bar{x}) \overline{\alpha}(d\bar{x})\, dt \right) \sigma(\bar{y}) \zeta(ds, d\bar{y})
$$

for all $0 < \varepsilon \le \varepsilon'_{\text{tub}}$, and set

$$
\begin{aligned}
I_2^\varphi(0) := -\tfrac{1}{2} \int_{s=0}^{\infty} \int_{\bar{y} \in \partial M} &\left(\int_{t=s}^{\infty} \int_{\bar{x} \in \partial M} p_{\bar{y}}^R(t-s, \bar{x}) \varphi(t, \bar{x})\, \overline{\alpha}(d\bar{x})\, dt \right) \\
&\times \sigma(\bar{y}) \zeta(ds, d\bar{y}).
\end{aligned} \tag{4.9}
$$

Then, in place of (4.5), we have that

$$\begin{aligned}
&\mathbb{E}\Big[\big|I_2^{\varphi}(\varepsilon_1) - I_2^{\varphi}(\varepsilon_2)\big|^2\Big] \\
&= \tfrac{1}{4}\int_{s=0}^{\infty}\int_{\bar{y}\in\partial M}\bigg(\int_{t=s}^{\infty}\int_{\bar{x}\in\partial M}\Big(p_{\bar{y}}^{R}\big(t-s,\theta_{\varepsilon_1}(\bar{x})\big) - p_{\bar{y}}^{R}\big(t-s,\theta_{\varepsilon_2}(\bar{x})\big)\Big) \\
&\qquad\qquad \times \varphi(t,\bar{x})\overline{\alpha}(d\bar{x})\,dt\bigg)^2 |\sigma(\bar{y})|^2\overline{\alpha}(d\bar{y})\,ds \qquad (4.10)\\
&\le \tfrac{1}{4}\sup_{\substack{t'\ge 0\\ x\in M}}|\varphi(t',x)|^2 \sup_{\bar{z}\in\partial M}|\sigma(\bar{z})|^2 \\
&\quad\times\int_{s=0}^{T}\int_{\bar{y}\in\partial M}\bigg(\int_{t=0}^{s}\int_{\bar{x}\in\partial M}\big|p_{\bar{y}}^{R}\big(t,\theta_{\varepsilon_1}(\bar{x})\big) - p_{\bar{y}}^{R}\big(t,\theta_{\varepsilon_2}(\bar{x})\big)\big|\,\overline{\alpha}(d\bar{x})\,dt\bigg)^2 \\
&\quad\times\overline{\alpha}(d\bar{y})\,ds
\end{aligned}$$

for all $0 \le \varepsilon_1 < \varepsilon_2 \le \varepsilon'_{\mathrm{tub}}$. Using the fact that $\varepsilon \mapsto \theta_\varepsilon(\bar{x})$ is a geodesic of unit speed, we have the bound

$$\begin{aligned}
&\int_{\bar{x}\in\partial M}\big|p_{\bar{y}}^{R}\big(s,\theta_{\varepsilon_1}(\bar{x})\big) - p_{\bar{y}}^{R}\big(s,\theta_{\varepsilon_2}(\bar{x})\big)\big|\,\overline{\alpha}(d\bar{x}) \qquad (4.11)\\
&\qquad\le \int_{\eta=\varepsilon_1}^{\varepsilon_2}\int_{\bar{y}\in\partial M}\big\|\nabla p_{\bar{y}}^{R}\big(s,\theta_\eta(\bar{x})\big)\big\|\overline{\alpha}(d\bar{x})\,d\eta,
\end{aligned}$$

which holds for all $0 < s \le T$, all $\bar{y}$ in ∂M and all $0 \le \varepsilon_1 < \varepsilon_2 \le \varepsilon'_{\mathrm{tub}}$. We can bound the innermost integral with Proposition 3.4. Fix any $0 < \varsigma < \frac{1}{2}$. We can use claim (b) of Proposition 3.4 with $\lambda_1 = 1 + \varsigma$ and $\lambda_2 = 2\varsigma$ to see that

$$\big\|\nabla p_{\bar{y}}^{R}\big(s,\theta_\eta(\bar{x})\big)\big\| \le K_2'(T,1+\varsigma,2\varsigma)s^{\varsigma/2-1}d^{-(n-1)+\varsigma}\big(\theta_\eta(\bar{x}),\bar{y}\big)\eta^{-2\varsigma}$$

for all $\bar{x}$ and $\bar{y}$ in ∂M, all $0 < \eta \le \varepsilon'_{\mathrm{tub}}$ and all $0 < s \le T$. Inserting this in (4.11) and then using Lemma 3.5, we calculate that

$$\begin{aligned}
&\int_{\eta=\varepsilon_1}^{\varepsilon_2}\int_{\bar{z}\in\partial M}\big\|\nabla p_{\bar{y}}^{R}\big(s,\theta_\eta(\bar{x})\big)\big\|\overline{\alpha}(d\bar{x})\,d\eta \qquad (4.12)\\
&\qquad\le K_2'(T,1+\varsigma,2\varsigma)s^{\varsigma/2-1}\Xi_\varsigma\bigg\{\int_{\eta=\varepsilon_1}^{\varepsilon_2}\eta^{-2\varsigma}\,d\eta\bigg\}
\end{aligned}$$

for any $0 < s \le T$, any $\bar{y}$ in ∂M and any $0 \le \varepsilon_1 < \varepsilon_2 \le \varepsilon'_{\mathrm{tub}}$. Use now a calculation similar to (4.7) to bound the term in braces. Then insert (4.12) into (4.11) and thence into (4.10). This gives us that for all $0 \le \varepsilon_1 < \varepsilon_2 \le \varepsilon'_{\mathrm{tub}}$,

$$\begin{aligned}
\mathbb{E}\Big[\big|I_2^{\varphi}(\varepsilon_1) - I_2^{\varphi}(\varepsilon_2)\big|^2\Big] &\le \tfrac{1}{4}\sup_{\substack{t'\ge 0\\ \bar{x}\in\partial M}}|\varphi(t',\bar{x})|^2 \sup_{\bar{z}\in\partial M}|\sigma(\bar{z})|^2\big\{K_2'(T,1+\varsigma,2\varsigma)\Xi_\varsigma\big\}^2 \\
&\quad\times\overline{\alpha}(\partial M)\Big\{\big(\varsigma/2\big)^{-2}(1+\varsigma)^{-1}T^{1+\varsigma}\Big\}(1-2\varsigma)^{-2} \\
&\quad\times|\varepsilon_2-\varepsilon_1|^{2(1-2\varsigma)}.
\end{aligned}$$

Another appeal to Kolmogorov's theorem combined with the fact that I_2^φ is Gaussian shows that there is a continuous version of I_2^φ, thus completing the proof of (2.7) and thus of condition $a.2$.

Finally, we prove condition $a.3$. The proofs of condition $a.2$ not only showed that the limits of (2.5) and (2.7) exist, it also identified them. In light of the expressions for these limits given by (4.4) and (4.9), we need to show that

$$\begin{aligned}
-\frac{1}{2}\int_{s=0}^{\infty}\int_{\bar{y}\in\partial M}&\left(\int_{t=s}^{\infty}\int_{x\in M^\circ}p_{\bar{y}}^{R}(t-s,x)\left(\frac{\partial\varphi}{\partial t}+\mathcal{L}^*\varphi\right)(t,x)\alpha(dx)\,dt\right)\\
&\times\sigma(\bar{y})\zeta(ds,d\bar{y})\\
=\frac{1}{4}\int_{s=0}^{\infty}\int_{\bar{y}\in\partial M}&\left(\int_{t=s}^{\infty}\int_{\bar{x}\in\partial M}p_{\bar{y}}^{R}(t-s,\bar{x})(\bar{\mathcal{L}}^*\varphi)(t,\bar{x})\bar{\alpha}(d\bar{x})\,dt\right)\\
&\times\sigma(\bar{y})\zeta(ds,d\bar{y})\\
+\frac{1}{2}\int_{s=0}^{\infty}\int_{\bar{y}\in\partial M}&\varphi(s,\bar{y})\sigma(\bar{y})\zeta(ds,d\bar{y}).
\end{aligned}\tag{4.13}$$

A simple integration by parts shows that

$$\begin{aligned}
&\int_{t=s}^{\infty}\int_{x\in M^\circ}p_{y}^{R}(t-s,x)\left(\frac{\partial\varphi}{\partial t}+\mathcal{L}^*\varphi\right)(t,x)\alpha(dx)\,dt\\
&\quad=-\frac{1}{2}\int_{t=s}^{\infty}\int_{\bar{x}\in\partial M}p_{y}^{R}(t-s,\bar{x})(\overline{\mathcal{L}}^*\varphi)(t,\bar{x})\bar{\alpha}(d\bar{x})\,dt+\frac{1}{2}\varphi(s,y)
\end{aligned}\tag{4.14}$$

for every $s\geq 0$ and y in M, and thus by the linearity of stochastic integration, (4.13) must follow. This shows that indeed condition $a.3$ holds. □

The following argument proves that (4.2) defines the unique solution of the stochastic PDE (4.1).

PROPOSITION 4.2. *The random field u_l of* (4.2) *is the unique solution of* (4.1) *in the sense of Definition* 2.2. *More exactly, if a random field $\tilde{u}_l$ solves* (4.1) *in the sense of Definition* 2.2, *then*

$$\mathbb{P}\{\tilde{u}_l(t,x)=u_l(t,x)\text{ for all }t\geq 0\text{ and }x\in M^\circ\}=1.$$

PROOF. Because the stochastic PDE (4.1) is linear, it suffices to show that any random field $\widehat{u}_l$ that solves

$$\frac{\partial\widehat{u}_l}{\partial t}=\mathcal{L}\widehat{u}_l,\qquad \widehat{u}_l(0,\cdot)\equiv 0,\qquad \overline{\mathcal{L}}\widehat{u}_l|_{\mathbb{R}_+\times\partial M}\equiv 0\tag{4.15}$$

in the sense of Definition 2.2, is the identically zero random field; that is,

$$\mathbb{P}\{\widehat{u}_l(t,x)=0\text{ for all }t\geq 0\text{ and }x\in M^\circ\}=1.$$

Definition 2.2 makes sense here upon setting $f \equiv 0$, $u_\circ \equiv 0$ and $\sigma \equiv 0$ in (I.1). Although all of the coefficients in (4.15) are deterministic, we must allow for random solutions because we are within the framework of Definition 2.2. In view of the continuity requirement of condition $a.1$, it is clearly sufficient to show the simpler property that

$$\mathbb{P}\{\widehat{u}(s,y) = 0\} = 1 \tag{4.16}$$

for each $s > 0$ and y in M°. We can do this with a fairly simple classical adjoint argument. For each $\eta > 0$, define ξ_η in $C^\infty(\mathbb{R}_+ \times M)$ by

$$\xi_\eta(t,x) := \begin{cases} C_\eta \exp\left[-\left(1 - \dfrac{d^2(x,y) + (t-s)^2}{\eta^2}\right)^{-1}\right], & \text{if } d^2(x,y) + (t-s)^2 \le \eta^2, \\ 0, & \text{else,} \end{cases}$$

where C_η is the unique constant such that $\int_{t=0}^{\infty} \int_{x \in M} \xi_\eta(t,x)\,dt\,\alpha(dx) = 1$. Then, as η tends to 0, ξ_η tends to a Dirac measure on $(\mathbb{R}_+ \times M, \mathcal{B}(\mathbb{R}_+ \times M))$ that is concentrated at (s,y). Now fix a $T > s$ and for each $\eta > 0$, let φ_η in $C^\infty(\mathbb{R}_+ \times M)$ be a solution of

$$\frac{\partial \varphi_\eta}{\partial t} + \mathcal{L}^* \varphi_\eta = \xi_\eta, \qquad \varphi_\eta(T,\cdot) \equiv 0, \qquad \overline{\mathcal{L}}^* \varphi_\eta|_{\mathbb{R}_+ \times \partial M} \equiv 0;$$

we may get such a φ by convolving ξ_η against $p^{R,*}$ as given by (3.12) and using the adjoint equation (3.13). Inserting this in (2.8) with $f \equiv 0$, $u_\circ \equiv 0$ and $\sigma \equiv 0$, we conclude that

$$\lim_{\varepsilon \to 0} \int_{t=0}^{\infty} \int_{x \in M_\varepsilon} \widehat{u}(t,x)\xi_\eta(t,x)\alpha(dx)\,dt = 0, \qquad \mathbb{P}\text{-a.s.}$$

Clearly $\operatorname{supp} \xi_\eta \subset (0,\infty) \times M_\varepsilon$ for ε and η small enough, so $\mathbb{P}$-a.s.

$$\int_{(t,x) \in \operatorname{supp} \xi_\eta} \widehat{u}(t,x)\xi_\eta(t,x)\alpha(dx)\,dt = \lim_{\varepsilon \to 0} \int_{(t,x) \in \operatorname{supp} \xi_\eta} \widehat{u}(t,x)\xi_\eta(t,x)\alpha(dx)\,dt = 0$$

for each $\eta > 0$ small enough. We finally get (4.16) by letting η tend to 0 through a countable sequence and using the continuity of $\widehat{u}$ required by condition $a.1$. □

5. Boundary layer degeneracies for the linear problems. We now characterize the boundary layer behavior for the stochastic PDE (4.1). From the results of Section 2, ζ is a generalized function that is locally the nth derivative of a continuous function. Thus we expect degeneracies in the boundary layer, and in this section we shall characterize them. Our approach of choice will be entirely probabilistic, as compared to exploiting the above-mentioned interpretation of ζ as a generalized function; this retains more closely the probabilistic nature of the problem.

To have an idea of where to start, we consider the prototypical stochastic PDE

$$\frac{\partial \widehat{u}_l}{\partial t} = \tfrac{1}{2}\widehat{\Delta}\widehat{u}_l, \qquad \widehat{u}_l(0,\cdot) \equiv 0, \qquad \left.\frac{\widehat{\partial u_l}}{\partial x_n}\right|_{\mathbb{R}_+ \times \partial H_n} = \widehat{\zeta}, \qquad t > 0,\ x \in H_n^\circ \tag{5.1}$$

on the half-space, where $\widehat{\Delta}$ is given by (1.1) and $\widehat{\zeta}$ is a standard space–time white noise field on $\mathbb{R}_+ \times \partial H_n$, which we may also write as

$$\widehat{\zeta} = \frac{\partial^n \widehat{W}}{\partial t\, \partial x_1\, \partial x_2 \cdots \partial x_{n-1}} \tag{5.2}$$

with $\widehat{W}$ being a standard Brownian sheet on $\mathbb{R}_+ \times \partial H_n$. This last fact of course corresponds exactly to (2.1). By classical results, the unique solution of this stochastic PDE, which does not grow too fast (we need not be more specific), is

$$\widehat{u}_l\big(t, (\overline{x}, x_n)\big) = -\tfrac{1}{2} \int_{s=0}^{t} \int_{\overline{y} \in \mathbb{R}^{n-1}} \widehat{p}^R_{(\overline{y},0)}\big(t-s, (\overline{x}, x_n)\big)\widehat{\zeta}(ds, d\overline{y})$$

for all $t > 0$ and all $(\overline{x}, x_n)$ in H_n°, where $\widehat{p}^R$ is as in (3.4). Consider the behavior of $\widehat{u}_l$ near the boundary as we vary x_n. An easy calculation shows that for all $t > 0$ and $(\overline{x}, x_n)$ in H_n°,

$$\begin{aligned}
\mathbb{E}\Big[\big|\widehat{u}_l\big(t, (\overline{x}, x_n)\big)\big|^2\Big] &= \frac{1}{4} \int_{s=0}^{t} \int_{\overline{y} \in \mathbb{R}^{n-1}} \big|\widehat{p}^R_{(\overline{y},0)}\big(t-s, (\overline{x}, x_n)\big)\big|^2 \, d\overline{y}\, ds \\
&= \int_{s=0}^{t} \int_{\overline{y} \in \mathbb{R}^{n-1}} \exp\left[-\frac{\|\overline{x}-\overline{y}\|^2 + x_n^2}{s}\right](2\pi s)^{-n}\, d\overline{y}\, ds \\
&= 2 \int_{s=0}^{t} \exp\left[-\frac{x_n^2}{s}\right](4\pi s)^{-(n+1)/2}\, ds \\
&= \frac{1}{x_n^{n-1}} \frac{4}{(8\pi)^{n/2}} \int_{u = x_n\sqrt{2/t}}^{\infty} \frac{\exp(-u^2/2)}{\sqrt{2\pi}} u^{n-2}\, du.
\end{aligned}$$

Thus asymptotically $\mathbb{E}[|\widehat{u}_l(t, (\overline{x}, x_n))|^2] \sim K/x_n^{n-1}$ at the boundary, so we could hope that $\mathbb{P}$-a.s. the mapping $x_n \mapsto \widehat{u}_l(t, (\overline{x}, x_n))$ has an algebraic singularity of order $(n-1)/2$ for each $t > 0$ and $\overline{x}$ in $\mathbb{R}^{n-1}$ (see [13], Section 1.3). More exactly, we conjecture that for each $t > 0$ and $\overline{x}$ in $\mathbb{R}^{n-1}$,

$$\lim_{x_n \to 0^+} x_n^\gamma \widehat{u}_l\big(t, (\overline{x}, x_n)\big) = 0 \tag{5.3}$$

for each $\gamma > (n-1)/2$ (recall that we are considering the case $n \geq 2$). This will serve as a model of the sought-for boundary layer results for (4.2) in the direction perpendicular to the boundary.

Turning now to the behavior of the boundary layer of (5.1) in the direction parallel to the boundary, we study the regularity of the mapping $(t, \overline{x}) \mapsto \widehat{u}_l(t, (\overline{x}, x_n))$ for x_n very close to 0. In view of (5.1) and the fact that the differential operator $\partial^n/\partial t\, \partial x_1\, \partial x_2 \cdots \partial x_{n-1}$ commutes with the operators $\partial/\partial t$, $\widehat{\Delta}$, and $\partial/\partial x_n$, we might formally apply an inverse operator $(\partial^n/\partial t\, \partial x_1\, \partial x_2 \cdots \partial x_{n-1})^{-1}$ to (5.1) and use (5.2) to get a stochastic PDE with continuous boundary data. More precisely, we may solve the stochastic PDE

$$\frac{\partial \widehat{u}'_l}{\partial t} = \frac{1}{2}\widehat{\Delta}\widehat{u}'_l, \qquad \widehat{u}'_l(0, \cdot) \equiv 0, \qquad \frac{\partial \widehat{u}'_l}{\partial x_n}\big(t, (\overline{x}, 0)\big) = \widehat{W}, \qquad t > 0,\ x \in H_n^\circ \tag{5.4}$$

(subject to some unspecified growth conditions that ensure uniqueness) and then show that $\partial^n \widehat{u}_l' / \partial t\, \partial x_1\, \partial x_2 \cdots \partial x_{n-1}$ also solves (5.1). Thus

$$\widehat{u}_l = \frac{\partial^n \widehat{u}_l'}{\partial t\, \partial x_1\, \partial x_2 \cdots \partial x_{n-1}}, \tag{5.5}$$

where we note that $\widehat{u}_l'$ is continuous on all of $\mathbb{R}_+ \times H_n^\circ$ and, in particular, at the boundary $\mathbb{R}_+ \times \partial H_n$. This sort of regularity result, that the solution of (5.1) is a certain number of derivatives of a function that is continuous up to and at the boundary, is what we seek for the variation of u_l of (4.2) parallel to the boundary.

Returning to (4.1), we see that the obvious analog of (5.3) is the following result.

PROPOSITION 5.1. *For each $T > 0$ and $\gamma > (n-1)/2$,*

$$\limsup_{\substack{x \to \partial M \\ 0 \le t \le T}} \big(\mathrm{dist}(x, \partial M)\big)^\gamma |u_l(t,x)| = 0, \qquad \mathbb{P}\text{-a.s.} \tag{5.6}$$

PROOF. For each $\gamma > (n-1)/2$, define

$$V_{1,\gamma}(t, \overline{x}, \varepsilon) := \varepsilon^\gamma u_l\big(t, \theta_\varepsilon(\overline{x})\big) \tag{5.7}$$

for all $t \ge 0$, $\overline{x} \in \partial M$, and $0 < \varepsilon \le \varepsilon_{\mathrm{tub}}'$. Fix $\gamma > (n-1)/2$ and choose any γ' such that $(n-1)/2 < \gamma' < \min\{\gamma, (n-1)/2 + 1\}$. For each x in M within distance $\varepsilon_{\mathrm{tub}}'$ of ∂M, necessarily $x = \theta_{\mathrm{dist}(x, \partial M)}(\overline{x})$ for some $\overline{x}$ in ∂M, so

$$\begin{aligned}\big(\mathrm{dist}(x, \partial M)\big)^\gamma |u_l(t,x)| &= \big(\mathrm{dist}(x, \partial M)\big)^\gamma \big|u_l\big(t, \theta_{\mathrm{dist}(x,\partial M)}(\overline{x})\big)\big| \\ &\le \big(\mathrm{dist}(x, \partial M)\big)^{\gamma - \gamma'} \big|V_{1,\gamma'}\big(t, \overline{x}, \mathrm{dist}(x, \partial M)\big)\big|.\end{aligned}$$

Because $\gamma - \gamma'$ is positive, the result will follow from the boundedness of $V_{1,\gamma'}$ on $[0,T] \times \partial M \times [0, \varepsilon_{\mathrm{tub}}']$. That $V_{1,\gamma'}$ has a version that is continuous at the boundary and hence is bounded near ∂M, follows from the next result, which we prove in Appendix B. □

LEMMA 5.2. *For any $T > 0$ and $(n-1)/2 < \gamma < (n-1)/2 + 1$,*

$$\sup_{\substack{0 < t \le T, \overline{x} \in \partial M \\ 0 < \varepsilon \le \varepsilon_{\mathrm{tub}}'}} \mathbb{E}\big[|V_{1,\gamma}(t, \overline{x}, \varepsilon)|^2\big] < \infty. \tag{5.8}$$

Furthermore, for any $0 < \beta < \gamma - (n-1)/2$, there is a constant K such that for any $i = 1, 2, \dots, \overline{i}$,

$$\begin{aligned}&\mathbb{E}\big[|V_{1,\gamma}(t_1, \overline{x}_1, \varepsilon_1) - V_{1,\gamma}(t_2, \overline{x}_2, \varepsilon_2)|^2\big] \\ &\qquad \le K\big\{|t_2 - t_1| + \overline{d}_i(\overline{x}_1, \overline{x}_2) + |\varepsilon_2 - \varepsilon_1|\big\}^\beta\end{aligned} \tag{5.9}$$

for all $(t_1, \overline{x}_1, \varepsilon_1)$ *and* $(t_2, \overline{x}_2, \varepsilon_2)$ *in* $(0, T] \times \partial_i M \times (0, \varepsilon'_{\text{tub}}]$.

Invoking again the Garsia–Rodemich–Rumsey results, we conclude from this that as a function of all three arguments, $V_{1,\gamma}$ has a version that is Hölder continuous of exponent $\beta'/2$ for each $0 < \beta' < \beta$. We leave the relevant calculations to the reader, noting only for clarity that the domain of $V_{1,\gamma}$ is an $n+1$-dimensional manifold. The reader should also convince himself or herself that there is no problem caused by the fact that we do not consider $t = 0$ or $\varepsilon = 0$ in Lemma 5.2 (this is done for convenience of proof). One may simply define $V_{1,\gamma}(t, \overline{x}, \varepsilon)$ for $t = 0$ or $\varepsilon = 0$ by using Lemma 5.2 and the fact that $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is complete. These arguments complete the proof of (5.6). □

We interpret this result as showing that the boundary layer of u_l of (4.1) has an algebraic singularity of order at most $(n-1)/2$ in the direction perpendicular to the boundary.

This leaves us with the task of finding the analog of (5.5) in order to describe the variation of u_l in the direction parallel to the boundary. We would like to show that locally

(5.10) $$u_l = \frac{\partial^n u'_l}{\partial t\, \partial x_1\, \partial x_2 \cdots \partial x_{n-1}},$$

where u'_l is some random field that is continuous at the boundary and where $x_1, x_2, \dots, x_{n-1}$ are some carefully chosen local coordinates on ∂M. Unfortunately, the random field $\widehat{u}_l$ of (5.4) came from a good guess, and no such guess is apparent here. Note, however, that heuristically we may rewrite (5.10) as $u'_l = (\partial^n / \partial t\, \partial x_1\, \partial x_2 \cdots \partial x_{n-1})^{-1} u_l$, so instead of trying to explicitly solve (5.10), we may try to simply show that the $dt\, dx_1\, dx_2 \cdots dx_{n-1}$ integral is continuous near the boundary. More precisely, we hope that

(5.11) $$\int_{(s, \bar{\xi}) \in A} u_l\big(s, \theta_\varepsilon(\overline{x})\big) ds\, \overline{\alpha}(d\bar{\xi})$$

is continuous in ε at $\varepsilon = 0$ for all sufficiently general sets A in $\mathcal{B}(\mathbb{R}_+ \times \partial M)$. Because it is reasonable to require that the sets A be geometrically meaningful, we shall fix some constant ϱ such that

$$0 < \varrho < \overline{\varepsilon_{injr}}$$

and take A of the form

(5.12) $$A = [0, t] \times \overline{B}(\overline{x}, r)$$

for $t > 0$, $\overline{x}$ in ∂M and $0 \le r \le \varrho$, where for any $\overline{x}$ in any $\partial_i M$ and any $0 < r \le \varrho$,

$$\overline{B}(\overline{x}, r) := \big\{ \overline{z} \in \partial_i M \colon \overline{d}_i(\overline{z}, \overline{x}) \le r \big\}.$$

Here $\overline{\varepsilon_{injr}}$ is as we defined in Section 1. The restriction that r be less than or equal to ϱ instead of $\overline{\varepsilon_{injr}}$ is enforced because we will want to locally map $\overline{B}(\overline{x}, r)$

into $\mathbb{R}^{n-1}$, and such a mapping may break down if r is near $\overline{\varepsilon_{injr}}$. For additional notational simplicity, we define

$$U(t,\bar{x},r,\varepsilon;T,\varphi) := \int_{s=0}^{t}\int_{\bar{y}\in\bar{B}(\bar{x},r)} \varphi\big(s,\theta_\varepsilon(\bar{y})\big)ds\,\overline{\alpha}(d\bar{y})$$

for each $T > 0$, each continuous real-valued mapping φ whose domain contains $[0,T]\times M^\circ$ and each $0 \le t \le T$, $\bar{x}\in\partial M$, $0\le r\le\varrho$ and $0<\varepsilon\le\varepsilon_{\text{tub}}$. This is simply the integral (5.11) with A of the form (5.12) and with u_l replaced by a general function φ defined at least up to the finite horizon T. Although the true parallel to (5.10) is that $U(\cdot,\cdot,\cdot,\cdot;T,u_l)$ be continuous for each $T>0$, we shall once again be content with something simpler, namely:

PROPOSITION 5.3. *For every $T>0$, $\mathbb{P}$-a.s,*

$$\limsup_{\substack{\varepsilon\to 0\\ 0<t\le T,\bar{x}\in\partial M\\ 0\le r<\varrho}} |U(t,\bar{x},r,\varepsilon;T,u_l)|<\infty.$$

PROOF. Similarly to (5.7), we fix a $T>0$ and define

$$V_2(t,\bar{x},r,\varepsilon) := U(t,\bar{x},r,\varepsilon;T,u_l)$$

for each $0<t\le T$, $\bar{x}$ in any $\partial_i M$, $0<r\le\varrho$ and $0<\varepsilon\le\varepsilon'_{\text{tub}}$. We intend to show that V_2 has a modification that is jointly continuous in all four arguments. This follows from the next lemma.

LEMMA 5.4. *For any $T>0$,*

$$\sup_{\substack{0<t\le T,\bar{x}\in\partial M\\ 0<r\le\varrho,0<\varepsilon\le\varepsilon'_{\text{tub}}}} \mathbb{E}\big[|V_2(t,\bar{x},r,\varepsilon)|^2\big]<\infty. \tag{5.13}$$

Furthermore, for any $0<\beta<1$, there is a constant K such that for all $i=1,2,\dots,\bar{i}$,

$$\begin{aligned}&\mathbb{E}\big[|V_2(t_1,\bar{x}_1,r_1,\varepsilon_1)-V_2(t_2,\bar{x}_2,r_2,\varepsilon_2)|^2\big]\\ &\quad\le K\big\{|t_2-t_1|+\overline{d}_i(\bar{x}_1,\bar{x}_2)+|r_2-r_1|+|\varepsilon_2-\varepsilon_1|\big\}^\beta\end{aligned} \tag{5.14}$$

for all $(t_1,\bar{x}_1,r_1,\varepsilon_1)$ and $(t_2,\bar{x}_2,r_2,\varepsilon_2)$ in $(0,T]\times\partial_i M\times(0,\varrho]\times(0,\varepsilon'_{\text{tub}}]$.

Invoking again standard estimates on Gaussian fields, we conclude from this that V_2 as a function of all four variables has a continuous version that is Hölder continuous of exponent $\beta'/2$ for each $0<\beta'<\beta$. The details are left to the reader, with the note that the domain of V_2 is an $n+2$-dimensional manifold. Similarly to the proof of Proposition 5.1, we have not included $t=0$, $r=0$ or $\varepsilon=0$ in the statement of Lemma 5.4. Finally, although Lemma 5.4 is sufficient for our purposes, we do not claim that it has the best possible Hölder exponents. □

This completes our study of the boundary layer of u_l. For future reference, we put these results into the description of a functional space. For each $T > 0$ and $\gamma > (n-1)/2$, define the vector spaces

$$C_\gamma(T) := \left\{ \varphi \in C\big([0,T] \times M^\circ\big) \colon \limsup_{\substack{x \to \partial M \\ 0 \le t \le T}} \big(\operatorname{dist}(x, \partial M)\big)^\gamma |\varphi(t,x)| = 0 \right\}$$

$$\cap \left\{ \varphi \in C\big([0,T] \times M^\circ\big) \colon \limsup_{\substack{\varepsilon \to 0 \\ 0 \le t \le T, \bar{x} \in \partial M \\ 0 \le r \le \varrho}} |U(t, \bar{x}, r, \varepsilon; T, \varphi)| < \infty \right\}$$

and

$$(5.15) \qquad C_\gamma := \big\{ \varphi \in C(\mathbb{R}_+ \times M^\circ) \colon \varphi|_{[0,T] \times M^\circ} \in C_\gamma(T) \text{ for all } T > 0 \big\}.$$

The implication of Propositions 5.1 and 5.3 is that for each $\gamma > (n-1)/2$, $C_\gamma(T)$ is a natural space in which the solution u_l of (4.1) has values:

THEOREM 5.5. *For each $\gamma > (n-1)/2$, the random field u_l is $\mathbb{P}$-a.s. in C_γ.*

It will be important for the upcoming section that $C_\gamma(T)$ actually be a Banach space. One may check that for each $T > 0$, the function

$$\|\varphi\|_{C_\gamma(T)} := \limsup_{\substack{x \to \partial M \\ 0 \le t \le T}} \big(\operatorname{dist}(x, \partial M)\big)^\gamma |\varphi(t,x)| + \limsup_{\substack{\varepsilon \to 0 \\ 0 \le t \le T, \bar{x} \in \partial M \\ 0 \le r \le \varrho}} |U(t, \bar{x}, r, \varepsilon; T, \varphi)|,$$

defined for each φ in $C_\gamma(T)$, is a norm on $C_\gamma(T)$ under which $C_\gamma(T)$ is complete, so $(C_\gamma(T), \|\cdot\|_{C_\gamma(T)})$ is a Banach space. We also note that (5.15) defines a family of seminorms on C_γ; namely, for each $T > 0$, we may define

$$\|\varphi\|_{C_\gamma(T)} := \|\varphi|_{[0,T] \times M^\circ}\|_{C_\gamma(T)}, \qquad \varphi \in C_\gamma.$$

6. The nonlinear problem. We return to the fully nonlinear SRDE (I.1). We search for the solution u of (I.1) as a classical nonlinear transformation of the solution u_l of the linear stochastic PDE (4.1). Of course the boundary-layer analysis of the previous section will play an important role in this, as we need to understand the interaction of the nonlinear transformation with the boundary-layer degeneracies. A result of this study will be the identification of a class of "admissible" nonlinearities f that do not magnify the boundary-layer degeneracies.

We begin with the integral equation that corresponds to (3.3) and that the solution of (I.1) should solve for each $t > 0$ and x in M°:

$$u(t,x) = -\tfrac{1}{2} \int_{s=0}^{t} \int_{\bar{y} \in \partial M} p^R_{\bar{y}}(t-s, x) \sigma(\bar{y}) \zeta(ds, d\bar{y})$$

$$+\int_{y\in M^\circ} p_y^R(t,x)u_\circ(y)\alpha(dy) \tag{6.1}$$
$$+\int_{s=0}^{t}\int_{y\in M^\circ} p_y^R(t-s,x)f\big(y,u(s,y)\big)\alpha(dy)\,ds$$

for every $t > 0$ and x in M°. Note that the first term on the right-hand side of this equation is exactly u_l—recall (4.2). Upon replacing the first term of (6.1) by u_l, we see that

$$u = Bu_l, \tag{6.2}$$

where B is the functional mapping that takes any φ in C_γ into the solution of

$$(B\varphi)(t,x) = \varphi(t,x) + \int_{y\in M^\circ} p_y^R(t,x)u_\circ(y)\alpha(dy)$$
$$+\int_{s=0}^{t}\int_{y\in M^\circ} p_y^R(t-s,x)f\big(y,\varphi(s,y)\big)\alpha(dy)\,ds, \qquad t\ge 0,\ x\in M^\circ. \tag{6.3}$$

We make this precise, finding conditions on f under which $B\varphi$ is well defined for all φ in C_γ for any fixed $\gamma > (n-1)/2$, and such that u as given by (6.2) solves (I.1) in the sense of Definition 2.2.

We shall require that, for each fixed $\gamma > (n-1)/2$, B maps C_γ to itself, that is, that the solution of the nonlinear SRDE (I.1) has the same boundary layer behavior as the solution of the linear problem (4.1). This is an important but natural restriction; if the nonlinearity f is "small" in some sense, it should not cause qualitative changes in the boundary layer. To get a better idea of what this condition implies about f, we add a nonlinear term to our model SRDE (5.1); that is, we study

$$\frac{\partial\widehat{u}}{\partial t} = \frac{1}{2}\widehat{\Delta}\widehat{u} + f(x,\widehat{u}), \qquad \widehat{u}(0,\cdot)\equiv 0, \qquad \left.\frac{\partial\widehat{u}}{\partial x_n}\right|_{\mathbb{R}_+\times\partial H_n} = \widehat{\zeta}, \qquad t>0,\ x\in H_n^\circ \tag{6.4}$$

The analogue to (6.1) is that

$$\widehat{u}(t,x) = \widehat{u}_l(t,x) + \int_{s=0}^{t}\int_{\bar{y}\in\mathbb{R}^{n-1}}\int_{y_n>0}\widehat{p}^R_{(\bar{y},y_n)}(t-s,x)$$
$$\times f\Big((\bar{y},y_n),\widehat{u}\big(s,(\bar{y},y_n)\big)\Big)dy_n\,d\bar{y}\,ds \tag{6.5}$$

for all $t > 0$ and x in H_n°, where $\widehat{p}^R_{(\bar{y},y_n)}$ is as in (3.4). The boundary behavior of $f(s,\widehat{u})$ will be assumed to be something like

$$\Big|f\Big((\bar{y},y_n),\widehat{u}\big(s,(\bar{y},y_n)\big)\Big) \le K\,|\,y_n\Big|^\eta \tag{6.6}$$

for some constant η for all $s \geq 0$ and all $(\bar{y}, y_n)$ in H_n°, so a natural condition, which is sufficient for the integral in (6.5) to be well defined, is that

$$\int_{s=0}^{t} \int_{\bar{y} \in \mathbb{R}^{n-1}} \int_{y_n > 0} \left| \widehat{p}^{R}_{(\bar{y}, y_n)}(t-s, x) \right| |y_n|^{\eta} \, dy_n \, d\bar{y} \, ds$$

be finite. From the explicit formula (3.4) for $\widehat{p}^R$, this is true if and only if

$$\text{(6.7)} \qquad \int_{s=0}^{t} \int_{\bar{y}_n > 0} \left| \exp\left[-\frac{|x_n - y_n|^2}{2(t-s)} \right] + \exp\left[-\frac{|x_n + y_n|^2}{2(t-s)} \right] \right| \times \left(2\pi(t-s)\right)^{-1/2} |y_n|^{\eta} \, dy_n \, ds$$

is finite, and from this we can easily see that η *should be greater than* -1. If the boundary behavior of the nonlinearity f is

$$\left| f\left((\bar{y}, y_n), u\right) \right| \leq K |y_n|^{\eta_1} |u|^{\eta_2}$$

for all $(\bar{y}, y_n)$ in H_n° close enough to ∂H_n and all u of large enough absolute value, and if $\widehat{u}$ of (6.4) has boundary behavior like (5.3), then the only way that we can get (6.6) for some $\eta > -1$ is to have

$$\text{(6.8)} \qquad \eta_1 - \eta_2 \gamma > -1.$$

This suggests that the natural, albeit rather odd-looking, growth condition for the nonlinearity f in (I.1), where u is to be considered as an element of C_γ, is:

$nl1_\gamma$. There is a constant $\overline{F}$ and two exponents η_1 and η_2, related by $\eta_1 - \eta_2\gamma > -1$, such that for all x in M° and all u in $\mathbb{R}$ with $|u| > 1$,

$$|f(x, u)| \leq \overline{F}\left(\operatorname{dist}(x, \partial M)\right)^{\eta_1} |u|^{\eta_2}.$$

We may now turn around and assess the consequences of this condition upon the last terms of (6.1) or (6.5), over and above simply ensuring that these integrals should be well defined. If we assume that (6.8) holds and set $\eta = \eta_1 - \eta_2\gamma$ in (6.7), we find that the expression of (6.7) *has no boundary-layer degeneracies*; that is, it is well behaved for x_n near 0. Thus, at least in (6.5), the natural condition that ensures that the last term of (6.5) is well defined, also makes this last term nondegenerate near the boundary, and thus implies that the boundary-layer degeneracies of (6.4) come only from $\widehat{u}_l$, the solution of (5.1).

We shall in a moment rigorously carry out this sort of calculation for (6.1)–(6.3), but to get the full existence and uniqueness result for the solution of (6.3), we will need to have a continuity condition on f, in the same way that one needs both growth and continuity requirements on the coefficients of finite-dimensional SDE's. We will here require:

$nl2_\gamma$. There is a constant $\bar{f}$ such that for all x in M° and all u and v in $\mathbb{R}$,

$$|f(x, u) - f(x, v)| \leq \bar{f}\left(\operatorname{dist}(x, \partial M)\right)^{\gamma} |u - v|.$$

Under this condition, the mapping $u \mapsto f(x,u)$ is Lipschitz continuous for every x in M°, but the Lipschitz constant is very small for x near ∂M. An important consequence of this condition is that for any elements φ_1 and φ_2 of C_γ,

$$\sup_{0 \le s \le t} \left| f\big(s, \varphi_1(s,x)\big) - f\big(s, \varphi_2(s,x)\big) \right| \le \bar{f} \|\varphi_1 - \varphi_2\|_{C_\gamma(t)} \tag{6.9}$$

for each $t \ge 0$.

We can better understand how these ideas fit into our study of B by separately considering the different terms in (6.3). Define

$$(T_1 u_\circ)(t,x) := \int_{y \in M^\circ} p_y^R(t,x) u_\circ(y) \alpha(dy)$$

for all $t \ge 0$ and x in M. Also, for each φ in C_γ, define

$$(T_2 \varphi)(t,x) := \int_{s=0}^{t} \int_{y \in M^\circ} p_y^R(t-s,x) f\big(s, \varphi(s,y)\big) \alpha(dy)\, ds$$

for all such t and x. If these two integrals are well defined, then we may rewrite (6.3) as

$$(B\varphi)(t,x) = \varphi(t,x) + (T_1 u_\circ)(t,x) + \big(T_2(B\varphi)\big)(t,x) \tag{6.10}$$

for all $t \ge 0$ and x in M°. We may further rewrite this as

$$\varphi(t,x) = (B\varphi)(t,x) - (T_1 u_\circ)(t,x) - \big(T_2(B\varphi)\big)(t,x) \tag{6.11}$$

for all such t and x, from which we see that B should be invertible with inverse

$$\big(B^{-1}\varphi\big)(t,x) = \varphi(t,x) - (T_1 u_\circ)(t,x) - (T_2 \varphi)(t,x) \tag{6.12}$$

for all such t and x. The following two results show that $T_1 u_\circ$ and T_2 are sufficiently well behaved that these ideas work. One may think of the second of these two results as the partial analog of our foregoing analysis of (6.4):

LEMMA 6.1. *The function $T_1 u_\circ$ is a well-defined element of $C(\mathbb{R}_+ \times M \sim \{(0,\bar{x}) : \bar{x} \in \partial M\})$ and of C_γ.*

PROOF. This standard result may be seen from the bounds of Propositions 3.1–3.3. □

LEMMA 6.2. *If conditions $nl1_\gamma$ and $nl2_\gamma$ hold for a $\gamma > (n-1)/2$, then T_2 is a well-defined mapping from C_γ to $C(\mathbb{R}_+ \times M)$ and thus to C_γ. Furthermore, for each $T \ge 0$, there is a constant $K_{T,\gamma} > 0$ such that for any φ_1 and φ_2 in C_γ,*

$$\|T_2\varphi_1 - T_2\varphi_2\|_{C_\gamma(t)} \le K_{T,\gamma} \int_{s=0}^{t} \|\varphi_1 - \varphi_2\|_{C_\gamma(s)}\, ds \tag{6.13}$$

for all $0 \le t \le T$.

PROOF. The proof that T_2 is a well-defined mapping from C_γ to $C(\mathbb{R}_+ \times M)$ follows from estimates like (6.7). For any φ in C_γ and all $t \ge 0$ and x in M, we can use claim (a) of Proposition 3.3 to see that

$$\begin{aligned}&\int_{s=0}^{t}\int_{y\in M^\circ} |p_y^R(t-s,x)|\left|f\big(s,\varphi(s,y)\big)\right|\alpha(dy)\,ds\\ &\quad\le \int_{s=0}^{t}\int_{y\in M^\circ} \kappa_1^{-1}(T)\exp\left[-\kappa_1(T)\frac{d^2(x,y)}{s}\right]s^{-n/2}\\ &\qquad\times \overline{F}\big(\operatorname{dist}(y,\partial M)\big)^{\eta_1-\eta_2\gamma}\alpha(dy)\,ds\,\|\varphi\|_{C_\gamma(t)}\end{aligned} \tag{6.14}$$

and that the last integral is finite. Thus $(T_2\varphi)(t,x)$ is well defined for all φ in C_γ and all $t \ge 0$ and x in M. The continuity of $(T_2\varphi)(t,x)$ in t and x follows from fairly standard arguments involving the uniform integrability of the integrand of the right side (6.14) as s, x and y vary (see [11], Lemma 1.3.1). The bound (6.13) stems from (6.9) and the estimate

$$|(T_2\varphi_1)(t,x)-(T_2\varphi_2)(t,x)| \le \int_{s=0}^{t}\left(\overline{f}\int_{y\in M^\circ}|p_y^R(t-s,x)|\alpha(dy)\right)\|\varphi_1-\varphi_2\|_{C_\gamma(s)}\,ds,$$

which holds for every φ_1 and φ_2 in C_γ and every $t \ge 0$ and x in M. By a bound similar to that of (6.14), the term in parentheses is integrable with uniformly bounded integral as $t-s$ and x vary over $[0,T]\times M$. □

These two results allow us to show that the mapping B defined by (6.3) is well behaved. In particular, they allow us to solve (6.10) by a Picard iteration scheme:

PROPOSITION 6.3. *For any fixed* $\gamma > (n-1)/2$, *if the function* f *satisfies conditions* $nl1_\gamma$ *and* $nl2_\gamma$, *then* B *is an invertible mapping of* C_γ *into itself with inverse given by* (6.12).

PROOF. For any φ in C_γ, we approximate (6.10) by

$$\begin{aligned}(B_0\varphi)(t,x) &:= \varphi(t,x)+(T_1u_0)(t,x),\\ (B_{n+1}\varphi)(t,x) &:= \varphi(t,x)+(T_1u_0)(t,x)+\big(T_2(B_n\varphi)\big)(t,x)\end{aligned} \tag{6.15}$$

for all $t \ge 0$ and all x in M°. Because C_γ is a vector space, Lemmas 6.1 and 6.2 ensure that $\{B_n\varphi\}$ is a well-defined sequence in C_γ. Furthermore, from (6.13) and standard calculations,

$$\begin{aligned}\sum_{n=0}^{\infty}\|B_{n+1}\varphi-B_n\varphi\|_{C_\gamma(t)} &\le \|T_2(B_0\varphi)\|_{C_\gamma(t)}\sum_{n=0}^{\infty}\frac{(K_{t,\gamma}t)^n}{n!}\\ &= \|T_2(B_0\varphi)\|_{C_\gamma(t)}\exp[K_{t,\gamma}t]\end{aligned}$$

for each $t \geq 0$, so there is some element $B_\infty\varphi$ of C_γ such that

$$\lim_{n \to \infty} \|B_n\varphi - B_\infty\varphi\|_{C_\gamma(t)} = 0$$

for each $t \geq 0$. Upon letting n tend to infinity in the second equation of (6.15) and using (6.13), we see that thus $B_\infty\varphi$ satisfies (6.10). If ψ in C_γ is any other solution of (6.10), then for any $0 \leq t \leq T$,

$$\|B_\infty\varphi - \psi\|_{C_\gamma(t)} \leq K_{T,\gamma} \int_{s=0}^{t} \|B_\infty\varphi - \psi\|_{C_\gamma(s)}\, ds,$$

so by Gronwall's lemma, $\|B_\infty\varphi - \psi\|_{C_\gamma(T)} = 0$. Hence $B_\infty\varphi$ is the unique solution of (6.10), and we may simply label it as $B\varphi$.

These calculations have shown that B is a well-defined mapping from C_γ to itself. The invertibility of B is clear from the explicit formula (6.11). □

We are almost finished. Proposition 6.3 implies that the random field u defined by combining (4.2), (6.2) and (6.3) is a well-defined element of C_γ. The remaining step is to show that (6.2)–(6.3), which was suggested by analogy with (3.3) and (6.1), actually gives the unique solution of (I.1) in the sense of Definition 2.2:

THEOREM 6.4. *If the nonlinearity f satisfies conditions $nl1_\gamma$ and $nl2_\gamma$ for some fixed $\gamma > (n-1)/2$, then there is a random field u in C_γ that solves* (I.1) *in the sense of Definition* 2.2. *This solution is unique in the sense that if $\widetilde{u}$ is any other random element of C_γ that solves* (I.1) *in the sense of Definition* 2.2, *then*

$$\mathbb{P}\{\widetilde{u}(t,x) = u(t,x) \text{ for all } t \geq 0 \text{ and } x \in M^\circ\} = 1.$$

Finally, this solution is exactly $u = Bu_l$, where u_l is given by (4.2).

PROOF. We first show that (6.2) gives a solution of (I.1) in the sense of Definition 2.2. Condition $a.1$ is true thanks to the continuity results of Proposition 4.1 and Lemmas 6.1 and 6.2. In light of Proposition 4.1, conditions $a.2$ and $a.3$ will be true if the following five limits exist for every φ in $C_c^\infty(\mathbb{R}_+ \times M)$ and every ψ in C_γ:

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_{t=0}^{\infty} \int_{x \in M_\varepsilon} (T_1 u_0)(t,x)\varphi(t,x)\alpha(dx)\, dt,\\
&\lim_{\varepsilon \to 0} \int_{t=0}^{\infty} \int_{x \in M_\varepsilon} (T_2 \psi)(t,x)\varphi(t,x)\alpha(dx)\, dt,\\
&\lim_{\varepsilon \to 0} \int_{t=0}^{\infty} \int_{x \in M_\varepsilon} f\big(x, \psi(t,x)\big)\varphi(t,x)\alpha(dx)\, dt, \qquad (6.16)\\
&\lim_{\varepsilon \to 0} \int_{t=0}^{\infty} \int_{\bar{x} \in \partial M} (T_1 u_0)\big(t, \theta_\varepsilon(\bar{x})\big)\varphi(t,\bar{x})\overline{\alpha}(d\bar{x})\, dt,\\
&\lim_{\varepsilon \to 0} \int_{t=0}^{\infty} \int_{\bar{x} \in \partial M} (T_2 \psi)\big(t, \theta_\varepsilon(\bar{x})\big)\varphi(t,\bar{x})\overline{\alpha}(d\bar{x})\, dt.
\end{aligned}$$

The first two and last two limits follow from the continuity of $T_1 u_\circ$ and $T_2\psi$ on $\mathbb{R}_+ \times M$. In fact, we may easily identify these limits:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{t=0}^{\infty}\int_{x\in M_\varepsilon}(T_1u_0)(t,x)\varphi(t,x)\alpha(dx)\,dt\\
&\qquad=\int_{y\in M^\circ}\int_{t=0}^{\infty}\int_{x\in M^\circ}p_y^R(t,x)\varphi(t,x)\alpha(dx)\,dt\,u_0(y)\alpha(dy),\\
&\lim_{\varepsilon\to 0}\int_{t=0}^{\infty}\int_{x\in M_\varepsilon}(T_2\psi)(t,x)\varphi(t,x)\alpha(dx)\,dt\\
&\qquad=\int_{s=0}^{\infty}\int_{y\in M^\circ}\int_{t=s}^{\infty}\int_{x\in M^\circ}p_y^R(t-s,x)\varphi(t,x)\alpha(dx)\,dt\\
&\qquad\qquad\times f\big(y,\psi(s,y)\big)\alpha(dy)\,ds,\\
&\lim_{\varepsilon\to 0}\int_{t=0}^{\infty}\int_{\bar{x}\in\partial M}(T_1u_0)\big(t,\theta_\varepsilon(\bar{x})\big)\varphi(t,\bar{x})\overline{\alpha}(d\bar{x})\,dt\\
&\qquad=\int_{y\in M^\circ}\int_{t=0}^{\infty}\int_{\bar{x}\in\partial M}p_y^R(t,\bar{x})\varphi(t,\bar{x})\overline{\alpha}(d\bar{x})\,dt\,u_0(y)\alpha(dy),\\
&\lim_{\varepsilon\to 0}\int_{t=0}^{\infty}\int_{\bar{x}\in\partial M}(T_2\psi)\big(t,\theta_\varepsilon(\bar{x})\big)\varphi(t,\bar{x})\overline{\alpha}(d\bar{x})\,dt\\
&\qquad=\int_{s=0}^{\infty}\int_{y\in M^\circ}\int_{t=s}^{\infty}\int_{\bar{x}\in\partial M}p_y^R(t-s,\bar{x})\varphi(t,\bar{x})\overline{\alpha}(d\bar{x})\,dt\\
&\qquad\qquad\times f\big(y,\psi(s,y)\big)\alpha(dy)\,ds.
\end{aligned}\tag{6.17}$$

The first two of these limits follow from dominated convergence and the absolute integrability of the right-hand sides [the calculations are similar to our study of (4.3)]. The last two follow from calculations similar to our analysis of (4.8) [one can directly use (4.11) and (4.12)]. Returning now to the third limit of (6.16), we see that it exists thanks to a calculation similar to (6.14):

$$\begin{aligned}
&\int_{t=0}^{\infty}\int_{y\in M^\circ}|\varphi(t,x)|\big|f\big(x,\psi(s,x)\big)\big|\alpha(dx)\,ds\\
&\qquad\le\sup_{\substack{t\ge 0\\ x\in N}}|\varphi(t,x)|\,\|\psi\|_{C_\gamma(T)}^{\eta_1}\int_{t=0}^{T}\int_{y\in M^\circ}\big(\mathrm{dist}(y,\partial M)\big)^{\eta_1-\eta_2\gamma}\alpha(dy)\,ds,
\end{aligned}$$

where $T>0$ is any number such that supp $\varphi\subset[0,T]\times M$, and where the integral on the right-hand side is finite. We thus see that indeed

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{t=0}^{\infty}\int_{x\in M_\varepsilon}f\big(x,\psi(t,x)\big)\varphi(t,x)\alpha(dx)\,dt\\
&\qquad=\int_{t=0}^{\infty}\int_{x\in M^\circ}f\big(x,\psi(t,x)\big)\varphi(t,x)\alpha(dx)\,dt,
\end{aligned}\tag{6.18}$$

where the right side makes sense. Finally, consider condition $a.3$. In light of Proposition 4.1, (6.17) and (6.18) and some simple calculations, condition $a.3$ follows from (4.14) [set $s = 0$ in (4.14) to take care of the contribution of $T_1 u_0$]. This completes the proof that (6.2) gives a solution of (I.1) in the sense of Definition 2.2.

Finally, consider the uniqueness question. If u in C_γ solves (I.1) in the sense of Definition 2.2, then we may define $\widehat{u} := B^{-1}u$. By Proposition 6.3, $\widehat{u}$ is in C_γ, and by collecting together (6.12), (6.17) and (6.18), $\widehat{u}$ must actually solve (4.1). Proposition 4.2 then implies that $\widehat{u} = u_l$ $\mathbb{P}$-a.s., so $u = Bu_l$. □

The picture is now complete. If the nonlinearity f satisfies conditions $nl1_\gamma$ and $nl2_\gamma$, then (I.1) has a unique solution in C_γ.

7. Conclusion. This completes our study of the equation (I.1). We have developed the basic theory for the Robin problem with white noise perturbations on the boundary. We have the existence and uniqueness results and we have a characterization of the solution given by (4.2), (6.2) and (6.3). In addition, we have a description of the boundary layer behavior for (I.1). We know the degeneracies of u both in the normal and tangential directions at the boundary.

There are a number of ways to build on the results presented here. Using the functional space C_γ of Section 5, one can consider boundary perturbations of the form

$$(\nu, \nabla u) + \beta(x)u|_{\mathbb{R}_+ \times \partial M} = \sigma_t(u)\zeta,$$

where for every t, $\sigma_t\colon C_\gamma(t) \to \mathbb{R}$ is some mapping with good enough behavior. Of course, with this boundary condition, the solution of both problems (I.1) and (4.1) would be non-Gaussian, so the estimates of Appendix B would need some major modifications. Another question one could consider involves the "correctness" of the white-noise perturbations ζ as the limit of a fast-oscillating noise; that is, central limit theorem type results (the case $n = 1$ was considered in [10]), and of course one could investigate large deviations for (I.1) if the random perturbation has the form $\varepsilon\zeta$ for small $\varepsilon > 0$. The question of random perturbations of PDE's by boundary noise is a new field with many interesting problems.

APPENDIX A

The Robin Kernel. We give here an idea of the proofs of Propositions 3.1–3.3. Although the construction and analysis of p^R using the parametrix method is classical, we have been unable to find a reference that gives the estimates we need, namely, Propositions 3.2 and 3.3. Because these estimates are crucial for our calculations, we shall briefly point out how to get these estimates from standard constructions and arguments. We shall in particular rely on [8], but other relevant references are [4, 5, 11 and 15].

To begin, we construct the double of M, which we shall denote by $\widetilde{M}$. Then $\widetilde{M}$ is a compact manifold without boundary and M is a regular domain in $\widetilde{M}$.

We extend the metric tensor $(\cdot,\cdot)$, the vector field b and the function c to all of $\widetilde{M}$ in any smooth way, denoting these extensions as $(\cdot,\cdot)^{\sim}$, $\widetilde{b}$ and $\widetilde{c}$. All objects related to $\widetilde{M}$ instead of M will have tildes. The manifold $\widetilde{M}$ together with $(\cdot,\cdot)^{\sim}$ is a Riemannian manifold. The metric $(\cdot,\cdot)^{\sim}$ induces gradient and Laplace–Beltrami operators $\widetilde{\nabla}$ and $\widetilde{\Delta}$ on $C^\infty(\widetilde{M})$ and a volume measure $\widetilde{\alpha}$. We can then define a second-order differential operator $\widetilde{\mathcal{L}}$ on $\widetilde{M}$ by attaching tildes to all geometric objects in (2.2). Classical techniques (see succeeding text) allow us to construct the heat kernel $\widetilde{p}$ for $\widetilde{\mathcal{L}}$ on $\widetilde{M}$. For each $\widetilde{y}$ in $\widetilde{M}$, $\widetilde{p}_{\widetilde{y}}$ is a solution of the PDE

$$\frac{\partial \widetilde{p}_{\widetilde{y}}}{\partial t} = \widetilde{\mathcal{L}}\widetilde{p}_{\widetilde{y}}, \qquad \lim_{t\to 0} \widetilde{p}_{\widetilde{y}}(t,\cdot) = \widetilde{\delta}_{\widetilde{y}} \tag{A.1}$$

with $\widetilde{\delta}_{\widetilde{y}}$ denoting the Dirac measure on $(\widetilde{M}, \mathcal{B}(\widetilde{M}))$ concentrated at $\widetilde{y}$. Note that $(\widetilde{\mathcal{L}}f)(x) = (\mathcal{L}f)(x)$ for any f in $C^\infty(\widetilde{M})$ and all x in M, so for each y in M, $\widetilde{p}_y$ must also satisfy

$$\frac{\partial \widetilde{p}_{y}}{\partial t} = \mathcal{L}\widetilde{p}_{y}, \qquad \lim_{t\to 0} \widetilde{p}_{y}(t,\cdot) = \delta_{y} \tag{A.2}$$

for all y in M°, where here δ_y is (as in Section 3) the Dirac measure on $(M, \mathcal{B}(M))$ concentrated at y. The heat kernel $\widetilde{p}$ will be the starting point of our construction of p^R.

As we noted previously, the construction and asymptotics of the heat kernel (A.1) is a classical problem. The relevant results are that the solution of (A.1) exists and that there is some function φ in $C^\infty(\mathbb{R}_+ \times \widetilde{M} \times \widetilde{M})$ such that

$$\widetilde{p}_{\widetilde{y}}(t,\widetilde{x}) = \exp\left[-\widetilde{d}^2(\widetilde{x},\widetilde{y})/(2t)\right](2\pi t)^{-n/2}\varphi(t,\widetilde{x},\widetilde{y}), \qquad t>0,\ \widetilde{x}\in\widetilde{M}, \tag{A.3}$$

for all $\widetilde{y}$ in $\widetilde{M}$ and such that $\varphi(0,\widetilde{x},\widetilde{x}) = 1$ for all $\widetilde{x}$ in $\widetilde{M}$. One can copy the arguments of [8], Chapter 6, everywhere replacing the Laplacian by $\widetilde{\mathcal{L}}$. The dominant asymptotic behavior of (A.3) is the same as that of Proposition 3.2, except that $\widetilde{d}$ replaces d. Of course if we replace M by H_n and $\widetilde{M}$ by $\mathbb{R}^n$, then because H_n is convex, $\widetilde{d}_{\mathbb{R}^n}(x,y) = d_{H_n}(x,y)$ for all x and y in H_n. We can extend this special case to our general M and $\widetilde{M}$ by expanding the metric tensor $(\cdot,\cdot)$ in Fermi coordinates. This yields a ϖ with $0 < \varpi \le 1$ such that $\varpi d(x,y) \le \widetilde{d}(x,y) \le \varpi^{-1}d(x,y)$ for all x and y in M. Thus it is more or less sufficient, at least for the purposes of Proposition 3.3, that p^R also have dominant asymptotics like those of (A.3).

We now start to construct the Robin kernel p^R_y. The dominant term should be $\widetilde{p}_y$, but recalling the model (3.4), we shall first add to $\widetilde{p}_y$ its reflection across ∂M. More specifically, for any y in M such that $\text{dist}(y,\partial M) \le \varepsilon'_{\text{tub}}$, necessarily $y = E(\text{dist}(y,\partial M)\nu(\bar{y}))$ for some $\bar{y}$ in ∂M, where E is the exponential map of Section 2. Letting now $\widetilde{E}$ denote the exponential map on $\widetilde{M}$ and considering $\nu(\bar{y})$ as an element of of $T\widetilde{M}$, we set $\iota(y) := \widetilde{E}(-\text{dist}(y,\partial M)\nu(\bar{y}))$. We also let $\psi'\colon M \to [0,1]$ be any C^∞ function such that $\psi'(y) = 1$ if $\text{dist}(y,\partial M) \le \varepsilon'_{\text{tub}}/4$

and $\psi'(y) = 0$ if $\operatorname{dist}(y, \partial M) \geq \varepsilon'_{\text{tub}}/2$. Finally, we set

$$p_y^{R,\dagger}(t,x) := \widetilde{p}_y(t,x) + \psi'(y)\widetilde{p}_{\iota(y)}(t,x), \qquad t > 0,\ x \in M,$$

for each y in M. Note that $p_y^{R,\dagger}$ also satisfies (A.2) for each y in M°. We then search for p_y^R in the form

$$\text{(A.4)}\quad \widetilde{p}_y^R(t,x) = p_y^{R,\dagger}(t,x) + \int_{s=0}^{t}\int_{\bar{z}\in\partial M} \widetilde{p}_{\bar{z}}(t-s,x)\bar{f}_y(s,\bar{z})\overline{\alpha}(d\bar{z})\,ds, \quad t > 0,\ x \in M,$$

for some sufficiently regular function $\bar{f}_y\colon \mathbb{R}_+ \times \partial M \to \mathbb{R}$. By the jump relation ([8], Theorem 7.1.1), $\bar{f}_y$ should then satisfy

$$\bar{f}_y(t,\bar{x}) = \left(\overline{\mathcal{L}}p_y^{R,\dagger}\right)(t,\bar{x}) + \int_{s=0}^{t}\int_{\bar{z}\in\partial M} (\overline{\mathcal{L}}\widetilde{p}_{\bar{z}})(t-s,\bar{x})\bar{f}_y(s,\bar{z})\overline{\alpha}(d\bar{z})\,ds$$

for all $t > 0$ and $\bar{x}$ in ∂M. This suggests the recursion

$$\bar{g}_y^0(t,\bar{x}) := \left(\overline{\mathcal{L}}p_y^{R,\dagger}\right)(t,\bar{x})$$

$$\text{(A.5)}\quad \bar{g}_y^{j+1}(t,\bar{x}) := \int_{s=0}^{t}\int_{\bar{z}\in\partial M} (\overline{\mathcal{L}}\widetilde{p}_{\bar{z}})(t-s,\bar{x})\bar{g}_y^j(s,\bar{z})\overline{\alpha}(d\bar{z})\,ds,$$
$$t > 0,\ \bar{x} \in M^\circ, n = 0, 1, \ldots,$$

with $\bar{f}$ being defined as

$$\text{(A.6)}\qquad \bar{f}_y(t,\bar{x}) := \sum_{j=0}^{\infty} \bar{g}_y^j(t,\bar{x}), \qquad t > 0,\ \bar{x} \in \partial M.$$

A detailed study of this recursion will finally yield the estimates of Proposition 3.3. To simplify our calculations, let us now define

$$\mathcal{E}(t,x,y;v) := v^{-1}\exp\left[-v\widetilde{d}^2(x,y)/t\right], \qquad t > 0,\ x, y \in M,$$

for each $v > 0$. The preexponential term v^{-1} will make some of our formulae simpler. Note for future reference that for each $t > 0$ and x and y in M, the mapping $v \mapsto \mathcal{E}(t,x,y;v)$ is monotone decreasing on $(0,1]$. Let us also now fix a finite time horizon $T > 0$. We will only consider (A.4)–(A.6) for t in $(0, T]$, and henceforth all v's and η's will be positive constants that depend (only) on T. Finally, Γ will denote the standard gamma function.

Some straightforward modifications of the parametrix calculations of [8], Chapter 6.4, show that the sum in (A.6) converges absolutely for each t in $(0,T]$, $\bar{x}$ in ∂M and y in M, and that there is a $v_1 > 0$ such that

$$\text{(A.7)}\qquad |\bar{f}_y(t,\bar{x})| \leq \mathcal{E}(t,\bar{x},y;v_1)t^{-n/2}, \qquad t \in (0,T],\ \bar{x} \in \partial M,\ y \in M.$$

We give only the most relevant parts of these calculations. Like equation (2) of [8], Section 7.1, there is a constant $v_{1,1} > 0$ such that

$$\text{(A.8)}\quad \left|\overline{\mathcal{L}}p_y^{R,\dagger}(t,\bar{x})\right| \leq \mathcal{E}(t,\bar{x},y;v_{1,1})t^{-n/2} \quad\text{and}\quad \left|\overline{\mathcal{L}}\widetilde{p}_{\bar{z}}(t,\bar{x})\right| \leq \mathcal{E}(t,\bar{x},\bar{z};v_{1,1})t^{-n/2}$$

for all $t \in (0,T]$, $\bar{x} \in \partial M$, $y \in M$ and $\bar{z} \in \partial M$. These bounds can be shown by using Fermi coordinates. It is very important here that ν in (2.3) is *normal* to ∂M. Next note that there is a constant η_1 with $0 < \eta_1 < 1$ such that for any $\widetilde{\upsilon} \le \upsilon_{1,1}$,

$$\begin{aligned}
&\int_{s=0}^{t}\int_{\bar{z}\in\partial M} \mathcal{E}(t-s,x,\bar{z};\upsilon_{1,1})(t-s)^{-n/2}\mathcal{E}(s,\bar{z},y;\widetilde{\upsilon})s^{(j-n)/2}\overline{\alpha}(d\bar{z})\,ds\\
&\quad\le \widetilde{\upsilon}^{-(n+1)/2}\mathcal{E}(t,x,y;\eta_1\widetilde{\upsilon})t^{-(n-1)/2}\left\{\int_{s=0}^{t}(t-s)^{-1/2}s^{(j-1)/2}\,ds\right\}\\
&\quad= \frac{\Gamma(1/2)\Gamma(j/2+1/2)}{\widetilde{\upsilon}^{(n+1)/2}\Gamma(j/2+1)}\mathcal{E}(t,x,y;\eta\widetilde{\upsilon})t^{((j+1)-n)/2}
\end{aligned}\tag{A.9}$$

for all $t \in (0,T]$, and all $x,y \in M$. The analogous calculation on H_n is simple, and the constant η_1 compensates for the error incurred by mapping (A.9) to the calculation on H_n. We use this calculation to show that for each $j = 0,1,\dots,$ there is a positive constant $\upsilon_{1,2,j}$ such that

$$|\bar{g}^j_y(t,\bar{x})| \le \mathcal{E}(t,\bar{x},y;\upsilon_{1,2,j})t^{(j-n)/2}, \qquad 0<t\le T,\ \bar{x}\in\partial M,\ y\in M. \tag{A.10}$$

Then the preexponential singularities of the $\bar{g}^j$'s get weaker and weaker as j grows. The cost of this is the constant η_1, which, because it is strictly less than 1, also weakens the exponential decay for small time. We shall use (A.10) to bound $\bar{g}^j$ only for $j = 0,1,\dots,n$. For $j = n,\ n+1,\ n+2,\dots,$ another calculation replaces (A.9), and this other calculation preserves the exponential decay for small time. There is a second constant $\eta_2 > 0$ such that for any $j = n,\ n+1,\ \dots,$

$$\begin{aligned}
&\int_{s=0}^{t}\int_{\bar{z}\in\partial M} \mathcal{E}(t-s,x,\bar{z};\upsilon_{1,1})(t-s)^{-n/2}\mathcal{E}(s,\bar{z},y;\upsilon_{1,2,n})s^{(j-n)/2}\overline{\alpha}(d\bar{z})\,ds\\
&\quad\le \mathcal{E}(t,\bar{x},y;\upsilon_{1,2,n})\left(\frac{\upsilon_{1,1}-\upsilon_{1,2,n}}{\upsilon_{1,2,n}}\right)\\
&\qquad\times\int_{s=0}^{t}\int_{\bar{z}\in\partial M}\mathcal{E}(t-s,\bar{x},\bar{z};\upsilon_{1,1}-\upsilon_{1,2,n})(t-s)^{-(n-1)/2}\overline{\alpha}(d\bar{z})\\
&\qquad\times(t-s)^{-1/2}s^{(j-n)/2}\,ds\\
&\quad\le \frac{\eta_2\Gamma(1/2)}{(\upsilon_{1,1}-\upsilon_{1,2,n})^{(n-1)/2}}\frac{\Gamma((j-n+2)/2)}{\Gamma((j-n+3)/2)}\mathcal{E}(t,\bar{x},y;\upsilon_{1,2,n})t^{((j+1)-n)/2}
\end{aligned}\tag{A.11}$$

for all $t \in (0,T]$ and all $x,\ y \in M$. The first exponential term on the right side of the first inequality comes from [8], Lemma 6.4.3. Note that because η_1 in (A.9) is strictly less than 1, $\upsilon_{1,2,n} < \upsilon_{1,1}$. We use this calculation to bound the $\bar{g}^j$'s for $j = n+1, n+2,\dots.$ The short-time exponential decay of all these $\bar{g}^j$'s is thus $G(\cdot,\cdot,\cdot;\upsilon_{1,2,n})$ and the preexponential terms decay to zero like $\upsilon^j/\Gamma((j-n+3)/2)$ (see [8], Section 7.2). This, in conjunction with (A.10), shows the convergence of (A.6) and yields the bound (A.7) for some υ_1.

For convenience, we now define

$$I_y(t,x) := \int_{s=0}^{t}\int_{\bar z\in\partial M}\tilde p_{\bar z}(t-s,\bar x)\bar f_y(s,\bar z)\overline{\alpha}(d\bar z)\,ds \tag{A.12}$$

for all $t \in (0,T]$ and x, $y \in M$ such that the right side is well defined. To show that I *is* well defined for all $t \in (0,T]$ and $x, y \in M$, and to also get the bounds of Proposition 3.3, we first need some corresponding bounds on $p^{R,\dagger}$ and $\tilde p$. Appealing again to local coordinates, we can see that there is a $\upsilon_2 > 0$ such that

$$\begin{aligned} &\left|p_y^{R,\dagger}(t,x)\right| \le \mathcal{E}(t,x,y;\upsilon_2)t^{-n/2}, \left|\tilde p_y(t,x)\right| \le \mathcal{E}(t,x,y;\upsilon_2)t^{-n/2},\\ &\left\|\nabla p_y^{R,\dagger}(t,x)\right\| \le \mathcal{E}(t,x,y;\upsilon_2)t^{-n/2-1/2}, \left\|\nabla\tilde p_y(t,x)\right\| \le \mathcal{E}(t,x,y;\upsilon_2)t^{-n/2-1/2},\\ &\left|\frac{\partial p_y^{R,\dagger}}{\partial t}(t,x)\right| \le \mathcal{E}(t,x,y;\upsilon_2)t^{-n/2-1}, \left|\frac{\partial\tilde p_y}{\partial t}(t,x)\right| \le \mathcal{E}(t,x,y;\upsilon_2)t^{-n/2-1} \end{aligned} \tag{A.13}$$

for all $t \in (0,T]$, $\bar x \in \partial M$, $y \in M$, and $\bar z \in \partial M$. These bounds, along with (A.7), give us the integrability of (A.12) and most of Proposition 3.2.

Idea of proof of existence of p^R and ideas of part of proof of Proposition 3.3. To do a number of calculations simultaneously, let us let j be 0, 1 or 2, and fix any λ' such that $0 \le \lambda' \le \min\{1,j\}$. Calculations similar to those proving Proposition 3.4 show that

$$\begin{aligned} \mathcal{E}(t,x,\bar z;\upsilon_2)t^{-n/2-j/2} &\le \left(\frac{\upsilon_2/2}{\tilde d^2(x,\bar z)}\right)^{(j-\lambda')/2}\mathcal{E}\left(t,x,\bar z;\frac{\upsilon_2}{2}\right)t^{-(n-1)/2-(\lambda'+1)/2}\\ &\quad\times\left(\frac{\tilde d^2(x,\bar z)}{(\upsilon_2/2)t}\right)^{(j-\lambda')/2}\mathcal{E}\left(t,x,\bar z;\frac{\upsilon_2}{2}\right)\\ &\le \left(\frac{\upsilon_2}{2}\right)^{(j-\lambda')/2}C(j-\lambda')\big(\operatorname{dist}(x,\partial M)\big)^{\lambda'-j}\\ &\quad\times\mathcal{E}\left(t,x,\bar z;\frac{\upsilon_2}{2}\right)t^{-(n-1)/2-(\lambda'+1)/2} \end{aligned} \tag{A.14}$$

for all $t \in (0,T]$, all $x \in M$ and all $\bar z \in \partial M$. Here $C(j-\lambda')$ is defined by (3.8). A calculation similar to (A.9) now shows that for some $\eta_3 > 0$,

$$\begin{aligned} &\int_{s=0}^{t}\int_{\bar z\in\partial M}\mathcal{E}\left(t-s,x,\bar z;\frac{\upsilon_2}{2}\right)(t-s)^{-(n-1)/2-(\lambda'+1)/2}\\ &\qquad\times\mathcal{E}(s,\bar z,y;\upsilon_1)s^{-n/2}\overline{\alpha}(d\bar z)\,ds\\ &\quad\le\frac{\eta_3\Gamma(1/2)\Gamma((1-\lambda')/2)}{\upsilon_4^{(n+1)/2}\Gamma(1-\lambda'/2)}\mathcal{E}(t,x,y;\eta_1\upsilon_3)t^{-n/2+(1-\lambda')/2} \end{aligned} \tag{A.15}$$

for all $t \in (0,T]$ and all x, $y \in M$, where $\upsilon_3 := \min\{\upsilon_1,\upsilon_2/2\}$. From (A.4), (A.12), (A.13) and (A.14), this is exactly what we need to get some of our desired results.

Taking $j = \lambda' = 0$, we get that the integral of (A.12) is indeed well defined and that for some v_4,

$$|I_y(t,x)| \leq \mathcal{E}(t,x,y;v_4)t^{-n/2}, \qquad t \in (0,T],\ x,\ y \in M.$$

Combine this, the first inequality of (A.13) and (A.14) to get claim (a) of Proposition 3.3. To get claim (b), at least for $0 < \lambda < 1$, take $j = 1$ and $\lambda' = 1 - \lambda$. Then by taking the gradient of I_y and moving the gradient operation inside the integrals in (A.12) and using (A.14) and (A.15), for some $v_5 > 0$,

$$\|\nabla I_y(t,x)\| \leq \left(v_2/2\right)^{\lambda} C(\lambda)\left(\mathrm{dist}(x,\partial M)\right)^{-\lambda}\mathcal{E}(t,x,y;v_5)t^{-n/2-1/2}$$

for all $t \in (0,T]$ and $x,\ y \in M$. Combine this and the third inequality of (A.13) to get claim (b) of Proposition 3.3 for $0 < \lambda < 1$. The case $\lambda \geq 1$ follows by calculations similar to the proof of claim (a) of Proposition 3.4. To get the proof of claim (c), at least for $1 < \lambda < 2$, take $j = 2$ and $\lambda' = 2 - \lambda$. Take the time derivative of I_y and pass the derivative inside the integrals in (A.12) [recall that $\lim_{s\to t^-} \tilde{p}_{\bar{z}}(t-s,x) = 0$ for all $t \in (0,T]$, $x \in M^\circ$ and $\bar{z} \in \partial M$]. This shows that for some $v_6 > 0$,

$$\left|\frac{\partial I_y}{\partial t}(t,x)\right| \leq \left(\frac{v_2}{2}\right)^{\lambda} C(\lambda)\left(\mathrm{dist}(x,\partial M)\right)^{-\lambda}\mathcal{E}(t,x,y;v_6)t^{-n/2-1}$$

for all $t \in (0,T]$ and $x,\ y \in M$. Combine this and the fifth inequality of (A.13) to get claim (c) of Proposition 3.3 for $1 < \lambda < 2$. The calculations for $\lambda \geq 2$ follow similarly to those for claim (b) with $\lambda \geq 1$. Note that if we want to take $0 \leq \lambda \leq 1$ in claim (c) of Proposition 3.3, then if we set $\lambda' = 2 - \lambda$, the expression $(t-s)^{-(\lambda'+1)/2}$ is not an integrable function of s on $(0,t)$, so (A.15) fails. Thus our proof of claim (c) of Proposition 3.3 is only partially complete. However, a careful analysis of our entire effort in this paper shows that this is a restriction only for calculations (B.10)–(B.12) when $n = 2$ and $\gamma \leq 1$. □

The same arguments that are at the end of [8], Section 6.2, show that p^R has the regularity of Proposition 3.1. By the Chapman–Kolmogorov equation, $p^{R,*}$ as given by (3.12) must satisfy (3.13). Because we can apply the same entire preceding argument to the PDE (3.13), we also know that p^R is unique (see [5], Proposition 2.17). Finally, we can get the asymptotics of Proposition 3.2 from a slight modification of the foregoing estimates for p^R. For any fixed x in M°, (A.3) implies that $\tilde{p}_{\iota(y)}(t,x)$ and $I_y(t,x)$ both decay faster than $\exp[-\mathrm{dist}^2(x,\partial M)/(2t)]$ for any y in M. If $d(x,y) < \mathrm{dist}(x,\partial M)$, then the exact decay of $\tilde{p}_y$ as given by (A.3) clearly dominates and gives Proposition 3.2 [a needed auxiliary result is that $d(x,y) = \tilde{d}(x,y)$ if both x and y in M are close enough together and far enough from ∂M].

Our only remaining task is to plug up the hole in the proof of claim (c) of Proposition 3.3. Clearly it is sufficient to prove the claim for $\lambda = 0$ [recall the proof of (3.9)]. From (A.13), $p^{R,\dagger}$ has the correct behavior. We thus need to show that for some v_7,

$$\left|\frac{\partial I_y}{\partial t}(t,x)\right| \leq \mathcal{E}(t,x,y;v_7)t^{-n/2-1}, \qquad t \in (0,T],\ x,\ y \in M. \tag{A.16}$$

A direct differentiation under the integral sign in (A.12) is not sufficient; that led only to claim (c) of Proposition 3.3 with $\lambda > 1$. An alternate way of studying $\partial I_y/\partial t$ is to write it in terms of $\partial \bar{f}/\partial t$ instead. Things will work out if we first rewrite (A.12) as

$$I_y(t,x) = \int_{s=0}^{t/2}\int_{\bar{z}\in\partial M}\{\tilde{p}_{\bar{z}}(t-s,\bar{x})\bar{f}_y(s,\bar{z}) + \tilde{p}_{\bar{z}}(s,\bar{x})\bar{f}_y(t-s,\bar{z})\}\bar{\alpha}(d\bar{z})\,ds \tag{A.17}$$

for all $t \in (0,T]$ and $x,\ y \in M$. Then

$$\begin{aligned}\frac{\partial I_y}{\partial t}(t,x) &= 2\int_{\bar{z}\in\partial M}\tilde{p}_{\bar{z}}\left(\frac{t}{2},\bar{x}\right)\bar{f}_y\left(\frac{t}{2},\bar{z}\right)\bar{\alpha}(d\bar{z})\\ &\quad+\int_{s=0}^{t/2}\int_{\bar{z}\in\partial M}\left\{\frac{\partial\tilde{p}_{\bar{z}}}{\partial t}(t-s,\bar{x})\bar{f}_y(s,\bar{z})\right.\\ &\qquad\left.+\tilde{p}_{\bar{z}}(s,\bar{x})\frac{\partial\bar{f}_y}{\partial t}(t-s,\bar{z})\right\}\bar{\alpha}(d\bar{z})\,ds\end{aligned} \tag{A.18}$$

for all such t, x and y. Collecting together (A.7), (A.13) and a calculation like (A.9), we see that (A.16) will hold for some $\upsilon_7 > 0$ if for some $\upsilon_8 > 0$,

$$\left|\frac{\partial\bar{f}_y}{\partial t}(t,x)\right| \le \mathcal{E}(t,x,y;\upsilon_8)t^{-n/2-1}, \qquad t\in(0,T],\ x,\ y\in M. \tag{A.19}$$

The proof of (A.8) shows that $\bar{g}^0$ obeys a bound like (A.19), so we shall try to use (A.5) to show that $\bar{f}$ does indeed have the bound (A.19) for some υ_8. The same calculations as in (A.17) and (A.18) show that for $j = 1,2,\dots,$

$$\begin{aligned}\frac{\partial\bar{g}_y^{j+1}}{\partial t}(t,x) &= 2\int_{\bar{z}\in\partial M}\bar{\mathfrak{L}}\tilde{p}_{\bar{z}}\left(\frac{t}{2},\bar{x}\right)\bar{g}_y^j\left(\frac{t}{2},\bar{z}\right)\bar{\alpha}(d\bar{z})\\ &\quad+\int_{s=0}^{t/2}\int_{\bar{z}\in\partial M}\left\{\frac{\partial(\bar{\mathfrak{L}}\tilde{p}_{\bar{z}})}{\partial t}(t-s,\bar{x})\bar{g}_y^j(s,\bar{z})\right.\\ &\qquad\left.+\bar{\mathfrak{L}}\tilde{p}_{\bar{z}}(s,\bar{x})\frac{\partial\bar{g}_y^j}{\partial t}(t-s,\bar{z})\right\}\bar{\alpha}(d\bar{z})\,ds\end{aligned} \tag{A.20}$$

for all $t\in(0,T]$, $\bar{x}\in\partial M$ and $y\in M$. Collect together the bounds on the $\bar{g}^j$'s for $j = 0,1,\dots,n+1$ of (A.10), (A.8) and a bound on $\partial(\bar{\mathfrak{L}}\tilde{p})/\partial t$ like (A.19). A bound on (A.20) using calculations like (A.9) shows that for $j = 0,1,\dots,n+1$, there is a $\upsilon_{8,1,j} > 0$ such that

$$\left|\frac{\partial\bar{g}_y^j}{\partial t}(t,x)\right| \le \mathcal{E}(t,x,y;\upsilon_{8,1,j})t^{(j-n)/2-1}, \qquad t\in(0,T],\ x,\ y\in M.$$

Use these bounds for all $j = 0,1,\dots,n+1$. Then for $j = n+2, n+3,\dots,$ use the calculation

$$\frac{\partial\bar{g}_y^{j+1}}{\partial t}(t,x) = \int_{s=0}^{t}\int_{\bar{z}\in\partial M}\bar{\mathfrak{L}}\tilde{p}_{\bar{z}}(s,\bar{x})\frac{\partial\bar{g}_y^j}{\partial t}(t-s,\bar{z})\bar{\alpha}(d\bar{z}).$$

This is exactly the same recursion as (A.5), so the calculations (A.11) and the comments following it directly show that for all $j = n+2, n+3, \dots$, $\partial \overline{g}^j_y/\partial t$ is bounded by $\mathcal{E}(t,x,y;v_{8,1,n+1})$ and the preexponential terms tend to zero in a summable way. This completes the proof of (A.19) and thus of (A.16) and thus completes the whole proof of claim (c) of Proposition 3.3.

APPENDIX B

Proofs of Lemmas 5.2 and 5.4. We here give the proofs of Lemmas 5.2 and 5.4. Although these calculations are very technical, they are essential. The basic idea of these lemmas, and thus perhaps the most illuminating approach to proving them, is that locally p^R looks very much like (3.4). In this direction, we should use the bounds of Proposition 3.3. Brevity, on the other hand, obliges us to cut some corners and use some cruder bounds, namely, those of Proposition 3.4. The ambitious reader may at his or her leisure make the calculations given here much more precise. A particular suggested refinement might be to see how the continuity of the estimates of Lemma 5.4 could be improved depending on which argument—t, $\overline{x}$, r or ε—is varied.

Let us begin with some comments that will be useful in the proof of both lemmas. Some of our calculations will be simpler with the notation

$$\|\sigma\| := \sup_{\bar{y}\in\partial M} |\sigma(\overline{y})|.$$

Second, the following auxiliary result will help us in two specific parts of our proofs, namely, the continuity of $V_{1,\gamma}$ in $\overline{x}$ and the continuity of V_2 in $\overline{x}$ and r.

LEMMA B.1. *For every $\eta > 0$ there is a positive constant $\hbar^1_\eta$ such that*

$$\begin{aligned}\int_{\bar{y}\in\partial M} & d^{-(n-1)+\eta}\big(\theta_\varepsilon(\overline{x}_1),\overline{y}\big)d^{-(n-1)+\eta}\big(\overline{y},\theta_\varepsilon(\overline{x}_2)\big)\overline{\alpha}(d\overline{y})\\ &\le \hbar^1_\eta d^{-(n-1)+2\eta}\big(\theta_\varepsilon(\overline{x}_1),\theta_\varepsilon(\overline{x}_2)\big)\end{aligned}$$

for all $0 \le \varepsilon \le \varepsilon'_{\mathrm{tub}}$ and for all $\overline{x}_1$ and $\overline{x}_2$ in ∂M such that $\overline{x}_1 \ne \overline{x}_2$.

PROOF. This is a generalization of equation (3) of [8], Section 7.1. Here, as there, we leave the proof to the reader. □

We first prove Lemma 5.2. We can use (4.2) to rewrite (5.7) as

$$\text{(B.1)}\quad \begin{aligned}V_{1,\gamma}(t,\overline{x},\varepsilon) &= -\tfrac{1}{2}\int_{s=0}^{t}\int_{\bar{y}\in\partial M}\varepsilon^\gamma p^R_{\bar{y}}\big(t-s,\theta_\varepsilon(\overline{x})\big)\sigma(\overline{y})\zeta(ds,d\overline{y})\\ &= -\tfrac{1}{2}\int_{s=0}^{\infty}\int_{\bar{y}\in\partial M}\varepsilon^\gamma \chi_{\{s\le t\}}p^R_{\bar{y}}\big(t-s,\theta_\varepsilon(\overline{x})\big)\sigma(\overline{y})\zeta(ds,d\overline{y})\end{aligned}$$

for all $t \ge 0$, all $\overline{x}$ in ∂M and all $0 < \varepsilon \le \varepsilon'_{\mathrm{tub}}$, where $\chi_{\{s\le t\}}$ is the indicator function of the set $\{s \le t\}$. Thus

$$\mathbb{E}\left[|V_{1,\gamma}(t,\bar{x},\varepsilon)|^2\right] \le \tfrac{1}{4}\|\sigma\|^2 \int_{s=0}^{t} \int_{\bar{y}\in\partial M} \left\{\varepsilon^{\gamma} p_{\bar{y}}^{R}\left(t-s,\theta_\varepsilon(\bar{x})\right)\right\}^2 \overline{\alpha}(d\bar{y})\,ds \tag{B.2}$$

for all $(t,\bar{x},\varepsilon)$ in $(0,T] \times \partial M \times (0,\varepsilon'_{\text{tub}}]$ and

$$\begin{aligned}
&\mathbb{E}\left[|V_{1,\gamma}(t_1,\bar{x}_1,\varepsilon_1) - V_{1,\gamma}(t_2,\bar{x}_2,\varepsilon_2)|^2\right] \\
&\quad = \tfrac{1}{4}\int_{s=0}^{\infty}\int_{\bar{y}\in\partial M} \Big\{\varepsilon_2^{\gamma}\chi_{\{s\le t_2\}} p_{\bar{y}}^{R}\left(t_2-s,\theta_{\varepsilon_2}(\bar{x}_2)\right) \\
&\qquad\qquad - \varepsilon_1^{\gamma}\chi_{\{s\le t_1\}} p_{\bar{y}}^{R}\left(t_1-s,\theta_{\varepsilon_1}(\bar{x}_1)\right)\Big\}^2 |\sigma(\bar{y})|^2 \alpha(d\bar{y})\,ds \\
&\quad \le \tfrac{1}{4}\|\sigma\|^2 \int_{s=0}^{\infty}\int_{\bar{y}\in\partial M} \Big\{\varepsilon_2^{\gamma}\chi_{\{s\le t_2\}} p_{\bar{y}}^{R}\left(t_2-s,\theta_{\varepsilon_2}(\bar{x}_2)\right) \\
&\qquad\qquad - \varepsilon_1^{\gamma}\chi_{\{s\le t_1\}} p_{\bar{y}}^{R}\left(t_1-s,\theta_{\varepsilon_1}(\bar{x}_1)\right)\Big\}^2 \alpha(d\bar{y})\,ds
\end{aligned} \tag{B.3}$$

for all $(t_1,\bar{x}_1,\varepsilon_1)$ and $(t_2,\bar{x}_2,\varepsilon_2)$ in $[0,T] \times \partial M \times (0,\varepsilon'_{\text{tub}}]$.

It will turn out that our study of the variation of $V_{1,\gamma}$ in $\bar{x}$ will require yet another auxiliary result in addition to Lemma B.1:

LEMMA B.2. *Fix $\eta > 0$. Then there is a positive constant $\hbar_\eta^2$ such that if $\{\vartheta(r): r \ge 0\}$ is a geodesic in any $\partial_i M$ and ϑ has unit speed,*

$$\int_{r=0}^{R} d^{\eta-1}\Big(\theta_\varepsilon\big(\vartheta(r)\big), \theta_\varepsilon\big(\vartheta(0)\big)\Big)dr \le \hbar_\eta^2 R^\eta$$

for any $0 \le R \le \varrho$ and any $0 \le \varepsilon \le \varepsilon'_{\text{tub}}$.

PROOF. By using Fermi coordinates and fields one can expand the mapping $(\varepsilon, r) \mapsto d^{\eta-1}(\theta_\varepsilon(\vartheta(r)), \theta_\varepsilon(\vartheta(0)))$. This allows us to find a positive $\hbar_\eta^{2,\dagger}$ such that

$$\int_{r=0}^{R} d^{\eta-1}\Big(\theta_\varepsilon\big(\vartheta(r)\big), \theta_\varepsilon\big(\vartheta(0)\big)\Big)dr \le \hbar_\eta^{2,\dagger}\int_{r=0}^{R} r^{\eta-1}\,dr = \hbar_\eta^{2,\dagger}\eta^{-1}R^\eta$$

for all ϑ, R and ε as in the statement of the lemma. Take $\hbar_\eta^2 := \hbar_\eta^{2,\dagger}\eta^{-1}$. □

With these representations and results in hand, we can commence with the proof of Lemma 5.2.

PROOF OF LEMMA 5.2. We shall first show the boundedness result (5.8) and then show the continuity estimate (5.9). By virtue of the triangle inequality, we can show the continuity estimate (5.9) by separately considering the variation of t, $\bar{x}$ and ε. To simplify the various parts of the proof, let us now fix for the rest of the proof some $\varsigma > 0$ such that

$$\varsigma < \min\{\gamma - (n-1)/2 - \beta, (n-1)/2\}, \tag{B.4}$$

where β is as in the statement of Lemma 5.2. □

Boundedness. Fix $0 < t \le T$, $\bar{x}$ in ∂M and any $0 < \varepsilon \le \varepsilon'_{\mathrm{tub}}$. From (B.2) and a simple change of variables,

$$\mathbb{E}\left[|V_{1,\gamma}(t,\bar{x},\varepsilon)|^2\right] \le \tfrac{1}{4}\|\sigma\|^2 \int_{s=0}^{t}\int_{\bar{y}\in\partial M}\left\{\varepsilon^{\gamma}p_{\bar{y}}^{R}\left(s,\theta_\varepsilon(\bar{x})\right)\right\}^2\overline{\alpha}(d\bar{y})\,ds. \tag{B.5}$$

We bound the integrand using claim (a) of Proposition 3.4 with $\lambda_1 = n/2 + 1/2 + \varsigma$ and $\lambda_2 = \gamma$. Then $\lambda_1 \le n$ by (B.4), and $\lambda_2 - \lambda_1 = (\gamma - (n-1)/2 - \varsigma) - 1$ and $-n + \lambda_1 = -(n-1)/2 + \varsigma$. Consequently,

$$\begin{aligned}\left|\varepsilon^{\gamma}p_{\bar{y}}^{R}\left(s,\theta_\varepsilon(\bar{x})\right)\right| &\le K_1'(T, n/2 + 1/2 + \varsigma, \gamma)s^{((\gamma-(n-1)/2-\varsigma)-1)/2}\\ &\qquad\times d^{-(n-1)/2+\varsigma}\left(\theta_\varepsilon(\bar{x}),\bar{y}\right).\end{aligned} \tag{B.6}$$

Inserting this in (B.5), we get that

$$\begin{aligned}&\int_{s=0}^{t}\int_{\bar{y}\in\partial M}\left\{\varepsilon^{\gamma}p_{\bar{y}}^{R}\left(s,\theta_\varepsilon(\bar{x})\right)\right\}^2\overline{\alpha}(d\bar{y})\,ds\\ &\quad\le \left\{K_1'(T, n/2+1/2+\varsigma,\gamma)\right\}^2\left\{\int_{s=0}^{t}s^{(\gamma-(n-1)/2-\varsigma)-1}\,ds\right\}\\ &\qquad\times\left\{\int_{\bar{y}\in\partial M}d^{-(n-1)+2\varsigma}\left(\theta_\varepsilon(\bar{x}),\bar{y}\right)\overline{\alpha}(d\bar{y})\right\}\\ &\quad\le\left\{K_1'(T,n/2+1/2+\varsigma,\gamma)\right\}^2\\ &\qquad\times\Xi_{2\varsigma}\left\{\left(\gamma-(n-1)/2-\varsigma\right)^{-1}t^{\gamma-(n-1)/2-\varsigma}\right\}.\end{aligned} \tag{B.7}$$

Because $t^{\gamma-(n-1)/2-\varsigma} \le T^{\gamma-(n-1)/2-\varsigma}$, (5.8) follows by combining (B.5) and (B.7).

Variation in t. Fix next $0 < t_1 < t_2 \le T$, $\bar{x}$ in ∂M and $0 < \varepsilon \le \varepsilon'_{\mathrm{tub}}$. We may then rewrite (B.3) using some simple transformations as

$$\begin{aligned}&\mathbb{E}\left[|V_{1,\gamma}(t_1,\bar{x},\varepsilon) - V_{1,\gamma}(t_2,\bar{x},\varepsilon)|^2\right]\\ &\quad\le \tfrac{1}{4}\|\sigma\|^2\int_{s=0}^{t_1}\int_{\bar{y}\in\partial M}\left\{\varepsilon^{\gamma}p_{\bar{y}}^{R}\left(t_2-s,\theta_\varepsilon(\bar{x})\right)\right.\\ &\qquad\qquad\left.-\varepsilon^{\gamma}p_{\bar{y}}^{R}\left(t_1-s,\theta_\varepsilon(\bar{x})\right)\right\}^2\overline{\alpha}(d\bar{y})\,ds\\ &\qquad+\tfrac{1}{4}\|\sigma\|^2\int_{s=t_1}^{t_2}\int_{\bar{y}\in\partial M}\left\{\varepsilon^{\gamma}p_{\bar{y}}^{R}\left(t_2-s,\theta_\varepsilon(\bar{x})\right)\right\}^2\overline{\alpha}(d\bar{y})\,ds\\ &\quad=\tfrac{1}{4}\|\sigma\|^2\int_{s=0}^{t_1}\int_{\bar{y}\in\partial M}\left\{\varepsilon^{\gamma}p_{\bar{y}}^{R}\left(t_2-t_1+s,\theta_\varepsilon(\bar{x})\right)\right.\\ &\qquad\qquad\left.-\varepsilon^{\gamma}p_{\bar{y}}^{R}\left(s,\theta_\varepsilon(\bar{x})\right)\right\}^2\overline{\alpha}(d\bar{y})\,ds\end{aligned} \tag{B.8}$$

$$+\tfrac{1}{4}\|\sigma\|^2\int_{s=0}^{t_2-t_1}\int_{\bar{y}\in\partial M}\left\{\varepsilon^\gamma p_{\bar{y}}^R\big(s,\theta_\varepsilon(\bar{x})\big)\right\}^2\overline{\alpha}(d\bar{y})\,ds.$$

The last term is the easiest to bound. From (B.7) we can see that

$$\begin{aligned}&\int_{s=0}^{t_2-t_1}\int_{\bar{y}\in\partial M}\left\{\varepsilon^\gamma p_{\bar{y}}^R\big(s,\theta_\varepsilon(\bar{x})\big)\right\}^2\overline{\alpha}(d\bar{y})\,ds\\ &\quad\le\left\{K_1'\big(T,n/2+1/2+\varsigma,\gamma\big)\right\}^2\Xi_{2\varsigma}\\ &\qquad\times\big(\gamma-(n-1)/2-\varsigma\big)^{-1}T^{\gamma-(n-1)/2-\beta-\varsigma}|t_2-t_1|^\beta.\end{aligned}\tag{B.9}$$

Returning now to the first integral in the last equality of (B.8), we can use the fundamental theorem of calculus to express the difference there in terms of $\partial p_{\bar{y}}^R/\partial t$. However, we can bound $\partial p_{\bar{y}}^R/\partial t$ by Proposition 3.4, similarly to the proof of boundedness. Take $\lambda_1 = n/2 + 1/2 + \varsigma$ and $\lambda_2 = \gamma$ in claim (c) of Proposition 3.4. Then calculations similar to those of (B.6) give us the bound

$$\begin{aligned}&\left|\varepsilon^\gamma\left(p_{\bar{y}}^R\big(t_2-t_1+s,\theta_\varepsilon(\bar{x})\big)-\varepsilon^\gamma p_{\bar{y}}^R\big(s,\theta_\varepsilon(\bar{x})\big)\right)\right|\\ &\quad\le K_3'(T,n/2+1/2+\varsigma,\gamma)\left\{\int_{u=s}^{t_2-t_1+s}u^{(\gamma-(n-1)/2-\varsigma-3)/2}\,du\right\}\\ &\qquad\times d^{-(n-1)/2+\varsigma}\big(\theta_\varepsilon(\bar{x}),\bar{y}\big).\end{aligned}\tag{B.10}$$

Because $\gamma < (n-1)/2 + 1$ (we required this in the statement of the lemma) and β and ς are positive, $\gamma - (n-1)/2 - \beta - \varsigma - 1 < 0$, so the mapping $u \mapsto u^{(\gamma-(n-1)/2-\beta-\varsigma-1)/2}$ is decreasing on $(0,\infty)$. Thus

$$\begin{aligned}&\int_{u=s}^{t_2-t_1+s}u^{(\gamma-(n-1)/2-\varsigma-3)/2}\,du\\ &\quad=\int_{u=s}^{t_2-t_1+s}u^{(\gamma-(n-1)/2-\beta-\varsigma-1)/2}u^{\beta/2-1}\,du\\ &\quad\le s^{(\gamma-(n-1)/2-\beta-\varsigma-1)/2}\int_{u=s}^{t_2-t_1+s}u^{\beta/2-1}\,du.\end{aligned}\tag{B.11}$$

Moreover, because the mapping $u \mapsto u^{\beta/2-1}$ is similarly decreasing on $(0,\infty)$, we have in a way similar to (4.7) that

$$\int_{u=s}^{t_2-t_1+s}u^{\beta/2-1}\,du\le\int_{u=0}^{t_2-t_1}u^{\beta/2-1}\,du=\big(\beta/2\big)^{-1}|t_2-t_1|^{\beta/2}.\tag{B.12}$$

Collect together (B.10)–(B.12) and use them to show the bound

$$\begin{aligned}&\int_{s=0}^{t_1}\int_{\bar{y}\in\partial M}\left\{\varepsilon^\gamma p_{\bar{y}}^R\big(t_2-t_1+s,\theta_\varepsilon(\bar{x})\big)-\varepsilon^\gamma p_{\bar{y}}^R\big(s,\theta_\varepsilon(\bar{x})\big)\right\}^2\overline{\alpha}(d\bar{y})\,ds\\ &\quad\le\left\{K_3'\big(T,n/2+1/2+\varsigma,\gamma\big)\right\}^2\Xi_{2\varsigma}\big(\gamma-(n-1)/2-\beta-\varsigma\big)^{-1}\\ &\qquad\times T^{\gamma-(n-1)/2-\beta-\varsigma}\big(\beta/2\big)^{-2}|t_2-t_1|^\beta.\end{aligned}\tag{B.13}$$

The combination of this and (B.9) gives us the desired bound for the variation in t.

Variation in ε. We next vary ε; the variation of $\overline{x}$ involves more complicated calculations. Fix $0 < t \le T$, $\overline{x}$ in some $\partial_i M$ and $0 < \varepsilon_1 < \varepsilon_2 \le \varepsilon'_{\text{tub}}$. From (B.3) and an obvious change of variables,

$$\begin{aligned} &\mathbb{E}\big[|V_{1,\gamma}(t,\overline{x},\varepsilon_2) - V_{1,\gamma}(t,\overline{x},\varepsilon_1)|^2\big] \\ &\quad \le \tfrac{1}{4}\|\sigma\|^2 \int_{s=0}^{t} \int_{\overline{y}\in\partial M} \Big\{\varepsilon_2^\gamma p_{\overline{y}}^R\big(s,\theta_{\varepsilon_2}(\overline{x})\big) - \varepsilon_1^\gamma p_{\overline{y}}^R\big(s,\theta_{\varepsilon_1}(\overline{x})\big)\Big\}^2 \overline{\alpha}(d\overline{y})\,ds. \end{aligned} \tag{B.14}$$

Again, we will bound the integrand by using the fundamental theorem of calculus and Proposition 3.4. We have that

$$\begin{aligned} &\big|\varepsilon_2^\gamma p_{\overline{y}}^R\big(s,\theta_{\varepsilon_2}(\overline{x})\big) - \varepsilon_1^\gamma p_{\overline{y}}^R\big(s,\theta_{\varepsilon_1}(\overline{x})\big)\big| \\ &\quad \le \int_{\eta=\varepsilon_1}^{\varepsilon_2} \Big\{\gamma\eta^{\gamma-1}\big|p_{\overline{y}}^R\big(t,\theta_\eta(\overline{x})\big)\big| + \eta^\gamma \,\|\dot\theta_\eta(\overline{x})\|\big\|\nabla p_{\overline{y}}^R\big(s,\theta_\eta(\overline{y})\big)\big\|\Big\}\,d\eta. \end{aligned} \tag{B.15}$$

Note that $\|\dot\theta_\eta(\overline{x})\| = 1$ for all $0 \le \eta \le \varepsilon'_{\text{tub}}$ because the curve $\eta \mapsto \theta_\eta(\overline{x})$ is a geodesic of unit speed. We next want to use claims (a) and (b) of Proposition 3.4 to combine ε^γ and $\varepsilon^{\gamma-1}$ with $|p_{\overline{y}}^R(t,\theta_\eta(\overline{x}))|$ and $\|\nabla p_{\overline{y}}^R(s,\theta_\eta(\overline{y}))\|$. After this, we want to use calculations somehow like those of (B.9) and (B.13).

To make all of this work out, let us bound (B.15) by using claim (a) of Proposition 3.3 with $\lambda_1 = n/2 + 1/2 + \varsigma$ and $\lambda_2 = \gamma - \beta/2$, and claim (b) of Proposition 3.4 with $\lambda_1 = n/2 + 1/2 + \varsigma$ and $\lambda_2 = \gamma - \beta/2 + 1$. Then

$$\begin{aligned} &\big|\varepsilon_2^\gamma p_{\overline{y}}^R\big(s,\theta_{\varepsilon_2}(\overline{x})\big) - \varepsilon_1^\gamma p_{\overline{y}}^R\big(s,\theta_{\varepsilon_1}(\overline{x})\big)\big| \\ &\quad \le \Big\{K_1'(T, n/2 + 1/2 + \varsigma, \gamma - \beta/2) + K_2'(T, n/2 + 1/2 + \varsigma, \gamma - \beta/2 + 1)\Big\} \\ &\qquad \times s^{(\gamma - (n-1)/2 - \beta/2 - \varsigma - 1)/2} \int_{\eta=\varepsilon_1}^{\varepsilon_2} d^{-(n-1)/2+\varsigma}\big(\theta_\eta(\overline{x}), \overline{y}\big)\eta^{\beta/2-1}\,d\eta. \end{aligned} \tag{B.16}$$

If we forget about the $d^{-(n-1)/2+\varsigma}(\theta_\eta(\overline{x}),\overline{y})$ term in the integral, we get a term like $|\varepsilon_2 - \varepsilon_1|^{\beta/2}$ by a calculation similar to that of (B.12). Inserting this, (B.14) would then give us the desired variation on the order of $|\varepsilon_2 - \varepsilon_1|^\beta$. On the other hand, if we forget about the $\eta^{\beta/2-1}$ term in the integral in (B.16), we could use Lemma 3.5 as we did in (B.9) and (B.13). We can use Cauchy–Schwarz to do both:

$$\begin{aligned} &\int_{\eta=\varepsilon_1}^{\varepsilon_2} d^{-(n-1)/2+\varsigma}\big(\theta_\eta(\overline{x}), \overline{y}\big)\eta^{\beta/2-1}\,d\eta \\ &\quad = \int_{\eta=\varepsilon_1}^{\varepsilon_2} d^{-(n-1)/2+\varsigma}\big(\theta_\eta(\overline{x}), \overline{y}\big)\eta^{\beta/4-1/2}\eta^{\beta/4-1/2}\,d\eta \\ &\quad \le \left\{\int_{\eta=\varepsilon_1}^{\varepsilon_2} d^{-(n-1)+2\varsigma}\big(\theta_\eta(\overline{x}),\overline{y}\big)\eta^{\beta/2-1}\,d\eta\right\}^{1/2}\left\{\int_{\eta=\varepsilon_1}^{\varepsilon_2} \eta^{\beta/2-1}\,d\eta\right\}^{1/2}. \end{aligned}$$

Insert this bound into (B.16) and thence into (B.14). This allows us to compute

$$\begin{aligned}
&\mathbb{E}\left[|V_{1,\gamma}(t,\overline{x},\varepsilon_2) - V_{1,\gamma}(t,\overline{x},\varepsilon_1)|^2\right] \\
&\quad \le \tfrac{1}{4}\|\sigma\|^2 \Big\{K_1'\big(T, n/2 + 1/2 + \varsigma, \gamma - \beta/2\big) \\
&\qquad + K_2'\big(T, n/2 + 1/2 + \varsigma, \gamma - \beta/2 + 1\big)\Big\}^2 \\
&\qquad \times \left\{\int_{s=0}^{t} s^{\gamma - (n-1)/2 - \beta/2 - \varsigma - 1}\, ds\right\} \\
&\qquad \times \left\{\int_{\eta=\varepsilon_1}^{\varepsilon_2}\int_{\overline{y}\in\partial M} d^{-(n-1)+2\varsigma}\big(\theta_\eta(\overline{x}), \overline{y}\big)\eta^{\beta/2 - 1}\overline{\alpha}(d\overline{y})\, d\eta\right\} \\
&\qquad \times \left\{\int_{\eta=\varepsilon_1}^{\varepsilon_2} \eta^{\beta/2-1}\, d\eta\right\} \\
&\quad \le \tfrac{1}{4}\|\sigma\|^2 \Big\{K_1'\big(T, n/2 + 1/2 + \varsigma, \gamma - \beta/2\big) \\
&\qquad\qquad + K_2'\big(T, n/2 + 1/2 + \varsigma, \gamma - \beta/2 + 1\big)\Big\}^2 \\
&\qquad \times \left\{\big(\gamma - (n-1)/2 - \beta/2 - \varsigma\big)^{-1} T^{\gamma - (n-1)/2 - \beta/2 - \varsigma}\right\}\Xi_{2\varsigma} \\
&\qquad \times \left\{\int_{\eta=\varepsilon_1}^{\varepsilon_2} \eta^{\beta/2-1}\, d\eta\right\}^2.
\end{aligned}$$

Using a computation analogous to (B.12) yields the bound

$$\begin{aligned}
&\mathbb{E}\left[|V_{1,\gamma}(t,\overline{x},\varepsilon_2) - V_{1,\gamma}(t,\overline{x},\varepsilon_1)|^2\right] \\
&\quad \le \tfrac{1}{4}\|\sigma\|^2 \Big\{K_1'\big(T, n/2 + 1/2 + \varsigma, \gamma - \beta/2\big) \\
&\qquad\qquad + K_1'\big(T, n/2 + 1/2 + \varsigma, \gamma - \beta/2 + 1\big)\Big\}^2 \\
&\qquad \times \left\{\big(\gamma - (n-1)/2 - \beta/2 - \varsigma\big)^{-1} T^{\gamma - (n-1)/2 - \beta/2 - \varsigma}\right\} \\
&\qquad \times \Xi_{2\varsigma}\big(\beta/2\big)^{-2}|\varepsilon_2 - \varepsilon_1|^{\beta},
\end{aligned}$$

which is what we want.

Variation in $\overline{x}$. Finally we vary $\overline{x}$. Thanks to (5.8), we can restrict our attention to $\overline{x}_1$ and $\overline{x}_2$ in some $\partial_i M$ that are within $\overline{d}_i$-distance ϱ of each other. From (5.8) there clearly exists some K such that (5.9) holds for all $(t, \overline{x}_1, \varepsilon)$ and $(t, \overline{x}_2, \varepsilon)$ in $[0,T] \times \partial_i M \times (0, \varepsilon'_{\text{tub}}]$ such that $\overline{d}_i(\overline{x}_1, \overline{x}_2) \ge \varrho$. Thus we fix $0 < t \le T$, $0 < \varepsilon \le \varepsilon'_{\text{tub}}$ and $\overline{x}_1$ and $\overline{x}_2$ in some $\partial_i M$ such that $\overline{d}_i(\overline{x}_1, \overline{x}_2) \le \varrho$. From (B.3),

$$\begin{aligned}
&\mathbb{E}\left[|V_{1,\gamma}(t,\overline{x}_2,\varepsilon) - V_{1,\gamma}(t,\overline{x}_1,\varepsilon)|^2\right] \\
&\quad \le \tfrac{1}{4}\|\sigma\|^2 \int_{s=0}^{t}\int_{\overline{y}\in\partial M}\left\{\varepsilon^{\gamma} p^R_{\overline{y}}\big(s, \theta_\varepsilon(\overline{x}_2)\big) - \varepsilon^{\gamma} p^R_{\overline{y}}\big(s, \theta_\varepsilon(\overline{x}_1)\big)\right\}^2 \overline{\alpha}(d\overline{y})\, ds.
\end{aligned} \tag{B.17}$$

Instead of directly using the fundamental theorem of calculus to represent the difference in the integrand, let us start with a slightly simpler manipulation.

We can break up the integral on the right of (B.17) as

$$\int_{s=0}^{t}\int_{\bar{y}\in\partial M}\left\{\varepsilon^{\gamma}p_{\bar{y}}^{R}\big(s,\theta_\varepsilon(\bar{x}_2)\big)-\varepsilon^{\gamma}p_{\bar{y}}^{R}\big(s,\theta_\varepsilon(\bar{x}_1)\big)\right\}^2\overline{\alpha}(d\bar{y})\,ds = I_2(t,\bar{x}_1,\bar{x}_2,\varepsilon)-I_1(t,\bar{x}_1,\bar{x}_2,\varepsilon), \tag{B.18}$$

where for $j=1$ or $j=2$,

$$I_j(t,\bar{x}_1,\bar{x}_2,\varepsilon) := \int_{s=0}^{t}\int_{\bar{y}\in\partial M}\left\{\varepsilon^{\gamma}p_{\bar{y}}^{R}\big(s,\theta_\varepsilon(\bar{x}_2)\big)-\varepsilon^{\gamma}p_{\bar{y}}^{R}\big(s,\theta_\varepsilon(\bar{x}_1)\big)\right\}\left\{\varepsilon^{\gamma}p_{\bar{y}}^{R}\big(s,\theta_\varepsilon(\bar{x}_j)\big)\right\}\overline{\alpha}(d\bar{y})\,ds.$$

Now let us use the fundamental theorem of calculus to separately bound I_1 and I_2. As in Lemma B.2, we let $\{\vartheta(r)\colon 0\le r\le \bar{d}_i(\bar{x}_1,\bar{x}_2)\}$ be the geodesic in $\partial_i M$ that has unit speed and for which $\vartheta(0)=\bar{x}_1$ and $\vartheta(\bar{d}_i(\bar{x}_1,\bar{x}_2))=\bar{x}_2$ [because $\bar{d}_i(\bar{x}_1,\bar{x}_2)\le\varrho<\overline{\varepsilon_{injr}}$, this geodesic must exist and be unique]. Then define $\varphi_\varepsilon(r):=\theta_\varepsilon(\vartheta(r))$ for all $0\le r\le\bar{d}_i(\bar{x}_1,\bar{x}_2)$ and all $0\le\varepsilon\le\varepsilon'_{\text{tub}}$. The curve $\{\varphi_\varepsilon(r)\colon 0\le r\le\bar{d}_i(\bar{x}_1,\bar{x}_2)\}$ joins $\theta_\varepsilon(\bar{x}_1)$ and $\theta_\varepsilon(\bar{x}_2)$. Note that for all $0\le\varepsilon\le\varepsilon'_{\text{tub}}$, $\dot{\varphi}_\varepsilon(r)=D\theta_\varepsilon(\dot{\vartheta}(r))$ for all $0\le r\le\bar{d}_i(\bar{x}_1,\bar{x}_2)$, where $D\theta_\varepsilon$ is the differential of the map $\bar{x}\mapsto\theta_\varepsilon(\bar{x})$. Note also that $\hbar':=\sup_{\bar{x}\in\partial M,\,0\le\varepsilon\le\varepsilon'_{\text{tub}}}\|D\theta_\varepsilon|_{T_{\bar{x}}M}\|_{\text{op}}$ is finite, where $\|\cdot\|_{\text{op}}$ is the operator norm. Because ϑ has unit speed, we have that $\sup_{0\le r\le\bar{d}_i(\bar{x}_1,\bar{x}_2)}\|\dot{\varphi}_\varepsilon(r)\|\le\hbar'$ for all $0\le\varepsilon\le\varepsilon'_{\text{tub}}$. Collecting together all of this and now using the fundamental theorem of calculus, we see that for $j=1$ or $j=2$,

$$|I_j(t,\bar{x}_1,\bar{x}_2,\varepsilon)| \le\hbar'\int_{s=0}^{t}\int_{\bar{y}\in\partial M}\int_{r=0}^{\bar{d}_i(\bar{x}_1,\bar{x}_2)}\left\{\varepsilon^{\gamma}\big\|\nabla p_{\bar{y}}^{R}\big(s,\varphi_\varepsilon(r)\big)\big\|\right\}\left\{\varepsilon^{\gamma}\big|p_{\bar{y}}^{R}\big(s,\theta_\varepsilon(\bar{x}_j)\big)\big|\right\} \times dr\,\overline{\alpha}(d\bar{y})\,ds. \tag{B.19}$$

To show that the I_j's are on the order of $\bar{d}_i^{\beta}(\bar{x}_1,\bar{x}_2)$, we shall use Lemma B.2. To make all of this work, we bound (B.19) by using claims (a) and (b) of Proposition 3.4 with $\lambda_1=n/2+\beta/2$ and $\lambda_2=\gamma$. Then $\lambda_2-\lambda_1=\gamma-(n-1)/2-\beta/2-1/2$ and $-n+\lambda_1=-(n-1)/2+(\beta-1)/2$, and so

$$\begin{aligned}&\left\{\varepsilon^{\gamma}\big\|\nabla p_{\bar{y}}^{R}\big(s,\varphi_\varepsilon(r)\big)\big\|\right\}\left\{\varepsilon^{\gamma}\big|p_{\bar{y}}^{R}\big(s,\theta_\varepsilon(\bar{x}_j)\big)\big|\right\}\\ &\quad\le K_1'(T,n/2+\beta/2,\gamma)K_2'(T,n/2+\beta/2,\gamma)s^{\gamma-(n-1)/2-\beta/2-1}\\ &\qquad\times d^{-(n-1)/2+(\beta-1)/2}\big(\varphi_\varepsilon(r),\bar{y}\big)d^{-(n-1)/2+(\beta-1)/2}\big(\bar{y},\theta_\varepsilon(\bar{x}_j)\big).\end{aligned}$$

Insert this in (B.19). Integrate in s and use Lemma B.1 to integrate in $\bar{y}$. Finally, use Lemma B.2 to integrate in r. If $\bar{x}_j=\bar{x}_1$, then $r\mapsto\vartheta(r)$ is a geodesic in $\partial_i M$ with unit speed and which starts at $\bar{x}_1$. If $\bar{x}_j=\bar{x}_2$, then $r\mapsto\vartheta(\bar{d}_i(\bar{x}_1,\bar{x}_2)-r)$ is a geodesic in $\partial_i M$ with unit speed and which starts at $\bar{x}_2$. This gives the following inequalities:

$$
\begin{aligned}
&|I_j(t,\overline{x}_1,\overline{x}_2,\varepsilon)| \\
&\quad \le \hbar' K_1'(T,n/2+\beta/2,\gamma)K_2'(T,n/2+\beta/2,\gamma) \\
&\qquad \times \left\{ \int_{s=0}^{t} s^{\gamma-(n-1)/2-\beta/2-1}\, ds \right\} \\
&\qquad \times \left\{ \hbar^2_{(n-2+\beta)/2} \int_{r=0}^{\bar{d}_i(\bar{x}_1,\bar{x}_2)} d^{\beta-1}\big(\varphi_\varepsilon(r),\theta_\varepsilon(\overline{x}_i)\big)\, dr \right\} \qquad \text{(B.20)}\\
&\quad \le \hbar' K_1'(T,n/2+\beta/2,\gamma)K_2'(T,n/2+\beta/2,\gamma) \\
&\qquad \times \left\{ \big(\gamma-(n-1)/2-\beta/2\big)^{-1} T^{\gamma-(n-1)/2-\beta/2} \right\} \\
&\qquad \times \hbar^1_{(n-2+\beta)/2} \hbar^2_\beta \overline{d}_i^{\,\beta}(\overline{x}_1,\overline{x}_2).
\end{aligned}
$$

Combining (B.17), (B.18) and (B.20), we find that

$$
\begin{aligned}
&\mathbb{E}\big[|V_{1,\gamma}(t,\overline{x}_2,\varepsilon) - V_{1,\gamma}(t,\overline{x}_1,\varepsilon)|^2\big] \\
&\quad \le \tfrac{1}{2}\|\sigma\|^2 \hbar' K_1'(T,n/2+\beta/2,\gamma)K_2'(T,n/2+\beta/2,\gamma) \\
&\qquad \times \left\{ \big(\gamma-(n-1)/2-\beta/2\big)^{-1} T^{\gamma-(n-1)/2-\beta/2} \right\} \hbar^1_{(n-2+\beta)/2} \hbar^2_\beta \overline{d}_i^{\,\beta}(\overline{x}_1,\overline{x}_2).
\end{aligned}
$$

This completes the proof of Lemma 5.2. □

We next prove Lemma 5.4. Using the stochastic Fubini theorem and a calculation similar to (B.1), we see that

$$
\begin{aligned}
V_2(t,\overline{x},r,\varepsilon) &= -\tfrac{1}{2} \int_{s=0}^{t} \int_{\bar{y}\in\bar{B}(\bar{x},r)} \left\{ \int_{u=0}^{s} \int_{\bar{z}\in\partial M} p^R_{\bar{z}}\big(s-u,\theta_\varepsilon(\overline{y})\big)\sigma(\overline{z})\zeta(du,d\overline{z}) \right\} \\
&\qquad \times \overline{\alpha}(d\overline{y})\, ds \\
&= -\tfrac{1}{2} \int_{u=0}^{t} \int_{z\in\partial M} \left\{ \int_{s=u}^{t} \int_{\bar{y}\in\bar{B}(\bar{x},r)} p^R_{\bar{z}}\big(s-u,\theta_\varepsilon(\overline{y})\big)\overline{\alpha}(d\overline{y})\, ds \right\} \\
&\qquad \times \sigma(\overline{z})\zeta(du,d\overline{z}) \\
&= -\tfrac{1}{2} \int_{u=0}^{\infty} \int_{z\in\partial M} \left\{ \chi_{\{u\le t\}} \int_{s=0}^{t-u} \int_{\bar{y}\in\bar{B}(\bar{x},r)} p^R_{\bar{z}}\big(s,\theta_\varepsilon(\overline{y})\big)\overline{\alpha}(d\overline{y})\, ds \right\} \\
&\qquad \times \sigma(\overline{z})\zeta(du,d\overline{z})
\end{aligned}
$$

for all $t > 0$, $\overline{x}$ in any $\partial_i M$, all $0 < r \le \varrho$ and all $0 < \varepsilon \le \varepsilon'_{\text{tub}}$. Thus, like (B.2),

$$
\begin{aligned}
&\mathbb{E}\big[|V_2(t,\overline{x},r,\varepsilon)|^2\big] \\
&\quad \le \tfrac{1}{4}\|\sigma\|^2 \int_{u=0}^{t} \int_{\bar{z}\in\partial M} \left\{ \int_{s=0}^{t-u} \int_{\bar{y}\in\bar{B}(\bar{x},r)} p^R_{\bar{z}}\big(s,\theta_\varepsilon(\overline{y})\big)\overline{\alpha}(d\overline{y})\, ds \right\}^2 \overline{\alpha}(d\overline{z})\, du
\end{aligned}
\qquad \text{(B.21)}
$$

for all $(t,\overline{x},r,\varepsilon)$ in any $(0,T]\times\partial_iM\times(0,\varrho]\times(0,\varepsilon'_{\rm tub}]$ and

$$
\begin{aligned}
&\mathbb{E}\left[|V_2(t_2,\overline{x}_2,r_2,\varepsilon_2)-V_2(t_1,\overline{x}_1,r_1,\varepsilon_1)|^2\right]\\
&\quad\le \tfrac14\|\sigma\|^2\int_{u=0}^{\infty}\int_{\bar z\in\partial M}\biggl\{\chi_{\{u\le t_2\}}\int_{s=0}^{t_2-u}\int_{\bar y\in\bar B(\bar x_2,r_2)}p^R_{\bar z}\bigl(s,\theta_{\varepsilon_2}(\overline{y})\bigr)\overline{\alpha}(d\overline{y})\,ds\\
&\qquad\qquad -\chi_{\{u\le t_1\}}\int_{s=0}^{t_1-u}\int_{\bar y\in\bar B(\bar x_1,r_1)}p^R_{\bar z}\bigl(s,\theta_{\varepsilon_1}(\overline{y})\bigr)\\
&\qquad\qquad\times\overline{\alpha}(d\overline{y})\,ds\biggr\}^2\overline{\alpha}(d\overline{z})\,du
\end{aligned}\tag{B.22}
$$

for all $(t_1,\overline{x}_1,r_1,\varepsilon_1)$ and $(t_2,\overline{x}_2,r_2,\varepsilon_2)$ in any $(0,T]\times\partial_iM\times(0,\varrho]\times(0,\varepsilon'_{\rm tub}]$.

An auxiliary result that will be necessary in these calculations is the following lemma.

LEMMA B.3. *There is a positive constant* $\hbar^3$ *such that*:

(a) *If* $\overline{x}$ *is in any* ∂_iM *and* $0\le r_1<r_2\le\varrho$,

$$\overline{\alpha}\bigl(\overline{B}(\overline{x},r_1)\,\triangle\,\overline{B}(\overline{x},r_2)\bigr)\le\hbar^3|r_2-r_1|.$$

(b) *If* $\overline{x}_1$ *and* $\overline{x}_2$ *are in any* ∂_iM *and* $0<r\le\varrho$,

$$\overline{\alpha}\bigl(\bar B(\overline{x}_1,r)\,\triangle\,\overline{B}(\overline{x}_2,r)\bigr)\le\hbar^3\overline{d}_i(\overline{x}_1,\overline{x}_2).$$

PROOF. Let us first prove part (a). For any $\overline{x}$ in any ∂_iM, we can expand the volume form $\overline{\alpha}_0$ in normal coordinates. Mapping the measure $\overline{\alpha}$ into a measure on $(\mathbb{R}^{n-1},\mathcal{B}(\mathbb{R}^{n-1}))$ through such normal coordinates, we see that there is a nonnegative constant $\hbar^{3,\dagger}$ such that for all $\overline{x}$ in any ∂_iM and all $0\le r_1\le r_2\le\varrho$,

$$\overline{\alpha}\bigl(\overline{B}(\overline{x},r_1)\,\triangle\,\overline{B}(\overline{x},r_2)\bigr)\le\hbar^{3,\dagger}\int_{r=r_1}^{r_2}r^{n-2}\,dr\le\hbar^{3,\dagger}\varrho^{n-2}|r_2-r_1|.\tag{B.23}$$

This proves part (a) of the lemma. To prove part (b), consider two cases: when $\overline{d}_i(\overline{x}_1,\overline{x}_2)\ge r$ and when $\overline{d}_i(\overline{x}_1,\overline{x}_2)<r$. If $\overline{d}_i(\overline{x}_1,\overline{x}_2)\ge r$, then by (B.23),

$$
\begin{aligned}
\overline{\alpha}\bigl(\overline{B}(\overline{x}_1,r)\,\triangle\,\overline{B}(\overline{x}_2,r)\bigr)&\le\overline{\alpha}\bigl(\overline{B}(\overline{x}_1,r)\bigr)+\overline{\alpha}\bigl(\overline{B}(\overline{x}_2,r)\bigr)\\
&\le 2\hbar^{3,\dagger}\varrho^{n-2}r\le 2\hbar^{3,\dagger}\varrho^{n-2}\overline{d}_i(\overline{x}_1,\overline{x}_2).
\end{aligned}\tag{B.24}
$$

On the other hand, if $\overline{d}_i(\overline{x}_1,\overline{x}_2)<r$, then for any $\overline{z}$ in $\overline{B}(\overline{x}_1,r)\sim\overline{B}(\overline{x}_2,r)$,

$$r\le\overline{d}_i(\overline{x}_2,\overline{z})\le\overline{d}_i(\overline{x}_2,\overline{x}_1)+\overline{d}_i(\overline{x}_1,\overline{z}).$$

Comparing the outer two terms, we see that $\overline{d}_i(\overline{x}_1,\overline{z})\ge r-\overline{d}_i(\overline{x}_1,\overline{x}_2)$. Because also $d_i(\overline{x}_1,\overline{z})\le r$ as $\overline{z}$ is in $\overline{B}(\overline{x}_1,r)$, we see that

$$\overline{B}(\overline{x}_1,r)\sim\overline{B}(\overline{x}_2,r)\subset\overline{B}(\overline{x}_1,r)\sim\overline{B}\bigl(\overline{x}_1,r-\overline{d}_i(\overline{x}_1,\overline{x}_2)\bigr).\tag{B.25}$$

Similarly,

$$\overline{B}(\overline{x}_2,r)\sim\overline{B}(\overline{x}_1,r)\subset\overline{B}(\overline{x}_2,r)\sim\overline{B}\big(\overline{x}_2,r-\overline{d}_i(\overline{x}_1,\overline{x}_2)\big). \tag{B.26}$$

Collecting together (B.23), (B.25) and (B.26), we thus have that if $\overline{d}_i(\overline{x}_1,\overline{x}_2)<r$,

$$\begin{aligned}\overline{\alpha}\big(\overline{B}(\overline{x}_1,r)\triangle\overline{B}(\overline{x}_2,r)\big) &= \overline{\alpha}\big(\overline{B}(\overline{x}_1,r)\sim\overline{B}(\overline{x}_2,r)\big)+\overline{\alpha}\big(\overline{B}(\overline{x}_2,r)\sim\overline{B}(\overline{x}_1,r)\big)\\ &\le \overline{\alpha}\Big(\overline{B}(\overline{x}_1,r)\sim\overline{B}\big(\overline{x}_1,r-\overline{d}_i(\overline{x}_1,\overline{x}_2)\big)\Big)\\ &\quad+\overline{\alpha}\Big(\overline{B}(\overline{x}_2,r)\sim\overline{B}\big(\overline{x}_2,r-\overline{d}_i(\overline{x}_1,\overline{x}_2)\big)\Big)\\ &\le 2\hbar^{3,\dagger}\varrho^{n-2}\overline{d}_i(\overline{x}_1,\overline{x}_2).\end{aligned}$$

The combination of this and (B.24) gives the full proof of part (b) of the lemma. □

We now can give the following:

PROOF OF LEMMA 5.4. As in the proof of Lemma 5.2, we first show boundedness, that is, (5.13), and then continuity, that is, (5.14). We prove the continuity by separately varying t, $\overline{x}$, r and ε. As much as possible, we shall try to follow arguments similar to those of the proof of Lemma 5.2. Similarly to (B.4), we shall fix for the rest of the proof some ς such that

$$0<\varsigma<1-\beta, \tag{B.27}$$

where β is as in the statement of Lemma 5.4.

Boundedness. Fix a $0<t\le T$, $\overline{x}$ in some $\partial_i M$, $0<r\le\varrho$ and $0<\varepsilon\le\varepsilon'_{\mathrm{tub}}$. From (B.21),

$$\begin{aligned}&\mathbb{E}\big[|V_2(t,\overline{x},r,\varepsilon)|^2\big]\\ &\quad\le \tfrac14\|\sigma\|^2\int_{u=0}^{t}\int_{\bar z\in\partial M}\left\{\int_{s=0}^{u}\int_{\bar y\in\bar B(\bar x,r)}\big|p^R_{\bar z}\big(s,\theta_\varepsilon(\overline{y})\big)\big|\overline{\alpha}(d\overline{y})\,ds\right\}^2\\ &\qquad\times\overline{\alpha}(d\overline{z})\,du.\end{aligned} \tag{B.28}$$

We shall use Proposition 3.4 to bound this quantity. To do so, take $\lambda_1=1+\varsigma$ and $\lambda_2=0$. Using these choices of λ_1 and λ_2 in claim (a) of Proposition 3.4, we have that

$$\begin{aligned}&\int_{s=0}^{u}\int_{\bar y\in\bar B(\bar x,r)}\big|p^R_{\bar z}\big(s,\theta_\varepsilon(\overline{y})\big)\big|\overline{\alpha}(d\overline{y})\,ds\\ &\quad\le K_1'(T,1+\varsigma,0)\left\{\int_{s=0}^{u}s^{-(1+\varsigma)/2}\,ds\right\}\\ &\qquad\times\left\{\int_{\bar y\in\partial M}d^{-(n-1)+\varsigma}\big(\theta_\varepsilon(\overline{y}),\overline{z}\big)\overline{\alpha}(d\overline{y})\right\}\\ &\quad\le K_1'(T,1+\varsigma,0)\Xi_\varsigma\left(\frac{1-\varsigma}{2}\right)^{-1}u^{(1-\varsigma)/2}.\end{aligned} \tag{B.29}$$

Thus

$$\begin{aligned}
&\int_{u=0}^{t}\int_{\bar z\in\partial M}\left\{\int_{s=0}^{u}\int_{\bar y\in\bar B(\bar x,r)}\left|p_{\bar z}^R\big(s,\theta_\varepsilon(\bar y)\big)\right|\overline{\alpha}(d\bar y)\,ds\right\}^2\overline{\alpha}(d\bar z)\,du\\
&\quad\le\left\{K_1'(T,1+\varsigma,0)\Xi_\varsigma\left(\frac{1-\varsigma}{2}\right)^{-1}\right\}^2\overline{\alpha}(\partial M)\left\{\int_{u=0}^{t}u^{1-\varsigma}\,du\right\}\\
&\quad=\left\{K_1'(T,1+\varsigma,0)\Xi_\varsigma\left(\frac{1-\varsigma}{2}\right)^{-1}\right\}^2\overline{\alpha}(\partial M)(2-\varsigma)^{-1}t^{2-\varsigma}.
\end{aligned}\tag{B.30}$$

Because $t^{2-\varsigma}\le T^{2-\varsigma}$, (5.13) follows from (B.28) and (B.30).

Variation in t. Fix $0<t_1<t_2\le T$, $\bar x$ in any $\partial_i M$, $0<r\le\varrho$ and $0<\varepsilon\le\varepsilon'_{\text{tub}}$. Then from (B.22),

$$\begin{aligned}
&\mathbb{E}\left[|V_1(t_2,\bar x,r,\varepsilon)-V_2(t_1,\bar x,r,\varepsilon)|^2\right]\\
&\le\tfrac14\|\sigma\|^2\int_{u=0}^{t_1}\int_{\bar z\in\partial M}\left\{\int_{s=t_1-u}^{t_2-u}\int_{\bar y\in\bar B(\bar x,r)}\left|p_{\bar z}^R\big(s,\theta_\varepsilon(\bar y)\big)\right|\overline{\alpha}(d\bar y)\,ds\right\}^2\\
&\qquad\times\overline{\alpha}(d\bar z)\,du\\
&\quad+\tfrac14\|\sigma\|^2\int_{u=t_1}^{t_2}\int_{\bar z\in\partial M}\left\{\int_{s=0}^{t_2-u}\int_{\bar y\in\bar B(\bar x,r)}\left|p_{\bar z}^R\big(s,\theta_\varepsilon(\bar y)\big)\right|\overline{\alpha}(d\bar y)\,ds\right\}\\
&\qquad\times\overline{\alpha}(d\bar z)\,du\\
&=\tfrac14\|\sigma\|^2\int_{u=0}^{t_1}\int_{\bar z\in\partial M}\left\{\int_{s=u}^{t_2-t_1+u}\int_{\bar y\in\bar B(\bar x,r)}\left|p_{\bar z}^R\big(s,\theta_\varepsilon(\bar y)\big)\right|\overline{\alpha}(d\bar y)\,ds\right\}^2\\
&\qquad\times\overline{\alpha}(d\bar z)\,du\\
&\quad+\tfrac14\|\sigma\|^2\int_{u=0}^{t_2-t_1}\int_{\bar z\in\partial M}\left\{\int_{s=0}^{u}\int_{\bar y\in\bar B(\bar x,r)}\left|p_{\bar z}^R\big(s,\theta_\varepsilon(\bar y)\big)\right|\overline{\alpha}(d\bar y)\,ds\right\}\\
&\qquad\times\overline{\alpha}(d\bar z)\,du.
\end{aligned}\tag{B.31}$$

The last term is the easiest to bound: replace t by t_2-t_1 in (B.30) to see that

$$\begin{aligned}
&\int_{u=0}^{t_2-t_1}\int_{\bar z\in\partial M}\left\{\int_{s=0}^{u}\int_{\bar y\in\bar B(\bar x,r)}\left|p_{\bar z}^R\big(s,\theta_\varepsilon(\bar y)\big)\right|\overline{\alpha}(d\bar y)\,ds\right\}^2\overline{\alpha}(d\bar z)\,du\\
&\quad\le\left\{K_1'(T,1+\varsigma,0)\Xi_\varsigma\left(\frac{1-\varsigma}{2}\right)^{-1}\right\}^2\overline{\alpha}(\partial M)(2-\varsigma)^{-1}T^{2-\beta-\varsigma}|t_2-t_1|^{\beta}.
\end{aligned}\tag{B.32}$$

A slightly different modification of (B.29) and (B.30) allows us to also bound the penultimate term in the last equality of (B.31):

$$\int_{s=u}^{t_2-t_1+u}\int_{\bar y\in\bar B(\bar x,r)}\left|p_{\bar z}^R\big(s,\theta_\varepsilon(\bar y)\big)\right|\overline{\alpha}(d\bar y)\,ds$$

$$\leq K_1'(T, 1+\varsigma, 0)\Xi_\varsigma\left\{\int_{s=u}^{t_2-t_1+u} u^{-(1+\varsigma)/2}\,du\right\}.$$

Using a calculation similar to (B.12) and inserting this into (B.31), we find that

$$\begin{aligned}
&\int_{u=0}^{t_1}\int_{\bar z\in\partial M}\left\{\int_{s=u}^{t_2-t_1+u}\int_{\bar y\in\bar B(\bar x,r)}\left|p_{\bar z}^R(s,\theta_\varepsilon(\bar y))\right|\overline{\alpha}(d\bar y)\,ds\right\}^2\overline{\alpha}(d\bar z)\,du\\
&\qquad\leq\left\{K_1'(T,1+\varsigma,0)\Xi_\varsigma\left(\frac{1-\varsigma}{2}\right)^{-1}\right\}^2\overline{\alpha}(\partial M)T|t_2-t_1|^{1-\varsigma} \qquad\text{(B.33)}\\
&\qquad\leq\left\{K_1'(T,1+\varsigma,0)\Xi_\varsigma\left(\frac{1-\varsigma}{2}\right)^{-1}\right\}^2\overline{\alpha}(\partial M)T^{2-\beta-\varsigma}|t_2-t_1|^{\beta}.
\end{aligned}$$

The passage to the last inequality used the fact that $\varsigma < 1-\beta$ to see that $|t_2-t_1|^{1-\varsigma}\leq T^{1-\beta-\varsigma}|t_2-t_1|^{\beta}$. Equations (B.32) and (B.33) yield the desired variation in t.

Variation in ε. We skip over the variation in $\bar x$ and r. Fix $0<t\leq T$, $\bar x$ in any $\partial_i M$, $0<r\leq\varrho$ and $0<\varepsilon_1<\varepsilon_2\leq\varepsilon'_{\text{tub}}$. From (B.22),

$$\begin{aligned}
&\mathbb{E}\left[|V_2(t,\bar x,r,\varepsilon_2)-V_2(t,\bar x,r,\varepsilon_1)|^2\right]\\
&\qquad\leq\tfrac14\|\sigma\|^2\int_{u=0}^{t}\int_{\bar z\in\partial M}\left\{\int_{s=0}^{u}\int_{\bar y\in\bar B(\bar x,r)}\left|p_{\bar z}^R(s,\theta_{\varepsilon_2}(\bar y))\right.\right. \qquad\text{(B.34)}\\
&\qquad\qquad\left.\left.-\,p_{\bar z}^R(s,\theta_{\varepsilon_1}(\bar y))\right|\overline{\alpha}(d\bar y)\,ds\right\}^2\overline{\alpha}(d\bar z)\,du.
\end{aligned}$$

As we did in the corresponding part of the proof of Lemma 5.2, we use the fundamental theorem of calculus and claim (b) of Proposition 3.4 to bound the difference within the absolute value signs in (B.34). To make everything work out in the end, take $\lambda_1 = 1+\varsigma$ and $\lambda_2 = 1-\beta/2$. Then by using claim (b) of Proposition 3.4 with these values of λ_1 and λ_2, we have that

$$\begin{aligned}
&\left|p_{\bar z}^R(s,\theta_{\varepsilon_2}(\bar y))-p_{\bar z}^R(s,\theta_{\varepsilon_1}(\bar y))\right|\\
&\qquad\leq K_2'(T,1+\varsigma,1-\beta/2)s^{-(1+(\beta/2+\varsigma))/2}\int_{\eta=\varepsilon_1}^{\varepsilon_2} d^{-(n-1)+\varsigma}(\theta_\eta(\bar y),\bar z)\eta^{\beta/2-1}\,d\eta.
\end{aligned}$$

Note that because of (B.27), $\beta/2+\varsigma<1$. Integrate s and $\bar y$ as indicated by (B.34). Interchange the η and $\bar y$ integrals and use Lemma 3.5 to integrate over $\bar y$. This yields that

$$\begin{aligned}
&\int_{s=0}^{u}\int_{\bar y\in\bar B(\bar x,r)}\left|p_{\bar z}^R(s,\theta_{\varepsilon_2}(\bar y))-p_{\bar z}^R(s,\theta_{\varepsilon_1}(\bar y))\right|\overline{\alpha}(d\bar y)\,ds\\
&\qquad\leq K_2'\left(T,1+\varsigma,1-\frac{\beta}{2}\right)\Xi_\varsigma\left(\frac{1-(\beta/2+\varsigma)}{2}\right)^{-1}u^{(1-(\beta/2+\varsigma))/2}\\
&\qquad\quad\times\left\{\int_{\eta=\varepsilon_1}^{\varepsilon_2}\eta^{\beta/2-1}\,d\eta\right\}.
\end{aligned}$$

Using an argument like (B.12), we can bound the term in braces. Insert the result back into (B.34) to see that

$$\begin{aligned}
&\mathbb{E}\left[|V_2(t,\overline{x},r,\varepsilon_2) - V_2(t,\overline{x},r,\varepsilon_1)|^2\right] \\
&\quad \le \tfrac{1}{4}\left\{K_2'(T,1+\varsigma,1-\beta/2)\Xi_\varsigma\left(\frac{1-(\beta/2+\varsigma)}{2}\right)^{-1}\right\}^2 \overline{\alpha}(\partial M) \\
&\qquad \times\left(2-\left(\frac{\beta}{2}+\varsigma\right)\right)^{-1} T^{2-(\beta/2+\varsigma)}\left(\frac{\beta}{2}\right)^{-2}|\varepsilon_2-\varepsilon_1|^\beta,
\end{aligned}$$

which shows the desired variation in ε.

Variation in $\overline{x}$ and r. Finally, we vary $\overline{x}$ and r. Note that if $\varepsilon_1 = \varepsilon_2$ and $t_1 = t_2$ in (B.22), then the size of the term in braces in the last inequality of (B.22) measures the difference caused by integrating $\overline{y}$ over $\overline{B}(\overline{x}_1, r_1)$ as compared to integrating $\overline{y}$ over $\overline{B}(\overline{x}_2, r_2)$. Motivated by this observation, we shall consider simultaneously the variation in $\overline{x}$ and r. Fix $0 < t \le T$ and $0 < \varepsilon \le \varepsilon'_{\mathrm{tub}}$. To vary $\overline{x}$, we take any $\overline{x}_1$ and $\overline{x}_2$ in any $\partial_i M$, fix any $0 < r \le \varrho$ and define subsets of ∂M:

$$A_1 := \overline{B}(\overline{x}_1, r) \quad\text{and}\quad A_2 := \overline{B}(\overline{x}_2, r). \tag{B.35}$$

If, on the other hand, we wish to vary r, we fix any $\overline{x}$ in any $\partial_i M$, take any $0 < r_1 < r_2 \le \varrho$ and then define

$$A_1 := \overline{B}(\overline{x}, r_1) \quad\text{and}\quad A_2 := \overline{B}(\overline{x}, r_2). \tag{B.36}$$

With either of these choices of A_1 and A_2, we can rewrite the right-hand side of (B.22) using the trick of (B.18):

$$\begin{aligned}
&\int_{u=0}^{t}\int_{\overline{z}\in\partial M}\left\{\int_{s=0}^{t-u}\left(\int_{\overline{y}\in A_1} p_{\overline{z}}^R(s,\theta_\varepsilon(\overline{y}))\overline{\alpha}(d\overline{y})\right.\right. \\
&\qquad\qquad \left.\left. - \int_{\overline{y}\in A_2} p_{\overline{z}}^R(s,\theta_\varepsilon(\overline{y}))\overline{\alpha}(d\overline{y})\right)ds\right\}^2 \overline{\alpha}(d\overline{z})\,du \\
&\quad = I_1(t, A_1, A_2, \varepsilon) - I_2(t, A_1, A_2, \varepsilon),
\end{aligned}\tag{B.37}$$

where

$$\begin{aligned}
I_j(t, A_1, A_2, \varepsilon) := &\int_{u=0}^{t}\int_{\overline{z}\in\partial M}\left\{\int_{s_1=0}^{u}\left(\int_{\overline{y}_1\in A_1} p_{\overline{z}}^R(s_1,\theta_\varepsilon(\overline{y}_1))\overline{\alpha}(d\overline{y}_1)\right.\right. \\
&\qquad \left.\left. - \int_{\overline{y}_1\in A_2} p_{\overline{z}}^R(s_1,\theta_\varepsilon(\overline{y}_1))\overline{\alpha}(d\overline{y}_1)\right)ds_1\right\} \\
&\times\left\{\int_{s_2=0}^{u}\int_{\overline{y}_2\in A_j} p_{\overline{z}}^R(s_2,\theta_\varepsilon(\overline{y}_2))\overline{\alpha}(d\overline{y}_2)\,ds_2\right\} \\
&\times\overline{\alpha}(d\overline{z})\,du
\end{aligned}\tag{B.38}$$

for $j = 1$ or $j = 2$. We will separately bound I_1 and I_2. A simple argument involving the linearity and monotonicity of integration shows that

$$\begin{aligned}&\left|\int_{\bar{y}_1 \in A_1} p^R_{\bar{z}}(s_1, \theta_\varepsilon(\bar{y}_1))\overline{\alpha}(d\bar{y}_1) - \int_{\bar{y}_1 \in A_2} p^R_{\bar{z}}(s_1, \theta_\varepsilon(\bar{y}_1))\overline{\alpha}(d\bar{y}_1)\right| \\ &\qquad \le \int_{\bar{y}_1 \in A_1 \triangle A_2} \left|p^R_{\bar{z}}(s_1, \theta_\varepsilon(\bar{y}_1))\right| \overline{\alpha}(d\bar{y}_1).\end{aligned} \tag{B.39}$$

Ultimately, we will want to use Lemma B.3 to bound this sort of integral. First, however, we use Proposition 3.4 to bound the last term of (B.38) and the right side of (B.39). To use claim (a) of Proposition 3.4, set $\lambda_1 = 1 + \varsigma$ and $\lambda_2 = 0$. Then we have the following two inequalities:

$$\begin{aligned}&\int_{\bar{y}_1 \in A_1 \triangle A_2} \left|p^R_{\bar{z}}(s_1, \theta_\varepsilon(\bar{y}_1))\right| \overline{\alpha}(d\bar{y}_1) \\ &\qquad \le K_1'(T, 1+\varsigma, 0) s_1^{-(1+\varsigma)/2} \int_{\bar{y}_1 \in A_1 \triangle A_2} d^{-(n-1)+\varsigma}(\theta_\varepsilon(\bar{y}_1), \bar{z})\overline{\alpha}(d\bar{y}_1), \\ &\int_{\bar{y}_2 \in A_j} \left|p^R_{\bar{z}}(s_2, \theta_\varepsilon(\bar{y}_2))\right| \overline{\alpha}(d\bar{y}_2) \\ &\qquad \le K_1'(T, 1+\varsigma, 0) s_2^{-(1+\varsigma)/2} \int_{\bar{y}_2 \in A_j} d^{-(n-1)+\varsigma}(\theta_\varepsilon(\bar{y}_2), \bar{z})\overline{\alpha}(d\bar{y}_2).\end{aligned}$$

We can insert these into (B.38) and reorder the integrals to get that for $j = 1$ or $j = 2$,

$$\begin{aligned}&|I_j(t, A_1, A_2, \varepsilon)| \\ &\quad \le \{K_1'(T, 1+\varsigma, 0)\}^2 \left\{\int_{u=0}^{t} \left(\int_{s=0}^{u} s^{-(1+\varsigma)/2}\, ds\right)^2 du\right\} \\ &\quad\quad \times \int_{\bar{y}_1 \in A_1 \triangle A_2} \int_{\bar{y}_2 \in A_j} \int_{\bar{z} \in \partial M} d^{-(n-1)+\varsigma}(\theta_\varepsilon(\bar{y}_1), \bar{z}) d^{-(n-1)+\varsigma}(\bar{z}, \theta_\varepsilon(\bar{y}_2)) \\ &\qquad\qquad \times \overline{\alpha}(d\bar{z})\overline{\alpha}(d\bar{y}_2)\overline{\alpha}(d\bar{y}_1).\end{aligned}$$

Use Lemma B.1 and Lemma 3.5 to bound the innermost two integrals. We can then use Lemma B.3. This calculation shows that

$$\begin{aligned}&\int_{\bar{y}_1 \in A_1 \triangle A_2} \int_{\bar{y}_2 \in A_j} \int_{\bar{z} \in \partial M} d^{-(n-1)+\varsigma}(\theta_\varepsilon(\bar{y}_1), \bar{z}) d^{-(n-1)+\varsigma}(\bar{z}, \theta_\varepsilon(\bar{y}_2)) \\ &\qquad\qquad \times \overline{\alpha}(d\bar{z})\, \overline{\alpha}(d\bar{y}_2)\, \overline{\alpha}(d\bar{y}_1) \\ &\quad \le \hbar^1_\varsigma \int_{\bar{y}_1 \in A_1 \triangle A_2} \int_{\bar{y}_2 \in A_j} d^{-(n-1)+2\varsigma}(\theta_\varepsilon(\bar{y}_1), \theta_\varepsilon(\bar{y}_2))\overline{\alpha}(d\bar{y}_2)\, \overline{\alpha}(d\bar{y}_1) \\ &\quad \le \hbar^1_\varsigma \Xi_{2\varsigma}\, \overline{\alpha}(A_1 \triangle A_2) \\ &\quad \le \hbar^1_\varsigma \Xi_{2\varsigma} \hbar^3 \begin{cases} \overline{d}_i(\overline{x}_1, \overline{x}_2), & \text{if } A_1 \text{ and } A_2 \text{ are given by (B.35)}, \\ |r_2 - r_1|, & \text{if } A_1 \text{ and } A_2 \text{ are given by (B.36)}. \end{cases}\end{aligned}$$

Thus

$$
\text{(B.40)}\quad
\begin{aligned}
&|I_j(t, A_1, A_2, \varepsilon)| \\
&\quad \le \{K_1'(T, 1+\varsigma, 0)\}^2 \left\{ \left(\frac{1-\varsigma}{2}\right)^{-2} (2-\varsigma)^{-1} T^{2-\varsigma} \right\} \\
&\qquad \times \hbar^1 \Xi_{2\varsigma} \hbar^3 \begin{cases} \overline{d}_i(\overline{x}_1, \overline{x}_2), & \text{if } A_1 \text{ and } A_2 \text{ are given by (B.35)}, \\ |r_2 - r_1|, & \text{if } A_1 \text{ and } A_2 \text{ are given by (B.36)}. \end{cases}
\end{aligned}
$$

Finally, combine (B.22), (B.37) and (B.40) to get the proper variation in $\overline{x}$ and r: for any $\overline{x}_1$ and $\overline{x}_2$ in any $\partial_i M$ and any $0 < r \le \varrho$,

$$
\begin{aligned}
&\mathbb{E}\left[|V_2(t, \overline{x}_1, r, \varepsilon) - V_2(t, \overline{x}_2, r, \varepsilon)|^2\right] \\
&\quad \le \frac{1}{2}\|\sigma\|^2 \{K_1'(T, 1+\varsigma, 0)\}^2 \left\{ \left(\frac{1-\varsigma}{2}\right)^{-2} (2-\varsigma)^{-1} T^{2-\varsigma} \right\} \\
&\qquad \times \hbar_\varsigma^1 \Xi_{2\varsigma} \hbar^3 \left(\overline{\mathrm{rad}}(\partial M)\right)^{1-\beta} \overline{d}_i^\beta(\overline{x}_1, \overline{x}_2)
\end{aligned}
$$

and for any $\overline{x}$ in any $\partial_i M$ and any $0 < r_1 < r_2 \le \varrho$,

$$
\begin{aligned}
&\mathbb{E}\left[|V_2(t, \overline{x}_1, r, \varepsilon) - V_2(t, \overline{x}_2, r, \varepsilon)|^2\right] \\
&\quad \le \frac{1}{2}\|\sigma\|^2 \{K_1'(T, 1+\varsigma, 0)\}^2 \left\{ \left(\frac{1-\varsigma}{2}\right)^{-2} (2-\varsigma)^{-1} T^{2-\varsigma} \right\} \\
&\qquad \times \hbar_\varsigma^1 \Xi_{2\varsigma} \hbar^3 \varrho^{1-\beta} |r_2 - r_1|^\beta.
\end{aligned}
$$

This completes the proof of Lemma 5.4. □

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