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A THEOREM ON MAJORIZING MEASURES

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Let (T,d) be a metric space and $\varphi:\mathbb{R}_+\to\mathbb{R}$ an increasing, convex function with $\varphi(0)=0$. We prove that if m is a probability measure m on T which is majorizing with respect to d,φ, that is, $\mathcal{S}:=\sup_{x\in T}\int_0^{D(T)}\varphi^{-1}(\frac{1}{m(B(x,\varepsilon))})\,d\varepsilon<\infty$, then

$$\mathbf{E}\sup_{s,t\in T}|X(s)-X(t)|\le 32\mathcal{S}$$

for each separable stochastic process $X(t)$, $t\in T$, which satisfies $\mathbf{E}\varphi(\frac{|X(s)-X(t)|}{d(s,t)})\le 1$ for all $s,t\in T$, $s\neq t$. This is a strengthening of one of the main results from Talagrand [*Ann. Probab.* **18** (1990) 1–49], and its proof is significantly simpler.

1. Introduction. In this paper, (T,d) is a fixed metric space and m a fixed probability measure (defined on Borel subsets) on T. We assume that $\mathrm{supp}(m)=T$. For $x\in T$ and $\varepsilon\ge 0$, $B(x,\varepsilon)$ denotes the closed ball with center at x and radius ε [i.e., $B(x,\varepsilon)=\{y\in T: d(x,y)\le\varepsilon\}$]. Let $D(T)$ be the diameter of T, that is, $D(T)=\sup\{d(s,t):s,t\in T\}$. We define $C(T)$ as to be the space of all continuous functions on T and $\mathcal{B}(T)$ as to be the space of all Borel and bounded functions on T.

For $a,b\ge 0$ we denote by $\mathcal{G}_{a,b}$ the class of all functions $\varphi:\mathbb{R}_+\to\mathbb{R}$ which are increasing, continuous, which satisfy $\varphi(0)=0$ and such that

$$x\le a+b\frac{\varphi(xy)}{\varphi(y)}\qquad \text{for all } x\ge 0,\ y\ge\varphi^{-1}(1). \tag{1.1}$$

For a fixed function $\varphi\in\mathcal{G}_{a,b}$ we define

$$\sigma(x):=\int_0^{D(T)}\varphi^{-1}\left(\frac{1}{m(B(x,\varepsilon))}\right)d\varepsilon,$$
$$\bar{\mathcal{S}}:=\int_T\sigma(u)m(du),$$
$$\mathcal{S}:=\sup_{x\in T}\sigma(x).$$

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We say that m is a *majorizing measure* if $\mathcal{S} < \infty$. In the sequel we will use the convention that $0/0 = 0$.

The following theorem is the main result of the paper:

THEOREM 1.1. *If φ is a Young function and m is a majorizing measure on T, then, for each separable stochastic process $X(t)$, $t \in T$, such that*

$$\sup_{s,t\in T} \mathbf{E}\varphi\left(\frac{|X(s)-X(t)|}{d(s,t)}\right) \le 1, \tag{1.2}$$

the following inequality holds:

$$\mathbf{E} \sup_{s,t\in T} |X(s)-X(t)| \le 32\mathcal{S}.$$

This is a generalization of Theorem 4.6 from Talagrand [3]. The method we use in this paper is new and the proof is simpler. Contrary to Talagrand's result, it works for all Young functions φ, in particular for $\varphi(x) \equiv x$. The author arrived at the idea of chaining with balls of given measure by studying [4] (see also [5]).

Our main tool needed to obtain Theorem 1.1 will be a Sobolev-type inequality.

THEOREM 1.2. *Suppose $\varphi \in \mathcal{G}_{a,b}$ and $R \ge 2$. Then there exists a probability measure ν on $T \times T$ such that, for each bounded, continuous function f on T, the inequality*

$$\left|f(t) - \int_T f(u)m(du)\right| \le aA\sigma(t) + bB\bar{\mathcal{S}} \int_{T\times T} \varphi\left(\frac{|f(u)-f(v)|}{d(u,v)}\right)\nu(du,dv),$$

holds for all $t \in T$, where $A = \frac{R^3}{(R-1)(R-2)}$, $B = \frac{R^2}{R-1}$.

An immediate consequence of Theorem 1.2 is the following corollary:

COROLLARY 1.1. *If $\varphi \in \mathcal{G}_{a,b}$ and $R \ge 2$ then there exists a probability measure ν on $T \times T$ such that, for all $f \in C(T)$,*

$$\sup_{s,t\in T} |f(s)-f(t)| \le 2aA\mathcal{S} + 2bB\bar{\mathcal{S}} \int_{T\times T} \varphi\left(\frac{|f(u)-f(v)|}{d(u,v)}\right)\nu(du,dv),$$

where $A = \frac{R^3}{(R-1)(R-2)}$, $B = \frac{R^2}{R-1}$.

REMARK 1.1. In terms of absolutely summing operators, Corollary 1.1 means that the embedding of the Banach space of Lipschitz functions on T into the Banach space of continuous and bounded functions on T is φ-absolutely summing, as defined by Assouad [1].

Each increasing, convex function φ with $\varphi(0) = 0$ (Young function) is in $\mathcal{G}_{1,1}$. Choosing $R = 4$, $a = b = 1$, Corollary 1.1 yields the following:

COROLLARY 1.2. *If φ is a Young function then there exists a probability measure ν on $T \times T$ such that, for all $f \in C(T)$,*

$$\sup_{s,t\in T} |f(s) - f(t)| \le 32\mathcal{S}\left(\frac{2}{3} + \frac{1}{3}\int_{T\times T} \varphi\left(\frac{|f(u) - f(v)|}{d(u,v)}\right)\nu(du, dv)\right).$$

REMARK 1.2. For a Young function, it is usually possible to choose better constants than $a = b = 1$. For example, the function $\varphi(x) \equiv x$ is in $\mathcal{G}_{0,1}$. Setting $R = 2$, $a = 0$, $b = 1$ in Corollary 1.1, we obtain that there exists a probability measure ν on $T \times T$ such that

$$\sup_{s,t\in T} |f(s) - f(t)| \le 8\bar{\mathcal{S}} \int_{T\times T} \frac{|f(u) - f(v)|}{d(u,v)}\nu(du, dv) \qquad \text{for all } f \in C(T).$$

The result is of interest if $\bar{\mathcal{S}} < \infty$, which is valid for a larger class of measures than majorizing measures.

We use Corollary 1.2 to prove the main result (Theorem 1.1).

2. Proofs and generalizations.

PROOF OF THEOREM 1.2. We can assume that $D(T) < \infty$, otherwise $\sigma(x) = \infty$, for all $x \in T$ and there is nothing to prove. There exists $k_0 \in \mathbb{Z}$ such that

$$R^{k_0} \le \varphi^{-1}(1) < R^{k_0+1}.$$

For $x \in T$ and $k > k_0$ we define

$$r_k(x) := \min\left\{\varepsilon \ge 0 : \varphi^{-1}\left(\frac{1}{m(B(x,\varepsilon))}\right) \le R^k\right\}. \tag{2.1}$$

If $k = k_0$, we put $r_{k_0}(x) := D(T)$.

LEMMA 2.1. *For $k \ge k_0$, functions r_k are 1-Lipschitz.*

PROOF. Indeed, r_{k_0} is constant, and if $k > k_0$ then for each $s, t \in T$ we obtain from the definition

$$\varphi^{-1}\left(\frac{1}{m(B(s, r_k(t) + d(s,t)))}\right) \le \varphi^{-1}\left(\frac{1}{m(B(t, r_k(t)))}\right) \le R^k.$$

Hence $r_k(s) \le r_k(t) + d(s,t)$ and similarly $r_k(t) \le r_k(s) + d(s,t)$, which means r_k is 1-Lipschitz. $\square$

We have

$$\begin{aligned}
&\sum_{k\geq k_0} r_k(x)(R^k - R^{k-1})\\
&\quad \leq \sum_{k\geq k_0}\bigl(r_k(x)-r_{k+1}(x)\bigr)R^k + \limsup_{k\to\infty} r_{k+1}(x)R^{k+1}\\
&\quad \leq \sum_{k\geq k_0}\int_{r_{k+1}(x)}^{r_k(x)} \varphi^{-1}\biggl(\frac{1}{m(B(x,\varepsilon))}\biggr)\,d\varepsilon\\
&\qquad + \limsup_{k\to\infty}\int_0^{r_{k+1}(x)} \varphi^{-1}\biggl(\frac{1}{m(B(x,\varepsilon))}\biggr)\,d\varepsilon\\
&\quad = \int_0^{D(T)} \varphi^{-1}\biggl(\frac{1}{m(B(x,\varepsilon))}\biggr)\,d\varepsilon.
\end{aligned}$$

Consequently,

$$\sum_{k\geq k_0} r_k(x)R^k \leq \frac{R}{R-1}\sigma(x). \tag{2.2}$$

Let us denote $B_k(x) := B(x, r_k(x))$.

For each $k \geq k_0$, we define the linear operator $S_k : \mathcal{B}(T) \to \mathcal{B}(T)$ by the formula

$$S_k f(x) := \fint_{B_k(x)} f(u)m(du) := \frac{1}{m(B_k(x))}\int_{B_k(x)} f(u)m(du).$$

If $f, g \in \mathcal{B}(T)$, $k \geq k_0$, we can easily check that:

1. $S_k 1 = 1$;
2. if $f \leq g$ then $S_k f \leq S_k g$, hence $|S_k f| \leq S_k|f|$;
3. $S_{k_0} f = \int_T f(u)m(du)$ and hence $S_k S_{k_0} f = S_{k_0} f$;
4. if $f \in C(T)$ then $\lim_{k\to\infty} S_k f(x) = f(x)$.

The last property holds true since $\lim_{k\to\infty} r_k(x) = 0$.

LEMMA 2.2. *If $m > k \geq k_0$ then*

$$S_m S_{m-1}\cdots S_{k+1} r_k \leq \sum_{i=k}^{m} 2^{i-k} r_i. \tag{2.3}$$

PROOF. First we will show that for $i, j \geq k_0$,

$$S_i r_j \leq r_i + r_j. \tag{2.4}$$

Indeed, due to Lemma 2.1, we obtain $r_j(v) \leq r_i(u) + r_j(u)$ for each $v \in B_i(u) = B(u, r_i(u))$. Since $S_i r_j(u) = \fint_{B_i(u)} r_j(v)m(dv)$, it implies (2.4).

We will prove Lemma 2.2 by induction on m. For $m = k + 1$, inequality (2.3) has the form $S_{k+1} r_k \le r_k + 2r_{k+1}$, and it follows by (2.4). Suppose that, for $m - 1$ such that $m - 1 > k \ge k_0$, it is

$$S_{m-1} S_{m-2} \cdots S_{k+1} r_k \le \sum_{i=k}^{m-1} 2^{i-k} r_i .$$

Applying (2.4) to the above inequality, we get

$$S_m S_{m-1} \cdots S_{k+1} r_k \le S_m \sum_{i=k}^{m-1} 2^{i-k} r_i \le \sum_{i=k}^{m-1} 2^{i-k} (r_i + r_m) \le \sum_{i=k}^{m} 2^{i-k} r_i . \qquad \square$$

Observe that

$$\begin{aligned} \sum_{k=k_0}^{m-1} \left(\sum_{i=k}^{m} 2^{i-k} r_i \right) R^k &= \sum_{k=k_0}^{m-1} \sum_{i=k}^{m} \left(\frac{2}{R} \right)^{i-k} r_i R^i \\ &\le \sum_{j=0}^{\infty} \left(\frac{2}{R} \right)^j \sum_{i=k_0}^{m} r_i R^i \\ &\le \frac{R}{R-2} \sum_{i=k_0}^{\infty} r_i R^i . \end{aligned} \tag{2.5}$$

By the properties 1–4 of the operators S_k, $k \ge k_0$, we get

$$\begin{aligned} \left| f(t) - \int_T f(u) m(du) \right| &= \lim_{m \to \infty} |S_m f - S_m S_{m-1} \cdots S_{k_0} f|(t) \\ &= \lim_{m \to \infty} \left| \sum_{k=k_0}^{m-1} S_m \cdots S_{k+2} S_{k+1} (I - S_k) f \right| (t) \\ &\le \lim_{m \to \infty} \sum_{k=k_0}^{m-1} S_m \cdots S_{k+2} |S_{k+1} (I - S_k) f|(t). \end{aligned} \tag{2.6}$$

We can easily check that

$$S_{k+1}(I - S_k) f(w) = \fint_{B_{k+1}(w)} \fint_{B_k(u)} \big(f(u) - f(v) \big) m(dv) m(du),$$

which gives

$$|S_{k+1}(I - S_k) f|(w) \le \fint_{B_{k+1}(w)} \fint_{B_k(u)} |f(u) - f(v)| m(dv) m(du).$$

Condition (1.1) implies that, for $v \in B_k(u)$,

$$\frac{|f(u) - f(v)|}{R^{k+1} d(u,v)} \le a + \frac{b}{\varphi(R^{k+1})} \varphi\left(\frac{|f(u) - f(v)|}{d(u,v)} \right). \tag{2.7}$$

For each $v \in B_k(u)$, we have that $d(u,v) \le r_k(u)$, and for $w \in T$ it is $m(B_{k+1}(w)) \ge \frac{1}{\varphi(R^{k+1})}$. Thus, for $v \in B_k(u)$, the following inequality holds:

$$|f(u) - f(v)| \le ar_k(u)R^{k+1} + bm(B_{k+1}(w))r_k(u)R^{k+1}\varphi\left(\frac{|f(u)-f(v)|}{d(u,v)}\right).$$

Consequently,

$$|S_{k+1}(I - S_k)f|(w) \le aR^{k+1}S_{k+1}r_k(w) \\ + b\int_T r_k(u)R^{k+1} ⨍_{B_k(u)} \varphi\left(\frac{|f(u)-f(v)|}{d(u,v)}\right)m(dv)m(du).$$

By Lemma 2.2, $S_m \cdots S_{k+2}S_{k+1}r_k \le \sum_{i=k}^m 2^{i-k}r_i$, therefore,

$$S_m \cdots S_{k+2}|S_{k+1}(I - S_k)f|(t) \\ \le aR\sum_{i=k}^m 2^{i-k}r_i(t)R^k \\ + bR\int_T r_k(u)R^k ⨍_{B_k(u)} \varphi\left(\frac{|f(u)-f(v)|}{d(u,v)}\right)m(dv)m(du).$$

Using (2.5), (2.6) and then (2.2) we obtain

$$\left|f(t) - \int_T f(u)m(du)\right| \\ \le a\frac{R^2}{R-2}\sum_{k=k_0}^\infty r_k(t)R^k \\ + bR\sum_{k=k_0}^\infty \int_T r_k(u)R^k ⨍_{B_k(u)} \varphi\left(\frac{|f(u)-f(v)|}{d(u,v)}\right)m(dv)m(du) \\ \le aA\sigma(t) + bR\sum_{k=k_0}^\infty \int_T r_k(u)R^k ⨍_{B_k(u)} \varphi\left(\frac{|f(u)-f(v)|}{d(u,v)}\right)m(dv)m(du),$$

where $A = \frac{R^3}{(R-1)(R-2)}$. Let ν be a probability measure on $T \times T$ defined by

$$\nu(g) := \frac{1}{M}\sum_{k=k_0}^\infty \int_T r_k(u)R^k ⨍_{B_k(u)} g(u,v)m(dv)m(du) \qquad \text{for } g \in \mathcal{B}(T \times T),$$

where $M = \sum_{k=k_0}^\infty \int_T r_k(u)R^k m(du)$. By (2.2) we obtain an inequality $M \le \frac{R}{R-1}\int_T \sigma(u)m(u) = \frac{R}{R-1}\bar{\mathcal{S}}$ and thus

$$\left|f(t) - \int_T f(u)m(du)\right| \le aA\sigma(t) + bB\bar{\mathcal{S}}\int_{T\times T}\varphi\left(\frac{|f(u)-f(v)|}{d(u,v)}\right)\nu(du,dv),$$

where $B = \frac{R^2}{R-1}$. Theorem 1.2 is proved. □

There is a standard way to strengthen the obtained inequalities. We provide it here for the sake of completeness:

THEOREM 2.1. *Let $\psi : \mathbb{R}_+ \to \mathbb{R}$ be an increasing, continuous function with $\psi(0) = 0$, and $\alpha, \beta \geq 0$ such that*

$$\psi(x) \leq \alpha + \beta \frac{\varphi(xy)}{\varphi(y)} \qquad \text{for all } x \geq 0,\ y \geq 0, \tag{2.8}$$

where $\varphi \in \mathcal{G}_{a,b}$. Then, for each bounded, continuous functions f on T, the following inequality holds:

$$\sup_{t\in T} \psi\left(\frac{|f(t) - \int_T f(u)m(du)|}{K}\right) \leq \alpha + \beta \int_{T\times T} \varphi\left(\frac{|f(u) - f(v)|}{d(u,v)}\right)\nu(du,dv),$$

where $K = (aA + bB)\mathcal{S}$, and A, B, ν are as in Theorem 1.2.

PROOF. Given function f, let c be chosen in such a way that

$$\psi(c) = \alpha + \beta \int_{T\times T} \varphi\left(\frac{|f(u) - f(v)|}{d(u,v)}\right)\nu(du,dv).$$

By (2.8) we get, for all $u, v \in T$,

$$\big(\psi(c) - \alpha\big)\varphi\left(\frac{|f(u) - f(v)|}{cd(u,v)}\right) \leq \beta\varphi\left(\frac{|f(u) - f(v)|}{d(u,v)}\right).$$

Hence

$$\int_{T\times T} \varphi\left(\frac{|f(u) - f(v)|}{cd(u,v)}\right)\nu(du,dv) \leq \frac{\beta}{\psi(c) - \alpha} \int_{T\times T} \varphi\left(\frac{|f(u) - f(v)|}{d(u,v)}\right)\nu(du,dv) = 1.$$

Therefore, by Theorem 1.2, we obtain

$$\begin{aligned} \frac{1}{c} \sup_{t\in T} \left| f(t) - \int_T f(u)m(du)\right| &\leq aA\sigma(t) + bB\bar{\mathcal{S}} \int_{T\times T} \varphi\left(\frac{|f(u) - f(v)|}{cd(u,v)}\right)\nu(du,dv) \\ &\leq (aA + bB)\mathcal{S} = K, \end{aligned}$$

which is the same as $\sup_{t\in T} \frac{|f(t)-\int_T f(u)m(du)|}{K} \le c$. Since ψ is increasing, we get

$$\sup_{t\in T} \psi\left(\frac{|f(t)-\int_T f(u)m(du)|}{K}\right) = \psi\left(\sup_{t\in T} \frac{|f(t)-\int_T f(u)m(du)|}{K}\right) \le \psi(c) = \alpha + \beta \int_{T\times T} \varphi\left(\frac{|f(u)-f(v)|}{d(u,v)}\right)\nu(du,dv). \qquad \square$$

REMARK 2.1. Similarly, we can prove that, for each $f \in C(T)$, the following inequality holds:

$$\sup_{s,t\in T} \psi\left(\frac{|f(s)-f(t)|}{2K}\right) \le \alpha + \beta \int_{T\times T} \varphi\left(\frac{|f(u)-f(v)|}{d(u,v)}\right)\nu(du,dv).$$

Each Young function satisfies (1.1) with $a = 1$, $b = 1$. The minimal constant $K = (A + B)\mathcal{S} = \frac{2R^2}{R-2}\mathcal{S}$ is equal to $16\mathcal{S}$ and is attained for $R = 4$. Let us consider functions $\varphi_p(x) \equiv x^p$, $p \ge 1$. The condition (1.1) is satisfied if and only if $(aq)^{1/q}(bp)^{1/p} \ge 1$, where $q = \frac{p}{p-1}$. Elementary calculations show that by choosing

$$R_p = 2 + \frac{1}{q}\left(\left(3q - \frac{q}{p}\right)^{1/2} + 1\right),$$

$$a_p = \frac{1}{q}\left(3q - \frac{q}{p}\right)^{-1/(2p)},$$

$$b_p = \frac{1}{p}\left(3q - \frac{q}{p}\right)^{1/(2q)},$$

we obtain the minimal constant $K_p := 2(\frac{3p-1}{p})(3q - \frac{q}{p})^{1/(2q)}\mathcal{S}$.

Since $\varphi_p(x) \equiv x^p$ satisfies (2.8) for $\alpha = 0, \beta = 1$, we can conclude the above considerations with the following proposition:

PROPOSITION 2.1. *If m is a majorizing measure on T, then there exists a probability measure ν on $T \times T$ such that*

$$\sup_{s,t\in T} |f(s)-f(t)|^p \le (2K_p)^p \int_{T\times T} \left(\frac{|f(u)-f(v)|}{d(u,v)}\right)^p \nu(du,dv),$$

for all $f \in C(T)$, where $K_p = 2(\frac{3p-1}{p})(3q - \frac{q}{p})^{1/(2q)}\mathcal{S}$.

3. An application to sample boundedness. The theorems from the preceding section allow us to prove results concerning the boundedness of stochastic processes. In this paper we consider only separable processes. For such a process $X(t)$, $t \in T$, we have

$$\mathbf{E} \sup_{t\in T} X(t) := \sup_{F\subset T} \mathbf{E} \sup_{t\in F} X(t),$$

where the supremum is taken over all finite sets $F \subset T$.

THEOREM 3.1. *Suppose $\varphi \in \mathcal{G}_{a,b}$ is a Young function, and $R \geq 2$. For each process $X(t)$, $t \in T$, which satisfies (1.2), the following inequality holds*:

$$\mathbf{E} \sup_{s,t\in T} |X(s) - X(t)| \leq 2aA\mathcal{S} + 2bB\bar{\mathcal{S}},$$

where $A = \frac{R^3}{(R-1)(R-2)}$, $B = \frac{R^2}{R-1}$.

PROOF. Our argument follows the proof of Theorem 2.3, [3]. The process $X(t)$ $t \in T$, is defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Take any point $t_0 \in T$. Condition (1.2) implies $\mathbf{E}|X(t) - X(t_0)| < \infty$, for all $t \in T$.

We define $Y(t) := X(t) - X(t_0)$. Necessarily, $\mathbf{E}|Y(t)| < \infty$, for all $t \in T$, condition (1.2) holds and $\mathbf{E}\sup_{s,t\in T} |X(s) - X(t)| = \mathbf{E}\sup_{s,t\in T} |Y(s) - Y(t)|$. First, we suppose that $\mathcal{F}$ is finite. We may identify points in each atom of $\mathcal{F}$, so we can assume that Ω is finite. Let us observe that

$$|Y(s,\omega) - Y(t,\omega)| \leq d(s,t)\varphi^{-1}\big(1/\mathbf{P}(\{\omega\})\big),$$

so trajectories of Y are Lipschitz and consequently continuous. Using Corollary 1.1, the Fubini theorem and condition (1.2), we obtain

$$\begin{aligned}\mathbf{E} \sup_{s,t\in T} |Y(s) - Y(t)| &\leq 2aA\mathcal{S} + 2bB\bar{\mathcal{S}} \int_{T\times T} \mathbf{E}\varphi\left(\frac{|Y(u) - Y(v)|}{d(u,v)}\right)\nu(du,dv) \\ &= 2aA\mathcal{S} + 2bB\bar{\mathcal{S}}.\end{aligned}$$

In the general case, we have to show that, for any finite $F \subset T$,

$$\mathbf{E} \sup_{s,t\in F} |Y(s) - Y(t)| \leq 2aA\mathcal{S} + 2bB\bar{\mathcal{S}}, \tag{3.1}$$

so we may assume that $\mathcal{F}$ is countably generated. There exists an increasing sequence $\mathcal{F}_n$ of finite σ-fields whose union generates $\mathcal{F}$. Since $\mathbf{E}|Y(t)| < \infty$, it is possible to define $Y_n(t) = \mathbf{E}(Y(t)|\mathcal{F}_n)$. Jensen's inequality shows that

$$\mathbf{E}\varphi\left(\frac{|Y_n(s) - Y_n(t)|}{d(s,t)}\right) \leq \mathbf{E}\varphi\left(\frac{|Y(s) - Y(t)|}{d(s,t)}\right) \leq 1.$$

We get (3.1) since $Y_n(t) \to Y(t)$, $\mathbf{P}$-a.s. and in L_1 for each $t \in F$. $\square$

Each Young function $\varphi \in \mathcal{G}_{1,1}$ and $\bar{\mathcal{S}} \leq \mathcal{S}$, so choosing $R = 4$, $a = b = 1$ in Theorem 3.1, we obtain Theorem 1.1.

REMARK 3.1. Our assumption that φ is a Young function is not necessary. Suppose we have an arbitrary function $\varphi \in \mathcal{G}_{a,b}$ and $R \geq 2$. For each process $X(t)$, $t \in T$ which satisfies (1.2), the following inequality holds:

$$\mathbf{E} \sup_{s,t \in T} |X(s) - X(t)| \leq 4K,$$

where $K = (aA + bB)\mathcal{S}$, $A = \frac{R^3}{(R-1)(R-2)}$, $B = \frac{R^2}{R-1}$.

PROOF. Following the proof of Theorem 11.9 from [2], for every finite $F \subset T$, there exists a measurable map $f : T \to F$ such that $d(f(t), x) \leq 2d(t, x)$, for all $t \in T, x \in F$.

We define $\mu_F = f(m)$ so that μ_F is supported by F. Thus, $f(B(x, \varepsilon)) \subset B_F(x, 2\varepsilon)$, and finally we get $m(B(x, \varepsilon)) \leq \mu_F(B_F(x, 2\varepsilon))$. Since the process X is continuous on F, similarly as in the proof of Theorem 3.1, we get

$$\begin{aligned}
&\mathbf{E} \sup_{s,t \in F} |X(s) - X(t)| \\
&\quad \leq 2(aA + bB) \sup_{x \in F} \int_0^{D(F)} \varphi^{-1}\left(\frac{1}{\mu_F(B(x, \varepsilon))}\right) d\varepsilon \\
&\quad \leq 2(aA + bB) \sup_{x \in F} \int_0^{D(F)} \varphi^{-1}\left(\frac{1}{m(B(x, 1/2\varepsilon))}\right) d\varepsilon \leq 4K. \qquad \square
\end{aligned}$$

The method presented in Theorem 2.1 allows us to obtain the following result:

THEOREM 3.2. *Let φ, ψ be as in Theorem 2.1. For each process which satisfies (1.2), the following inequality holds:*

$$\mathbf{E} \sup_{s,t \in T} \psi\left(\frac{|X(s) - X(t)|}{2K}\right) \leq \alpha + \beta,$$

where $K = (aA + bB)\mathcal{S}$, $A = \frac{R^3}{(R-1)(R-2)}$, $B = \frac{R^2}{R-1}$.

REMARK 3.2. In the case of function $\varphi_p(x) = x^p$, $p \geq 1$, following Remark 2.1, we obtain

$$\left\| \sup_{s,t \in T} |X(s) - X(t)| \right\|_p \leq 2K_p.$$

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