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STRONG SOLUTIONS TO THE STOCHASTIC QUANTIZATION EQUATIONS

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We prove the existence and uniqueness of a strong solution of the stochastic quantization equation in dimension 2 for almost all initial data with respect to the invariant measure. The method is based on a fixed point result in suitable Besov spaces.

1. Introduction. In this article, we consider stochastic quantization equations in space dimension 2 with periodic boundary conditions. These are reaction–diffusion equations driven by a space–time white noise. It is well known that the solution is not expected to be a smooth process and the nonlinear term is modified thanks to a renormalization.

More precisely, let $G = [0, 2\pi]^2$ and $H = L^2(G)$. We are concerned with the equation set

$$\begin{aligned} dX &= \big(AX + :p(X):\big)\,dt + dW(t), \\ X(0) &= x, \end{aligned} \tag{1.1}$$

where $A : D(A) \subset H \to H$ is the linear operator

$$Ax = \Delta x - x, \qquad D(A) = H^2_\#(G)$$

and $H^2_\#(G)$ is the subspace of $H^2(G)$ of all functions which are periodic together with their first derivatives. Moreover, $p(\xi) = \sum_{h=0}^n a_k \xi^k$ is a polynomial of odd degree $n \geq 3$, with $a_n < 0$, and $:p(x):$ means the renormalization of $p(x)$ whose definition we shall recall in Section 2. Finally W is a cylindrical Wiener process defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values on $L^2(G)$.

Formally, (1.1) is a gradient system with an invariant Gibbs measure ν defined as

$$\nu(dx) = ce^{(:q(x):,1)}\mu(dx), \tag{1.2}$$

where q is a primitive of p, c is a normalization constant and μ is the Gaussian invariant measure of the free field. The measure ν is well defined by the important Nelson estimate; see [15], Chapter V2.

In [14], Chapter V, this problem was set to find a dynamic that has invariant measure ν. This problem was considered by several authors, beginning with

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Jona Lasinio and Mitter [11] who used a Girsanov transform; see also [3, 10].

Using the Dirichlet form approach, this problem was studied by Albeverio and Röckner [1] and by Liskevich and Röckner [12], who studied strong uniqueness for a class of infinite dimensional Dirichlet operators; see also [8] for similar results.

Finally, the problem was considered by Mikulevicius and Rozovskii [13] in their general theory of martingale solutions for stochastic partial differential equations (PDEs).

In all these articles (with the exception of [1] and [13]) a modified equation was considered of the kind

$$\begin{aligned} &dX = \big((-A)^{1-\varepsilon} X + (-A)^{-\varepsilon} :p(X):\big)\,dt + (-A)^{-\varepsilon/2}\,dW(t),\\ &X(0) = x, \end{aligned} \tag{1.3}$$

where ε is a positive number subject to different restrictions. Notice that the invariant measure that corresponds to (1.3) is still ν. This modification allows smoothing of the nonlinear term and, in some cases, using the Girsanov transform.

In [13] the problem (1.1) also was considered in its original form, showing that it has a stationary weak solution thanks to a compactness method. Note also that in this latter paper, it was shown that if $\varepsilon = 0$, then the law of the stationary solution is singular with respect to the law of the linear stationary solution ([13], Theorem 4.1). Therefore, it seems hopeless to use the Girsanov transform in this case.

Up to now, no article has considered the existence of pathwise solutions except for special nonlinearities of the type $:f(x):$, where f is the Fourier transform of a complex measure with compact support (e.g., $f(x) = \cos x$; see [2]). (The case $\varepsilon = 0$ also can be considered here.)

In [13] only solutions in law are constructed. In the present article we construct a strong (in the probability sense) solution for the original problem (1.1) for μ-almost every initial data x. We do not consider the modified equation (1.3) although our results easily extend to this case. Since many qualitative properties can be derived more easily from strong solutions than from weak solutions, we think that our result can be used to further study (1.1).

Let us explain our method, which is similar to what we used in [6] for the two-dimensional Navier–Stokes equations. One of the main difficulties when dealing with renormalized products is that they do not depend continuously on their arguments, so that it is not straightforward to use a fixed point argument or even to get strong uniqueness. The trick here is to split the unknown into two parts: we set $X = Y + z$, where $z(t)$ is the stochastic convolution

$$z(t) = \int_{-\infty}^{t} e^{(t-s)A}\,dW(s).$$

It is classical knowledge that z is a stationary solution to the linear version of (1.1). Then the observation is that Y is much smoother than X and that we can write

$$:X^k: = \sum_{l=0}^{k} C_k^l Y^l :z^{k-l}: \tag{1.4}$$

so that problem (1.1) becomes

$$\begin{aligned} \frac{dY}{dt} &= AY + \sum_{k=0}^{n} a_k \sum_{l=0}^{k} C_k^l Y^l :z^{k-l}:, \\ Y(0) &= x - z(0). \end{aligned} \tag{1.5}$$

Since the law of $z(t)$ is equal to μ for any $t \in \mathbb{R}$, we can define $:z^n:$ in the classical way through the formula

$$\mathbb{E}[g(:z^n:)] = \int_{\mathcal{H}} g(:x^n:)\mu(dx),$$

where g is any Borel bounded real function. The main advantage of considering (1.5) is that now the nonlinear term is a continuous function with respect to the unknown. However, the price to pay is to work with Besov spaces which are well suited to define the product with the distribution $:z^n:$. We will show that Y is sufficiently smooth so that the products in (1.4) are well defined. Then we can solve (4.2) by a fixed point on a suitable Besov space.

2. Notations and preliminaries. We denote by $\{e_k\}_{k\in\mathbb{Z}^2}$ the complete orthonormal system in $H = L^2(G)$, $G = [0, 2\pi]^2$, defined as

$$e_k(\xi) = \frac{1}{2\pi} e^{i\langle k,\xi\rangle}, \qquad k \in \mathbb{Z}^2.$$

If $x \in H$, we define $(x_k)_{k\in\mathbb{Z}^2}$ by

$$x = \sum_{k\in\mathbb{Z}^2} x_k e_k.$$

For any $N \in \mathbb{N}$ and $x \in H$, we set

$$x_N = P_N x = \sum_{|k|\le N} \langle x, e_k\rangle e_k.$$

Then we set $\mathcal{H} := \mathbb{C}^{\mathbb{Z}^2}$ and denote by μ the product measure on $\mathcal{H}$,

$$\mu = \bigtimes_{k\in\mathbb{Z}^2} \mathcal{N}\big(0, (1 + |k|^2)^{-1}\big) := \mathcal{N}(0, C),$$

where $\mathcal{N}(0, (1 + |k|^2)^{-1})$ is the Gaussian measure on $\mathbb{C}$ with mean 0 and variance $(1 + |k|^2)^{-1}$.

We identify H with $\ell^2(\mathbb{Z}^2)$ through the isomorphism

$$x \in H \mapsto \{x_k\}_{k\in\mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2).$$

Finally, we recall the definition of *white noise*. We denote by H_0 the linear space spanned by $\{e_k\}_{k\in\mathbb{Z}^2}$ and, for any $z \in H_0$, we set

$$W_z(x) = \langle x, C^{-1/2}z\rangle, \qquad x \in \mathcal{H}.$$

Then $W_z \in L^2(\mathcal{H}, \mu)$ and we have

$$\int_{\mathcal{H}} W_z(x) W_{z'}(x)\mu(dx) = \langle z, z'\rangle, \qquad z, z' \in H_0. \tag{2.1}$$

Therefore, the mapping

$$H_0 \to L^2(\mathcal{H}, \mu), \qquad z \mapsto W_z,$$

is an isomorphism and, consequently, it can be extended to all H. Thus W_f is a well-defined element of $L^2(\mathcal{H}, \mu)$ for any $f \in H$.

Moreover, H_n, $n = 0, 1, \dots,$ are the *Hermite polynomials* defined by the formula $F(t, \lambda) = e^{-t^2/2+t\lambda} = \sum_{n=0}^{\infty}(t^n/\sqrt{n!})H_n(\lambda)$, $t \geq 0$, $\lambda \in \mathbb{R}$. Well-known results are

$$\int_{\mathcal{H}} e^{W_z(x)}\mu(dx) = e^{|z|^2/2}, \qquad z \in H_0 \tag{2.2}$$

and

$$\int_{\mathcal{H}} H_n(W_f(z))H_m(W_g(z))\mu(dz) = \delta_{n,m}[\langle f, g\rangle]^n, \tag{2.3}$$

where $f, g \in H$, with $|f| = |g| = 1$, and $n, m \in \mathbb{N}\cup\{0\}$. Then $H_n(W_f), H_m(W_g) \in L^2(\mathcal{H}, \mu)$.

Let us introduce the renormalized power. We have for any $N \in \mathbb{N}$,

$$x_N(\xi) = \rho_N W_{\eta_N(\xi)}(x) \qquad \text{for } x\mu \text{ a.e. in } H,$$

where

$$\rho_N = \frac{1}{2\pi}\left[\sum_{|k|\leq N} \frac{1}{1+|k|^2}\right]^{1/2}$$

and

$$\eta_N(\xi) = \frac{1}{\rho_N}\sum_{|k|\leq N} \frac{\overline{e_k(\xi)}}{\sqrt{1+|k|^2}} e_k.$$

Now, for any $n \in \mathbb{N}$, we set

$$:x_N^n:(\xi) = \sqrt{n!}\rho_N^n H_n\big(W_{\eta_N(\xi)}(x)\big) \qquad \text{for } x\mu \text{ a.e. in } H;$$

the sequence $x \mapsto :x_N^n:$ is Cauchy in $L^2(\mathcal{H}, \mu; H^s(G))$ for any $s < 0$ (see [15], Chapter V1) and we have denoted by $H^s(G)$ the classical Sobolev spaces. Thus it has a limit denoted by $:x^n:$. Finally, we set $:p(X): = \sum_{k=0}^n a_k :X^k:$.

We are going to solve (1.1) in the Besov spaces $\mathcal{B}_{p,r}^s(G)$. Let us recall their definition.

For any $q \in \mathbb{N}$, we define $\delta_q = P_{2^q} - P_{2^{q-1}}$, so that

$$\delta_q u = \sum_{2^{q-1} < |k| \le 2^q} u_k e_k.$$

For $\sigma \in \mathbb{R}$, $p \ge 1$ and $r \ge 1$, we define

$$\mathcal{B}_{p,r}^\sigma(G) = \left\{ u : \sum_{q \in \mathbb{N}} 2^{rq\sigma} |\delta_q u|_{L^p(G)}^r < +\infty \right\},$$

which is a Banach space with the norm

$$|u|_{\mathcal{B}_{p,r}^\sigma(G)} = \left(\sum_{q \in \mathbb{N}} 2^{rq\sigma} |\delta_q u|_{L^p(G)}^r \right)^{1/r}.$$

The following result is crucial in our argument and is the main motivation for working in Besov spaces; see [4, 5].

PROPOSITION 2.1. *Let $p, r \ge 1$, $\alpha + \beta > 0$, $\alpha < 2/p$ and $\beta < 2/p$. Then if $u \in \mathcal{B}_{p,r}^\alpha(G)$ and $v \in \mathcal{B}_{p,r}^\beta(G)$, we have $uv \in \mathcal{B}_{p,r}^\gamma(G)$, where $\gamma = \alpha + \beta - \frac{2}{p}$ and*

$$|uv|_{B_{p,r}^\gamma(G)} \le c |u|_{B_{p,r}^\alpha(G)} |v|_{B_{p,r}^\beta(G)}. \tag{2.4}$$

3. Some technical lemmas. We recall the following result for the reader's convenience.

LEMMA 3.1. *For all $\xi, \eta \in \mathbb{R}$ and $n \in \mathbb{N}$, we have*

$$\sqrt{n!} H_n(\xi + \eta) = \sum_{k=0}^n C_n^k \sqrt{k!} H_k(\xi) \eta^{n-k}, \tag{3.1}$$

where

$$C_n^k = \frac{n!}{k!(n-k)!}.$$

PROOF. Recalling the definition of the Hermite polynomials, we have

$$e^{-t^2/2 + t(\xi+\eta)} = \sum_{n=0}^\infty \frac{t^n}{\sqrt{n!}} H_n(\xi + \eta)$$

and

$$\begin{aligned} e^{-t^2/2+t\xi} e^{t\eta} &= \sum_{n=0}^{\infty} \frac{t^n}{\sqrt{n!}} H_n(\xi) \sum_{n=0}^{\infty} \frac{t^n \eta^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{\sqrt{k!}(n-k)!} H_k(\xi) \eta^{n-h} \right) t^n. \end{aligned}$$

Now, identifying coefficients we get

$$\frac{1}{\sqrt{n!}} H_n(\xi + \eta) = \sum_{k=0}^{n} \frac{1}{\sqrt{k!}(n-k)!} H_k(\xi) \eta^{n-k}, \qquad n \in \mathbb{N},$$

and the conclusion follows. $\square$

LEMMA 3.2. *Let $k, r, p \geq 1, \sigma < 0$. Then for $n \in \mathbb{N}$ the mapping*

$$z \mapsto :z^n:$$

belongs to $L^k(\mathcal{H}, \mu, \mathcal{B}^{\sigma}_{p,r}(G))$. Moreover, when $N \to \infty$, $:z_N^n:$ converges to $:z^n:$ in $L^k(\mathcal{H}, \mu, \mathcal{B}^{\sigma}_{p,r}(G))$.

PROOF. Let $q \in \mathbb{N}$. Then

$$\begin{aligned} &\int_{\mathcal{H}} |\delta_q :z_N^n:|^p_{L^p(G)} \mu(dz) \\ &\quad = \int_{\mathcal{H}} \int_G \left| \sum_{2^{q-1} < |h| \leq 2^q} (:z_N^n:, e_h) e_h(\xi) \right|^p d\xi \, \mu(dz) \\ &\quad = \int_G \left[\int_{\mathcal{H}} \left| \sum_{2^{q-1} < |h| \leq 2^q} (:z_N^n:, e_h) e_h(\xi) \right|^p \mu(dz) \right] d\xi \\ &\quad \leq (p-1)^{pn/2} \int_G \left[\int_{\mathcal{H}} \left| \sum_{2^{q-1} < |h| \leq 2^q} (:z_N^n:, e_h) e_h(\xi) \right|^2 \mu(dz) \right]^{p/2} d\xi \end{aligned}$$

since

$$\sum_{2^{q-1} < |h| \leq 2^q} (:z_N^n:, e_h) e_h \in L^2_n(\mathcal{H}, \mu),$$

where $L^2_n(\mathcal{H}, \mu)$ is the Wiener chaos of order n, by the Nelson ultracontractivity estimate; see [15].

However,

$$\int_{\mathcal{H}}\Bigg|\sum_{2^{q-1}<|h|\le 2^q}(:z_N^n:, e_h)e_h(\xi)\Bigg|^2\mu(dz)$$
$$=\int_{\mathcal{H}}\int_{G\times G}\sum_{2^{q-1}<|h_1|,|h_2|\le 2^q}:z_N^n(\xi_1)::z_N^n(\xi_2):\overline{e_{h_1}(\xi_1)}$$
$$\times e_{h_1}(\xi)e_{h_2}(\xi_2)\overline{e_{h_2}(\xi)}\,d\xi_1\,d\xi_2\,\mu(dz).$$

We have

$$\int_{\mathcal{H}}:z_N^n:(\xi_1):z_N^n(\xi_2):\mu(dz)$$
$$=n!\big(\eta_N(\xi_1),\eta_N(\xi_2)\big)^n$$
$$=n!\Bigg(\sum_{|h|<N}\frac{1}{1+|h|^2}\overline{e_h(\xi_1)}e_h(\xi_2)\Bigg)^n$$

since

$$:z_N^n:(\xi_i)=\sqrt{n!}\rho_N^n H_n\big(W_{\eta_N(\xi_i)}(z)\big),\qquad i=1,2,$$

where

$$\eta_N(\xi_i)=\rho_N^{-1}\sum_{|h|<N}\frac{1}{1+|h|^2}\overline{e_h(\xi_i)}e_h.$$

We deduce that

$$\int_{\mathcal{H}}\Bigg|\sum_{2^{q-1}<|h|\le 2^q}(:z_N^n:, e_h)e_h(\xi)\Bigg|^2\mu(dz)$$
$$=\sum_{2^{q-1}<|h_1|,|h_2|\le 2^q}n!\int_{G\times G}\Bigg(\sum_{|h|<N}\frac{1}{1+|h|^2}\overline{e_h(\xi_1)}e_h(\xi_2)\Bigg)^n \tag{3.2}$$
$$\times\overline{e_{h_1}(\xi_1)}e_{h_2}(\xi_2)e_{h_1}(\xi)\overline{e_{h_2}(\xi)}\,d\xi_1\,d\xi_2$$
$$=\sum_{2^{q-1}<|h|\le 2^q}n!\alpha_h^n(N),$$

where $\alpha_h^n(N)$ are defined by

$$\gamma_N^n(\xi)=\Bigg(\sum_{|h|\le N}\frac{1}{1+|h|^2}e_h(\xi)\Bigg)^n=\sum_{h\in\mathbb{Z}^2}\alpha_h^n(N)e_h(\xi).$$

We have in fact

$$\int_{G\times G}\sum_{h\in\mathbb{Z}^2}\alpha_h^n(N)\overline{e_h(\xi_1)}e_h(\xi_2)\overline{e_{h_1}(\xi_1)}e_{h_2}(\xi_2)\,d\xi_1\,d\xi_2$$

$$=\sum_{h\in\mathbb{Z}^2}\alpha_h^n(N)\delta_{h+h_1=0}\delta_{h+h_2=0}$$

and (3.2) follows.

Therefore, we have

$$\begin{aligned}\int_{\mathcal{H}}&|\delta_q:z_N^n:|_{L^p(G)}^p\mu(dz)\\&\le(p-1)^{np/2}(n!)^{p/2}|G|\Bigg(\sum_{2^{q-1}<|h|\le 2^q}\alpha_h^n(N)\Bigg)^{p/2}\\&\le c(p,n)|G|\Bigg(\sum_{2^{q-1}<|h|\le 2^q}(\alpha_h^n(N))^2\Bigg)^{p/4}2^{pq/2}.\end{aligned}\tag{3.3}$$

We claim that

$$\int_{\mathcal{H}}|\delta_q:z_N^n:|_{L^p(G)}^p\mu(dz)\le c(p,n,s)2^{-qs}\qquad\forall s<0.\tag{3.4}$$

Fix $\tilde{s}<0$. Then we have

$$\begin{aligned}\sum_{2^{q-1}<|h|\le 2^q}(\alpha_h^n(N))^2&\le 2^{-2(q-1)(1+\tilde{s})}\sum_{h\in\mathbb{Z}^2}|h|^{2(1+\tilde{s})}(\alpha_h^n(N))^2\\&\le 2^{-2(q-1)(1+\tilde{s})}|\gamma_N^n|_{H^{1+\tilde{s}}(G)}^2.\end{aligned}$$

Consequently,

$$\begin{aligned}\Bigg(&\sum_{2^{q-1}<|h|\le 2^q}(\alpha_h^n(N))^2\Bigg)^{p/4}2^{p(q-1)/2}\\&\le 2^{-p(q-1)(1+\tilde{s})/2+p(q-1)/2}|\gamma_N^n|_{H^{1+\tilde{s}}(G)}^{p/2}\\&=2^{-p(q-1)\tilde{s}/2}|\gamma_N^n|_{H^{1+\tilde{s}}(G)}^{p/2}.\end{aligned}$$

On the other hand, we have

$$|\gamma_N^n|_{H^{1+\tilde{s}}(G)}\le c(n,\tilde{s})|\gamma_N|_{H^{\beta_n}(G)}^n\le c(n,\tilde{s})$$

since

$$\beta_n=1+\frac{\tilde{s}}{2^{n-1}}<1,$$

as proved by recurrence. Therefore, setting $\tilde{s} = 2s/p$ yields (3.4).

Let now $k \ge 1$, $r \ge 1$, $p \ge 1$ and $\sigma < 0$. Then, using the Jensen and Hölder inequalities, we find

$$\begin{aligned}
&\int_{\mathcal{H}} |{:}z_N^n{:}|^k_{\mathcal{B}^\sigma_{p,r}} \mu(dz) \\
&\quad = \int_{\mathcal{H}} \left(\sum_{q\in\mathbb{N}} 2^{qr\sigma} |\delta_q {:}z_N^n{:}|^r_{L^p(G)} \right)^{k/r} \mu(dz) \\
&\quad \le \left(\int_{\mathcal{H}} \left(\sum_{q\in\mathbb{N}} 2^{qr\sigma} |\delta_q {:}z_N^n{:}|^r_{L^p(G)} \right)^{k} \mu(dz) \right)^{1/r} \\
&\quad \le \left(\int_{\mathcal{H}} \left(\sum_{q\in\mathbb{N}} 2^{qrk\sigma/(2(k-1))} \right)^{k-1} \sum_{q\in\mathbb{N}} 2^{qrk\sigma/2} |\delta_q {:}z_N^n{:}|^{rk}_{L^p(G)} \mu(dz) \right)^{1/r} \\
&\quad \le c(r,k,\sigma,s,n) \left(\int_{\mathcal{H}} \sum_{q\in\mathbb{N}} 2^{qrk\sigma/2} |\delta_q {:}z_N^n{:}|^{rk}_{L^p(G)} \mu(dz) \right)^{1/r}.
\end{aligned}$$

We can assume that $rk \ge p$ (otherwise we will choose the larger one). Then we have, taking into account (3.4),

$$\begin{aligned}
&\int_{\mathcal{H}} |{:}z_N^n{:}|^k_{\mathcal{B}^\sigma_{p,r}(G)} \mu(dz) \\
&\qquad \le c(p,r,k,\sigma) \left[\int_{\mathcal{H}} \sum_{q\in\mathbb{N}} 2^{qrk\sigma/2} |\delta_q {:}z_N^n{:}|^{rk}_{L^p(G)} \mu(dz) \right]^{1/r} \\
&\qquad \le c(p,r,k,\sigma) \left[\sum_{q\in\mathbb{N}} 2^{qrk\sigma/2} 2^{-qs} \right]^{1/r}
\end{aligned}$$

for any $s < 0$. Therefore, choosing $s = rk\sigma/4$, we find

$$\int_{\mathcal{H}} |{:}z_N^n{:}|^k_{\mathcal{B}^\sigma_{p,r}(G)} \mu(dz) \le c(p,r,k,\sigma).$$

It is now easy to conclude the proof. □

LEMMA 3.3. *Let $g \in \mathcal{B}^\alpha_{p,r}(G)$ and $h \in \mathcal{B}^s_{p,r}(G)$. Assume that $s < 0$, $\alpha = \frac{2}{p} + 2s$, $-s < \frac{2}{p(2n+1)}$, $p \ge 1$ and $r \ge 1$. Then for $l = 0, \dots, n-1$, $g^l h \in \mathcal{B}^{(2l+1)s}_{p,r}(G)$ and a constant $c(s,\alpha,n,p,r)$ exists such that*

$$|g^l h|_{B^{(2l+1)s}_{p,r}(G)} \le c(s,\alpha,n,p,r) |g|^l_{B^\alpha_{p,r}(G)} |h|_{B^s_{p,r}(G)}.$$

Moreover,

$$(P_N g)^l P_N h \to g^l h \qquad \text{in } \mathcal{B}^{(2l+1)s}_{p,r}(G).$$

PROOF. The result is clear for $l = 0$ and follows from Proposition 2.1 for $l = 1$. We prove the general case by recurrence. Assume that the result is true for some $l \geq 1$. Then applying Proposition 2.1 with $u = g$ and $v = g^l h$ gives the results for $l + 1$. $\square$

As already mentioned, we can define the probability measure on $\mathcal{H}$,

$$\nu(dx) = ce^{-U(x)}\mu(dx)$$

with

$$U(x) = -\big(:q(x):, 1\big),$$

where q is a primitive of p and c is a normalization constant. Furthermore,

$$e^{-U} \in L^k(\mathcal{H}; \mu) \qquad \forall k \geq 1. \tag{3.5}$$

COROLLARY 3.4. *Let x be a random variable with law ν and let z be a random variable with law μ. Assume in addition that $y := x - z \in L^a(\Omega, \mathcal{B}^\alpha_{p,r}(G))$ with*

$$a \geq n, \qquad \alpha = \frac{2}{p} + 2s, \qquad -\frac{2}{p(2n+1)} < s < 0,\ r \geq 1,\ p \geq 1.$$

Then

$$:p(x): = \sum_{k=0}^{n} a_k \sum_{l=0}^{k} C_k^l y^l :z^{k-l}:,$$

$\mathbb{P}$-almost surely.

PROOF. By Lemma 3.2 it follows that

$$:z_N^l: \to :z^l: \qquad \text{in } L^k(\Omega; \mathcal{B}^s_{p,r})\ \forall s < 0,\ p \geq 1,\ r \geq 1,\ k \geq 1,\ l \in \mathbb{N}.$$

In a similar way, using the identity

$$\mathbb{E}\big(|:p(x_N):|^k_{\mathcal{B}^s_{p,r}(G)}\big) = \int_{\mathcal{H}} |:p(x_N):|^k_{\mathcal{B}^s_{p,r}(G)} e^{-U(x)}\mu(dx) \leq C,$$

by (3.5) and the Hölder inequality, we get

$$:p(x_N): \to :p(x): \qquad \text{in } L^k\big(\Omega, \mathcal{B}^s_{p,r}(G)\big)\ \forall s < 0,\ p \geq 1,\ r \geq 1,\ k \geq 1.$$

By Lemma 3.3, we have for all $k \geq 1$, $l = 1, \dots, n-1$,

$$y_N^l :z_N^k: \to y^l :z^k: \qquad \text{in } L^{a/(l+1)}\big(\Omega, \mathcal{B}^{(2l+1)s}_{p,r}(G)\big)$$

$$\text{and so in } L^{a/n}\big(\Omega, \mathcal{B}^{(2n+1)s}_{p,r}(G)\big)$$

and

$$y_N^n \to y^n \qquad \text{in } L^{a/n}\big(\Omega, \mathcal{B}_{p,r}^{(2n+1)s}(G)\big).$$

Now, by Lemma 3.1 we have

$$\begin{aligned} :p(x_N): &= \sum_{k=0}^{n} a_k :x_N^k: = \sum_{k=0}^{n} a_k :(y_N + z_N)^k: \\ &= \sum_{k=0}^{n} a_k \sqrt{k!}\rho_n^k H_k\big(W_{\eta_N(\xi)}(y_N) + W_{\eta_N(\xi)}(z_N)\big) \\ &= \sum_{k=0}^{n} a_k \sum_{l=0}^{k} C_k^l y_N^l :z_N^{l-k}: \end{aligned}$$

and we obtain the result by letting $N \to \infty$. □

COROLLARY 3.5. *Let $y \in L^p(0, T; \mathcal{B}_{p,r}^{\alpha}(G))$ and $h \in L^p(0, T; \mathcal{B}_{p,r}^{s}(G))$, with $\alpha = 2/p + 2s$, $p \geq n$, $-2/(p(2n+1)) < s < 0$ and $r \geq 1$. Then for all $l = 1, \dots, n-1$, we have $hy^l \in L^{p/(l+1)}(0, T; \mathcal{B}_{p,r}^{s(2l+1)}(G))$ and*

$$|hy^l|_{L^{p/(l+1)}(0,T;\mathcal{B}_{p,r}^{s(2l+1)}(G))} \leq c(p, \alpha, n, r, s)|h|_{L^p(0,T;\mathcal{B}_{p,r}^{s}(G))}|y|^l_{L^p(0,T;\mathcal{B}_{p,r}^{\alpha}(G))}.$$

The proof is a straightforward consequence of Lemma 3.3.

LEMMA 3.6. *Let $f \in L^{p/n}(0, T; \mathcal{B}_{p,r}^{(2n-1)s}(G))$ and $p \geq n$, $s < 0$, $\alpha = 2/p + 2s$ such that*

$$(n-1)s + 1 - \frac{n}{p} > 0.$$

Then

$$\int_0^t e^{(t-\tau)A} f(\tau)\, d\tau \in C\big([0, T]; \mathcal{B}_{p,r}^{s}(G)\big) \cap L^p\big(0, T; \mathcal{B}_{p,r}^{\alpha}(G)\big)$$

and

$$\begin{aligned} &\left| \int_0^t e^{(t-\tau)A} f(\tau)\, d\tau \right|_{C([0,T];\mathcal{B}_{p,r}^{s}(G))\cap L^p(0,T;\mathcal{B}_{p,r}^{\alpha}(G))} \\ &\qquad \leq c(p, n, s, \alpha) T^{\varepsilon} |f|_{L^{p/n}(0,T;\mathcal{B}_{p,r}^{(2n-1)s}(G))}, \end{aligned}$$

where $\varepsilon = 1 + (n-1)s - \frac{n}{p}$.

PROOF. Since

$$|e^{tA}x|_{\mathcal{B}_{p,r}^{s}(G)} \leq ct^{(n-1)s}|x|_{\mathcal{B}_{p,r}^{(2n-1)s}(G)}, \qquad x \in \mathcal{B}_{p,r}^{(2n-1)s}(G),$$

we have

$$\left|\int_0^t e^{(t-\tau)A} f(\tau)\,d\tau\right|_{\mathcal{B}^s_{p,r}(G)} \le c\int_0^t (t-\tau)^{(n-1)s}|f(\tau)|_{\mathcal{B}^{(2n-1)s}_{p,r}(G)}\,d\tau.$$

Now, by the Hölder inequality it follows that

$$\begin{aligned}\left|\int_0^t e^{(t-\tau)A} f(\tau)\,d\tau\right|_{C([0,T];\mathcal{B}^s_{p,r}(G))} &\le c\left(\int_0^T \tau^{\gamma s(n-1)}\,d\tau\right)^{1/\gamma}|f|_{L^{p/n}(0,T;\mathcal{B}^{(2n-1)s}_{p,r}(G))}\\ &= cT^{(n-1)s+1/\gamma}|f|_{L^{p/n}(0,T;\mathcal{B}^{(2n-1)s}_{p,r}(G))}\\ &= cT^{\varepsilon}|f|_{L^{p/n}(0,T;\mathcal{B}^{(2n-1)s}_{p,r}(G))},\end{aligned}$$

where $\frac{1}{\gamma}+\frac{n}{p}=1$.

In a similar way we find

$$\left|\int_0^t e^{(t-\tau)A} f(\tau)\,d\tau\right|_{B^{\alpha}_{p,r}(G)} \le c\int_0^t (t-\tau)^{((2n-1)s-\alpha)/2}|f(\tau)|_{B^{(2n-1)s}_{p,r}(G)}\,d\tau,$$

and by the Hausdorff–Young inequality ($|f*g|_{L^r}\le |f|_{L^p}|g|_{L^q}$ if $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$, $p,q,r\in[1,+\infty]$), it follows that

$$\begin{aligned}\left|\int_0^t e^{(t-\tau)A} f(\tau)\,d\tau\right|_{L^p(0,T;\mathcal{B}^{\alpha}_{p,r}(G))} &\le c\left(\int_0^T \tau^{\tilde\gamma[((2n-3)s)/2-1/p]}\,d\tau\right)^{1/\tilde\gamma}|f|_{L^{p/n}(0,T;\mathcal{B}^{(2n-1)s}_{p,r}(G))}\\ &\le cT^{((2n-3)s)/2-1/p+1/\tilde\gamma}|f|_{L^{p/n}(0,T;\mathcal{B}^{(2n-1)s}_{p,r}(G))}\\ &= cT^{((2n-3)s)/2-n/p+1}|f|_{L^{p/n}(0,T;\mathcal{B}^{(2n-1)s}_{p,r}(G))},\end{aligned}$$

where $\frac{1}{\tilde\gamma}+\frac{n}{p}=1+\frac{1}{p}$. □

4. Construction of the solution and main result. We want to solve the problem

$$\begin{aligned} dX &= \big(AX + :p(X):\big)\,dt + dW(t),\\ X(0) &= x. \end{aligned} \tag{4.1}$$

In view of Corollary 3.4, we know that if X is a stationary solution of (4.1) with invariant law ν and if $Y = X - z$, where

$$z(t) = \int_{-\infty}^t e^{(t-s)A}\,dW(s)$$

is sufficiently regular, then (4.1) is equivalent to

$$\frac{dY}{dt} = AY + \sum_{k=0}^{n} a_k \sum_{l=0}^{k} C_k^l Y^l :z^{k-l}:,$$
$$Y(0) = x - z(0). \tag{4.2}$$

This argument, as well as Corollary 3.4, remains valuable under the weaker assumption that $\nu_t = \mathcal{L}(X(t))$ is absolutely continuous with respect to μ and has a density in $L^p(\mathcal{H}, \mu)$ with $p > 1$. (Note that, as is the case for stationary solutions, it is expected that if X is a solution to (4.1), the law of the *process* X is singular with respect to the law of a linear solution [13]. However, as is the case for stationary solutions, this does not contradict the possibility that for each t, the law of the random variable $X(t)$ is absolutely continuous with respect to μ.) This motivates the following definition.

DEFINITION 4.1. X is a solution of (4.1) if and only if $X = Y + z$ and Y is a mild solution of (4.2).

We now state the main result of this article.

THEOREM 4.2. *Let* $\alpha = \frac{2}{p} + 2s$, $p > n, r \geq 1$ *and*

$$0 > s > \max\left\{-\frac{2}{p(2n+1)}, -\frac{1}{n-1}\left(1 - \frac{n}{p}\right)\right\}.$$

Then for all ν*-almost every* x *there exists for any* $T \geq 0$ *a unique solution of* (4.1) *such that*

$$Y \in C\big([0, T], \mathcal{B}^s_{p,r}(G)\big) \cap L^p\big(0, T, \mathcal{B}^\alpha_{p,r}(G)\big).$$

REMARK 4.3. It is not difficult to see that if we take a random initial data with law ν then Theorem 4.2 provides a solution Y such that $Y + z$ is a stationary solution of (4.1).

Before proving the theorem, we state and prove a local existence result.

PROPOSITION 4.4. *Let* $\alpha = \frac{2}{p} + 2s$, $p > n, r \geq 1$ *and*

$$0 > s > \max\left\{-\frac{2}{p(2n+1)}, -\frac{1}{n-1}\left(1 - \frac{n}{p}\right)\right\}.$$

Then for all $x \in \mathcal{B}^s_{p,r}(G)$ *and a.e.* $\omega \in \Omega$, *there exists* $T^*(x, \omega)$ *and a unique solution of* (4.1) *such that*

$$Y \in C\big([0, T^*(x,\omega)], \mathcal{B}^s_{p,r}(G)\big) \cap L^p\big(0, T^*(x,\omega), \mathcal{B}^\alpha_{p,r}(G)\big).$$

PROOF. We solve the integral equation

$$Y(t)=e^{tA}\big(x-Z(0)\big)+\int_0^t e^{(t-\tau)A}\sum_{k=0}^n a_k\sum_{l=0}^k C_k^l Y^l(\tau) :z^{k-l}(\tau):\,d\tau \tag{4.3}$$

using a fixed point argument in the space

$$\mathcal{E}_T=C\big([0,T],\mathcal{B}^s_{p,r}(G)\big)\cap L^p\big(0,T,\mathcal{B}^\alpha_{p,r}(G)\big).$$

Clearly

$$:z^{k-l}:\in L^p\big(0,T,\mathcal{B}^s_{p,r}(G)\big)\qquad \text{a.s. for } 1\le p<\infty.$$

We have, in fact, by Lemma 3.2 and stationarity of z,

$$\mathbb{E}\bigg(\int_0^T |:z^{k-l}(\tau):|^p_{\mathcal{B}^s_{p,r}(G)}\,d\tau\bigg)=T\int_{\mathcal{H}}|:x^{k-l}:|^p_{\mathcal{B}^s_{p,r}(G)}\mu(dx).$$

Moreover, by Corollary 3.5 and Lemma 3.6, if $Y\in\mathcal{E}_T$, we have

$$\int_0^t e^{(t-\tau)A}\sum_{k=0}^n a_k\sum_{l=0}^k C_k^l Y^l(\tau) :z^{k-l}(\tau):\,d\tau\in\mathcal{E}_T$$

and

$$\begin{aligned}
&\bigg|\int_0^t e^{(t-\tau)A}\sum_{k=0}^n a_k\sum_{l=0}^k C_k^l Y^l(\tau) :z^{k-l}(\tau):\,d\tau\bigg|_{\mathcal{E}_T}\\
&\quad\le cT^\varepsilon\sum_{k=0}^n|a_k|\Bigg(\sum_{l=0}^{k-1}|Y^l :z^{k-l}:|_{L^{p/(l+1)}(0,T,\mathcal{B}^{(2l+1)s}_{p,r})}\\
&\qquad\qquad+|Y^k|_{L^{p/k}(0,T,\mathcal{B}^{(2k-1)s}_{p,r})}\Bigg)\\
&\quad\le cT^\varepsilon\sum_{k=0}^n|a_k|\Bigg(\sum_{l=0}^{k-1}|Y|^l_{L^p(0,T,\mathcal{B}^\alpha_{p,r})}|:z^{k-l}:|_{L^p(0,T,\mathcal{B}^s_{p,r})}\\
&\qquad\qquad+|Y|^{k-1}_{L^p(0,T,\mathcal{B}^\alpha_{p,r})}|Y|_{L^p(0,T,\mathcal{B}^s_{p,r})}\Bigg)\\
&\quad\le c(s,p,r,T,n,a_0,\dots,a_n,\omega)T^\varepsilon\big(|Y|^n_{\mathcal{E}_T}+1\big).
\end{aligned}$$

Moreover,

$$\big|e^{tA}\big(x-z(0)\big)\big|_{\mathcal{E}_T}\le c|x-z(0)|_{\mathcal{B}^s_{p,r}}.$$

Consequently, we can find an invariant ball in $\mathcal{E}_T$ for the fixed point iteration for $T\le T^*(x,z(0),z)=T^*(x,\omega)$. The same argument shows that the iteration mapping is a strict contraction in $\mathcal{E}_T$. $\square$

As far as global existence is concerned, it remains to find an a priori estimate in $\mathcal{B}^s_{p,r}(G)$. The following computations can be justified by using a Galerkin approximation which has an invariant measure close to ν (such as in [13]). Let $X(t,x)$ be the solution of (4.1). We have

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-\tau)A} :p(X(\tau,x)): \, d\tau + z(t) - e^{tA}z(0)$$

and so

$$\begin{aligned}|X(t,x)|_{\mathcal{B}^s_{p,r}(G)} \le c|x|_{\mathcal{B}^s_{p,r}(G)} + c\int_0^t |:p(X(\tau,x)):|_{\mathcal{B}^s_{p,r}(G)}\,d\tau \\ + |z(t)|_{\mathcal{B}^s_{p,r}(G)} + |z(0)|_{\mathcal{B}^s_{p,r}(G)}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\sup_{t\in[0,T]} |X(t,x)|_{\mathcal{B}^s_{p,r}(G)} \\ &\qquad \le c|x|_{\mathcal{B}^s_{p,r}(G)} + c\int_0^T |:p(X(\tau,x)):|_{\mathcal{B}^s_{p,r}(G)}\,d\tau + 2\sup_{t\in[0,T]} |z(t)|_{\mathcal{B}^s_{p,r}}.\end{aligned}$$

Now it follows that

$$\begin{aligned}&\mathbb{E}\Big(\sup_{t\in[0,T]} |X(t,x)|_{\mathcal{B}^s_{p,r}(G)}\Big) \\ &\qquad \le c|x|_{\mathcal{B}^s_{p,r}(G)} + c\int_0^T \mathbb{E}|:p(X(\tau,x)):|_{\mathcal{B}^s_{p,r}(G)}\,d\tau \\ &\qquad\quad + 2\mathbb{E}\Big(\sup_{t\in[0,T]} |z(t)|_{\mathcal{B}^s_{p,r}(G)}\Big)\end{aligned}$$

and, consequently,

$$\begin{aligned}&\int_{\mathcal{H}} \mathbb{E}\Big(\sup_{t\in[0,T]} |X(t,x)|_{\mathcal{B}^s_{p,r}(G)}\Big)\nu(dx) \\ &\qquad \le c\int_{\mathcal{H}} |x|_{\mathcal{B}^s_{p,r}(G)}\nu(dx) + c\int_0^T\int_{\mathcal{H}} \mathbb{E}|:p(X(\tau,x)):|_{\mathcal{B}^s_{p,r}(G)}\nu(dx)\,d\tau \\ &\qquad\quad + 2\mathbb{E}\Big(\sup_{t\in[0,T]} |z(t)|_{\mathcal{B}^s_{p,r}(G)}\Big).\end{aligned}$$

Since $\nu(dx) = e^{-U(x)}\mu(dx)$ and

$$\int_{\mathcal{H}} |x|^k_{\mathcal{B}^s_{p,r}(G)}\mu(dx) < +\infty \qquad \forall k \in \mathbb{N},$$

we have

$$\int_{\mathcal{H}} |x|_{\mathcal{B}^s_{p,r}(G)}\nu(dx) < +\infty.$$

Since ν is invariant, we have

$$\begin{aligned}\int_0^T \int_{\mathcal{H}} \mathbb{E}|{:}p(X(\tau,x)){:}|_{\mathcal{B}^s_{p,r}(G)}\nu(dx)\,d\tau &= T\int_{\mathcal{H}} |{:}p(x){:}|_{\mathcal{B}^s_{p,r}(G)}e^{-U(x)}\mu(dx)\\ &\le T|e^{-U}|_{L^2(\mathcal{H},\mu)}\bigg[\int_{\mathcal{H}} |{:}p(x){:}|^2_{\mathcal{B}^s_{p,r}(G)}\mu(dx)\bigg]^{1/2} < +\infty\end{aligned}$$

by Lemma 3.2.

Finally, it is not difficult to see, using the factorization method (see [9], Section 5.3), that

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |z(t)|_{\mathcal{B}^s_{p,r}(G)}\Big) < +\infty.$$

In conclusion,

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |X(t,x)|_{\mathcal{B}^s_{p,r}(G)}\Big) < +\infty$$

for ν-almost all x and then the global existence for ν-almost all x follows. This ends the proof of Theorem 4.2. □

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