The Annals of Probability
2004, Vol. 32, No. 1A, 216–242

LOWER TAIL PROBABILITIES FOR GAUSSIAN PROCESSES

BY WENBO V. LI[1] AND QI-MAN SHAO[2]

University of Delaware and University of Oregon

Let $X = (X_t)_{t\in S}$ be a real-valued Gaussian random process indexed by S with mean zero. General upper and lower estimates are given for the lower tail probability $\mathbb{P}(\sup_{t\in S}(X_t - X_{t_0}) \le x)$ as $x \to 0$, with $t_0 \in S$ fixed. In particular, sharp rates are given for fractional Brownian sheet. Furthermore, connections between lower tail probabilities for Gaussian processes with stationary increments and level crossing probabilities for stationary Gaussian processes are studied. Our methods also provide useful information on a random pursuit problem for fractional Brownian particles.

1. Introduction.

Let $X = (X_t)_{t\in S}$ be a real-valued Gaussian random process indexed by S with mean zero. The main aim of this paper is to determine the rate of lower tail probability

$$\mathbb{P}\Big(\sup_{t\in S}(X_t - X_{t_0}) \le x\Big) \qquad \text{as } x \to 0 \tag{1.1}$$

with $t_0 \in S$ fixed. There are various motivation for the study of (1.1) other than its own importance. Let us mention five concrete examples and their related applications, along with some consequences of our general results.

Our first example comes from the most visited sites of symmetric stable processes. In their deep study of the most visited sites of symmetric stable processes, based on a remarkable connection between the local time of a stable process and the fractional Brownian motion, Bass, Eisenbaum and Shi (2000) used the key probability estimate

$$\mathbb{P}\Big(\sup_{0\le t\le 2}\big(B_\alpha(t) - B_\alpha(1)\big) \le x\Big) \le cx^{3/2} \tag{1.2}$$

for $0 < \alpha \le 1$ and $x > 0$ small, where $\{B_\alpha(t), t \ge 0\}$ is the fractional Brownian motion of order α, that is, $\mathbb{E}(B_\alpha(t) - B_\alpha(t))^2 = |t-s|^\alpha$ for $s, t \ge 0$. The arguments for the proof of (1.2) in Bass, Eisenbaum and Shi (2000) involve a clever application of Slepian's lemma which reduces the problem to the consideration of the probability that planar Brownian motion spends a unit of time in a certain cone. Recently, (1.2) is generalized in Marcus (2000) to a larger class of Gaussian processes with stationary increments. Our Corollary 2.1 with proper modification

Received June 2000; revised January 2003.

[1]Supported in part by NSF Grant DMS-02-04513.

[2]Supported in part by NSF Grant DMS-01-03487.

AMS 2000 subject classifications. Primary 60G15; secondary 60G17, 60G60.

Key words and phrases. Lower tail probability, Gaussian processes.

implies that the probability in (1.2) decays at polynomial rate. After submision of this paper, we learned the works of Molchan (1999, 2000) which give exact polynomial decay rate in (1.2). The motivation comes from an analysis of the fractal nature of solutions of Berger's equation with random initial data.

Our second example comes from the study of Brownian sheet in $\mathbb{R}^2$. In Csaki, Khoshnevisan and Shi (2000), the following result for two-dimensional Brownian sheet $W(s,t)$ is proved: for $x > 0$ small

$$-c_2 \log^2(1/x) \le \log \mathbb{P}\Big(\sup_{0 \le s,t \le 1} W(s,t) \le x \Big) \le -c_1 \frac{\log^2(1/x)}{\log\log(1/x)} \tag{1.3}$$

for some positive constants c_1 and c_2. Various connections and applications of the estimate (1.3) are also discussed in the paper. Obtaining a sharp correct bound by removing the $\log\log$ term in (1.3) is actually our original motivation of this paper. See (1.9) for the statement for d-dimensional Brownian sheet.

Our third example comes from Brownian pursuits. Let $W_0, W_1, \dots, W_n$ be independent standard Brownian motion, starting at 0, and define the stopping time

$$\tau_n = \inf\{t > 0 : W_i(t) - 1 = W_0(t) \text{ for some } 1 \le i \le n\}.$$

Then τ_n can be viewed as a capture time in a random pursuit setting. Assume that a Brownian prisoner escapes, running along the path of W_0. In his pursuit, there are n independent Brownian policemen. These policemen run along the paths of $W_1, \dots, W_n$, respectively. At the outset, the prisoner is ahead of the policemen by 1 unit of distance. Then, τ_n represents the capture time when the fastest of the policemen catches the prisoner. In their studies on coupling various stochastic processes, Bramson and Griffeath (1991) raised the question: for which n is $\mathbb{E}\tau_n < \infty$. It is known that

$$\mathbb{P}\{\tau_n > t\} \sim c t^{-\gamma_n} \qquad \text{as } t \to \infty, \tag{1.4}$$

where γ_n is determined by the first eigenvalue of the Dirichlet problem for the Laplace–Beltrami operator on a subset of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Of course, $\gamma_1 = 1/2$ by the reflection principle, and the analysis in Bramson and Griffeath (1991) shows that $\gamma_2 = 3/4$, $\gamma_3 < 1$. Further, they conjecture that $\gamma_4 > 1$ as their simulation suggests that $\gamma_4 \approx 1.032$. Very recently, Li and Shao (2001b) show that $\gamma_5 > 1$ by using some distribution identities and the Faber–Krahn isoperimetric inequality. Early, using closely related independent stationary Ornstein–Uhlenbeck processes and the theory of large deviations, Kesten (1991) showed that

$$0 < \liminf_{n\to\infty} \gamma_n / \log n \le \limsup_{n\to\infty} \gamma_n / \log n \le 1/4 \tag{1.5}$$

and conjectured the existence of $\lim_{n\to\infty} \gamma_n / \log n$. It is shown by developing a new normal comparison inequality in Li and Shao (2002) that in fact $\lim_{n\to\infty} \gamma_n / \log n = 1/4$. To see the connection with (1.1) from the point of view of the theory of Gaussian processes, we note that estimating the tail of τ_n, that is,

$P(\tau_n > t)$ as $t \to \infty$, is the same as estimating the lower tail probability for the Gaussian process $X(k,s) = W_k(s) - W_0(s)$ indexed by $(k,s) \in \{1,\dots,n\} \times [0,1]$, that is, $\mathbb{P}(\max_{1\le k\le n} \sup_{0\le s\le 1}(W_k(s) - W_0(s)) \le x)$ as $x \to 0$. In fact, for any $t > 0$, by the Brownian scaling

$$\begin{aligned}\mathbb{P}(\tau_n > t) &= \mathbb{P}\Big(\max_{1\le k\le n}\sup_{0\le s\le t}\big(W_k(s) - W_0(s)\big) < 1\Big)\\ &= \mathbb{P}\Big(\max_{1\le k\le n}\sup_{0\le s\le 1}\big(W_k(s) - W_0(s)\big) < t^{-1/2}\Big).\end{aligned}$$

We prove in Theorem 4.1 a lower bound for the above probability for the analogy fractional Brownian motion pursuit problem and the proof is based on pure Gaussian techniques developed in Theorem 2.1.

Our fourth example comes from the study of real zeros of random polynomials. Let $\{Z_i, i \ge 0\}$ be independent standard normal random variables. In their study on the probability that the random polynomial $\sum_{i=0}^n Z_i x^i$ does not have real root in $\mathbb{R}$, Dembo, Poonen, Shao and Zeitouni (2002) obtain

$$\mathbb{P}\left(\sum_{i=0}^n Z_i x^i \le 0 \ \forall x \in \mathbb{R}^1\right) = n^{-b+o(1)}$$

as $n \to \infty$ through even integers, where

$$b = -4 \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\Big(\sup_{0\le t\le T} X_t \le 0\Big) \tag{1.6}$$

and X_t is a centered stationary Gaussian process with

$$\mathbb{E}X_s X_t = \frac{2e^{-|t-s|/2}}{1+e^{-|t-s|}} \qquad \text{for } s,t \ge 0. \tag{1.7}$$

Furthermore, they proved that $0.4 < b \le 2$. Here we find a very closed relation between (1.6) and the lower tail probability for a connected dual Gaussian process. See Proposition 3.3, which also contains the improvement $b < 1.29$.

Our fifth example comes from an old problem of the first passage time for the so-called Slepian process. Let $S(t)$, $t \ge 0$, be the Slepian process, which is the Gaussian process with mean zero and covariance $\mathbb{E}S(t)S(s) = (1 - |t-s|)\mathbb{1}_{\{|t-s|\le 1\}}$. It is easy to see that $S(t)$ can be represented in terms of the standard Wiener process $W(t)$ by

$$S(t) = W(t) - W(t+1), \qquad t \ge 0.$$

The first passage probability

$$Q_a(T) = \mathbb{P}\Big(\sup_{0\le t\le T} S(t) \le a\Big)$$

was studied by many authors. In particular, Slepian (1961) found a simple expression when $T \le 1$ and Shepp (1971) gave an explicit but very hard to evaluate formula in terms of a T-fold integral for an integer T and a $(2[T]+2)$-fold integral for a noninteger T. The question of finding bounds on

$$\lim_{n\to\infty} n^{-1} \log \mathbb{P}\Big(\sup_{0\le t\le n} S(t) \le a\Big) \tag{1.8}$$

(assuming the limit exists) was asked in Shepp (1971) since the formula of $Q_a(T)$ is apparently not suited for either numerical calculation or asymptotic estimation. As a simple consequence of Proposition 3.1, we have the limit in (1.8) exists.

From all these examples, we find that it is of sufficient interest to have estimates of lower tail probabilities for general Gaussian processes. The paper is organized as follows. Section 2 presents general upper and lower bounds of (1.1). The bounds are sharp for many well-known Gaussian processes. Probably the most amazing consequence of our general estimates is that for $x > 0$ small

$$-c_2 \log^d(1/x) \le \log \mathbb{P}\Big(\sup_{\mathbf{t}\in[0,1]^d} W(\mathbf{t}) \le x\Big) \le -c_1 \log^d(1/x) \tag{1.9}$$

for the d-dimensional Brownian sheet $W(\mathbf{t})$, and

$$-c_2 \log(1/x) \le \log \mathbb{P}\Big(\sup_{\mathbf{t}\in[0,1]^d} L(\mathbf{t}) \le x\Big) \le -c_1 \log(1/x)$$

for the d-dimensional Lévy Brownian sheet $L(\mathbf{t})$. Motivated by the fourth and the fifth examples, we consider in Section 3 the level crossing probability for stationary Gaussian process with positive correlation function

$$\mathbb{P}\Big(\sup_{0\le t\le T} X_t \le a\Big)$$

as $T \to \infty$ for fixed $a \in \mathbb{R}$. Connections are made with the lower tail probability. In particular, we show that for the fractional Brownian motion $\{B_\alpha(t), t \ge 0\}$ of order α, $0 < \alpha < 2$, the limit

$$c_\alpha := -\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\Big(\sup_{0\le t\le T} \frac{B_\alpha(e^t)}{e^{t\alpha/2}} \le 0\Big)$$

exists and moreover, as $x \to 0$,

$$\mathbb{P}\Big(\sup_{0\le t\le 1} B_\alpha(t) \le x\Big) = x^{2c_\alpha/\alpha + o(1)}.$$

It then follows from Molchan (1999, 2000) that $c_\alpha = 1 - \alpha/2$.

To see how far our approach to the estimate of (1.1) can go, we discuss the fractional Brownian motion pursuit problem in Section 4. All proofs are given in Section 5.

Finally, we note the obvious fact that the bounds obtained on a specific process are of interest not merely because of the information they provide about the individual processes, but because by virtue of Slepian's inequality and a companion inequality in Li and Shao (2002), they lead to bounds for probabilities of Gaussian processes whose covariance functions are either dominated by, or dominate, the ones we know.

Throughout this paper, we use letter c and their modifications C, c_1, c_2, and so on, for various positive constants which may be different from a line to another, use $f \approx g$ to denote

$$c_1 g \le f \le c_2 g$$

for some positive constants c_1 and c_2, and $\log x$ for the natural logarithm.

2. Main results. Let $X = (X_t)_{t\in S}$ be a real-valued Gaussian random process indexed by S with mean zero. Define an L^2-metric induced by the process X as

$$d(s,t) = (\mathbb{E}|X_s - X_t|^2)^{1/2}, \qquad s, t \in S.$$

For every $\varepsilon > 0$ and a subset A of S, let $N(A, \varepsilon)$ denote the minimal number of open balls of radius ε for the metric d that are necessary to cover A. For $t \in S$ and $h > 0$, let $B(t,h) = \{s \in S : d(t,s) \le h\}$, and define a locally and uniformly Dudley type entropy (LUDE) integral

$$Q = \sup_{h>0}\sup_{t\in S}\int_0^\infty \bigl(\log N(B(t,h), \varepsilon h)\bigr)^{1/2}\,d\varepsilon. \tag{2.1}$$

This notion of entropy integral is finer than that usually employed and specially suited to the framework here. When Dudley type entropy (LUDE) condition $Q < \infty$ holds, it is clear that

$$\begin{aligned} D^{-1}\int_0^D \bigl(\log N(S,\varepsilon)\bigr)^{1/2}\,d\varepsilon &= \sup_{h>D}\sup_{t\in S}\int_0^\infty \bigl(\log N(B(t,h), \varepsilon h)\bigr)^{1/2}\,d\varepsilon \\ &\le Q < \infty, \end{aligned}$$

where $D = \sup_{s,t\in S} d(t,s)$ is the diameter of the set S, and hence X_t is sample bounded by the result of Dudley (1967) and Fernique (1964). Furthermore, for any $h_0 > 0$ fixed,

$$\begin{aligned} &\sup_{h>h_0}\sup_{t\in S}\int_0^\infty \bigl(\log N(B(t,h), \varepsilon h)\bigr)^{1/2}\,d\varepsilon \\ &\quad\le \sup_{h>h_0}\int_0^\infty \bigl(\log N(S, \varepsilon h)\bigr)^{1/2}\,d\varepsilon \\ &\quad= h_0^{-1}\int_0^\infty \bigl(\log N(S,\varepsilon)\bigr)^{1/2}\,d\varepsilon. \end{aligned}$$

Thus what really matters in the condition $Q < \infty$ is the situation for $h > 0$ small when Dudley's condition $\int_0^\infty (\log N(S,\varepsilon))^{1/2}\, d\varepsilon < \infty$ is satisfied.

For $\theta = 1000(1+Q)$, define

$$\begin{aligned} \mathcal{A}_{-1} &= \{t \in S : d(t,t_0) \le \theta^{-1} x\}, \\ \mathcal{A}_k &= \{t \in S : \theta^{k-1} x < d(t,t_0) \le \theta^k x\}, \qquad k = 0,1,2,\dots,L, \end{aligned}$$

where $L = 1 + [\log_\theta(D/x)]$. Let $N_k(x) := N(\mathcal{A}_k, \theta^{k-2} x)$ denote the minimal number of open balls of radius $\theta^{k-2}x$ for the metric d that are necessary to cover $\mathcal{A}_k$, $k = 0,1,\dots,L$, and let

$$N(x) = 1 + \sum_{0 \le k \le L} N_k(x).$$

We first present a general lower bound.

THEOREM 2.1. *Assume that $Q < \infty$ and*

$$\mathbb{E}\big((X_s - X_{t_0})(X_t - X_{t_0})\big) \ge 0 \qquad \textit{for } s,t \in S. \tag{2.2}$$

Then we have

$$\mathbb{P}\Big(\sup_{t\in S}(X_t - X_{t_0}) \le x\Big) \ge e^{-N(x)}. \tag{2.3}$$

It should be pointed out that the formulation given above are not the most general, but it is the best to show the idea of obtaining lower bounds in an abstract setting. Various modification of the proof of Theorem 2.1 allows one to handle the cases that $Q = \infty$ or $N(x)$ is difficult to estimate.

Our next theorem gives an upper bound for the probability under a different set of conditions.

THEOREM 2.2. *For $x > 0$, let $s_i \in S$, $i = 1,\dots,M$, be a sequence such that for every i,*

$$\sum_{j=1}^{M} \big|\mathrm{Corr}(X_{s_i} - X_{t_0}, X_{s_j} - X_{t_0})\big| \le 5/4 \tag{2.4}$$

and

$$d(s_i,t_0) = \big(\mathbb{E}|X_{s_i} - X_{t_0}|^2\big)^{1/2} \ge x/2. \tag{2.5}$$

Then

$$\mathbb{P}\Big(\sup_{t\in S}\big(X(t) - X(t_0)\big) \le x\Big) \le e^{-M/10}. \tag{2.6}$$

To match the lower bound given in Theorem 2.1, we provide the following guide line of selecting the sequence $\{s_i\}$ in Theorem 2.2. Let $q > 1$. For $k = 1, 2, \dots, L-1$, choose $s_{k,j}$, $j = 1, \dots, M_k$, such that

$$(1/2)q^k x \le d(s_{k,j}, t_0) \le q^k x.$$

Hopefully, when q is large, $\{s_{k,j}, 1 \le j \le M_k, 1 \le k < L\}$ satisfies (2.4).

Our next two theorems show that the bounds provided by Theorems 2.1 and 2.2 are sharp under certain regular conditions.

THEOREM 2.3. *Let $\{X(t), t \in [0,1]^d\}$ be a centered Gaussian process with $X(0) = 0$ and stationary increments, that is,*

$$\forall\, t, s \in [0,1]^d, \qquad \sigma^2(|t-s|) = \mathbb{E}(X_t - X_s)^2$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$. If there are $0 < \alpha \le \beta < 1$ such that $\sigma(h)/h^\alpha$ is nondecreasing and $\sigma(h)/h^\beta$ nonincreasing. Then there exist $0 < c_1 \le c_2 < \infty$ depending only on α, β and d such that for $0 < x < 1/2$,

$$\exp\bigl(-c_2 \log(1/x)\bigr) \le \mathbb{P}\Bigl(\sup_{t\in[0,1]^d} X(t) \le \sigma(x)\Bigr) \le \exp\bigl(-c_1 \log(1/x)\bigr). \tag{2.7}$$

THEOREM 2.4. *Let $\{X(t), t \in [0,1]^d\}$ be a centered Gaussian process with $X(0) = 0$ and*

$$\mathbb{E}(X_t X_s) = \prod_{i=1}^d \tfrac{1}{2}\bigl(\sigma^2(t_i) + \sigma^2(s_i) - \sigma^2(|t_i - s_i|)\bigr) \tag{2.8}$$

for $t = (t_1, \dots, t_d)$ and $s = (s_1, \dots, s_d)$. If there are $0 < \alpha \le \beta < 1$ such that $\sigma(h)/h^\alpha$ is nondecreasing and $\sigma(h)/h^\beta$ nonincreasing. Then there exist $0 < c_3 \le c_4 < \infty$ depending only on α, β and d such that for $0 < x < 1/2$,

$$\exp\bigl(-c_4 \log^d(1/x)\bigr) \le \mathbb{P}\Bigl(\sup_{t\in[0,1]^d} X(t) \le \sigma^d(x)\Bigr) \le \exp\bigl(-c_3 \log^d(1/x)\bigr). \tag{2.9}$$

Next we state a few corollaries of Theorems 2.3 and 2.4.

COROLLARY 2.1. *Let $\{L_\alpha(t), t \in [0,1]^d\}$ be the fractional Lévy's Brownian motion of order α, $0 < \alpha < 2$, that is, $L_\alpha(0) = 0$, $\mathbb{E}L_\alpha(t) = 0$ and $\mathbb{E}(L_\alpha(t) - L_\alpha(s))^2 = |t-s|^\gamma$, $0 < \gamma < 2$,*

$$\exp\bigl(-c_2 \log(1/x)\bigr) \le \mathbb{P}\Bigl(\sup_{t\in[0,1]^d} L_\gamma(t) \le x\Bigr) \le \exp\bigl(-c_1 \log(1/x)\bigr). \tag{2.10}$$

COROLLARY 2.2. *Let* $\{B_\alpha(t), t \in [0,1]^d\}$ *be the fractional Brownian sheet of order* α, $0 < \alpha < 2$, *that is*, $B_\alpha(0) = 0$, $\mathbb{E}B_\alpha(t) = 0$ *and*

$$\mathbb{E}\big(B_\alpha(t)B_\alpha(s)\big) = \prod_{i=1}^{d} \tfrac{1}{2}(t_i^\alpha + s_i^\alpha - |t_i - s_i|^\alpha)$$

for $t = (t_1, \dots, t_d)$ *and* $s = (s_1, \dots, s_d)$. *Then there exist* $0 < c_3 \le c_4 < \infty$ *depending only on* α *and* d *such that for* $0 < x < 1/2$,

$$\exp\big(-c_4 \log^d(1/x)\big) \le \mathbb{P}\Big(\sup_{t\in[0,1]^d} B_\alpha(t) \le x\Big) \le \exp\big(-c_3 \log^d(1/x)\big).$$

Since the lower tail probability of a Gaussian process determines the lim inf behavior of related sample path property, comparing Corollaries 2.1 and 2.2, we see that the Lévy fractional Brownian motion and the fractional Brownian sheet on $\mathbb{R}^d$, $d \ge 2$, have very different lower bound of sample path properties, though they share very similar upper bound of sample path properties. As an interesting application of Corollary 2.2, we mention the following boundary crossing result for the sample paths of Brownian sheet. The proof follows from the 0–1 law and the subsequence method [see the argument given in Csáki, Khoshnevisan and Shi (2000)].

COROLLARY 2.3. *There exists positive and finite constant* κ *such that*

$$\liminf_{T\to\infty} (\log\log T)^{-1/d} \log\Big(T^{-d/2} \sup_{t\in[0,T]^d} W(t)\Big) = -\kappa.$$

Based on the results given in Theorems 2.1 and 2.2, we would like to point out that the lower tail probability is very different from the small ball probability under the sup-norm, which considers the absolute value of the supremum of a Gaussian process. In particular, the small ball problem for Brownian sheet $W(t)$ on $\mathbb{R}^d$ (or B_1 in Corollary 2.2) under the sup-norm is still open for $d \ge 3$. The best-known results are

$$\log \mathbb{P}\Big(\sup_{\mathbf{t}\in[0,1]^2} |W(\mathbf{t})| \le x\Big) \approx -x^{-2}\log^3(1/x)$$

and

$$-c_2 x^{-2}\log^{2d-1}(1/x) \le \log \mathbb{P}\Big(\sup_{\mathbf{t}\in[0,1]^d} |W(\mathbf{t})| \le x\Big) \le -c_1 x^{-2}\log^{2d-2}(1/x)$$

for $d \ge 3$ as $x \to 0$. We refer to a recent survey paper of Li and Shao (2001a) for more information on the small ball probability and its applications.

3. Stationary Gaussian processes. Motivated by Example 4 on the probability that a random polynomial does not have a real zero and Example 5 on the first passage probability for Slepian process, we discuss the level crossing probability for stationary Gaussian processes in this section. It turns out the situation becomes much nicer.

PROPOSITION 3.1. *Let $\{X_t, t \ge 0\}$ be an almost surely continuous stationary Gaussian process with zero mean. Assume that*

$$\mathbb{E}X_0X_t \ge 0 \qquad \text{for } t \ge 0. \tag{3.1}$$

Then the limit

$$p(x) := \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\Big(\sup_{0\le t\le T} X_t \le x\Big) \tag{3.2}$$

exists for every $x \in \mathbb{R}^1$. Moreover, $p(x)$ is left continuous and

$$p(x) = \sup_{T>0} T^{-1} \log \mathbb{P}\Big(\sup_{0\le t\le T} X_t \le x\Big).$$

The existence of the limit $p(x)$ is ensured by subadditivity: since $\mathbb{E}X_0X_t \ge 0$, Slepian's lemma and the stationarity of X imply

$$\begin{aligned}\mathbb{P}\Big(\sup_{0\le t\le T+S} X_t \le x\Big) &\ge \mathbb{P}\Big(\sup_{0\le t\le T} X_t \le x\Big)\mathbb{P}\Big(\sup_{T\le t\le T+S} X_t \le x\Big)\\ &= \mathbb{P}\Big(\sup_{0\le t\le T} X_t \le x\Big)\mathbb{P}\Big(\sup_{0\le t\le S} X_t \le x\Big).\end{aligned} \tag{3.3}$$

The left continuity of $p(x)$ follows from the fact that $p(x)$ is nondecreasing and $\mathbb{P}(\sup_{0\le t\le T} X_t \le x)$ is continuous of x for each fixed T. The sup representation for $p(x)$ follows from the subadditivity relation (3.3).

For the question of Shepp (1971) mentioned in (1.8) on the Slepian process, we have the following.

PROPOSITION 3.2. *The limit*

$$Q_a := \lim_{T\to\infty} T^{-1} \log \mathbb{P}\Big(\sup_{0\le t\le T} S(t) \le a\Big) = \sup_{T>0} T^{-1} \log \mathbb{P}\Big(\sup_{0\le t\le T} S(t) \le a\Big) \tag{3.4}$$

exists for every $a \in \mathbb{R}$ and for any $T_0 > 0$,

$$T_0^{-1} \log \mathbb{P}\Big(\sup_{0\le t\le T_0} S(t) \le a\Big) \le Q(a) \le (T_0+1)^{-1} \log \mathbb{P}\Big(\sup_{0\le t\le T_0} S(t) \le a\Big). \tag{3.5}$$

The upper estimate follows from

$$
\begin{aligned}
&\mathbb{P}\Big(\sup_{0\le t\le T} S(t)\le a\Big)\\
&\quad\le \mathbb{P}\Big(\sup_{k(T_0+1)\le t\le (k+1)(T_0+1)-1} S(t)\le a, 0\le k\le [T/(T_0+1)]-1\Big)\\
&\quad= \prod_{k=0}^{[T/(T_0+1)]-1} \mathbb{P}\Big(\sup_{k(T_0+1)\le t\le (k+1)(T_0+1)-1} S(t)\le a\Big)\\
&\quad= \Big(\mathbb{P}\Big(\sup_{0\le t\le T_0} S(t)\le a\Big)\Big)^{[T/(T_0+1)]}
\end{aligned}
$$

for T large. Explicit and simple formula for $\mathbb{P}(\sup_{0\le t\le T_0} S(t)\le a)$ for $0<T_0\le 1$ is given in Slepian (1961).

Next, we make connections between lower tail probabilities and level crossing probabilities for stationary Gaussian processes. For the fractional Brownian motion B_α on $\mathbb{R}^1$ of order α, $0<\alpha<2$, let

$$X_\alpha(t)=e^{-t\alpha/2}B_\alpha(e^t).$$

It is easy to see that X_α is a centered stationary Gaussian process satisfying (3.1). Hence, by Proposition 3.1,

$$
\begin{aligned}
c_\alpha &:= -\lim_{T\to\infty}\frac{1}{T}\log\mathbb{P}\Big(\sup_{0\le t\le T} X_\alpha(t)\le 0\Big)\\
&= \sup_{T>0}\frac{1}{T}\log\mathbb{P}\Big(\sup_{0\le t\le T} X_\alpha(t)\le 0\Big)
\end{aligned}
\tag{3.6}
$$

exists. On the other hand, by Corollary 2.1,

$$
\begin{aligned}
-\infty &< \liminf_{x\downarrow 0}\frac{\log\mathbb{P}(\sup_{0\le t\le 1} B_\alpha(t)\le x)}{\log(1/x)}\\
&\le \limsup_{x\downarrow 0}\frac{\log\mathbb{P}(\sup_{0\le t\le 1} B_\alpha(t)\le x)}{\log(1/x)}<0.
\end{aligned}
$$

Our next theorem confirms that the limit of $\log\mathbb{P}(\sup_{0\le t\le 1} B_\alpha(t)\le x)/\log(1/x)$ as $x\to 0$ not only exists but also has a close relation with the constant c_α.

THEOREM 3.1. *We have, as $x\to 0$,*

$$\mathbb{P}\Big(\sup_{0\le t\le 1} B_\alpha(t)\le x\Big)=x^{2c_\alpha/\alpha+o(1)}\tag{3.7}$$

and hence $c_\alpha=1-\alpha/2$.

It is well known by the reflection principle that for the Brownian motion, that is, $\alpha = 1$,

$$\mathbb{P}\Big(\sup_{0\le t\le 1} W(t) \le x\Big) = \mathbb{P}\big(|W(1)| \le x\big) \sim (2/\pi)^{1/2} x$$

as $x \to 0$. Thus, Theorem 3.1 recovers the following well-known result:

$$\mathbb{P}\Big(\sup_{0\le t\le T} U(t) \le 0\Big) = \exp\big(-T/2 + o(T)\big)$$

as $T \to \infty$, where U is the Ornstein–Uhlenbeck process.

Similarly, one can verify that condition in Proposition 3.1 is satisfied for the stationary Gaussian process X_t in Example 4. Furthermore, we introduce a dual Gaussian process $\{Y(t), t \ge 0\}$ as follows. Let $Y(0) = 0$ and

$$Y(t) = \sqrt{2} t^2 \int_0^\infty W(u) e^{-ut}\, du$$

for $t > 0$, where W is the Brownian motion. It is easy to see that $\mathbb{E}Y(t) = 0$ and

$$\mathbb{E}Y(t)Y(s) = \frac{2st}{s+t} \qquad \text{for } s, t > 0.$$

Hence, $\{X_t, t \ge 0\}$ in (1.7) and $\{Y(e^t)/e^{t/2},\ t \ge 0\}$ have the same distribution. The Gaussian process $Y(t)$ given above has many amazing properties. Its sample path is infinite differentiable for $t > 0$ and it has the same scaling properties as Brownian motion, that is, $\{Y(ct), t \ge 0\} = \{c^{1/2} Y(t), t \ge 0\}$ in distribution for any $c > 0$ and $\{tY(t^{-1}), t \ge 0\} = \{Y(t), t \ge 0\}$ in distribution. Analogously to (3.7), we have an alternative formula for b in (1.6).

PROPOSITION 3.3. *Let b be the constant in* (1.6). *Then*

$$\mathbb{P}\Big(\sup_{0\le t\le 1} Y(t) \le x\Big) = x^{b/2+o(1)}$$

as $x \to 0$. Furthermore, $0.4 < b < 1.29$.

The bound for b given in Dembo, Poonen, Shao and Zeitouni (2002) is $0.4 < b \le 2$. The improvement $b < 1.29$ here follows from (3.3) and an estimate in Slepian (1962) for short interval. Other related estimates can be found in Newell and Rosenblatt (1962) and Strakhov and Kurz (1968). Since then, there appears to be little in the literature concerning the asymptotic behavior for large time T or lower level x. Closed formula for the distribution of the maximum of Gaussian processes is known only for a handful cases (six for stationary Gaussian processes) to the best of our knowledge. And some of them do not allow the asymptotic evaluation discussed here as our Example 5 shows.

On the other hand, large deviation results or probabilities of extreme values for stationary Gaussian processes are well explored [see, e.g., Pickands (1969) and Leadbetter, Lindgren and Rootzen (1983)]. In particular, for almost surely differentiable centered stationary Gaussian process with $\mathbb{E}X_0X_t = 1 - Ct^2 + o(t^2)$ as $t \to 0$, one has

$$\mathbb{P}\Big(\sup_{0\le t\le 1} X_t > x\Big) \sim \frac{(C/2)^{1/2}}{\pi} e^{-x^2/2}$$

as $x \to \infty$. Therefore, we would like to pose the following open questions:

1. If $\{X_t, t \ge 0\}$ is a centered differentiable stationary Gaussian process with positive correlation, what is the limit

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\Big(\sup_{0\le t\le T} X_t \le 0\Big)?$$

2. As mentioned in Proposition 3.3, $0.4 < b < 1.29$. What is the exact value of b? Note that the Gaussian process $X(t)$ in our fourth example is differentiable.
3. What is the exact value of the limit in (3.4) for the Slepian process?
4. Using the scaling property of the d-dimensional Brownian sheet $W(\mathbf{t})$, we see that (1.9) is equivalent to

$$-c_2 \le (\log T)^{-d} \log \mathbb{P}\Big(\sup_{\mathbf{t}\in[0,T]^d} W(\mathbf{t}) \le 1\Big) \le -c_1 \tag{3.8}$$

for large T. It seems hard to show the existence of a limit as $T \to \infty$. The possible connection similar to the one in Theorem 3.1 is not clear for the dual stationary sheet $X(\mathbf{t}) = \exp\{-\sum_{i=1}^d t_i/2\} W(e^{t_1}, \dots, e^{t_d})$.

Before the end of this section, we want to point out two well-known general approaches to $P(T) = \mathbb{P}(\sup_{0\le t\le T} X(t) \le 0)$ discussed in details in Slepian (1962). Unfortunately, none of them works well for problems we considered in this paper. First, $P(T)$ can be approached (and upper bounded) by

$$P_n(\mathbf{r}) = \mathbb{P}\Big(\max_{1\le i\le n} X(t_i) \le 0\Big), \qquad 0 = t_1 < t_2 < \cdots < t_n = T,$$

where $\mathbf{r} = (r_{ij})$ is the covariance matrix with $r_{ij} = \mathbb{E}X(t_i)X(t_j)$. Then $P_n(\mathbf{r})$ admits a simple geometric interpretation and is the fraction of the unit sphere in Euclidean n-space cut out by n-hyperplanes through the center of the sphere. The angle θ_{ij} between the normals to the ith and jth hyperplanes directed into the cutout region is given by $\cos\theta_{ij} = r_{ij}$, $1 \le i, j \le n$. Second, we have Rice's series representation under suitable smooth condition, namely

$$P(T) = \frac{1}{2} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_0^T \cdots \int_0^T q_n(t_1, \dots, t_n)\, dt_1 \cdots dt_n$$

where

$$q_n(t_1, \dots, t_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |y_1 \cdots y_n| p(0, \dots, 0, y_1, \dots, y_n)\, dy_1 \cdots dy_n$$

and $p(x_1, \dots, x_n, y_1, \dots, y_n)$ is the joint density for $X(t_1), \dots, X(t_n), X'(t_1), \dots, X'(t_n)$. Since the proof follows from the method of inclusion and exclusion, various bounds are available. Similar formulae also hold for $\mathbb{P}(\sup_{0 \le t \le T} X(t) \le x)$.

4. Capture time of the fractional Brownian motion pursuit. Throughout this section, $\{B_{k,\alpha}(t); t \ge 0\}$ $(k = 0, 1, 2, \dots, n)$ denote independent fractional Brownian motions of order $\alpha \in (0, 2)$ all starting from 0. Let

$$\tau_n := \tau_{n,\alpha} = \inf\Big\{t > 0 \colon \max_{1 \le k \le n} B_{k,\alpha}(t) = B_{0,\alpha}(t) + 1\Big\}.$$

Then, when is $\mathbb{E}(\tau_n)$ finite? As we have seen in Example 3, the question is the same as estimating the lower tail probabilities of $\max_{1 \le k \le n} \sup_{0 \le t \le 1} (B_{k,\alpha}(t) - B_{0,\alpha}(t))$. In fact, for any $s > 0$, by the fractional Brownian scaling,

$$\begin{aligned}
\mathbb{P}(\tau_n > s) &= \mathbb{P}\Big(\max_{1 \le k \le n} \sup_{0 \le t \le s} \big(B_{k,\alpha}(t) - B_{0,\alpha}(t)\big) < 1\Big) \\
&= \mathbb{P}\Big(\max_{1 \le k \le n} \sup_{0 \le t \le 1} \big(B_{k,\alpha}(t) - B_{0,\alpha}(t)\big) < s^{-\alpha/2}\Big).
\end{aligned}$$

A direct application of Theorem 2.1 gives a lower bound of $-c\,n \log(1/x)$ for the probability $\log \mathbb{P}(\max_{1 \le k \le n} \sup_{0 \le t \le 1} (B_{k,\alpha}(t) - B_{0,\alpha}(t)) < x)$ as $x \to 0$, where c is a positive and finite constant depending only on α. However, according to the bound of (1.5) due to Kesten (1991) for the Brownian motion case, the above lower bound is far from sharp when n is large. Kesten's method is based on large deviation results for independent stationary Ornstein–Uhlenbeck processes, which is hardly applicable for the fractional Brownian motion. Fortunately, the methods developed for proving the results in previous sections do give us a sharp lower bound even for this fractional Brownian motion pursuit problem.

Let

$$X_{k,\alpha}(t) = e^{-t\alpha/2} B_{k,\alpha}(t) \qquad \text{for } k = 0, 1, \dots, n.$$

Analogously to Proposition 3.1, the limit

$$c_{n,\alpha} := -\lim_{T \to \infty} \frac{1}{T} \log \mathbb{P}\Big(\sup_{0 \le t \le T} \max_{1 \le k \le n} \big(X_{k,\alpha}(t) - X_{0,\alpha}(t)\big) \le 0\Big) \tag{4.1}$$

exists.

THEOREM 4.1. *We have*

$$\mathbb{P}\Big(\max_{1 \le k \le n} \sup_{0 \le t \le 1} \big(B_{k,\alpha}(t) - B_{0,\alpha}(t)\big) < x\Big) = x^{2c_{n,\alpha}/\alpha + o(1)} \tag{4.2}$$

as $x \to 0$. *Moreover,*

$$0 < \liminf_{n\to\infty} \frac{c_{n,\alpha}}{\log n} \le \limsup_{n\to\infty} \frac{c_{n,\alpha}}{\log n} < \infty. \tag{4.3}$$

We will only prove the right-hand side of (4.3) here. The left-hand side of (4.3) requires new techniques and is given in Li and Shao (2002) together with a proof of Kesten's conjecture mentioned in Example 3 in the Introduction.

5. Proofs. The proof of lower bound relies on the Slepian lemma and large deviation results for Gaussian processes. The following comparison inequality [Lemma 3.1 in Shao (1999)] plays a key role in the proof of upper bounds. We present its simple proof for completeness.

LEMMA 5.1. *Let* $X = (X_1, \dots, X_n)$ *be distributed according to* $N(0, \Sigma_1)$ *and* $Y = (Y_1, \dots, Y_n)$ *according to* $N(0, \Sigma_2)$. *If* $\Sigma_2 - \Sigma_1$ *is positive semidefinite, then*

$$\forall C \subset \mathbb{R}^n, \qquad \mathbb{P}(Y \in C) \ge (|\Sigma_1|/|\Sigma_2|)^{1/2}\mathbb{P}(X \in C). \tag{5.1}$$

PROOF. Let f_X and f_Y be the joint density functions of X and Y, respectively. Since $\Sigma_2 - \Sigma_1$ is positive semidefinite, $\Sigma_1^{-1} - \Sigma_2^{-1}$ is positive semidefinite too [see, e.g., Bellman (1970), page 59]. Hence

$$\begin{aligned} f_Y(x) &= \frac{1}{(2\pi)^{n/2}|\Sigma_2|^{1/2}} \exp\left(-\frac{1}{2}x'\Sigma_2^{-1}x\right) \\ &\ge \frac{1}{(2\pi)^{n/2}|\Sigma_2|^{1/2}} \exp\left(-\frac{1}{2}x'\Sigma_1^{-1}x\right) \\ &= \left(\frac{|\Sigma_1|}{|\Sigma_2|}\right)^{1/2} f_X(x), \end{aligned}$$

which yields (5.1) immediately. □

PROOF OF THEOREM 2.1. Without loss of generality, assume that $X_{t_0} = 0$. Let $\mathcal{A}_{k,j}$, $j = 1, \dots, N_k(x)$, be the open balls of radius $\theta^{k-2}x$ for the metric d that cover $\mathcal{A}_k$, $k = 0, 1, \dots, L$. Then, by assumption (1.1) and the Slepian lemma,

$$\begin{aligned} &\mathbb{P}\left(\sup_{t\in S}(X_t - X_{t_0}) \le x\right) \\ &\qquad \ge \mathbb{P}\left(\sup_{t\in\mathcal{A}_{-1}} X_t \le x\right) \prod_{k=0}^{L} \prod_{j=1}^{N_k(x)} \mathbb{P}\left(\sup_{t\in\mathcal{A}_{k,j}} X_t \le x\right). \end{aligned}$$

By Dudley (1967) [see Theorem 6.1 of Ludoux (1996)], we have

$$
\begin{aligned}
\mathbb{E}\sup_{t\in\mathcal{A}_{-1}} X(t) &\le 42\int_0^{\theta^{-1}x} \big(\log N(\mathcal{A}_{-1},\varepsilon)\big)^{1/2}\,d\varepsilon \\
&\le 42\int_0^{\theta^{-1}x} \big(\log N\big(B(t_0,\theta^{-1}x),\varepsilon\big)\big)^{1/2}\,d\varepsilon \\
&= 42\theta^{-1}x\int_0^{1} \big(\log N\big(B(t_0,\theta^{-1}x),\varepsilon\theta^{-1}x\big)\big)^{1/2}\,d\varepsilon \\
&\le 42Q\theta^{-1}x \le x/2.
\end{aligned}
\tag{5.2}
$$

Hence

$$
\mathbb{P}\Big(\sup_{t\in\mathcal{A}_{-1}} X_t \le x\Big) = 1-\mathbb{P}\Big(\sup_{t\in\mathcal{A}_{-1}} X_t > x\Big) \ge 1/2.
$$

It suffices to show that

$$
\mathbb{P}\Big(\sup_{t\in\mathcal{A}_{k,j}} X_t \le 0\Big) \ge e^{-1} \tag{5.3}
$$

for every $1\le j\le N_k(x), 0\le k\le L$. Let $s_{k,j}$ be the center of $\mathcal{A}_{k,j}$. Then $d(s_{k,j},t_0)\ge\theta^{k-1}x$. Observe that

$$
\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in\mathcal{A}_{k,j}} X_t \le 0\Big) \\
&\quad\ge \mathbb{P}(X_{s_{k,j}} \le -\theta^{k-1}x/4) - \mathbb{P}\Big(\sup_{t\in\mathcal{A}_{k,j}} (X_t - X_{s_{k,j}}) > \theta^{k-1}x/4\Big) \\
&\quad\ge \mathbb{P}(Z\ge 1/4) - \mathbb{P}\Big(\sup_{t\in\mathcal{A}_{k,j}} (X_t - X_{s_{k,j}}) > \theta^{k-1}x/4\Big) \\
&\quad\ge e^{-1}+10^{-2} - \mathbb{P}\Big(\sup_{t\in\mathcal{A}_{k,j}} (X_t - X_{s_{k,j}}) > \theta^{k-1}x/4\Big),
\end{aligned}
\tag{5.4}
$$

where Z is the standard normal random variable. By the definition of $\mathcal{A}_{k,j}$ and similarly to (5.2), we have

$$
\sup_{t\in\mathcal{A}_{k,j}} d(t,s_{k,j}) \le \theta^{k-2}x
$$

and

$$
\mathbb{E}\sup_{t\in\mathcal{A}_{k,j}} (X_t - X_{s_{k,j}}) \le 42Q\theta^{k-2}x.
$$

Hence, it follows from the deviation estimate for Gaussian process [see Ledoux and Talagrand (1991)] that

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in\mathcal{A}_{k,j}}(X_t - X_{s_{k,j}}) > \theta^{k-1}x/4\Big)\\
&\quad\le \mathbb{P}\Big(\sup_{t\in\mathcal{A}_{k,j}}(X_t - X_{s_{k,j}}) > \mathbb{E}\sup_{t\in\mathcal{A}_{k,j}}(X_t - X_{s_{k,j}})\\
&\qquad\qquad + \theta^{k-1}x/4 - 42Q\theta^{k-2}x\Big) \qquad (5.5)\\
&\quad\le \mathbb{P}\Big(\sup_{t\in\mathcal{A}_{k,j}}(X_t - X_{s_{k,j}}) > \mathbb{E}\sup_{t\in\mathcal{A}_{k,j}}(X_t - X_{s_{k,j}}) + \theta^{k-1}x/5\Big)\\
&\quad\le 2\exp\big(-(\theta/5)^2/2\big) \le 10^{-2}.
\end{aligned}$$

This proves (5.3). □

PROOF OF THEOREM 2.2. Without loss of generality, assume again that $X_{t_0} = 0$. Let Z_i, $1 \le i \le M$, be i.i.d. standard normal random variables, Σ_1 be the covariance matrix of $\{X_{s_i}/(\mathbb{E}|X_{s_i}|^2)^{1/2}, 1 \le i \le M\}$ and Σ_2 be the covariance matrix of $\{3Z_i/2, 1 \le i \le M\}$. By (2.4), Σ_1 is a dominant principal diagonal matrix. Moreover, by Price (1951),

$$\det(\Sigma_1) \ge (1 - 1/4)^M. \qquad (5.6)$$

It follows from assumption (2.4) again that $\Sigma_2 - \Sigma_1$ is also a dominant principal diagonal matrix and hence is positive semidefinite. Thus, by Lemma 5.1,

$$\begin{aligned}
&\mathbb{P}\big((X_{s_i}/(\mathbb{E}X_{s_i}^2)^{1/2}, i \le M) \in G\big)\\
&\quad\le \big(\det(\Sigma_2)/\det(\Sigma_1)\big)^{1/2}\mathbb{P}\big((3Z_i/2, i \le M) \in G\big) \qquad (5.7)\\
&\quad\le 2^{M/2}\mathbb{P}\big((3Z_i/2, i \le M) \in G\big)
\end{aligned}$$

for all $G \subset \mathbb{R}^M$. In particular, we have

$$\begin{aligned}
\mathbb{P}\Big(\max_{i\le M} X_{s_i} \le x\Big) &= \mathbb{P}\Big(\bigcap_{i\le M}\{X_{s_i}/(\mathbb{E}X_{s_i}^2)^{1/2} \le x/(\mathbb{E}X_{s_i}^2)^{1/2}\}\Big)\\
&\le \mathbb{P}\Big(\bigcap_{i\le M}\{X_{s_i}/(\mathbb{E}X_{s_i}^2)^{1/2} \le 1/2\}\Big)\\
&\le 2^{M/2}\mathbb{P}\Big(\max_{i\le M} 3Z_i/2 \le 1/2\Big)\\
&= \big(2^{1/2}\mathbb{P}(Z \le 1/3)\big)^M\\
&\le e^{-M/10},
\end{aligned}$$

as desired. $\square$

PROOF OF THEOREM 2.3. First note that (2.2) is satisfied. This can be verified as follows. If $|t-s| \le \max(|t|, |s|)$, then clearly (2.2) holds. If $|t-s| > \max(|t|, |s|)$, we have

$$\begin{aligned}
2\mathbb{E}(X_t X_s) &= \mathbb{E}X_t^2 + \mathbb{E}X_s^2 - \mathbb{E}(X_t - X_s)^2 \\
&= \sigma^2(|t|) + \sigma^2(|s|) - \sigma^2(|t-s|) \\
&= \sigma^2(|t-s|)\bigl(\sigma^2(|t|)/\sigma^2(|t-s|) + \sigma^2(|s|)/\sigma^2(|t-s|) - 1\bigr) \\
&\ge \sigma^2(|t-s|)\bigl(|t|^2/|t-s|^2 + |s|^2/|t-s|^2 - 1\bigr) \\
&= \sigma^2(|t-s|)(|t|^2 + |s|^2 - |t-s|^2)/|t-s|^2 \ge 0
\end{aligned}$$

by the assumption that $\sigma(t)/t$ is nonincreasing and $\sigma(h)$ is nondecreasing. For $t \in [0,1]^d$ and $0 < h \le \sigma(1)$, we have

$$B(t,h) = \{s \in [0,1]^d : \sigma(|s-t|) \le h\} = \{s \in [0,1]^d : |s-t| \le \sigma^{-1}(h)\},$$

where σ^{-1} is the inverse of σ. It is easy to see that there is a constant K such that

$$N\bigl(B(t,h), \varepsilon h\bigr) \le K\bigl(\sigma^{-1}(h)/\sigma^{-1}(\varepsilon h)\bigr)^d \le K\varepsilon^{-d/\alpha} \tag{5.8}$$

where we use the assumption that $\sigma(h)/h^\alpha$ is nondecreasing. Hence Q is finite. Similarly to the establishment of (5.8), it is easy to see that $N_k(x) \le K\theta^{2d/\alpha}$ for $k = 0, 1, \dots, L$. Note that $L \le 1 + \log_\theta(\sigma(1)/\sigma(x)) \le 1 + \log_\theta(1/x^\beta)$. This proves the lower bound of (2.7) by Theorem 2.1.

We next prove the right-hand side of (2.7). Without loss of generality, we can assume $d = 1$. In fact, we have

$$\mathbb{P}\Bigl(\sup_{t\in[0,1]^d} X_t \le \sigma(x)\Bigr) \le \mathbb{P}\Bigl(\sup_{t\in[0,1]} X_{t,0,\dots,0} \le \sigma(x)\Bigr)$$

and $\sigma^2(|t-s|) = \mathbb{E}(X_{t,0,\dots,0} - X_{s,0,\dots,0})^2$ for $s, t \in [0,1]$. Let $\theta > 1$, which will be specified later. Put $s_i = \theta^i x$ for $i = 1, 2, \dots, L$ where $L := [\log_\theta(1/x)]$. We need to verify that (2.4) is satisfied when θ is large. For every $1 \le i < j \le L$, we have

$$\begin{aligned}
|\mathrm{Corr}(X_{s_i}, X_{s_j})| &= (1/2)|\sigma^2(s_i) + \sigma^2(s_j) - \sigma^2(s_j - s_i)|/\bigl(\sigma(s_i)\sigma(s_j)\bigr) \\
&\le \sigma(s_i)/\sigma(s_j) + 2\bigl(\sigma(s_j) - \sigma(s_j - s_i)\bigr)/\sigma(s_i) \\
&\le (s_i/s_j)^\alpha + 2\sigma(s_j)\bigl(1 - ((s_j - s_i)/s_j)^\beta\bigr)/\sigma(s_i) \\
&\le \theta^{(i-j)\alpha} + 2\sigma(s_j)(s_i/s_j)/\sigma(s_i) \\
&\le \theta^{(i-j)\alpha} + 2(s_j/s_i)^\beta(s_i/s_j) \\
&= \theta^{(i-j)\alpha} + 2\theta^{(i-j)(1-\beta)},
\end{aligned} \tag{5.9}$$

which follows immediately that (2.4) is satisfied when θ is large. Therefore, the right-hand side of (2.7) holds. $\square$

5.1. *Proof of Theorem* 2.4. We first prove the left-hand side of (2.9). Although it is possible to apply Theorem 2.1 directly, we would like to present an alternative proof, which may be of independent interest when it is difficult to estimate $N(B(t,h),\varepsilon h)$. Let $\theta > 1$ and $m \ge 2$ which will be specified later, and $a = a(x) = 1 + [m \log_\theta(1/x)]$, $b = b(x) = 1 + [\log_\theta(1/x)]$. Write $\mathbf{k} = (k_1, \dots, k_d)$ and use the notation $\theta^{\mathbf{k}} = (\theta^{k_1}, \dots, \theta^{k_d})$ and $\theta^{\mathbf{k-1}} = (\theta^{k_1-1}, \dots, \theta^{k_d-1})$.

By using the Slepian lemma, we have

$$\begin{aligned} &\mathbb{P}\Big(\sup_{t\in[0,1]^d} X_t \le \sigma^d(x)\Big) \\ &\quad \ge \mathbb{P}\Big(\sup_{t\in[x\theta^{-a},1]^d} X_t \le \sigma^d(x)\Big) \prod_{i=1}^d \mathbb{P}\Big(\sup_{t\in[0,1]^d, 0\le t_i < x\theta^{-a}} X_t \le \sigma^d(x)\Big) \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} &\mathbb{P}\Big(\sup_{t\in[x\theta^{-a},1]^d} X_t \le \sigma^d(x)\Big) \\ &\quad \ge \mathbb{P}\Big(\max_{-a\le k_i\le b, 1\le i\le d} \sup_{\theta^{\mathbf{k-1}}x \le t \le \theta^{\mathbf{k}}x} X_t \le \sigma^d(x)\Big) \\ &\quad \ge \prod_{l=-da}^{db} \prod_{k_1+\cdots+k_d=l, -a\le k_i\le b} \mathbb{P}\Big(\sup_{\theta^{\mathbf{k-1}}x\le t\le\theta^{\mathbf{k}}x} X_t \le \sigma^d(x)\Big). \end{aligned}$$

Writing $X_t - X_s = \sum_{i=1}^d (X_{t^{(i)}} - X_{t^{(i-1)}})$, where $t^{(i)} = (t_1, \dots, t_i, s_{i+1}, \dots, s_d)$, we have

$$\begin{aligned} \mathbb{E}(X_t - X_s)^2 &\le d \sum_{i=1}^d \mathbb{E}\big(X_{t^{(i)}} - X_{t^{(i-1)}}\big)^2 \\ &= d \sum_{i=1}^d \sigma^2(|t_i - s_i|) \Big\{\prod_{1\le j<i} \sigma^2(t_j)\Big\}\Big\{\prod_{i<j\le n} \sigma^2(s_j)\Big\} \\ &\le d^2 \max_{i\le d}\big(\sigma^2(|t_i - s_i|)/\big(\sigma^2(t_i)\vee\sigma^2(s_i)\big)\big) \prod_{i=1}^d \big(\sigma^2(t_i)\vee\sigma^2(s_i)\big). \end{aligned} \tag{5.11}$$

Since $\sigma(h)/h^\alpha$ is nondecreasing and $\sigma(h)/h^\beta$ is nonincreasing, $\sigma(\delta h) \le \delta^\alpha \sigma(h)$, $\sigma(\Delta h) \le \delta^\beta \sigma(h)$ for $0 < \delta \le 1 \le \Delta$. Hence for $1 \le i \le d$,

$$\begin{aligned} \sup_{t\in[0,1]^d, 0\le t_i\le x\theta^{-a}} \mathbb{E}X_t^2 &\le \sigma^2(x\theta^{-a})\sigma^{2(d-1)}(1) \\ &\le \theta^{-2a\alpha} x^{-2d} \sigma^{2d}(x) \le (1/4)\sigma^{2d}(x) \end{aligned}$$

for sufficient large m. Thus, by the Fernique (1964) inequality,

$$\mathbb{P}\Big(\sup_{t\in[0,1]^d, 0\le t_i\le x\theta^{-a}} X_t \le \sigma^d(x)\Big) \ge e^{-1}. \tag{5.12}$$

It remains to show that

$$\mathbb{P}\Big(\sup_{\theta^{\mathbf{k}-1}x\le t\le\theta^{\mathbf{k}}x} X_t \le \sigma^d(x)\Big) \ge e^{-1} \tag{5.13}$$

for all $-da \le l \le db$, $-a \le k_i \le b$ and $k_1 + \cdots + k_d = l$.

Let $s_k = \theta^{\mathbf{k}} x$. Following the proof of (5.4), we see that

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{\theta^{\mathbf{k}-\mathbf{1}}x\le t\le\theta^{\mathbf{k}}x} X_t \le 0\Big)\\
&\quad \ge \mathbb{P}\Big(X_{s_{\mathbf{k}}} < -\tfrac14 \prod_{i=1}^d \sigma\big(\theta^{k_i}x\big)\Big) - \mathbb{P}\Big(\sup_{\theta^{\mathbf{k}-\mathbf{1}}x\le t\le\theta^{\mathbf{k}}x} (X_t - X_{s_{\mathbf{k}}}) \ge \tfrac14 \prod_{i=1}^d \sigma(\theta^{k_i}x)\Big)\\
&\quad \ge e^{-1} + 10^{-2} - \mathbb{P}\Big(\sup_{\theta^{\mathbf{k}-\mathbf{1}}x\le t\le\theta^{\mathbf{k}}x} (X_t - X_{s_{\mathbf{k}}}) \ge \tfrac14 \prod_{i=1}^d \sigma(\theta^{k_i}x)\Big).
\end{aligned}$$

By (5.11) and the Fernique inequality again, along the same line of proving (5.5), we have

$$\mathbb{P}\Big(\sup_{\theta^{\mathbf{k}-\mathbf{1}}x\le t\le\theta^{\mathbf{k}}x} (X_t - X_{s_{\mathbf{k}}}) \ge \tfrac14 \prod_{i=1}^d \sigma\big(\theta^{k_i}x\big)\Big) < 10^{-2} \tag{5.14}$$

when $\theta > 1$ is close to 1. This proves the left-hand side of (2.9).

To prove the right-hand side of (2.9), put

$$\gamma = \min(\alpha, 1-\beta), \qquad \theta_0 = 2^{1/\gamma}.$$

Let $\theta > \theta_0$, $L = [\log_\theta(1/x)]$ and

$$s_{\mathbf{k}} = \theta^{\mathbf{k}} x^{1/d}, \qquad \mathbf{k} = (k_1, \ldots, k_d), \qquad 1 \le k_i \le L,$$

so that $d(s_{\mathbf{k}}, \mathbf{0}) \ge \theta^{k_1+\cdots+k_d} x \ge x/2$. Similar to the arguments in (5.9), we have

$$\begin{aligned}
\big|\mathrm{Corr}\big(X_{s_{\mathbf{k}}}, X_{s_{\mathbf{j}}}\big)\big| &= \prod_{i=1}^d \frac{\sigma^2(\theta^{k_i}x^{1/d}) + \sigma^2(\theta^{j_i}x^{1/d}) - \sigma^2(|\theta^{k_i} - \theta^{j_i}|x^{1/d})}{2\sigma(\theta^{k_i}x^{1/d})\sigma(\theta^{j_i}x^{1/d})}\\
&\le \prod_{i=1}^d \min\big(1, 2\theta^{-|k_i-j_i|\gamma}\big)\\
&\le \prod_{i=1}^d (\theta/\theta_0)^{-|k_i-j_i|\gamma}\\
&= (\theta/\theta_0)^{-\gamma\sum_{i=1}^d |k_i-j_i|}.
\end{aligned} \tag{5.15}$$

Therefore, for any given $\mathbf{k}$,

$$\begin{aligned}\sum_{1\le\mathbf{j}\le L}|\operatorname{Corr}(X_{\mathbf{k}},X_{\mathbf{j}})| &\le \sum_{1\le\mathbf{j}\le L}(\theta/\theta_0)^{-\gamma\sum_{i=1}^d|k_i-j_i|}\\ &\le 1+\frac{d2^d(\theta_0/\theta)}{(1-\theta_0/\theta)^d}\le 5/4\end{aligned}\tag{5.16}$$

when θ is sufficiently large. Now the right-hand side of (2.9) follows from Theorem 2.2.

PROOF OF THEOREM 3.1. It suffices to show that

$$\liminf_{x\downarrow 0}\frac{\log\mathbb{P}(\sup_{0\le t\le 1}B_\alpha(t)\le x)}{\log(1/x)}\ge -2c_\alpha/\alpha\tag{5.17}$$

and

$$\limsup_{x\downarrow 0}\frac{\log\mathbb{P}(\sup_{0\le t\le 1}B_\alpha(t)\le x)}{\log(1/x)}\le -2c_\alpha/\alpha.\tag{5.18}$$

By the scaling property of the fractional Brownian motion, for any $0<x<1/2$,

$$\begin{aligned}&\mathbb{P}\Big(\sup_{0\le t\le(2/\alpha)\log(1/x)}X_\alpha(t)\le 0\Big)\\ &\quad=\mathbb{P}\Big(\sup_{x^{2/\alpha}\le t\le 1}B_\alpha(t)\le 0\Big)\\ &\quad\le\mathbb{P}\Big(\sup_{x^{2/\alpha}\le t\le 1}B_\alpha(t)\le x\Big)\\ &\quad\le\frac{\mathbb{P}(\sup_{0<t\le 1}B_\alpha(t)\le x)}{\mathbb{P}(\sup_{0<t\le x^{2/\alpha}}B_\alpha(t)\le x)}\\ &\quad=\frac{\mathbb{P}(\sup_{0<t\le 1}B_\alpha(t)\le x)}{\mathbb{P}(\sup_{0<t\le 1}B_\alpha(t)\le 1)},\end{aligned}\tag{5.19}$$

where we have used the Slepian lemma in the last inequality. Now (5.17) follows from (3.6) and (5.19).

To prove (5.18), we consider two cases.

Case I. $1\le\alpha<2$. It is easy to see that, for $t\ge a$,

$$\mathbb{E}\big(B_\alpha(t)-B_\alpha(a)\big)B_\alpha(a)\ge 0.\tag{5.20}$$

Therefore, by the Slepian lemma,

$$
\begin{aligned}
&\mathbb{P}\Big(\sup_{0\le t\le (2/\alpha)\log(1/x)} X_\alpha(t)\le 0\Big)\\
&\qquad = \Big(\sup_{x^{2/\alpha}\le t\le 1} B_\alpha(t)\le 0\Big)\\
&\qquad \ge \mathbb{P}\Big(\sup_{x^{2/\alpha}\le t\le 1}\big(B_\alpha(t)-B_\alpha(x^{2/\alpha})\big)\le x,\, B_\alpha(x^{2/\alpha})\le -x\Big) \qquad (5.21)\\
&\qquad \ge \mathbb{P}\Big(\sup_{x^{2/\alpha}\le t\le 1}\big(B_\alpha(t)-B_\alpha(x^{2/\alpha})\big)\le x\Big)\mathbb{P}\big(B_\alpha(x^{2/\alpha})\le -x\big)\\
&\qquad \ge \mathbb{P}\Big(\sup_{0<t\le 1} B_\alpha(t)\le x\Big)\mathbb{P}\big(B_\alpha(1)\le -1\big).
\end{aligned}
$$

Hence (5.18) holds by (3.6).

Case II. $0<\alpha<1$. Let $0<\theta<1-\alpha$, and choose $m=m(\alpha,\theta)>2^{\alpha/(2\theta)}$ such that, for $0\le y\le (1/m)^{2/\alpha}$,

$$\theta+\alpha+\theta y^\alpha-\theta(1-y)^\alpha-\alpha(1-y)^{\alpha-1}\ge 0, \tag{5.22}$$

$$(2\theta+\alpha)-(\theta+\alpha)y^\theta-\theta y^{\theta+\alpha}+\theta y^\theta(1-y)^\alpha+\alpha y^\theta(1-y)^{\alpha-1}\ge 0 \tag{5.23}$$

and

$$\theta+\alpha-\theta(1-y)^\alpha-\alpha(1-y)^{\alpha-1}\le 0. \tag{5.24}$$

The existence of m can be verified with the Taylor expansion.

For $0<x<1/m$, put

$$b=x^{2/\alpha},\qquad Y(t)=t^\theta B_\alpha(t)-b^\theta B_\alpha(b)$$

for $t\ge m^{2/\alpha}b$. Then we have, for $t\ge m^{2/\alpha}b$,

$$h(t):=2t^\theta\mathbb{E}B_\alpha(t)B_\alpha(b)=t^\theta\big(t^\alpha+b^\alpha-(t-b)^\alpha\big)$$

and

$$
\begin{aligned}
h'(t)&=(\theta+\alpha)t^{\theta+\alpha-1}+\theta b^\alpha t^{\theta-1}-\theta t^{\theta-1}(t-b)^\alpha-\alpha t^\theta(t-b)^{\alpha-1}\\
&=t^{\theta+\alpha-1}\big(\theta+\alpha+\theta(b/t)^\alpha-\theta(1-b/t)^\alpha-\alpha(1-b/t)^{\alpha-1}\big)\\
&\ge 0
\end{aligned}
$$

by using (5.22). Thus, for $t\ge m^{2/\alpha}b$ and $m>2^{\alpha/(2\theta)}$,

$$
\begin{aligned}
&\mathbb{E}\big(t^\theta B_\alpha(t)-b^\theta B_\alpha(b)\big)B_\alpha(b)\\
&\qquad\ge \mathbb{E}\big(m^{2\theta/\alpha}b^\theta B_\alpha(m^{2/\alpha}b)-b^\theta B_\alpha(b)\big)B_\alpha(b)\ge 0.
\end{aligned} \tag{5.25}
$$

Next, for $t \geq m^{2/\alpha}b$, let

$$t^* = \big(\mathbb{E}Y^2(t)\big)^{1/\alpha} = \big(t^{2\theta+\alpha} + b^{2\theta+\alpha} - t^\theta b^\theta\big(t^\alpha + b^\alpha - (t-b)^\alpha\big)\big)^{1/\alpha}.$$

Then for $t \geq s \geq m^{2/\alpha}b$, we have $t^* \geq s^*$ by using (5.23) to show $dt^*/dt \geq 0$. And furthermore, for $t \geq s \geq m^{2/\alpha}b$,

$$\begin{aligned}
\mathbb{E}\big(Y(t)Y(s)\big) &= \tfrac{1}{2}\big(\mathbb{E}Y^2(t) + \mathbb{E}Y^2(s) - \mathbb{E}\big(Y(t) - Y(s)\big)^2\big) \\
&= \tfrac{1}{2}(t^{*\alpha} + s^{*\alpha} - |t^* - s^*|^\alpha) \\
&\quad + \tfrac{1}{2}\big(|t^* - s^*|^\alpha - t^{2\theta+\alpha} - s^{2\theta+\alpha} + t^\theta s^\theta\big(t^\alpha + s^\alpha - (t-s)^\alpha\big)\big) \\
&= \mathbb{E}B_\alpha(t^*)B_\alpha(s^*) + g(t,s)/2,
\end{aligned}$$

where

$$\begin{aligned}
g(t,s) &:= |t^* - s^*|^\alpha - t^{2\theta+\alpha} - s^{2\theta+\alpha} + t^\theta s^\theta\big(t^\alpha + s^\alpha - (t-s)^\alpha\big) \\
&\geq\ t^{*\alpha} - s^{*\alpha} - t^{2\theta+\alpha} - s^{2\theta+\alpha} + t^\theta s^\theta s^\alpha \\
&=\ s^\theta b^\theta\big(s^\alpha + b^\alpha - (s-b)^\alpha\big) - t^\theta b^\theta\big(t^\alpha + b^\alpha - (t-b)^\alpha\big) + t^\theta s^{\theta+\alpha} \\
&\geq\ s^\theta b^\theta\big(s^\alpha - (s-b)^\alpha\big) - t^\theta b^\theta\big(t^\alpha - (t-b)^\alpha\big) - t^\theta b^\theta b^\alpha + t^\theta s^{\theta+\alpha} \\
&\geq\ s^\theta b^\theta\big(s^\alpha - (s-b)^\alpha\big) - t^\theta b^\theta\big(t^\alpha - (t-b)^\alpha\big) \\
&\geq\ 0
\end{aligned}$$

since the function $t^\theta(t^\alpha - (t-b)^\alpha)$ is decreasing by (5.24). Consequently, we have

$$\mathbb{E}\big(Y(t)Y(s)\big) \geq \mathbb{E}B_\alpha(t^*)B_\alpha(s^*) \qquad \text{for } t \geq s \geq m^{2/\alpha}b. \tag{5.26}$$

Following the proof of Case I and by using (5.25), (5.26) and the Slepian lemma again, we obtain

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{0 \leq t \leq (2/\alpha)\log(1/(mx))} X_\alpha(t) \leq 0\Big) \\
&\quad = \mathbb{P}\Big(\sup_{(mx)^{2/\alpha} \leq t \leq 1} t^\theta B_\alpha(t) \leq 0\Big) \\
&\quad \geq \mathbb{P}\Big(\sup_{m^{2/\alpha}b \leq t \leq 1} Y(t) \leq xb^\theta, b^\theta B_\alpha(b) \leq -xb^\theta\Big) \\
&\quad \geq \mathbb{P}\Big(\sup_{m^{2/\alpha}b \leq t \leq 1} Y(t) \leq xb^\theta \mathbb{P}B_\alpha(b) \leq -x\Big) \\
&\quad \geq \mathbb{P}\Big(\sup_{m^{2/\alpha}b \leq t \leq 1} B_\alpha(t^*) \leq xb^\theta\Big)\mathbb{P}\big(B_\alpha(x^{2/\alpha}) \leq -x\big) \\
&\quad \geq \mathbb{P}\Big(\sup_{0 \leq t \leq 1} B_\alpha(t) \leq xb^\theta\Big)\mathbb{P}\big(B_\alpha(1) \leq -1\big),
\end{aligned}$$

where we used the fact that $1^* \le 1$ in the last inequality. Hence

$$\begin{aligned}
&\limsup_{x\downarrow 0} \frac{\log \mathbb{P}(\sup_{0\le t\le 1} B_\alpha(t) \le x)}{\log(1/x)} \\
&\quad = \limsup_{x\downarrow 0} \frac{\log \mathbb{P}(\sup_{0\le t\le 1} B_\alpha(t) \le x b^\theta)}{\log(1/(xb^\theta))} \\
&\quad \le \limsup_{x\downarrow 0} \frac{\log \mathbb{P}(\sup_{0\le t\le (2/\alpha)\log(1/(mx))} X_\alpha(t) \le 0)}{\log(1/(xb^\theta))} \\
&\quad = \frac{2c_\alpha}{\alpha + 2\theta}.
\end{aligned}$$

Letting $\theta \to 0$ yields (5.18), as desired. Finally, we have $c_\alpha = 1 - \alpha/2$ based on the main results in Molchan (1999, 2000). $\square$

PROOF OF PROPOSITION 3.3. Recall that $Y(t)$ has a scaling property, that is, $\{Y(at), t > 0\}$ and $\{a^{1/2}Y(t), t > 0\}$ have the same distribution. Thus, following the proof of (5.17),

$$\liminf_{x\downarrow 0} \frac{\log \mathbb{P}(\sup_{0\le t\le 1} Y(t) \le x)}{\log(1/x)} \ge -b/2.$$

We see that

$$\mathbb{E}\big(Y(t) - Y(a)\big)Y(a) = \frac{(t-a)a}{t+a} \ge 0$$

for $t \ge a$. Similar to the proof of (5.18) in Case I, we have

$$\limsup_{x\downarrow 0} \frac{\log \mathbb{P}(\sup_{0\le t\le 1} Y(t) \le x)}{\log(1/x)} \le -b/2.$$

To show $b < 1.29$, we need the following result of Slepian (1962). For any stationary Gaussian process Z_t with continuous covariance $r(\tau) = \mathbb{E}Z_t Z_{t+\tau} \sim 1 - \tau^2/2 + o(\tau^2)$ as $\tau \to 0$,

$$\mathbb{P}\Big(\sup_{0\le t\le T} Z_t \le 0\Big) \ge \frac{3}{8} - \frac{T}{4\pi} + \frac{1}{4\pi}\arcsin(r(T)). \tag{5.27}$$

This bound goes negative for relatively small values of T (at least before $T = 2\pi$). So we have to use (3.3) to handle large T. Now for X_t given in (1.7), the process X_{2t} satisfies the condition for (5.27) with $r(T) = 1/\cosh(T) \ge 0$. Hence for $T_0 = 3.1$ and $T = 2nT_0$ large, we have from (3.3)

$$\mathbb{P}\Big(\sup_{0\le t\le 2nT_0} X_t \le 0\Big) = \mathbb{P}\Big(\sup_{0\le t\le nT_0} X_{2t} \le 0\Big) \ge \mathbb{P}\Big(\sup_{0\le t\le T_0} X_{2t} \le 0\Big)^n.$$

Thus from (1.6) and (5.27),

$$\begin{aligned}
b &= -4 \lim_{n\to\infty} \frac{1}{2nT_0} \log \mathbb{P}\Big(\sup_{0\le t\le 2nT_0} X_t \le 0\Big) \\
&\le -\frac{2}{T_0} \log \mathbb{P}\Big(\sup_{0\le t\le T_0} X_{2t} \le 0\Big) \\
&\le -\frac{2}{T_0} \log\Big(\frac{3}{8} - \frac{T_0}{4\pi} + \frac{1}{4\pi} \arcsin\big(1/\cosh(T_0)\big)\Big) \\
&< 1.29.
\end{aligned}$$

This completes the proof of Proposition 3.3. □

PROOF OF THEOREM 4.1. Equation (4.2) follows directly from the proof of Theorem 3.1 with minor modification. To prove the right-hand side of (4.3), it suffices to show that

$$\mathbb{P}\Big(\sup_{0\le t\le 1} \max_{1\le k\le n} \big(B_k(t) + B_0(t)\big) \le x\Big) \ge r_{n,\alpha} x^{r_\alpha \log n}, \tag{5.28}$$

where $r_{n,\alpha}$ and r_α are some positive constants, and $B_i = B_{i,\alpha}$ for $0 \le i \le k$ for the sake of statement simplicity.

Let m be an integer such that

$$1 \le x e^{m\alpha/2} \le e^{\alpha/2}. \tag{5.29}$$

Then, by the Slepian lemma,

$$\begin{aligned}
&\mathbb{P}\Big(\max_{1\le k\le n} \sup_{0\le t\le 1} \big(B_k(t) + B_0(t)\big) < x\Big) \\
&\quad\ge \mathbb{P}\Big(\sup_{0\le t\le e^{-m}} \max_{1\le k\le n} \big(B_k(t) + B_0(t)\big) < x\Big) \\
&\qquad\times \prod_{i=1}^{m} \mathbb{P}\Big(\sup_{e^{-i}\le t\le e^{-i+1}} \max_{1\le k\le n} \big(B_k(t) + B_0(t)\big) < x\Big) \\
&\quad\ge \mathbb{P}\Big(\sup_{0\le t\le 1} \max_{1\le k\le n} \big(B_k(t) + B_0(t)\big) < x e^{m\alpha/2}\Big) \\
&\qquad\times \prod_{i=1}^{m} \mathbb{P}\Big(\sup_{e^{-i}\le t\le e^{-i+1}} \max_{1\le k\le n} \big(B_k(t) + B_0(t)\big) < 0\Big) \\
&\quad\ge \mathbb{P}\Big(\sup_{0\le t\le 1} \max_{1\le k\le n} \big(B_k(t) + B_0(t)\big) < 1\Big) \\
&\qquad\times \Big\{\mathbb{P}\Big(\sup_{1\le t\le e} \max_{1\le k\le n} \big(B_k(t) + B_0(t)\big) < 0\Big)\Big\}^m.
\end{aligned} \tag{5.30}$$

Observe that, for $a > 0$,

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{1\le t\le e}\max_{1\le k\le n}\big(B_k(t)+B_0(t)\big)<0\Big)\\
&\quad\ge \mathbb{P}\Big(\sup_{1\le t\le e}B_0(t)\le -a,\ \max_{1\le k\le n}\sup_{1\le t\le e}B_k(t)<a\Big)\\
&\quad= \mathbb{P}\Big(\sup_{1\le t\le e}B_0(t)\le -a\Big)\mathbb{P}\Big(\sup_{1\le t\le e}B_0(t)<a\Big)^n.
\end{aligned}$$

It is easy to see that

$$\mathbb{E}\big(\big(B_0(t)-(1/2)B_0(1)\big)B_0(1)\big)\ge 0$$

for $1\le t\le e$. Hence, by the Slepian lemma,

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{1\le t\le e}B_0(t)\le -a\Big)\\
&\quad\ge \mathbb{P}\Big(\sup_{1\le t\le e}B_0(t)-(1/2)B_0(1)\le a,\ B_0(1)\le -4a\Big)\\
&\quad\ge \mathbb{P}\Big(\sup_{1\le t\le e}B_0(t)-(1/2)B_0(1)\le a\Big)\mathbb{P}\big(B_0(1)\le -4a\big).
\end{aligned}$$

By the Fernique inequality, there exists $K_\alpha > 0$ such that

$$\mathbb{P}\Big(\sup_{1\le t\le e}B_0(t)-(1/2)B_0(1)\le a\Big)\ge 1-K_\alpha\exp\big(-a^2/(2e^\alpha)\big),$$

$$\mathbb{P}\Big(\sup_{1\le t\le e}B_0(t)\le a\Big)\ge 1-K_\alpha\exp\big(-a^2/(2e^\alpha)\big)$$

and

$$\mathbb{P}\big(B_\alpha(1)\le -4a\big)\ge \exp(-16a^2).$$

Now letting $a^2=2e^\alpha\log(K_\alpha n)$ yields

$$\mathbb{P}\Big(\sup_{1\le t\le e}\max_{1\le k\le n}\big(B_k(t)+B_0(t)\big)<0\Big)\ge \exp\big(-2e^\alpha\log(K_\alpha n)\big)$$

and hence

$$\Big(\mathbb{P}\Big(\sup_{1\le t\le e}\max_{1\le k\le n}\big(B_k(t)+B_0(t)\big)<0\Big)\Big)^m\ge \exp\big(-m2e^\alpha\log(K_\alpha n)\big)\ge x^{r_\alpha\log n}$$

for some $r_\alpha > 0$ and for all $0 < x < 1/2$. This proves (5.28). $\square$

REFERENCES

BASS, R., EISENBAUM, N. and SHI, Z. (2000). The most visited sites of symmetric stable processes. *Probab. Theory Related Fields* **116** 391–404.

BELLMAN, R. (1970). *Introduction to Matrix Analysis*. McGraw-Hill, New York. [Reprinted (1997) by SIAM.]

BRAMSON, M. and GRIFFEATH, D. (1991). Capture problems for coupled random walks. In *Random Walks, Brownian Motion and Interacting Particle Systems* (R. Durrett and H. Kesten, eds.) 153–188. Birkhäuser, Boston.

CSÁKI, E., KHOSHNEVISAN, D. and SHI, Z. (2000). Boundary crossings and the distribution function of the maximum of Brownian sheet. *Stochastic Process. Appl.* **90** 1–18.

DEMBO, A., POONEN, B., SHAO, Q.-M. and ZEITOUNI, O. (2002). Random polynomials having few or no real zeros. *J. Amer. Math. Soc.* **15** 857–892.

DUDLEY, R. M. (1967). The sizes of compact subsets of Hilbert space and continuity of Gaussian processes. *J. Funct. Anal.* **1** 290–330.

FERNIQUE, X. (1964). Continuité des processus Gaussiens. *C. R. Acad. Sci. Paris* **258** 6058–6060.

KESTEN, H. (1991). An absorption problem for several Brownian motions. In *Seminar on Stochastic Processes* (E. Çinlar, K. L. Chung and M. J. Sharpe, eds.) 59–72. Birkhäuser, Boston.

LEADBETTER, M. R., LINDGREN, G. and ROOTZEN, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer, New York.

LEDOUX, M. (1996). Isoperimetry and Gaussian analysis. *Lectures on Probability Theory and Statistics. Lecture Notes in Math.* **1648** 165–294. Springer, New York.

LEDOUX, M. and TALAGRAND, M. (1991). *Probability on Banach Spaces*. Springer, New York.

LI, W. V. and SHAO, Q.-M. (2001a). Gaussian processes: Inequalities, small ball probabilities and applications. In *Stochastic Processes*: *Theory and Methods* (C. R. Rao and D. Shanbhag, eds.) 533–598. North-Holland, Amsterdam.

LI, W. V. and SHAO, Q.-M. (2001b). Capture time of Brownian pursuits. *Probab. Theory Related Fields* **121** 30–48.

LI, W. V. and SHAO, Q.-M. (2002). A normal comparison inequality and its applications. *Probab. Theory Related Fields* **122** 494–508.

MARCUS, M. (2000). Probability estimates for lower levels of certain Gaussian processes with stationary increments. In *High Dimensional Probability II* (E. Gine, D. Mason and J. Wellner, eds.) 173–179. Birkhäuser, Boston.

MOLCHAN, M. (1999). Maximum of a fractional Brownian motion: Probabilities of small values. *Comm. Math. Phys.* **205** 97–111.

MOLCHAN, M. (2000). On the maximum of fractional Brownian motion. *Theory Probab. Appl.* **44** 97–102.

NEWELL, G. and ROSENBLATT, M. (1962). Zero crossing probabilities for Gaussian stationary processes. *Ann. Math. Stat.* **33** 1306–1313.

PICKANDS, J. (1969). Asymptotic properties of maximum in a stationary Gaussian processes. *Trans. Amer. Math. Soc.* **145** 75–86.

PRICE, G. (1951). Bounds for determinants with dominant principal diagonal. *Proc. Amer. Math. Soc.* **2** 497–502.

SHAO, Q.-M. (1999). A Gaussian correlation inequality and its applications to the existence of small ball constant. Preprint.

SLEPIAN, D. (1961). First passage time for a particular Gaussian process. *Ann. Math. Statist.* **32** 610–612.

SLEPIAN, D. (1962). The one-sided barrier problem for Gaussian noise. *Bell System Tech. J.* **41** 463–501.

STRAKHOV, N. and KURZ, L. (1968). An upper bound on the zero-crossing distribution. *Bell System Tech. J.* **47** 529–547.

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF DELAWARE
NEWARK, DELAWARE 19716
USA
E-MAIL: wli@math.udel.edu

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OREGON
EUGENE, OREGON 97403
USA
E-MAIL: shao@math.uoregon.edu