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STRONG LAW OF LARGE NUMBERS FOR SUMS OF PRODUCTS

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Let X, X_n, $n \geq 1$, be a sequence of independent identically distributed random variables. We give necessary and sufficient conditions for the strong law of large numbers

$$n^{-k/p} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} X_{i_1} X_{i_2} \dots X_{i_k} \to 0 \quad \text{a.s.}$$

for $k = 2$ without regularity conditions on X, for $k \geq 3$ in three cases: (i) symmetric X, (ii) $P\{X \geq 0\} = 1$ and (iii) regularly varying $P\{|X| > x\}$ as $x \to \infty$, without further conditions, and for general X and k under a condition on the growth of the truncated mean of X. Randomized, centered, squared and decoupled strong laws and general normalizing sequences are also considered.

1. Introduction. Let X, X_n, $n \geq 1$, be a sequence of independent identically distributed (i.i.d.) random variables. Define

$$S_{m,n}^{[k]} = \sum_{m < i_1 < i_2 < \cdots < i_k \leq n} X_{i_1} X_{i_2} \dots X_{i_k}, \qquad S_n^{[k]} = S_{0,n}^{[k]}, \qquad S_n = S_n^{[1]}. \tag{1.1}$$

Then $S_n^{[k]}/\binom{n}{k}$ are U-statistics. This paper concerns the strong law of large numbers (SLLN)

$$S_n^{[k]}/b_n^k = b_n^{-k} \sum_{i_1 < i_2 < \cdots < i_k \leq n} X_{i_1} X_{i_2} \dots X_{i_k} \to 0 \quad \text{a.s.} \tag{1.2}$$

and its randomized, centered, squared and decoupled versions, where $b_n = b(n)$, $n \geq 1$, and $b(t)$ is a positive continuous increasing function of t.

Let $h(x_1, \dots, x_k)$ be a measurable symmetric function: $h(x_1, \dots, x_k) = h(x_{i_1}, \dots, x_{i_k})$ for all permutations of $\{1, \dots, k\}$. The Hoeffding (1961) SLLN for U-statistics asserts that if $E|h(X_1, \dots, X_k)| < \infty$, then

$$\binom{n}{k}^{-1} \sum_{i_1 < i_2 < \cdots < i_k \leq n} h(X_{i_1}, \dots, X_{i_k}) \to Eh(X_1, \dots, X_k) \quad \text{a.s.}$$

[see also Serfling (1980)]. Under the condition $E|h(X_1, \dots, X_k)|^p < \infty$, $0 < p < 2$, the Marcinkiewicz–Zygmund strong law

$$n^{-k/p} \sum_{i_1 < i_2 < \cdots < i_k \leq n} h(X_{i_1}, \dots, X_{i_k}) \to 0 \quad \text{a.s.}$$

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was obtained by Sen (1974) for $p < 1$, by Teicher (1992) for the product $h(x_1, \dots, x_k) = \prod_{i=1}^{k} x_i$, under $EX = 0$ when $1 \le p < 2$, and by Giné and Zinn (1992) for general h, completely degenerate when $1 \le p < 2$. Assume $EX = 0$ whenever $E|X| < \infty$. By the Kolmogorov and Marcinkiewicz–Zygmund strong laws, (1.2) holds for $b_n = n^{1/p}$ and $k = 1$ if and only if (iff) $E|X|^p < \infty$. However, the case $k \ge 2$ is quite different. Giné and Zinn (1992) gave an example to show that the condition $E|X|^p < \infty$ is not necessary for (1.2) with $k = 2$ and $b_n = n^{1/p}$. For $k = 2$, Cuzick, Giné and Zinn (1995) recently obtained necessary and sufficient conditions for the SLLN (1.2) under certain regularity conditions on the sequence $\{b_n\}$ and the distribution of X (e.g., X symmetric, $P\{|X| > x\}$ regularly varying), and considered the almost sure convergence of normalized maxima of products and normalized sums of symmetrized, squared or decoupled products.

In this paper, we consider $k = 2$ as well as the case $k > 2$. For $k = 2$, necessary and sufficient conditions for the SLLN (1.2) are given without regularity conditions on X and under a mild condition

$$\text{(1.3)} \qquad \begin{aligned} &\text{either} \qquad \frac{b_n}{n^{1/p}} \le \frac{b_{n+1}}{(n+1)^{1/p}} \qquad \forall\, n \ge 1 \\ &\text{or} \qquad \lim_{t\to\infty} \frac{b(ct)}{b(t)} = c^{1/p} \qquad \forall\, c > 0, \end{aligned}$$

for some $0 < p < 2$, on the normalizing constants. For $t > 0$, define

$$\text{(1.4)} \qquad \begin{aligned} c_\alpha(t) &= \sup\left\{c > 1\colon\ E\left(\frac{|X|}{c} \wedge 1\right)^\alpha \ge \frac{1}{t}\right\}, \qquad \alpha > 0, \\ c_\infty(t) &= \lim_{\alpha\to\infty} c_\alpha(t), \end{aligned}$$

$$\text{(1.5)} \qquad \mu(t) = E\big[X \big| -t \le X \le t\big], \qquad \mu(t) = 0 \quad \text{if } P\{|X| \le t\} = 0,$$

$$\text{(1.6)} \qquad \nu^*(t) = \max_{0 \le x \le t} \nu(x), \qquad \nu(t) = \max\{c_2(t), |t\mu(c_2(t))|\},$$

where $\sup\varnothing = 1$. The function $c_\alpha(t)$ is increasing in t and decreasing in α. For $0 < \delta \le 1$, we observe $c_\alpha(\delta^\alpha t) \le \delta c_\alpha(t)$, as $E(|X| \wedge \delta c)^\alpha \le E(|X| \wedge c)^\alpha$. It is also useful to note that $P\{|X| \ge c_\alpha(t)\} \le 1/t$. Here and throughout the sequel, the following notation is used: $x^+ = x \vee 0$, $x_1 \vee \cdots \vee x_m = \max(x_1, \dots, x_m)$, $x_1 \wedge \cdots \wedge x_m = \min(x_1, \dots, x_m)$ and $u \sim v$ means $|u/v| + |v/u| = O(1)$ for any functions or sequences u and v (of n, x, t, etc.) as their argument tends to ∞.

THEOREM 1.1 (SLLN for $k = 2$). *Let M_0 be a positive constant and $c_n \sim \nu^*(n/M_0)$. Suppose* (1.3) *holds. Then*

$$\text{(1.2}'\text{)} \qquad b_n^{-2} \sum_{1\le i<j\le n} X_i X_j \to 0 \quad a.s.$$

iff the following three conditions hold:

$$c_n/b_n \to 0, \tag{1.7}$$

$$\sum_{n=1}^{\infty} P\left\{c_n|X_1| > b_n^2\right\} < \infty, \tag{1.8}$$

$$\sum_{n=1}^{\infty} nP\left\{|X_1X_2| > b_n^2,\ |X_1| \wedge |X_2| > c_n\right\} < \infty. \tag{1.9}$$

Theorem 1.1 is proved in Section 4. It follows from a Borel–Cantelli argument that (1.7), (1.8) and (1.9) together are essentially equivalent to

$$\xi_n^{[2]}/b_n^2 \to 0 \quad \text{a.s.}, \qquad \xi_n^{[2]} = \max_{1\le i<j\le n}(|X_i| \vee c_n)(|X_j| \vee c_n) \tag{1.10}$$

[cf. Theorems 4.1(ii) and 2.1]. This is the content of our conditions for the SLLN (1.2′). The function $\nu(t)$ describes the order of magnitude of certain percentiles of $|S_n|$ (cf. Lemma 4.4). It also gives the L^2-order of the sums of truncated X_i, as $\nu^2(n) \sim E(\sum_{i=1}^n X_i')^2$ for $X_i' = c_2(n) \wedge ((-c_2(n)) \vee X_i)$. Condition (1.7) holds for all $0 < M_0 < \infty$ iff

$$\lim_{n\to\infty} nE\left(\frac{|X|}{b_n} \wedge 1\right)^2 = 0,\ \lim_{n\to\infty} nE\left(\frac{X}{b_n}\right)I\{|X| \le b_n\} = 0 \tag{1.7′}$$

iff the weak law $S_n/b_n = o_P(1)$ holds. Therefore, Theorem 1.1 remains valid if (1.7) is replaced by (1.7′) or the weak law.

The connection between (1.2′) and (1.10) can be described with the following outline of the proof. The necessity of (1.10) can be obtained by a decoupling argument. For sets A of positive integers, define $S_A = \sum_{i\in A} X_i$ and $S_A^{[2]} = \sum_{A^2} X_iX_j$, where $A^2 = \{(i, j)\colon\ i < j, i \in A, j \in A\}$. Let $A_{1,n}$ be the odd integers in $[1, n]$ and $A_{2,n}$ the even ones. Since $S_n^{[2]} = S_{A_{1,n}}^{[2]} + S_{A_{2,n}}^{[2]} + S_{A_{1,n}}S_{A_{2,n}}$, the SLLN (1.2′) implies its decoupled version

$$S_{A_{1,n}}S_{A_{2,n}}/b_n^2 \to 0 \quad \text{a.s.} \tag{1.11}$$

By a recent result of Montgomery-Smith (1993) (cf. Theorem 4.3), (1.11) is equivalent to

$$\max_{1\le i,\ j\le mn} \left|S_{A_{1,i}}S_{A_{2,j}}\right|/b_n^2 \to 0 \quad \text{a.s.} \qquad \forall\ m = 1, 2, \ldots.$$

It will be shown in Lemma 4.4 that there exist positive δ_0 and m such that $\delta_0\nu^*(n/M_0)$ are bounded from above by certain percentiles of $|S_{nm}|$ for all n. Thus, the decoupled SLLN (1.11) implies $\max(c_n, X_{A_{1,n}}^{[1]})\max(c_n, X_{A_{2,n}}^{[1]})/b_n^2 \to 0$ a.s., which is equivalent to (1.10) and therefore implies (1.7)–(1.9). Here $X_A^{[1]} = \max_{i\in A}|X_i|$.

The sufficiency of our conditions is obtained by focusing on centered variables as well as the lifted maxima in (1.10). Let $M_1 > M_0$ and

$\mu_n = \mu(c_2(n/M_1))$ via (1.4) and (1.5). Since $X_i X_j = (X_i - \mu_n)(X_j - \mu_n) + (X_i + X_j)\mu_n - \mu_n^2$, the SLLN (1.2′) is a consequence of its centered version

$$b_n^{-2} \sum_{1 \le i < j \le n} (X_i - \mu_n)(X_j - \mu_n) \to 0 \quad \text{a.s.} \tag{1.12}$$

and $(|S_n| + |n\mu_n|)|n\mu_n|/b_n^2 \to 0$ a.s. Let $X_n^{[\ell]}$ be the ℓth largest among $\{|X_i|: 1 \le i \le n\}$. By the Mori (1977) theorem on the SLLN of lightly trimmed sums, $X_n^{[2]}/b_n \to 0$ a.s. implies $(|S_n - n\mu_n| - X_n^{[1]})/b_n \to 0$ a.s. This and (1.10) imply $(|S_n| + |n\mu_n|)|n\mu_n|/b_n^2 \to 0$ a.s., so that (1.2′) is a consequence of the centered SLLN (1.12). Let n_j be suitable integers satisfying $1 < \gamma_1 \le n_{j+1}/n_j \le \gamma_2 < \infty$. It follows from a martingale argument and the Borel–Cantelli lemma that (1.12) holds if

$$\sum_{j=1}^{\infty} b_{n_j}^{-4} E\left\{ \sum_{1 \le i_1 < i_2 \le n_j} (X_{i_1} - \mu_{n_j})(X_{i_2} - \mu_{n_j}) \right\}^2 I\{\xi_{n_j}^{[2]} \le b_{n_j}^2\} < \infty. \tag{1.13}$$

It turns out that, due to the appropriate levels of centering and (random) truncation, the cross-product terms in the expectation in (1.13) are of no larger order than the squared terms (Lemma 3.4), so that the SLLN (1.2′) is implied by

$$E(X_1 X_2)^2 \sum_{j=1}^{\infty} b_{n_j}^{-4} n_j^2 I\{\xi_{n_j}^{[2]} \le b_{n_j}^2\} < \infty. \tag{1.14}$$

Since $\sum_{\ell=j}^{\infty} b_{n_\ell}^{-4} n_\ell^2$ is of the same order as $b_{n_j}^{-4} n_j^2$ by (1.3), (1.14) holds if

$$E(X_1 X_2)^2 \sum_{j=1}^{\infty} b_{n_j}^{-4} n_j^2 I\{b_{n_{j-1}}^2 < \xi_{n_{j-1}}^{[2]},\ \xi_{n_j}^{[2]} \le b_{n_j}^2\} < \infty \tag{1.15}$$

(cf. Lemma 3.5). Finally, (1.14) and therefore the SLLN (1.2′) are obtained via the Borel–Cantelli lemma from (1.10) and the inequality (Lemma 3.3)

$$b_{n_j}^{-4} n_j^2 E(X_1 X_2)^2 I\{b_{n_{j-1}}^2 < \xi_{n_{j-1}}^{[2]},\ \xi_{n_j}^{[2]} \le b_{n_j}^2\} = O(1) P\{b_{n_{j-1}}^2 < \xi_{n_{j-1}}^{[2]}\}. \tag{1.16}$$

The main difference between our proof of (1.12) and the common proofs of the SLLN is that the X_i are truncated at random levels and that certain events about $\xi_n^{[2]}$ are kept throughout the calculation.

We also generalize the results of Cuzick, Giné and Zinn (1995) from $k = 2$ to $k \ge 3$ under weaker regularity conditions, especially for $P\{X \ge 0\} = 1$ and the case where $xP\{|X| > x\}$ is slowly varying as $x \to \infty$. The regularity conditions of Cuzick, Giné and Zinn (1995), Proposition 3.8, imply that the random variable X is "essentially symmetric" in the sense that the mean of the partial sums of truncated X_i does not have a larger order than their standard deviation at proper levels of truncation, whereas a single regularity condition is imposed in Theorem 2.3 on the magnitude of the truncated mean relative to $\{b_n\}$ which holds automatically for $k = 2$ and allows the mean of truncated sums to grow faster than the standard deviation. Without any condition on the distribution of X, the equivalence of symmetrized, centered and squared versions of (1.2) is established for general $k \geq 2$, and that of (1.2) and its decoupled version for $k = 2$.

One of the main concerns in Cuzick, Giné and Zinn (1995) is the equivalence of (1.2) and the strong law for the maxima of products

$$b_n^{-k} \max_{i_1<i_2<\cdots<i_k\leq n} |X_{i_1} X_{i_2} \dots X_{i_k}| \to 0 \quad \text{a.s.,} \tag{1.17}$$

which is always a consequence of (1.2). In Section 5 we show that (1.17) does not necessarily imply (1.2) even under quite strong conditions by giving an example such that $EX = 0$ and both $xP\{|X| > x\}$ and b_n are regularly varying at ∞. Under our regularity conditions on the mean of truncated X and the sequence $\{b_n\}$, we obtain the equivalence of (1.2) and the SLLN for the lifted maxima [i.e., the k-version of (1.10)]

$$b_n^{-k} \max_{i_1<i_2<\cdots<i_k\leq n} \prod_{j=1}^{k} \max\left(c_n, |X_{i_j}|\right) = b_n^{-k} \prod_{\ell=1}^{k} \max(c_n, X_n^{[\ell]}) \to 0 \quad \text{a.s.,} \tag{1.18}$$

with $c_n \sim \nu^*(n/M_0)$, but we still do not know whether (1.2) and (1.17) are equivalent when X is symmetric and $b_n = n^{1/p}$, $0 < p < 2$, even for $k = 2$.

The paper is organized as follows. The main results are stated in Section 2. The sufficiency of our conditions is proved in Section 3, where some general randomized and centered versions of (1.2) are also considered. The decoupled versions of (1.2) and (1.18) are considered in Section 4, where the necessity parts of the proofs are provided. Variables with a regularly varying $P\{|X| > x\}$ at ∞ are considered in Section 5 with some discussion.

2. Main results. In this section, the main results are stated concerning necessary and sufficient conditions for the strong law (1.2) and its randomized, centered and squared versions, and their relationship to each other and to (1.17). Our regularity and necessary and/or sufficient conditions are also explained here.

Consider conditions of the form

$$c_n/b_n \to 0, \tag{2.1}$$

$$\sum_{n=1}^{\infty} \sum_{\ell=1}^{k} n^{\ell-1} P\{c_n^{k-\ell} |X_1 \dots X_\ell| > \varepsilon b_n^k,\ |X_1| \wedge \cdots \wedge |X_\ell| > c_n\} < \infty, \tag{2.2}$$

where $\varepsilon > 0$ and $\{c_n,\ n \geq 1\}$ is a suitable sequence of positive constants. These conditions are the k-version of (1.7)–(1.9) and connected to (1.18) via the following result.

THEOREM 2.1 (SLLN for lifted maxima). *Let $c_n \sim c(n/M_0)$ for some positive increasing function $c(\cdot)$ such that $nP\{|X| > c(n)\} = O(1)$. Then* (1.18) *holds for all $0 < M_0 < \infty$ iff both* (2.1) *and* (2.2) *hold for all positive ε and M_0.*

Theorem 2.1 is a consequence of Theorem 4.1(ii). For $k = 2$, Cuzick, Giné and Zinn (1995), proof of Theorem 2.1′, showed that (1.17) holds iff (2.2) holds for $c_n = c_\infty(n)$ and all ε. By (1.4), $P\{|X| > c_\infty(n)\} \le 1/n \le P\{|X| \ge c_\infty(n)\}$. In most cases considered here, the sequence $\{c_n\}$ is of the form in Theorem 2.1 with $c(t) = \nu^*(t)$ or $c(t) = c_\alpha(t)$ via (1.4)–(1.6).

We shall first consider symmetrized, centered and squared versions of the SLLN. Let $\{\varepsilon_n\}$ be a Rademacher sequence independent of $\{X_n\}$, i.i.d. with $P\{\varepsilon_n = \pm 1\} = 1/2$.

THEOREM 2.2 (Symmetrized, centered and squared SLLN). *Let M_0 and M_1 be positive constants and n_j be positive integers with $1 < \inf_j n_{j+1}/n_j \le \sup_j n_{j+1}/n_j < \infty$. Let $\bar\mu_n = \mu(c_2(n_j/M_1))$ for $n_j \le n < n_{j+1}$ and $c_n \sim c_2(n/M_0)$ via* (1.4) *and* (1.5). *Suppose*

$$\sup_{n\ge 1} \frac{b_n^{2k}}{n^k} \sum_{m=n}^{\infty} \frac{m^{k-1}}{b_m^{2k}} < \infty. \tag{2.3}$$

Then (1.18) *and the following symmetrized, centered and squared versions of the SLLN are all equivalent to each other*:

$$b_n^{-k} \sum_{i_1<i_2<\cdots<i_k\le n} \varepsilon_{i_1}\varepsilon_{i_2}\dots\varepsilon_{i_k} X_{i_1}X_{i_2}\dots X_{i_k} \to 0 \quad a.s., \tag{2.4}$$

$$b_n^{-k} \sum_{i_1<i_2<\cdots<i_k\le n} (X_{i_1} - \bar\mu_n)(X_{i_2} - \bar\mu_n)\dots(X_{i_k} - \bar\mu_n) \to 0 \quad a.s., \tag{2.5}$$

$$b_n^{-2k} \sum_{i_1<i_2<\cdots<i_k\le n} |X_{i_1}|^2|X_{i_2}|^2\dots|X_{i_k}|^2 \to 0 \quad a.s. \tag{2.6}$$

Furthermore, (2.4) *holds* [*along with* (2.5), (2.6) *and* (1.18)] *iff both* (2.1) *and* (2.2) *hold for* (*some or all*) $\varepsilon > 0$.

REMARK. It will be shown in Theorem 3.1 that the centered SLLN (2.5) still holds when $\bar\mu_n$ is replaced by $\mu_n(c'_n)$ at the centering level $c_\infty(n/M) \le c'_n \le Mc_2(n/M_1)$ for some $0 < M < \infty$. Condition (2.1) holds with $c_n \sim c_2(n/M_0)$ for all $0 < M_0 < \infty$ iff

$$\lim_{n\to\infty} nE\left(\frac{|X|}{b_n} \wedge 1\right)^2 = 0 \tag{2.7}$$

iff the weak law $\{S_n - n\mu(b_n)\}/b_n = o_P(1)$ holds. Thus, condition (2.1) in Theorem 2.2 can be replaced by (2.7).

COROLLARY TO THEOREM 2.2 (SLLN for symmetric and positive X). *Let* $\varepsilon = 1$.

(i) *Suppose* (2.3) *holds and X is symmetric. Then the SLLN* (1.2) *holds iff both* (2.1) *and* (2.2) *hold for* $c_n = c_2(n)$.

(ii) *Suppose* $P\{X \geq 0\} = 1$ *and*

$$\sup_{n\geq 1} \frac{b_n^k}{n^k} \sum_{m=n}^{\infty} \frac{m^{k-1}}{b_m^k} < \infty. \tag{2.3$'$}$$

Then the SLLN (1.2) *holds iff both* (2.1) *and* (2.2) *hold for* $c_n = c_1(n)$. *In fact, for* $c_n = c_1(n)$, (2.1) *and* (2.2) *imply* (1.2) *without the condition* $P\{X \geq 0\} = 1$.

The proofs of (2.1) and (2.2) $\Rightarrow$ (2.4)–(2.6) are provided in Section 3, and those of (2.4) or (2.5) or (2.6) $\Rightarrow$ (1.18) $\Rightarrow$ (2.1) and (2.2) in Section 4. For $k = 2$, Cuzick, Giné and Zinn (1995), Lemma 4.5 and Proposition 4.7, proved (2.1) $\Rightarrow$ (2.4) $\Rightarrow$ (2.6), and provided somewhat different (but equivalent) necessary and sufficient conditions for (2.4) under slightly stronger regularity conditions on the normalizing sequence $\{b_n^k\}$.

Let $\mu(\cdot)$ be given by (1.5). Define the sums of products of centered variables

$$\begin{aligned} H_n^{[k]}(c) &= \sum_{i_1<i_2<\cdots<i_k\leq n} (X_{i_1} - \mu(c))(X_{i_2} - \mu(c))\dots(X_{i_k} - \mu(c)), \\ H_n^{[0]} &= 1. \end{aligned} \tag{2.8}$$

Since $X_i = (X_i - \mu(c)) + \mu(c)$, (1.1) can be decomposed into the sum

$$S_n^{[k]} = \sum_{\ell=0}^{k} \binom{n-\ell}{k-\ell} \mu_n^{k-\ell} H_n^{[\ell]}(c'_n) \tag{2.9}$$

for suitable constants c'_n, where $\mu_n = \mu(c'_n)$. Consider $c'_n = c_2(n)$ and conditions (2.1) and (2.2) with $c_n = \nu^*(n) \geq c'_n$. The strong law for the term with $\ell = k$ in (2.9) is essentially (2.5) in Theorem 2.2. The term with $\ell = 0$ is bounded by $|n\mu_n|^k \leq c_n^k$, which is $o(b_n^k)$ by (2.1). As discussed in the outline of the proof of Theorem 1.1 in Section 1, the term with $\ell = 1$ in (2.9) can be trimmed by (1.18) and then handled by Mori's (1977) theorem on the strong law of lightly trimmed sums. For general increasing $b_n \to \infty$ and $\varepsilon > 0$, Kiefer (1972) proved that $P\{X_n^{[k]} > \varepsilon b_n \text{ i.o.}\} = 0$ iff

$$\sum_{n=1}^{\infty} n^{k-1} P^k \{|X| > \varepsilon b_n\} < \infty, \tag{2.10}$$

which is a consequence of (2.1) and (2.2) in view of the terms with $\ell = k$ in (2.2). Conditions (2.10) and $n\mu(ab_n)/b_n \to 0$ for all $a > 0$ are sufficient for the Mori (1977) theorem, with the normalizing constants satisfying (1.3). Mori (1977) required an additional condition $b_{2n}/b_n = O(1)$, which was removed by Cuzick, Giné and Zinn (1995), Theorem 3.4, although (1.3) is still stronger than (2.3). It is a consequence of Theorem 5.1 that (2.10) is not sufficient for (1.2), even when X is symmetric with a regularly varying distribution function. For $k > 2$, we have to deal with intermediate terms in (2.9) for $2 \leq \ell \leq k - 1$. In our next theorem, an additional sufficient condition is imposed to control the growth of the mean of truncated variables, which is essentially a modified

(2.3) with respect to the SLLN for $H_n^{[\ell]}$ in (2.9) with the normalizing sequence $\{b_n^k/\nu^{k-\ell}(n)\}$.

THEOREM 2.3 (SLLN for $k \geq 2$). *Let δ_0 and M_j, $j = 0, 1, 2, 3$, be positive numbers.*

(i) *Let $c_n \geq \delta_0 \nu^*(n/M_0)$. Suppose* (1.3) *holds and*

$$(2.11)\quad \sum_{m=n}^{\infty} \frac{m}{b_m^{2k}} \left(\left[\nu^2\left(\frac{m}{M_1}\right) - M_2 c_2^2\left(\frac{m}{M_1}\right) \right]^+ \right)^{k-2} < \frac{M_3 c_n^{2(k-2)} n^2}{b_n^{2k}}, \qquad n \geq 1.$$

If (2.1) *and* (2.2) *hold for all $\varepsilon > 0$, then the SLLN* (1.2) *holds.*

(ii) *Let $c_n \geq \delta_0 \nu^*(n/M_0)$. Suppose* (2.3) *holds and*

$$(2.11')\quad \sum_{m=n}^{\infty} b_m^{-2k} \left(\left[\nu^2\left(\frac{m}{M_1}\right) - M_2 c_2^2\left(\frac{m}{M_1}\right) \right]^+ \right)^{k-1} < \frac{M_3 c_n^{2(k-1)} n}{b_n^{2k}}, \qquad n \geq 1.$$

If (2.1) *and* (2.2) *hold for $\varepsilon = 1$, then the SLLN* (1.2) *holds.*

(iii) *Let $c_n \sim \nu^*(n/M_0)$. Then the SLLN* (1.2) *implies the SLLN for lifted maxima* (1.18), *which then implies both* (2.1) *and* (2.2). *If* (2.3) *holds, then the SLLN* (1.2) *implies its symmetrized, centered and squared versions* (2.4)–(2.6).

REMARK. Condition (1.3) implies (2.3). For $k = 2$, (2.3) implies (2.11), so that Theorem 1.1 is a consequence of Theorem 2.3(i) and (iii), except for the redundancy of (2.2) for all $\varepsilon > 0$. Conditions (2.3) and (2.11′) imply (2.11) by the Hölder inequality [cf. (3.18)].

COROLLARY TO THEOREM 2.3. *Suppose either* (1.3) *and* (2.11) *hold or* (2.3) *and* (2.11′) *hold for some $c_n \sim \nu^*(n/M_0)$ with $0 < M_0 < \infty$. Then* (2.1) *and* (2.2) *for all $\varepsilon > 0 \Leftrightarrow$* (1.2) $\Leftrightarrow$ (1.18).

Parts (i) and (ii) of Theorem 2.3 are proved in Section 3 and part (iii) in Section 4. It will be shown in Section 5 that (2.11′) can be removed if $P\{|X| > x\}$ is regularly varying as $x \to \infty$. By the definition of $\nu(t)$ in (1.6), (2.11′) holds if $n\mu(c_2(n))/c_2(n) = O(1)$ as in Cuzick, Giné and Zinn (1995), Definition 3.6 and Proposition 3.8.

3. Sufficiency. In this section, we verify the sufficiency parts of Theorems 2.1–2.3. The main difference between our proofs and the common proofs of the SLLN is that the X_i are truncated at random levels and that certain events about the lifted maxima in (1.18) are kept throughout the calculation.

The sufficiency part of Theorem 2.2 concerning the symmetrized SLLN (2.4) and the centered SLLN (2.5) is a consequence of Theorem 3.1. Let Y, Y_n, $n \geq 1$, be i.i.d. random vectors independent of the sequence $\{X_n\}$, and let $h(y_1, \ldots, y_k)$ be a symmetric Borel function, completely degenerate and with a finite variance: $Eh(Y, y_2, \ldots, y_k) = 0$ and $E|h(Y_1, \ldots, Y_k)|^2 < \infty$.

THEOREM 3.1. *Let δ_0, M, M_0 and M_1 be positive numbers. Suppose* (2.3) *holds. Set $\mu_n = \mu(c'_n)$ for some $c_\infty(n/M) \le c'_n \le Mc_2(n/M_1)$. If* (2.1) *and* (2.2) *hold for some $c_n \ge \delta_0 c_2(n/M_0)$ and $\varepsilon > 0$, then*

$$b_n^{-k} \sum_{i_1<i_2<\cdots<i_k\le n} h(Y_{i_1}, \dots, Y_{i_k}) X_{i_1} X_{i_2} \dots X_{i_k} \to 0 \quad a.s. \tag{3.1}$$

and

$$b_n^{-k} \sum_{i_1<i_2<\cdots<i_k\le n} (X_{i_1} - \mu_n)(X_{i_2} - \mu_n) \cdots (X_{i_k} - \mu_n) \to 0 \quad a.s. \tag{3.2}$$

REMARK. In (2.4), $h(y_1, \dots, y_k) = y_1 \dots y_k$ and $Y_n = \varepsilon_n$. In (2.5), $c_2(n/M_1) \ge c'_n = c_2(n_j/M_1) \ge c_\infty(n/M)$ for $n_j \le n < n_{j+1}$ and $M > M_1 \sup n_{j+1}/n_j$.

We need some lemmas for the proofs. Let $X_{m,n}^{[1]} \ge X_{m,n}^{[2]} \ge \cdots \ge X_{m,n}^{[n-m]}$ be the order statistics of $|X_{m+1}|, \dots, |X_n|$, and $X_n^{[\ell]} = X_{0,n}^{[\ell]}$ as in (1.10). For positive c define as in (1.18) the lifted partial maxima of products

$$\begin{aligned} \xi_n^{[k]}(c) &= \xi_{0,n}^{[k]}(c), \\ \xi_{m,n}^{[k]}(c) &= \prod_{\ell=1}^{k} \left(c \vee X_{m,n}^{[\ell]}\right) = \max_{m<i_1<i_2<\cdots<i_k\le n} \prod_{j=1}^{k} \max\left(c, |X_{i_j}|\right). \end{aligned} \tag{3.3}$$

Our first lemma implies the sufficiency part of Theorem 2.1.

LEMMA 3.2. *Let $\varepsilon > 0$, $1 < \gamma < \infty$ and $c(\cdot)$ be an increasing function.*

(i) *For all integers $m_0 \ge 1$,* (2.1) *and* (2.2) *imply*

$$\sum_{n=1}^{\infty} n^{-1} P\left\{\xi_{m_0 n}^{[k]}(c_n) > \varepsilon b_n^k\right\} < \infty. \tag{3.4}$$

(ii) *If* (2.1) *and* (2.2) *hold for $c_n \ge c(n)$, then*

$$\sum_{j=1}^{\infty} P\left\{\xi_{n_{j+1}}^{[k]}(c(n_j/\sqrt{\gamma})) > \varepsilon b_{n_j}^k\right\} < \infty \tag{3.5}$$

for all sequences of positive integers $\{n_j\}$ such that $1 < \inf_j n_{j+1}/n_j \le \sup_j n_{j+1}/n_j < \infty$. Consequently, $P\{\xi_n^{[k]}(c(n/\gamma)) > \varepsilon b_n^k \text{ i.o.}\} = 0$.

PROOF. (i) By (3.3) and for $c_n^k < \varepsilon b_n^k$,

$$\begin{aligned} &P\left\{\xi_{m_0 n}^{[k]}(c_n) > \varepsilon b_n^k\right\} \\ &\quad \le \sum_{\ell=1}^{k} P\left\{c_n^{k-\ell} X_{m_0 n}^{[1]} \dots X_{m_0 n}^{[\ell]} > \varepsilon b_n^k,\ X_{m_0 n}^{[1]} \wedge \cdots \wedge X_{m_0 n}^{[\ell]} > c_n\right\} \\ &\quad \le \sum_{\ell=1}^{k} (m_0 n)^\ell P\left\{c_n^{k-\ell} |X_1 \dots X_\ell| > \varepsilon b_n^k,\ |X_1| \wedge \cdots \wedge |X_\ell| > c_n\right\}. \end{aligned}$$

(ii) For $m_0 n > n_{j+1}$ and $n > n_j/\sqrt{\gamma}$, $\xi^{[k]}_{n_{j+1}}(c(n_j/\sqrt{\gamma})) \le \xi^{[k]}_{m_0 n}(c(n))$, so that (3.4) implies (3.5). Take $n_{j+1}/n_j \le \sqrt{\gamma}$ in (3.5). Since $\xi^{[k]}_n(c(n/\gamma)) \le \xi^{[k]}_{n_{j+1}}(c(n_j/\sqrt{\gamma}))$ for $n_j \le n < n_{j+1}$, (3.5) implies $P\{\xi^{[k]}_n(c(n/\gamma)) > \varepsilon b_n^k \text{ i.o.}\} = 0$ by the Borel–Cantelli lemma. □

For $\alpha = 2$ and $c_n = c_2(n)$, Lemma 3.3 asserts that the conditional expectation of the sum of squares in (2.6), given $c_n \vee X^{[1]}_n, \dots, c_n \vee X^{[k]}_n$ is controlled by that of the square of the lifted maxima (3.3). It extends (1.16) to general k.

LEMMA 3.3. *Let $Y_{m,n} = Y_{m,n}(c_n) = g(c_n \vee X^{[1]}_{m,n}, \dots, c_n \vee X^{[k]}_{m,n})$ for some $c_n \ge c_\alpha(n/M_0)$ and a nonnegative Borel function $g(x_1, \dots, x_k)$. Then, for $0 \le \ell \le k \le n$,*

$$n^{k-\ell} E\left\{ Y_{0,n} \prod_{i=1}^{\ell} |X_i|^\alpha \right\} \le \left(\frac{M_0}{1 - M_0/n} + \frac{k}{1 - \ell/n} \right)^{\ell} E\big(\xi^{[\ell]}_{0,n}\big)^\alpha Y_{0,n}, \tag{3.6}$$

where $\xi^{[\ell]}_{m,n} = \xi^{[\ell]}_{m,n}(c_n)$ is given by (3.3). *In particular*,

$$\begin{aligned} &b_2^{-\alpha} (c^*)^{\alpha(k-\ell)} n^\ell E \left| \prod_{i=1}^{\ell} X_i \right|^\alpha I\big\{ b_1 < \xi^{[k]}_{0,n}(c_n), \xi^{[k]}_{0,n^*}(c^*) \le b_2 \big\} \\ &\quad \le \left(\frac{M_0}{1 - M_0/n} + \frac{k}{1 - \ell/n} \right)^{\ell} P\big\{ b_1 < \xi^{[k]}_{0,n}(c_n) \big\} \end{aligned} \tag{3.7}$$

for all $b_1 < b_2$, $c^* \ge c_\alpha(n/M_0)$ and $n^* \ge n$.

PROOF. Let $c'_n = c_\alpha(n/M_0)$ and $R^{[i]}_{m,n}$ be the rank of $|X_i|$ in $|X_{m+1}|, \dots, |X_n|$ in descending order, $m < i \le n$, $R^{[i]}_n = R^{[i]}_{0,n}$, with ties broken by randomization. For $0 \le \ell_1 \le \ell_2 \le k - \ell$, define

$$\begin{aligned} B^{[\ell_1,\ell_2]}_n = I\{ &|X_i| \le c'_n,\ 1 \le i \le \ell_1;\ c'_n < |X_i|,\ R^{[i]}_n > k,\ \ell_1 < i \le \ell_2; \\ &c'_n < |X_i|,\ R^{[i]}_n \le k,\ \ell_2 < i \le k - \ell \}. \end{aligned}$$

On the event with $B^{[\ell_1,\ell_2]}_n = 1$, $Y_{0,n} = Y_{\ell_2,n}$ and $|X_{\ell_1+1} \dots X_\ell| \le \xi^{[\ell-\ell_1]}_{\ell_2,n}$, so that

$$\begin{aligned} Y_{0,n} \prod_{i=1}^{\ell} |X_i|^\alpha \le{}& \left(\prod_{i=1}^{\ell_1} |X_i|^\alpha I\{|X_i| \le c'_n\} \right) \times \left(\prod_{i=\ell_1+1}^{\ell_2} I\{c'_n < |X_i|\} \right) \\ &\times \left(\big\{\xi^{[\ell-\ell_1]}_{\ell_2,n}\big\}^\alpha Y_{\ell_2,n} I\big\{ R^{[i]}_{\ell_2,n} \le k,\ \ell_2 < i \le k - \ell \big\} \right). \end{aligned}$$

Since X_i are i.i.d., the three factors on the right-hand side above are independent, so that

$$
\begin{aligned}
&E\left|\prod_{i=1}^{\ell} X_i\right|^{\alpha} Y_{0,n} B_n^{[\ell_1,\ell_2]} \\
&\quad \le (E|X|^{\alpha} I\{|X| \le c_n'\})^{\ell_1} \times (P\{|X| > c_n'\})^{\ell_2-\ell_1} \\
&\qquad \times \left(E\left\{\xi_{\ell_2,n}^{[\ell-\ell_1]}\right\}^{\alpha} Y_{\ell_2,n} I\left\{R_{\ell_2,n}^{[i]} \le k,\ \ell_2 < i \le k-\ell\right\}\right).
\end{aligned} \tag{3.8}
$$

Set $p_n' = P\{|X| > c_n'\}$. Since $c_n' = c_\alpha(n/M_0)$, $E|X|^{\alpha} I\{|X| \le c_n'\} = (c_n')^{\alpha}(M_0/n - p_n')$ by (1.4). Since the rank vector $(R_{\ell_2,n}^{[\ell_2+1]}, \dots, R_{\ell_2,n}^{[n]})$ is uniformly distributed given the order statistics $X_{\ell_2,n}^{[i]}$ (and therefore given $\xi_{\ell_2,n}^{[\ell-\ell_1]}$ and $Y_{\ell_2,n}$),

$$
E\left\{\xi_{\ell_2,n}^{[\ell-\ell_1]}\right\}^{\alpha} Y_{\ell_2,n} I\left\{R_{\ell_2,n}^{[i]} \le k,\ \ell_2 < i \le \ell\right\} \le \{k/(n-\ell_2)\}^{\ell-\ell_2} E\left\{\xi_{\ell_2,n}^{[\ell-\ell_1]}\right\}^{\alpha} Y_{\ell_2,n}.
$$

Since $c_n' \le c_n$ and $\xi_{m,n}^{[\ell]}$, $0 \le \ell \le k$, and $Y_{m,n}$ are functions of $c_n \vee X_{m,n}^{[\ell]}$, $0 \le \ell \le k$,

$$
\xi_{m,n}^{[\ell]} Y_{m,n} I\{|X_m| \le c'\} = \xi_{m-1,n}^{[\ell]} Y_{m-1,n} I\{|X_m| \le c_n'\}, \tag{3.9}
$$

so that $E\{\xi_{\ell_2,n}^{[\ell-\ell_1]}\}^{\alpha} Y_{\ell_2,n} \le (1-p_n')^{-\ell_2} E\{\xi_{0,n}^{[\ell-\ell_1]}\}^{\alpha} Y_{0,n}$. Inserting these inequalities into (3.8), we obtain

$$
\begin{aligned}
&E\left|\prod_{i=1}^{\ell} X_i\right|^{\alpha} Y_{0,n} B_n^{[\ell_1,\ell_2]} \\
&\quad \le \{(c_n')^{\alpha}(M_0/n - p_n')\}^{\ell_1} (p_n')^{\ell_2-\ell_1} \{k/(n-\ell_2)\}^{\ell-\ell_2} E\{\xi_{\ell_2,n}^{[\ell-\ell_1]}\}^{\alpha} Y_{\ell_2,n} \\
&\quad \le (M_0/n - p_n')^{\ell_1} (p_n')^{\ell_2-\ell_1} \{k/(n-\ell)\}^{\ell-\ell_2} (1-M_0/n)^{-\ell_2} E\{\xi_{0,n}^{[\ell]}\}^{\alpha} Y_{0,n}
\end{aligned}
$$

as $p_n' \le M_0/n$ and $(c_n')^{\ell_1} \xi_{\ell_2,n}^{[\ell-\ell_1]}(c_n) \le \xi_{\ell_2,n}^{[\ell]}(c_n)$ by (3.3) and the condition $c_n' \le c_n$. This gives (3.6) by the exchangeability of X_i, since

$$
\begin{aligned}
&n^{\ell} E\left|\prod_{i=1}^{\ell} X_i\right|^{\alpha} Y_{0,n} \\
&\quad = \sum_{0 \le \ell_1 \le \ell_2 \le \ell} \binom{\ell}{\ell_2}\binom{\ell_2}{\ell_1} n^{\ell} E\left|\prod_{i=1}^{\ell} X_i\right|^{\alpha} Y_{0,n} B_n^{[\ell_1,\ell_2]} \\
&\quad \le \sum_{0 \le \ell_1 \le \ell_2 \le \ell} \binom{\ell}{\ell_2}\binom{\ell_2}{\ell_1} \frac{(M_0/n - p_n')^{\ell_1}(p_n')^{\ell_2-\ell_1}}{(1-M_0/n)^{\ell_2}} \left(\frac{k}{n-\ell}\right)^{\ell-\ell_2} n^{\ell} E\{\xi_{0,n}^{[\ell]}\}^{\alpha} Y_{0,n} \\
&\quad = \left(\frac{M_0}{1-M_0/n} + \frac{k}{1-\ell/n}\right)^{\ell} E\{\xi_{0,n}^{[\ell]}\}^{\alpha} Y_{0,n}.
\end{aligned}
$$

For (3.7), $Y_{m,n} = P\{b_1 < \xi^{[k]}_{m,n}(c_n),\ \xi^{[k]}_{m,n^*}(c^*) \le b_2 | X_m, \dots, X_n\}$ is a function of $c'_n \vee X^{[\ell]}_{m,n}$ and $(c^*)^{k-\ell}\xi^{[\ell]}_{0,n}(c'_n) \le \xi^{[k]}_{0,n}(c^*)$, so that $b_2^{-\alpha}(c^*)^{\alpha(k-\ell)} \cdot E\{\xi^{[\ell]}_{0,n}(c'_n)\}^\alpha Y_{0,n}$ is bounded by

$$\begin{aligned} &b_2^{-\alpha}(c^*)^{\alpha(k-\ell)}E\{\xi^{[\ell]}_{0,n}(c'_n)\}^\alpha I\{b_1 < \xi^{[k]}_{0,n}(c_n), \xi^{[k]}_{0,n^*}(c^*) \le b_2\} \\ &\quad \le P\{b_1 < \xi^{[k]}_{0,n}(c_n)\}. \end{aligned}$$ □

Let $H^{[\ell]}_n(c)$ be the centered sum of products and $\xi^{[k]}_n(c)$ be the lifted maxima. For suitable $c_n \ge c'_n$, Lemma 3.4 asserts that $E\{H^{[\ell]}_n(c'_n)\}^2 I\{\xi^{[k]}_n(c_n) \le b\}$ is dominated by the expectation of the sum of the squared terms in its expansion and therefore by the maxima in Lemma 3.3. For $k = 2$, this gives (1.14) $\Rightarrow$ (1.13).

LEMMA 3.4. *Let $H^{[k]}_n(c)$ be given by (2.8) and $Y_{m,n} = Y_{m,n}(c_n)$ be as in Lemma 3.3 with $c_n \ge c_2(n/M_0)$. For $M_0 \le M_1$ and $M_0 < M_2$, set $c'_n = c_2(n/M_1)$ and $c''_n = c_2(n/M_2)$. Then, for $0 \le \ell_1 \le \ell_1 + \ell_2 = \ell \le k \le n/3$,*

$$\begin{aligned} &(c''_n)^{2\ell_1} E\big\{H^{[\ell_2]}_n(c'_n)\big\}^2 Y_{0,n} \\ &\quad \le \frac{3^{\ell_2+1}}{2}\left(\frac{2M_1}{1-M_1/n} + 2k\right)^{2\ell_2}\left(\frac{2}{M_2-M_0}\right)^{\ell_1} n^\ell E\left\{Y_{0,n}\prod_{i=1}^{\ell} X_i^2\right\}. \end{aligned}$$

PROOF. Let $H^{[\ell]}_n = H^{[\ell]}_n(c'_n)$. Expanding the square of (2.8), we obtain

$$E\{H^{[\ell]}_n\}^2 Y_{0,n} = \sum_{\ell_1=0}^{\ell} N_{n,\ell,\ell_1} E\left\{Y_{0,n}\prod_{i=1}^{\ell-\ell_1}(X_i-\mu_n)^2 \prod_{i=\ell-\ell_1+1}^{\ell+\ell_1}(X_i-\mu_n)\right\}, \tag{3.10}$$

where $N_{n,\ell,\ell_1} \le n^{\ell+\ell_1}$ and $\mu_n = \mu(c'_n)$. The first step is to control the cross-product terms in (3.10) with $1 \le \ell_1 \le \ell$.

Let $X'_i = (X_i - \mu(c'_n))I\{|X_i| > c'_n\}$ with $c'_n = c_2(n/M_1)$, and $Z_0 = g_0(X_1, \dots, X_{m_0})$ with a Borel function g_0 of $m_0 \le n-k$ variables. The proof is based on the following facts:

$$E\big(X_n - \mu(c'_n)\big)Z_0 Y_{0,n} = EX'_n Z_0 Y_{0,n}, \tag{3.11}$$

$$E\big(X_n - \mu(c'_n)\big)^2 |Z_0| Y_{0,n} \le 4EX_n^2|Z_0|Y_{0,n}, \tag{3.12}$$

$$(n-m_0)E\big|X'_nX'_{n-1}Z_0\big|Y_{0,n} \le 4\left(\frac{M_1}{1-M_1/n} + k\right)^2 EX_n^2|Z_0|Y_{0,n} \tag{3.13}$$

and

$$c_2^2\left(\frac{n}{M_2}\right)E|Z_0|Y_{0,n} \le \frac{n(n-m_0)}{(M_2-M_0)(n-m_0-k)}EX_n^2|Z_0|Y_{0,n}. \tag{3.14}$$

The proofs of (3.11)–(3.14) are given in the Appendix.

Coming back to (3.10), we find by repeated applications of (3.11), (3.12) and (3.13) with $n - m_0 \geq n - 2\ell \geq n/3$ that

$$
\begin{aligned}
&N_{n,\ell,\ell_1} E\Biggl\{Y_{0,n} \prod_{i=1}^{\ell-\ell_1} (X_i - \mu_n)^2 \prod_{i=\ell-\ell_1+1}^{\ell+\ell_1} (X_i - \mu_n)\Biggr\} \\
&\quad = N_{n,\ell,\ell_1} E\Biggl\{Y_{0,n} \prod_{i=1}^{\ell-\ell_1} (X_i - \mu_n)^2 \prod_{\ell-\ell_1+1}^{\ell+\ell_1} X_i'\Biggr\} \\
&\quad \leq n^{\ell+\ell_1} 4^{\ell-\ell_1} E\Biggl\{Y_{0,n} \prod_{i=1}^{\ell-\ell_1} X_i^2 \prod_{i=\ell-\ell_1+1}^{\ell+\ell_1} |X_i'|\Biggr\} \\
&\quad \leq \left(\frac{M_1}{1 - M_1/n} + k\right)^{2\ell_1} 3^{\ell_1} 4^{\ell} n^{\ell} E\Biggl\{Y_{0,n} \prod_{i=1}^{\ell} X_i^2\Biggr\},
\end{aligned}
$$

due to the exchangeability of X_i. Summing up over ℓ_1 in (3.10), we obtain

$$
E\{H_n^{[\ell]}\}^2 Y_{0,n} \leq \frac{3^{\ell+1}}{2}\left(\frac{2M_1}{1 - M_1/n} + 2k\right)^{2\ell} n^{\ell} E\Biggl\{Y_{0,n} \prod_{i=1}^{\ell} X_i^2\Biggr\}.
$$

Since $n - m_0 \leq 2(n - m_0 - k)$ for $m_0 \leq \ell \leq k \leq n/3$, it follows from (3.14) that

$$
\begin{aligned}
&(c_n'')^{2\ell_1} E\{H_n^{[\ell_2]}(c_n')\}^2 Y_{0,n} \\
&\quad \leq \frac{3^{\ell_2+1}}{2}\left(\frac{2M_1}{1 - M_1/n} + 2k\right)^{2\ell_2} n^{\ell_2} (c_n'')^{2\ell_1} E\Biggl\{Y_{0,n} \prod_{i=1}^{\ell_2} X_i^2\Biggr\} \\
&\quad \leq \frac{3^{\ell_2+1}}{2}\left(\frac{2M_1}{1 - M_1/n} + 2k\right)^{2\ell_2} \left(\frac{2}{M_2 - M_0}\right)^{\ell_1} n^{\ell} E\Biggl\{Y_{0,n} \prod_{i=1}^{\ell} X_i^2\Biggr\}. \qquad \square
\end{aligned}
$$

The following elementary lemma is quite useful in our proofs here. For $k = 2$, it gives (1.15) $\Rightarrow$ (1.14).

LEMMA 3.5. *Let η_j be nonnegative random variables and A_j be events. Then*

$$
\sum_{j=j_0}^{\infty} \eta_j I_{A_j} \leq I_{A_{j_0}} \sum_{i=j_0}^{\infty} \eta_i + \sum_{j=j_0}^{\infty} I_{A_j^c A_{j+1}} \sum_{i=j+1}^{\infty} \eta_i.
$$

In the rest of this section, M' denotes a finite positive constant which may change from one place to another.

PROOF OF THEOREMS 2.2 (Sufficiency) AND 3.1. Suppose (2.1) and (2.2) hold for some $c_n \geq \delta_0 c_2(n/M_0)$ and $\varepsilon = 1$. We shall prove (3.1), (3.2) and (2.6). Assume further $M_1 > M_0$ and $\delta_0 = 1$. Let $M_0 < M_2 < M_1$. By Lemma 3.2(i),

$$
\sum_{j=1}^{\infty} P\{\xi_{n_{j+1}}^{[k]}(c_{n_j}) > b_{n_j}^k\} < \infty \tag{3.15}
$$

for some n_j with $1 < \gamma_1 \le n_{j+1}/n_j \le (n_{j+1}-k)/(n_j-k) \le \gamma_2 = M_1/M_2$. For example, we may choose n_{j+1} such that $P\{\xi^{[k]}_{m_0 n_{j+1}}(c_{n_{j+1}}) > b^k_{n_{j+1}}\}$ is the smallest among $P\{\xi^{[k]}_{m_0 n}(c_n) > b^k_n\}$, $\gamma_1 n_j \le n \le \gamma_2 n_j - (\gamma_2 - 1)k$, for some $m_0 > \gamma_2$.

Set $c^*_j = c_2(n_j/M_1)$. Similar to (2.9), for $n_j \le n < n_{j+1}$, $H^{[k]}_n(c'_n)$ can be written as

$$\sum_{\ell=0}^{k} \binom{n-\ell}{k-\ell} \{\mu(c^*_j) - \mu(c'_n)\}^{k-\ell} H^{[\ell]}_n(c^*_j).$$

Since $c_\infty(n/M') \le c'_n \le M' c_2(n/M_1)$ and $c_\infty(n_j/M_1) \le c^*_j \le c_2(n/M_1)$ for $n_j \le n < n_{j+1}$,

$$\begin{aligned} n\big|EXI\{|X| \le c^*_j\} - EXI\{|X| \le c'_n\}\big| &\le nE|X|I\{c_\infty(n_j/M') \le |X| \le M'c_2(n_{j+1}/M_1)\} \\ &\le M'c_2(n_{j+1}/M_1)nP\{|X| > c_\infty(n_j/M')\} \le M'c_2(n_j/M_2), \end{aligned}$$

which implies $n|\mu(c^*_j) - \mu(c'_n)| \le M'c_2(n_j/M_2)$ by (1.5) as $P\{|X| > c'_n\} \le M'/n$ and $P\{|X| > c^*_j\} \le M_1/n_j$. Thus, with $c^{**}_j = c_2(n_j/M_2)$, (3.2) is a consequence of (2.1) and

$$\lim_{j\to\infty} b^{-k}_{n_j}\{c^{**}_j\}^{k-\ell} \max_{n_j \le n < n_{j+1}} \big|H^{[\ell]}_n(c^*_j)\big| = 0 \quad \text{a.s.}, \qquad 1 \le \ell \le k. \tag{3.16}$$

Define $T^{[k]}_n = \sum_n h(Y_{i_1}, \dots, Y_{i_k}) X_{i_1} X_{i_2} \dots X_{i_k}$, $V^{[k]}_n = \sum_n |X_{i_1} X_{i_2} \dots X_{i_k}|^2$,

$$J_1 = \sum_{j=j_0}^{\infty} b^{-2k}_{n_j} E\{T^{[k]}_{n_j}\}^2 I\{\xi^{[k]}_{n_j}(c_{n_j}) \le b^k_{n_j}\}$$

and $J^{[\ell]}_2$ and J_3 in the same manner with $\{T^{[k]}_{n_j}\}^2$ replaced by $\{(c^{**}_j)^{k-\ell} \cdot H^{[\ell]}_{n_j}(c^*_j)\}^2$ and $V^{[k]}_{n_j}$, respectively. Let $\mathscr{F}_n$ be the σ-algebra generated by all symmetric functions of (X_i, Y_i), $1 \le i \le n$, under the permutation group for the vectors. For $n_j \le n < n_{j+1}$,

$$E\big[T^{[k]}_{n_j} I\{\xi^{[k]}_{n_j}(c_{n_j}) \le b^k_{n_j}\}\big|\mathscr{F}_n\big] = T^{[k]}_n \binom{n}{k}^{-1} \binom{n_j}{k}$$

on the event $\{\xi^{[k]}_{n_{j+1}}(c_{n_j}) \le b^k_{n_j}\}$. Since $\{n_j!(n-k)!\}/\{(n_j-k)!n!\} \ge \gamma_2^{-k}$ and $b_n \ge b_{n_j}$ for $n_j \le n < n_{j+1}$, by the Doob inequality for the martingale on the left-hand side above

$$\begin{aligned} &P\Big\{\max_{n_j \le n < n_{j+1}} |T^{[k]}_n / b^k_n| \ge \varepsilon,\ \xi^{[k]}_{n_{j+1}}(c_{n_j}) \le b^k_{n_j}\Big\} \\ &\quad\le P\Big\{\max_{n_j \le n < n_{j+1}} \big|E\big[T^{[k]}_{n_j} I\{\xi^{[k]}_{n_j}(c_{n_j}) \le b^k_{n_j}\}\big|\mathscr{F}_n\big]\big| \ge \varepsilon b^k_{n_j}\gamma_2^{-k}\Big\} \\ &\quad\le 4\big(\varepsilon b^k_{n_j}\gamma_2^{-k}\big)^{-2} E\big[T^{[k]}_{n_j}\big]^2 I\{\xi^{[k]}_{n_j}(c_{n_j}) \le b^k_{n_j}\}, \qquad \forall\, \varepsilon > 0. \end{aligned}$$

Therefore, by (3.15) and the Borel–Cantelli lemma, $J_1 < \infty$ implies $P\{T_n^{[k]}/b_n^k \to 0\} = 1$ in (3.1). Similarly, (3.16) and (2.6) hold if $J_2^{[\ell]}$, $1 \le \ell \le k$, and J_3 are all finite. The martingale argument applies to (3.16) since the level of truncation c_j^* is the same for $n_j \le n < n_{j+1}$. Since $Eh(Y, y_2, \dots, y_k) = 0$ and $\{Y_i\}$ is independent of $\{X_i\}$, $J_1 = Eh^2(Y_1, \dots, Y_k)J_3$, so that it suffices to prove $J_2^{[\ell]} < \infty$, $1 \le \ell \le k$, and $J_3 < \infty$.

Since $\sum_{i=j}^{\infty} n_i^k/b_{n_i}^{2k} \le M' n_j^k/b_{n_j}^{2k}$ by (2.3), it follows from Lemmas 3.5 and 3.3 (with $\alpha = 2$ and $\ell = 0$) and (3.5) that, for large j_0,

$$\begin{aligned} J_3 &= \sum_{j=j_0}^{\infty} b_{n_j}^{-2k}\binom{n_j}{k} E\prod_{i=1}^{k} X_i^2 I\{\xi_{n_j}^{[k]}(c_{n_j}) \le b_{n_j}^k\} \\ &\le M' n_{j_0}^k + M' \sum_{j=j_0}^{\infty} \frac{n_{j+1}^k}{b_{n_{j+1}}^{2k}} E\prod_{i=1}^{k} X_i^2 I\{b_{n_j}^k < \xi_{n_j}^{[k]}(c_{n_j}),\ \xi_{n_{j+1}}^{[k]}(c_{n_{j+1}}) \le b_{n_{j+1}}^k\} \\ &\le M' + M' \sum_{j=j_0}^{\infty} P\{b_{n_j}^k < \xi_{n_j}^{[k]}(c_{n_j})\} < \infty. \end{aligned}$$

By Lemma 3.4 [with $(\ell_1, \ell_2, \ell) \leftrightarrow (\ell, k-\ell, k)$ and $Y_{0,n} \leftrightarrow I\{\xi_{n_j}^{[k]}(c_{n_j}) \le b_{n_j}^k\}$],

$$\begin{aligned} J_2^{[\ell]} &= \sum_{j=j_0}^{\infty} b_{n_j}^{-2k}(c_j^{**})^{2(k-\ell)} E\{H_{n_j}^{[\ell]}(c_j^*)\}^2 I\{\xi_{n_j}^{[k]}(c_{n_j}) \le b_{n_j}^k\} \\ &\le M' \sum_{j=j_0}^{\infty} b_{n_j}^{-2k} n_j^k E\prod_{i=1}^{k} X_i^2 I\{\xi_{n_j}^{[k]}(c_{n_j}) \le b_{n_j}^k\} \le M' J_3 < \infty. \end{aligned}$$

Although the proof here is only for $M_1 > M_0$ and $\delta_0 = 1$, it poses no problem as $\delta_0 c_2(n/M_0) \ge c_2(\delta_0^2 n/M_0)$ and the necessity part for (2.4) implies both (2.1) and (2.2) for all M_0 when $c_n \sim c_2(n/M_0)$. $\square$

PROOF OF THEOREM 2.3(i) AND (ii). We shall first prove part (i). Let $M_0 < M_1^* < \infty$ and n_j be arbitrary positive integers satisfying $1 < \gamma_1 \le n_{j+1}/n_j \le \gamma_2 < \infty$. Set $c_j^* = c_2(n_j/M_1^*)$. It follows from (2.9) that (2.1) is a consequence of

$$b_{n_j}^{-k}|n_j\mu(c_j^*)|^{k-\ell} \max_{n_j \le n < n_{j+1}} \left|H_n^{[\ell]}(c_j^*)\right| \to 0 \quad \text{a.s.}, \qquad 0 \le \ell \le k. \tag{3.17}$$

For $\ell = 0$, (3.17) follows from (2.1) and (1.6), as $M_0 < M_1^*$ implies

$$|n_j\mu(c_j^*)|/b_{n_j} \le M_1^*\nu(n_j/M_1^*)/b_{n_j} \le M_1^*\nu^*(n_j/M_0)/b_{n_j} \le M_1^*\delta_0^{-1}c_{n_j}/b_{n_j} \to 0.$$

Since $c_n/b_n \to 0$ by (2.1), it follows from (2.2) (with the terms for $\ell = k$) that (2.10) holds for all $\varepsilon > 0$, so that $nP\{|X| > \varepsilon_n b_n\} \to 0$ for some $\varepsilon_n \to 0+$. By (1.4), $nP\{|X| > c_2(n/M_0)\} \le M_0$. For $a > 0$ and large n these imply

$$\begin{aligned} |n\mu(ab_n)/b_n| &\le n|\mu(c_2(n/M_0))|/b_n + nE|X|I\{c_2(n/M_0) < |X| \le ab_n\}/b_n \\ &\le M'c_n/b_n + \varepsilon_n nP\{|X| > c_2(n/M_0)\} + anP\{|X| > \varepsilon_n b_n\} = o(1). \end{aligned}$$

Thus, the conditions for Mori's theorem are satisfied for the normalizing sequence $\{b(n/\gamma_2)\}$ as discussed in the paragraph before Theorem 2.3, so that $\sum_{k\le R_n^{[i]}\le n} X_i/b(n/\gamma_2)\to 0$ a.s., where $R_n^{[i]}$ is the rank of $|X_i|$ in $|X_1|,\dots,|X_n|$. By Lemma 3.2(ii), (2.1) and (2.2) for all $\varepsilon>0$ imply

$$|n_j\mu(c_j^*)|^{k-1}X_{n_{j+1}}^{[1]}/b_{n_j}^k \le M'\xi_{n_{j+1}}^{[k]}(c_2(n_j/M_1^*))/b_{n_j}\to 0\quad \text{a.s.}$$

Therefore, for $\ell=1$ the left-hand side of (3.17) is bounded by

$$\frac{|n_j\mu(c_j^*)|^{k-1}}{b_{n_j}^k}\left\{\max_{n_j\le n<n_{j+1}}\left|\sum_{k\le R_n^{[i]}\le n} X_i\right| + (k-1)X_{n_{j+1}}^{[1]} + n_{j+1}|\mu(c_j^*)|\right\}\to 0\quad \text{a.s.}$$

Hence, it suffices to show (3.17) for $2\le\ell\le k$.

Take $0<\delta_0\le 1$ without loss of generality. Set $M_1^*>M_0\delta_0^{-2}$ such that $M_1^*=m^*M_1$ for some integer $m^*\ge 1$. Define $\bar c(t)=(\nu^2(t)-M_2c_2^2(t))^+$. Since (1.3) implies (2.3), it follows from the Hölder inequality and (2.11) that, for $2\le\ell\le k$,

$$\begin{aligned}&\sum_{m=n}^{\infty}\frac{m^{k-1}}{b_m^{2k}}\left(\frac{\bar c(m/M_1)}{m}\right)^{k-\ell}\\ &\quad\le\left\{\sum_{m=n}^{\infty}\frac{m^{k-1}}{b_m^{2k}}\right\}^{1-1/p_\ell}\left\{\sum_{m=n}^{\infty}\frac{m^{k-1}}{b_m^{2k}}\left(\frac{\bar c(m/M_1)}{m}\right)^{p_\ell(k-\ell)}\right\}^{1/p_\ell}\\ &\quad\le M'\frac{n^\ell c_n^{2(k-\ell)}}{b_n^{2k}},\end{aligned}\tag{3.18}$$

where $p_\ell=(k-2)/(k-\ell)\ge 1$. Thus, we may choose $n_j=m^*m_j$ such that $m^*n_j\le n_{j+1}<2m^*n_j$,

$$\sum_{i=j}^{\infty}\frac{\{\bar c(n_i/M_1^*)\}^{k-\ell}n_i^\ell}{b_{n_i}^{2k}}<M'\frac{c_{n_j}^{2(k-\ell)}n_j^\ell}{b_{n_j}^{2k}},\qquad 2\le\ell\le k,\tag{3.19}$$

and such that, by Lemma 3.2(i) with $\varepsilon=1$,

$$\sum_{j=1}^{\infty}P\{\xi_{n_{j+1}}^{[k]}(c_{n_j})>b_{n_j}\}<\infty.\tag{3.15'}$$

This can be done by taking m_{j+1} to satisfy $a(m_{j+1};\ell)\le kn_j^{-1}\sum_{m=n_j}^{2n_j}a(m;\ell)$ for all $2\le\ell\le k$ with $a(m;\ell)$ being the summands on the left-hand side of (3.18), as $n_i/M_1^*=m_i/M_1$, and also to satisfy $a(n_{j+1})\le kn_j^{-1}\sum_{m=n_j}^{2n_j}a(m^*m)$ with $a(n)$ being the summands in (3.4).

By (3.15′) and the martingale argument in the proof of Theorem 2.2, for $2\le\ell\le k$, (3.17) is a consequence of

$$J=\sum_{\ell=2}^{k}\sum_{j=j_0}^{\infty}b_{n_j}^{-2k}E\big((n_j\mu(c_j^*))^{k-\ell}H_{n_j}^{[\ell]}(c_j^*)\big)^2 I\{\xi_{n_j}^{[k]}(c_{n_j})\le b_{n_j}^k\}<\infty.$$

Since $M_1^* > M_0/\delta_0^2$ and $c_j^* = c_2(n_j/M_1^*) \le c_2(\delta_0^2 n_j/M_0) \le c_{n_j} \wedge c_{n_{j+1}}$, it follows from Lemma 3.4, Lemma 3.5 and (3.19), Lemma 3.3 and then (3.15′) that

$$\begin{aligned} J &\le M' \sum_{\ell=2}^{k} \sum_{j=j_0}^{\infty} b_{n_j}^{-2k} \{c_j^* + \bar{c}(n_j/M_1^*)\}^{k-\ell} E\big(H_{n_j}^{[\ell]}(c_j^*)\big)^2 I\{\xi_{n_j}^{[k]}(c_{n_j}) \le b_{n_j}^k\} \\ &\le M' \sum_{\ell=2}^{k} \sum_{j=j_0}^{\infty} b_{n_j}^{-2k} \{\bar{c}(n_j/M_1^*)\}^{k-\ell} n_j^{\ell} E \prod_{i=1}^{\ell} X_i^2 I\{\xi_{n_j}^{[k]}(c_{n_j}) \le b_{n_j}^k\} \\ &\le M' + M' \sum_{\ell=2}^{k} \sum_{j=j_0}^{\infty} \frac{c_{n_{j+1}}^{2(k-\ell)} n_{n+1}^{\ell}}{b_{n_{j+1}}^{2k}} E \prod_{i=1}^{\ell} X_i^2 I\{b_{n_j}^k < \xi_{n_j}^{[k]}(c_{n_j}),\ \xi_{n_{j+1}}^{[k]}(c_{n_{j+1}}) \le b_{n_{j+1}}^k\} \\ &\le M' + M' \sum_{\ell=2}^{k} \sum_{j=j_0}^{\infty} P\{b_{n_j}^k < \xi_{n_j}^{[k]}(c_{n_j})\} < \infty. \end{aligned}$$

For part (ii), we verify (3.17) for $1 \le \ell \le k$ without using the Mori theorem. Since (2.11′) holds, the value $p_\ell = (k-1)/(k-\ell)$ is taken in (3.18). The rest of the proof is similar and omitted. Condition (1.3) can be replaced by (2.3) as it is not used after (3.18). Condition (2.2) is used only to obtain (3.15′) with $\varepsilon = 1$. □

4. Necessity. In this section we prove Theorems 1.1 and 2.1 and the necessity part of Theorems 2.2 and 2.3, through (1.2) ⇒ (1.18) ⇒ (2.1) and (2.2). Decoupled products are also considered. Our methods include decoupling, a Lévy-type inequality and certain bounds for the percentiles of $|S_n|$.

Let $\{\tilde{X}_n^{(\ell)},\ n \ge 1\}$, $\ell \ge 1$, be i.i.d. copies of the sequence $\{X_n\}$. Define

$$\tilde{X}_n^{(\ell)*} = \tilde{X}_{0,n}^{(\ell)*}, \qquad \tilde{X}_{m,n}^{(\ell)*} = \max\big(|\tilde{X}_{m+1}^{(\ell)}|, \dots, |\tilde{X}_n^{(\ell)}|\big),$$

$$\tilde{\xi}_n^{[k]}(c) = \tilde{\xi}_{0,n}^{[k]}(c), \qquad \tilde{\xi}_{m,n}^{[k]}(c) = \prod_{\ell=1}^{k} \big(c \vee \tilde{X}_{m,n}^{(\ell)*}\big), \qquad c > 0,$$

$$\tilde{S}_{m,n}^{(\ell)} = \sum_{i=m+1}^{n} \tilde{X}_i^{(\ell)}, \qquad \tilde{S}_n^{(\ell)} = \tilde{S}_{0,n}^{(\ell)}, \qquad \tilde{S}_n^{(\ell)*} = \max\big(|\tilde{S}_1^{(\ell)}|, \dots, |\tilde{S}_n^{(\ell)}|\big). \tag{4.1}$$

Consider the statements

$$\sum_{j=1}^{\infty} P\left\{\prod_{\ell=1}^{k} \tilde{S}_{n_j}^{(\ell)*} > \varepsilon b_{n_j}\right\} < \infty \qquad \forall\ \varepsilon > 0,\ 1 < \gamma_1 \le n_{j+1}/n_j \le \gamma_2 < \infty, \tag{4.2}$$

$$P\{\xi_n^{[k]}(c_n) > \varepsilon b_n^k \text{ i.o.}\} = 0, \tag{4.3}$$

$$P\{\tilde{\xi}_n^{[k]}(c_n) > \varepsilon b_n^k \text{ i.o.}\} = 0. \tag{4.4}$$

THEOREM 4.1. *Let b_n be an increasing sequence of constants.*

(i) *Let $c_n \sim \nu^*(n/M_0)$ and $\varepsilon > 0$. Then* (1.2) ⇒ (4.2) ⇒ (2.1) *and* (2.2).

(ii) *For* $\varepsilon > 0$ *and* $c_n > 0$, (4.3) *implies* (4.4). *If, in addition,* $nP\{|X| > c_n\} = O(1)$, *then* (4.4) *implies* (2.1) *and* (2.2). *Conversely, conditions* (2.1) *and* (2.2) *with* $c_n \geq c(n)$ *imply* $P\{\xi_n^{[k]}(c(n/\gamma)) > \varepsilon b_n^k \text{ i.o.}\} = 0$, *provided that* $c(\cdot)$ *is increasing and* $1 < \gamma < \infty$.

(iii) *The summability in* (4.2) *is equivalent to the decoupled SLLN* $b_n^{-k}\prod_{\ell=1}^k \tilde{S}_n^{(\ell)} \to 0$ *a.s. or the stronger* $b_n^{-k}\prod_{\ell=1}^k \tilde{S}_n^{(\ell)*} \to 0$ *a.s.*

REMARK. By Theorems 4.1(i) and (iii), 1.1 and 2.3 and Corollary to Theorem 2.2, the SLLN (1.2) is equivalent to its decoupled versions in Theorem 4.1(iii) under respective conditions.

For disjoint sets of positive integers $A_1, \dots, A_\ell$, define the sum of "cross-block" terms

$$S^{[k]}_{A_1\otimes\cdots\otimes A_\ell} = \sum_{i_1<i_2<\cdots<i_k} h(X_{i_1}, \dots, X_{i_k}) I\{(i_1, \dots, i_k) \in A_1 \otimes \cdots \otimes A_\ell\},$$

where $A_1 \otimes \cdots \otimes A_\ell$ is the set of vectors $(i_1, \dots, i_k)$ such that $\{i_1, \dots, i_k\} \subseteq \bigcup_{j=1}^\ell A_j$ and $\{i_1, \dots, i_k\} \cap A_j \neq \varnothing$ for all $1 \leq j \leq \ell$. For example, $S_A^{[k]}/\binom{|A|}{k}$ is the U-statistic based on the set of variables $\{X_i,\ i \in A\}$, where $|A|$ is the size of the set A.

PROPOSITION 4.2. *Let* A_j, $0 \leq j \leq \ell$, *be disjoint sets of positive integers and* $a_{\mathbf{i}}$ *be real numbers indexed by vectors* $\mathbf{i} = (i_1, \dots, i_k)$. *Then*

$$\sum_{\mathbf{i}\in\Lambda} a_{\mathbf{i}} I\{\mathbf{i} \in (A_1 \otimes \cdots \otimes A_\ell) \cup (A_0 \otimes A_1 \otimes \cdots \otimes A_\ell)\}$$
$$= \sum_{j=0}^{\ell} (-1)^{\ell-j} \sum_{0<m_1<\cdots<m_j\leq \ell} \sum_{\mathbf{i}\in\Lambda} a_{\mathbf{i}} I\big\{\{i_1, \dots, i_k\} \subseteq A_0 \cup A_{m_1} \cup \cdots \cup A_{m_j}\big\}$$

for all sets Λ *of finitely many vectors. In particular,*

$$S^{[k]}_{A_1\otimes\cdots\otimes A_k} = \sum_{j=0}^{k} (-1)^{k-j} \sum_{0<m_1<\cdots<m_j\leq k} S^{[k]}_{A_0\cup A_{m_1}\cup\cdots\cup A_{m_j}}. \tag{4.5}$$

This proposition, proved in the Appendix, gives one-sided decoupling when $\ell = k$. Giné and Zinn (1994), Lemma 1, obtained (4.5) for $A_0 = \varnothing$. The case $A_0 \neq \varnothing$ is useful for the application of the Borel–Cantelli lemma in our proofs.

The following Lévy-type inequality is a straightforward extension of Montgomery-Smith (1993).

THEOREM 4.3 [Montgomery-Smith (1993)]. *Let* $\tilde{S}_n^{(\ell)}$ *and* $\tilde{S}_n^{(\ell)*}$ *be given by* (4.1). *Then there exist universal constants* C_{k,m_0} *such that, for positive integers* k *and* m_0,

$$P\left\{\prod_{\ell=1}^k \tilde{S}^{(\ell)*}_{m_0 n} > t\right\} \leq C_{k,m_0} P\left\{C_{k,m_0}\left|\prod_{\ell=1}^k \tilde{S}_n^{(\ell)}\right| > t\right\}. \tag{4.6}$$

For $m_0 = k = 1$, (4.6) is Corollary 4 of Montgomery-Smith (1993). The general case is proved by taking conditional expectation of each copy $\{X_n^{(\ell)}\}$ given other copies.

Lemma 4.4 provides bounds for the percentiles of $|S_n|$. Its proof is provided in the Appendix.

LEMMA 4.4. *Let $c_\alpha(\cdot)$ and $\nu(\cdot)$ be given by* (1.4) *and* (1.6), *respectively.*

(i) *Suppose $EX^2 = \infty$. Then there exists a universal constant C such that, as $n \to \infty$,*

$$\sup_{-\infty<a<\infty} P\big\{|S_n - a| \le c_2(tn)/4\big\} \le (1 + o(1))C\sqrt{t/2}.$$

(ii) *If $P\big\{X \ge 0\big\} = 1$, then for, $tM > 1$,*

$$P\{S_n \le Mc_1(tn)\} \ge \min\{(1 - 1/(tn))^n, 1 - 1/(tM)\},$$
$$P\{S_n \le \delta c_1(tn)\} \le \exp\{-\delta \log(\delta t) + \delta - 1/t\}.$$

(iii) *There exists a universal constant C such that, for $\sqrt{t}(M - 1/t) > 1$ and $\delta \le 1/2$,*

$$P\{|S_n| \le M\nu(tn)\} \ge \min\left\{\left(1 - \frac{1}{tn}\right)^n, 1 - \frac{1}{t(M - 1/t)^2}\right\},$$

$$P\big\{|S_n| \le \delta\nu(tn)\big\} \le \max\left\{C\sqrt{\frac{t}{2}}\left(1 - \frac{1}{tn}\left(\frac{2\delta+1}{2\delta}\right)^2\right)^{-1/2}, 1 - \left(1 - \frac{1}{tn}\right)^n, t^{-1}\left(\frac{2\delta t}{1 - \delta t}\right)^2\right\}.$$

REMARK. The constant C is the same as the one in Esséen's (1968) upper bound of concentration functions, which implies that, for $L > 0$,

$$\sup_a P\{a \le S_n \le a + L\} \le CL[nE(|X^s| \wedge (2L))^2]^{-1/2}, \tag{4.7}$$

where $X^s = X_1 - X_2$.

PROOF OF THEOREM 4.1. We shall only prove (i) and (4.3) $\Rightarrow$ (4.4) $\Rightarrow$ (2.1) and (2.2) for (ii), as the last statement of (ii) is in Lemma 3.2 and part (iii) is a direct consequence of Theorem 4.3 and the Borel–Cantelli lemma.

Step 1. (4.3) $\Rightarrow$ (4.4). Let $A_{j,n} = A_j \cap \{1, \dots, n\}$, $A_j = j + A_0$ and $A_0 = \{mk\colon m = 0, 1, \dots\}$. Define $X_A^{[1]} = \max_{i \in A} |X_i|$. Then $\prod_{\ell=1}^k \big(c_n \vee X_{A_{\ell,n}}^{[1]}\big) \le \xi_n^{[k]}$.

Step 2. (4.4) $\Rightarrow$ (2.1) and (2.2). Set $\lambda = \sup_n nP\{|X| > c_n\} \in (0, \infty)$. Since $\tilde\xi_n^{[k]}(c_n) \ge c_n^k$, (2.1) holds. Since $\{(1 - e^{-\lambda})/\lambda\}np_n \le 1 - (1 - p_n)^n$ for $p_n n \le \lambda$,

$$\begin{aligned}
\{(1 - e^\lambda)/\lambda\}^\ell n^\ell P\{c_n^{k-\ell}|X_1 \dots X_\ell| > \varepsilon b_n^k,\ |X_1| \wedge \dots \wedge |X_\ell| > c_n\}& \\
\le P\big\{c_n^{k-\ell}\tilde X_n^{(1)*}\tilde X_n^{(2)*} \dots \tilde X_n^{(\ell)*} > \varepsilon b_n^k,\ \tilde X_n^{(1)*} \wedge \dots \wedge \tilde X_n^{(\ell)*} > c_n\big\}& \\
\le P\big\{\tilde\xi_n^{[k]}(c_n) > \varepsilon b_n^k\big\},&
\end{aligned}$$

so that (2.2) holds if $\sum_j P\{\tilde{\xi}_{n_j}^{[k]}(c_{n_j}) > \varepsilon b_{n_j}^k\} < \infty$ with n_j being the index at which $P\{\tilde{\xi}_n^{[k]}(c_n) > b_n\}$ is maximized over $2^j \le n < 2^{j+1}$. This summability condition holds by the Borel–Cantelli lemma, as $P\{\tilde{\tilde{\xi}}_{n_{j-2},n_j}(c_{n_j})/b_{n_j} > \varepsilon \text{ i.o.}\} = 0$.

Step 3. (1.2) $\Rightarrow$ (4.2). Let $n_j = (k+1)^j$ and $A_{\ell,j}$, $1 \le \ell \le k$, be disjoint subsets of $\{n\colon n_j \le n < n_{j+1}\}$ of size n_j. It follows from Proposition 4.2 that there exist 2^k sequences of i.i.d. variables $\{Y_n^{(m)},\ n \ge 1\}$, each a permutation of $\{X_n\}$, such that

$$b_{n_{j+1}}^{-1}\left|\prod_{\ell=1}^{k} S_{A_{\ell,j}}\right| = b_{n_{j+1}}^{-1}\left|S_{A_{1,j}\otimes\cdots\otimes A_{k,j}}^{[k]}\right| \le b_{n_{j+1}}^{-1}\sum_{m=1}^{2^k}\max_{n_j\le n<n_{j+1}}|S_n^{(m,k]}| \to 0 \quad \text{a.s.},$$

where $S_A = \sum_{i\in A} X_i$ and $S_n^{(m,k]}$ is the sum of products based on $Y_1^{(m)},\dots,Y_n^{(m)}$ as in (1.1). Since $A_{\ell,j}$, $1 \le \ell \le k$, $j \ge 1$, are mutually exclusive sets, by the Borel–Cantelli lemma

$$\sum_{j=1}^{\infty} P\left\{\prod_{\ell=1}^{k}|\tilde{S}_{n_j}^{(\ell)}| > \varepsilon b_{n_{j+1}}\right\} = \sum_{j=1}^{\infty} P\left\{\prod_{\ell=1}^{k}|S_{A_{\ell,j}}| > \varepsilon b_{n_{j+1}}\right\} < \infty \qquad \forall\, \varepsilon > 0,$$

which implies (4.2) by Theorem 4.3.

Step 4. (4.2) $\Rightarrow$ (4.4). By Lemma 4.4(iii) there exist constants C_0 and m_0 depending on M_0 only such that

$$P\{\nu^*(2n/M_0) \vee \tilde{X}_{2n}^{(\ell)*} > t\} \le C_0' P\{C_0'\tilde{S}_{m_0 n}^{(\ell)*} > t\} \le C_0 P\{C_0 \tilde{S}_n^{(\ell)*} > t\}$$

for all $t > 0$. Repeated applications of this inequality on each copy $\{\tilde{X}_n^{(\ell)}\}$ in the summands in (4.2) yield $\sum_j P\{\tilde{\xi}_{2n_j}^{[k]}(\nu^*(2n_j/M_0)) > \varepsilon\delta_0^k b_{n_j}^k\} < \infty$ for $n_j = 2^j$, which then implies (4.4) by the Borel–Cantelli lemma for $c_n \le \delta_0^{-1}\nu^*(n/M_0)$. □

PROOF OF THEOREMS 2.2 AND 2.3 (Necessity). Theorem 2.3(iii) follows from Theorem 4.1(i). Since $\nu^*(t) = c_2(t)$ for symmetric variables, Theorem 4.1(i) also implies (2.4) $\Rightarrow$ (2.1) and (2.2). Instead of (iii) in Step 4 of the proof of Theorem 4.1, we use Lemma 4.4(i) and (ii), respectively, to obtain (2.1) and (2.2) under (2.5) or (2.6). □

PROOF OF THEOREM 1.1. By Theorem 2.3(i) and (iii), we only need to show that (1.7)–(1.9) imply (2.1) and (2.2) for $k = 2$ and all $\varepsilon > 0$, as (1.3) $\Rightarrow$ (2.3) $\Rightarrow$ (2.11) for $k = 2$. It follows from Lemma 3.2(ii) that $P\{\xi_n^{[k]}(\nu^*(n/(2M_0))) \ge \varepsilon_0 b_n^k \text{ i.o.}\} = 0$ for some $\varepsilon_0 > 0$, which implies $X_n^{[1]}X_n^{[2]}/b_n^2 \le \xi_n^{[k]}(c_2(\gamma n/M_0))/b_n^2 \to 0$ a.s. by Theorem 2.2. Set $v_\gamma = \limsup_n \xi_n^{[k]}(\nu^*(\gamma n/(2M_0)))/b_n^2$. Since $\nu^*(\gamma t) \le \gamma\nu^*(t) + c_2(\gamma t)$ for $\gamma \ge 1$, by (2.1) and (1.3),

$$\begin{aligned}\frac{v_\gamma}{\gamma^2} \le v_1 &= \limsup_{n\to\infty} \xi_{2n}^{[k]}(\nu^*(n/M_0))/b_{2n}^2 = \limsup_{n\to\infty} \nu^*(n/M_0)X_{2n}^{[1]}/b_{2n}^2 \\ &\le \limsup_{n\to\infty} 2\nu^*(n/(2M_0))X_n^{[1]}/b_{2n}^2 \le 2v_1/2^{2/p} \le 2^{1-2/p}\varepsilon_0 < \infty,\end{aligned}$$

which implies $v_\gamma = v_1 = 0$ as $0 < p < 2$. Hence, (2.2) holds by Theorem 4.1(ii). □

5. Regularly varying distributions and discussion. In this section, we consider conditions (2.1), (2.2), (2.11) and (2.11′) based on their interpretation in the case

$$C_1 P\{|X| > x\} \le x^{-p} L(x) \le C_2 P\{|X| > x\}, \qquad x > x_0, \tag{5.1}$$

where $0 < C_1 < C_2 < \infty$ and $L(x)$ is a slowly varying function as $x \to \infty$. This condition is slightly weaker than the requirement that $P\{|X| > x\}$ be regularly varying as $x \to \infty$. We shall assume $EX = 0$ when $E|X| < \infty$, due to the strong law of Hoeffding (1961). Some discussion is given at the end.

THEOREM 5.1. *Suppose* (2.3) *and* (5.1) *hold for some* $0 < p < 2$, $EX = 0$ *if* $E|X| < \infty$, *and that* $b(t)$ *is regularly varying as* $t \to \infty$ *if* $p = 1$. *Let* $c_n = \nu^*(n)$. *Then* $\nu^*(t) = O(c_2(t))$ *as* $t \to \infty$ *and* (2.11′) *holds for* $p \ne 1$, *and* (2.1) *implies* (2.11′) *for* $p = 1$. *Consequently*, (2.1) *and* (2.2) *together are equivalent to each and all of the statements* (1.2), (1.18), (4.2), (4.3) *and* (4.4), *equivalent to* (1.17), (2.4), (2.5) *and* (2.6) *if* $p \ne 1$, *and equivalent to*

$$b_n^{-k} \sum_{i_1 < i_2 < \cdots < i_k \le n} |X_{i_1} X_{i_2} \dots X_{i_k}| \to 0 \quad a.s. \tag{5.2}$$

if $0 < p < 1$ *and* (2.3′) *holds. Furthermore, if* $\nu^*(t) = O(c_2(t))$ (*e.g.*, $p \ne 1$) *and*

$$\sup_{y_0 \le y \le x^{1/p+\varepsilon}} L(xL(y))/L(x) \sim \inf_{y_0 \le y \le x^{1/p+\varepsilon}} L(xL(y))/L(x) \sim 1, \tag{5.3}$$

then for $b_n = n^{1/p}$ (1.2) *holds iff*

$$\int_0^\infty \left(-\log\left[\min\left(tP\{|X|^p > t\}, \frac{1}{2} \right) \right] \right)^{k-1} (tP\{|X|^p > t\})^k \frac{dt}{t} < \infty. \tag{5.4}$$

REMARK. The last statement of Theorem 5.1 shows that (2.10) is not sufficient for (1.2). Condition (5.3) holds if $L(x) = \prod_{j=1}^m (\log_j x)^{-\beta_j}$, where $\log_1(x) = \{\log(x \vee 1)\} \vee 1$ and $\log_{j+1}(x) = \log_1(\log_j(x))$.

PROOF.

Step 1. Proofs for $p \ne 1$. By (5.1), $E(|X| \wedge x)^\alpha \sim x^{\alpha - p} L(x)$ for $p < \alpha$ and $E(|X| - x)^+ \sim x^{1-p} L(x)$ for $p > 1$. These and (1.4) imply $c_2^p(t)/L(c_2(t)) \sim t$, and together they imply

$$|n\mu(c_2(n))| \le nE\big[|X| \wedge c_2(n)\big] = O(1) n \big[c_2(n)\big]^{1-p} L(c_2(n)) = O(1) c_2(n)$$

for $0 < p < 1$ and

$$|n\mu(c_2(n))| \le nE(|X| - c_2(n))^+ + nc_2(n) P\{|X| > c_2(n)\} = O(1) c_2(n)$$

for $1 < p < 2$ and $EX = 0$. They also imply $c_\alpha(t) \sim c_2(t)$ for $p < \alpha \le \infty$ Thus, (2.11′) holds, as $c_2(n/M_1) \sim \nu(n/M_1)$. By Theorem 2.2, (1.2) $\Leftrightarrow$ (2.1) and (2.2) with $c_n = \nu^*(n)$ or $c_n = c_\alpha(n)$ for all $p < \alpha \le \infty$. Therefore, (2.1) and (2.2) are equivalent to (1.2) and (1.18) by Theorem 2.3, to (4.2), (4.3) and (4.4) by Theorem 4.1 and to (2.3), (2.4) and (2.5) by Theorem 2.2. Also, $c_2(n) \sim c_\infty(n)$ implies (1.17) $\Rightarrow$ (1.18), and $c_1(n) \sim c_2(n)$ implies (1.2) $\Rightarrow$ (5.2) by Corollary to Theorem 2.2(ii) under (2.3′).

Step 2. Prove (2.1) $\Rightarrow$ (2.11′) for $p = 1$. Since b_n is regularly varying, $b_n^k = n^{k/p'} L_0(n)$ for some p' and slowly varying function $L_0(n)$. Let $c_n = \nu^*(n)$. Since $\nu^*(n) \ge c_2(n) \sim nL(c_2(n))$ and $L(c_2(n))$ is slowly varying, (2.1) implies $p' \le 1$. Since $L(x)$ is slowly varying,

$$\begin{aligned}|\mu(c_2(m)) - \mu(c_2(n))| &\le c_2(n) P\{|X| \ge c_2(n)\} + \int_{c_2(n)}^{c_2(m)} P\{|X| \ge x\}\, dx\\ &\le c_2(n)/n + C_1^{-1} c_2^{-\delta}(n) \int_{c_2(n)}^{c_2(m)} x^{\delta-1} L(x)\, dx\\ &\le M'(L(c_2(n)) + \left[c_2(m)/c_2(n)\right]^{\delta} L(c_2(m))),\end{aligned}$$

where $M' = M'_\delta < \infty$ does not depend on m or n, and $0 < \delta < 1/(2k)$. Since both $L_0(n)$ and $L(c_2(n))$ are slowly varying as $n \to \infty$ and $c_2(n) \sim nL(c_2(n))$, (2.11′) holds for $c_n = \nu^*(n)$ and $M_1 = 1$, so that (1.2) $\Leftrightarrow$ (2.1) and (2.2).

Step 3. Prove (5.4) $\Leftrightarrow$ (2.1) and (2.2) for $b_n = n^{1/p}$, $p \ne 1$ and $0 < p < 2$. Let $c_n = c_\alpha(n)$ for some $\alpha > p$. By (5.3), $c_n^p \sim nL(n^{1/p})$, so that (2.1) holds iff $L(x) \to 0$ as $x \to \infty$. Since the finiteness of (2.2) depends only on the order of $P\{|X| > x\}$ for large x, we may further assume without loss of generality that $|X|$ has a density function $f(x) \sim x^{-p-1} L(x)$. Let A_ℓ be the event $\{|X_1| \wedge \cdots \wedge |X_{\ell-1}| > c_n,\ c_n^{k-i}|X_1 \dots X_i| \le b_n^k,\ 1 \le i < \ell\}$. Then $b_n^k c_n^{-k+\ell}/|X_1 \dots X_{\ell-1}|$ and $|X_i|$, $1 \le i < \ell$, are all between c_n and $b_n^k c_n^{-k+1}$ on A_ℓ. Since $L(x) \sim L(n^{1/p})$ for $c_n \le x \le b_n^k c_n^{-k+1}$ by (5.3), we have

$$\begin{aligned}P\{c_n^{k-\ell}|X_1 \dots X_\ell| > b_n^k, A_\ell\} &\sim \int_{A_\ell} \left(\frac{b_n^k c_n^{-k+\ell}}{x_1 \dots x_{\ell-1}}\right)^{-p} L(n^{1/p}) \prod_{i=1}^{\ell-1} f(x_i)\, dx_i\\ &\sim \left(\frac{c_n^{k-\ell}}{b_n^k}\right)^p L^\ell(n^{1/p}) \int_{A_\ell} (x_1 \dots x_{\ell-1})^{-1} \prod_{i=1}^{\ell-1} dx_i\\ &\sim \left(\frac{c_n^{k-\ell}}{b_n^k}\right)^p L^\ell(n^{1/p}) \left\{\log\left(\frac{b_n^k}{c_n^k}\right)\right\}^{\ell-1}\\ &\sim L^k(n^{1/p}) n^{-\ell} \left\{\log\left(\frac{1}{L(n^{1/p})}\right)\right\}^{\ell-1}.\end{aligned}$$

Since (2.1) $\Leftrightarrow$ $\lim_{x\to\infty} L(x) = 0$, (2.2) holds with $b_n = n^{1/p}$ iff

$$\int_{x_0}^{\infty} L^k(t^{1/p}) \left\{\log\left(\frac{1}{L(t^{1/p})}\right)\right\}^{k-1} \frac{dt}{t} < \infty,$$

which is equivalent to (5.4). □

EXAMPLE 5.2. Suppose (5.1) holds with $p = 1$ and

$$L(x) = \prod_{j=1}^{m} (\log_j x)^{-\beta_j}, \qquad b_n^k = n^k L_0(x) = n^k (\log_{m_0} x)^{-\beta_0} \tag{5.5}$$

for some $m \geq 1$ and $m_0 \geq 1$. Similar to the case of $p \neq 1$, we have

$$n^\ell P\{c_n^{k-\ell}|X_1 \dots X_\ell| > b_n^k, A_\ell\} \sim (c_n^k/b_n^k)[nL(n)/c_n]^\ell \{\log(b_n^k/c_n^k)\}^{\ell-1}.$$

By assumption, $c_2(n) \sim nL(n)$ and $\log_1(b_n^k/c_2^k(n)) \sim \log(L_0(n)/L^k(n)) \sim \log_{m_2}(n)$ for some $m_2 \geq 2$ if (2.1) holds. By Theorem 5.1, (2.4) holds iff

$$\int_0^\infty (tL_0(t))^{-1} L^k(t) (\log_{m_2} t)^{k-1}\, dt < \infty. \tag{5.6}$$

Suppose $P\{X \geq M\} = 1$ for some $-\infty < M < 0$ if $EX = 0$ and $M = 0$ if $E|X| = \infty$. Then $\nu^*(n) \sim |n\mu(c_2(n))| \sim nL(n)L_1(n)$ and (2.1) implies $\log(b_n^k/(\nu^*(n))^k) = O(\log_2 n)$, where $L_1(x) = \prod_{j=1}^{m_1} \log_j(x)$ with $m_1 = \min\{j \geq 1\colon \beta_j \neq 1\}$. Therefore, (1.2) holds iff

$$\int_0^\infty (tL_0(t)L_1(t))^{-1} (L(t)L_1(t))^k\, dt < \infty. \tag{5.7}$$

For example, if $m = m_0 = 1$, then (2.4) $\Leftrightarrow$ (5.6) $\Leftrightarrow$ $k\beta_1 - \beta_0 > 1$, while (1.2) $\Leftrightarrow$ (5.7) $\Leftrightarrow$ $k\beta_1 - \beta_0 > k$. For $\beta_0 = 0$ and $\beta_1 = 2/k$, $k \geq 2$, $E|X| = \infty$, (2.4) holds but (1.2) does not. The same is true for $\beta_0 = 2(k-1)$ and $\beta_1 = 2$ under $EX = 0$. The general case is more complicated, where c_n may fluctuate between $\pm nL(n)L_1(n)$.

REMARK 1. It is not clear whether the condition $P\{X \geq 0\} = 1$ can be completely removed from Corollary to Theorem 2.2(ii), even for $b_n = n^{1/p}$, $0 < p < 1$. By Theorem 5.1, (1.2) and (5.2) are equivalent under (5.1) for $0 < p < 1$. For $k = 1$, there is no need to center the variables and (1.2) is equivalent to (5.2) for $b_n = n^{1/p}$ by the Marcinkiewicz–Zygmund strong law of large numbers. In Example 5.2, (2.4) and (1.2) are not equivalent for certain parameter values in (5.5), so that (1.2) and (5.2) are not equivalent when X_i is replaced by $\varepsilon_i X_i$. However, (2.3′) does not hold.

REMARK 2. The problem in Remark 1 is also related to the question concerning the equivalence between (2.4) and (1.17). Suppose (2.3′) and (1.2) hold and (5.2) does not. Then (1.18) holds by Theorem 2.2, so that

$$b_n^{-k/2} \max_{i_1 < i_2 < \cdots < i_k \leq n} \sqrt{|X_{i_1} X_{i_2} \dots X_{i_k}|} \to 0 \quad \text{a.s.}$$

On the other hand, Theorem 2.2 also implies that

$$b_n^{-k/2} \sum_{i_1 < i_2 < \cdots < i_k \leq n} \varepsilon_{i_1} \varepsilon_{i_2} \dots \varepsilon_{i_k} \sqrt{|X_{i_1} X_{i_2} \dots X_{i_k}|} \to 0 \quad \text{a.s.}$$

does not hold. This would show that (2.4) and (1.17) are not equivalent for $\sqrt{|X|}$ and the normalizing sequence $b_n^{k/2}$.

APPENDIX

Here we prove (3.11)–(3.14), Proposition 4.2 and Lemma 4.4.

PROOF OF (3.11)–(3.14). Let $p_n = P\{|X| > c_n\}$ and $p'_n = P\{|X| > c'_n\}$. Similar to (3.9), we have

$$Y_{0,n} I\{|X_n| \le c'_n\} = Y_{0,n-1} I\{|X_n| \le c'_n\}. \tag{A.1}$$

This implies (3.11) due to the independence of $(Z_0, Y_{0,n-1})$ and X_n, since, by (1.5) $\mu(c'_n)$ is the conditional expectation of X_n given $|X_n| \le c'_n$. Similarly, (A.1) and (1.5) imply

$$\begin{aligned} &E(X_n - \mu(c'_n))^2 |Z_0| Y_{0,n} \\ &\quad \le EX_n^2 |Z_0| Y_{0,n} I\{|X_n| \le c'_n\} + E(X'_n)^2 |Z_0| Y_{0,n} I\{|X_n| > c'_n\}, \end{aligned}$$

so that (3.12) follows from

$$|X'_j| \le (|X_j| + c'_n) I\{|X_j| > c'_n\} \le 2|X_j| I\{|X_j| > c'_n\}. \tag{A.2}$$

Let $R^{[i]}_{m,n}$ be the rank of $|X_i|$ in $|X_{m+1}|, \ldots, |X_n|$ in descending order as in the proof of Lemma 3.3. By (A.2)

$$\begin{aligned} &E|X'_n X'_{n-1} Z_0| Y_{0,n} \\ &\quad \le 4E|X_n X_{n-1} Z_0| Y_{0,n} I\big\{R^{[n-1]}_{m_0,n} \le k,\ R^{[n]}_{m_0,n} \le k\big\} \\ &\qquad + 4E|X_n X_{n-1} Z_0| Y_{0,n} I\big\{R^{[n-1]}_{m_0,n} > k,\ R^{[n]}_{m_0,n} > k, \\ &\qquad\qquad |X_{n-1}| > c'_n,\ |X_n| > c'_n\big\} \\ &\qquad + 8E|X_n X_{n-1} Z_0| Y_{0,n} I\big\{R^{[n-1]}_{m_0,n} \le k < R^{[n]}_{m_0,n},\ |X_n| > c'_n\big\}. \end{aligned} \tag{A.3}$$

We shall derive (3.13) by bounding the three terms on the right-hand side above. Since $|X_n X_{n-1}| \le (X^{[1]}_{m_0,n})^2$ and $(R^{[n-1]}_{m_0,n}, R^{[n]}_{m_0,n})$ is uniformly distributed given $(Z_0, X^{[1]}_{m_0,n}, Y_{0,n})$,

$$\begin{aligned} &E|X_n X_{n-1} Z_0| Y_{0,n} I\{R^{[n-1]}_{m_0,n} \le k,\ R^{[n]}_{m_0,n} \le k\} \\ &\quad \le k(k-1)\{(n-m_0)(n-m_0-1)\}^{-1} E(X^{[1]}_{m_0,n})^2 |Z_0| Y_{0,n} \\ &\quad \le k(k-1)\{(n-m_0)(n-m_0-1)\}^{-1} E(X^2_{m_0+1} + \cdots + X_n^2)|Z_0| Y_{0,n} \\ &\quad \le k^2 (n-m_0)^{-1} EX_n^2 |Z_0| Y_{0,n}. \end{aligned}$$

Since $|X_n X_{n-1}| \le (X^{[1]}_{m_0,n-2})^2$ and $Y_{0,n} = Y_{0,n-2}$ when $R^{[n-1]}_{m_0,n}$ and $R^{[n]}_{m_0,n}$ are both greater than k, by the independence of (X_{n-1}, X_n) and $(X^{[1]}_{m_0,n-2}, Z_0,$

$Y_{0,n-2}$) we obtain

$$
\begin{aligned}
E|X_n X_{n-1} Z_0| Y_{0,n} I\{R^{[n-1]}_{m_0,n} > k,\ R^{[n]}_{m_0,n} > k,\ |X_{n-1}| > c'_n,\ |X_n| > c'_n\}& \\
\le E(X^{[1]}_{m_0,n-2})^2 |Z_0| Y_{0,n-2} (p'_n)^2& \\
\le \{p'_n/(1-p'_n)\}^2 E(X^2_{m_0+1} + \cdots + X^2_{n-2}) |Z_0| Y_{0,n-2} I\{|X_{n-1}| \vee |X_n| \le c'_n\}& \\
\le (n-m_0)\{p'_n/(1-p'_n)\}^2 E X^2_n |Z_0| Y_{0,n}.&
\end{aligned}
$$

Combining the above arguments,

$$
\begin{aligned}
E|X_n X_{n-1} Z_0| Y_{0,n} I\{R^{[n-1]}_{m_0,n} \le k,\ R^{[n]}_{m_0,n} > k,\ |X_n| > c'_n\}& \\
\le E(X^{[1]}_{m_0,n-1})^2 |Z_0| Y_{0,n-1} I\{R^{[n-1]}_{m_0,n} \le k,\ |X_n| > c'_n\}& \\
\le p'_n(1-p'_n)^{-1} k (n-1-m_0)^{-1} E(X^{[1]}_{m_0,n-1})^2 |Z_0| Y_{0,n}& \\
\le p'_n(1-p'_n)^{-1} k E X^2_n |Z_0| Y_{0,n}.&
\end{aligned}
$$

Adding the above three inequalities together, we obtain (3.13) by (A.3) and $np'_n \le M_1$.

Finally, let us prove (3.14). By (1.4), $c_2^2(t)/t \le E\{|X| \wedge c_2(t)\}^2$, so that by (A.1)

$$
\begin{aligned}
M_2 c_2^2(n/M_2) E|Z_0| Y_{0,n} &\le nE\{X^2_{n+1} \wedge c_2^2(n/M_2)\} |Z_0| Y_{0,n} \\
&\le nE X^2_{n+1} |Z_0| Y_{0,n} I\{|X_{n+1}| \le c_n\} \\
&\quad + n c_2^2(n/M_2) E|Z_0| Y_{0,n} P\{|X_{n+1}| > c_n\} \\
&\le nE X^2_{n+1} |Z_0| Y_{0,n+1} + n p_n c_2^2(n/M_2) E|Z_0| Y_{0,n}.
\end{aligned}
$$

Hence, we have (3.14) as $np_n \le M_0$ and

$$
\begin{aligned}
E X^2_{n+1} |Z_0| Y_{0,n+1} &= E X^2_{m_0+1} |Z_0| Y_{0,n+1} I\{R^{[n+1]}_{m_0+1,n+1} > k\} \\
&\quad + E X^2_{m_0+1} |Z_0| Y_{0,n+1} I\{R^{[n+1]}_{m_0+1,n+1} \le k\} \\
&\le E X^2_n |Z_0| Y_{0,n} + k(n-m_0)^{-1} E X^2_{n+1} |Z_0| Y_{0,n+1}. \qquad \square
\end{aligned}
$$

PROOF OF PROPOSITION 4.2. It suffices to show

$$
\begin{aligned}
&I\{\mathbf{i} \in (A_0 \otimes A_1 \otimes \cdots \otimes A_\ell) \cup (A_1 \otimes \cdots \otimes A_\ell)\} \\
&\quad = \sum_{j=0}^{\ell} (-1)^{\ell-j} \sum_{0<m_1<\cdots<m_j\le \ell} I\big\{\{i_i, \ldots, i_k\} \subseteq A_0 \cup A_{m_1} \cup \cdots \cup A_{m_j}\big\} \qquad \text{(A.4)}
\end{aligned}
$$

for all vectors $\mathbf{i} = (i_1, \ldots, i_k)$ with $\{i_1, \ldots, i_k\} \subseteq A_0 \cup A_1 \cup \cdots \cup A_\ell$. Let B_m be the indicator of the "event" $\{i_1, \ldots, i_k\} \cap A_m = \varnothing$. Since

$$
I\big\{\{i_1, \ldots, i_k\} \subseteq A_0 \cup A_{m_1} \cup \cdots \cup A_{m_j}\big\} = B_{m'_1} \ldots B_{m'_{j'}}
$$

such that $\{m'_1, \ldots, m'_{j'}\} = \{i, \ldots, \ell\} \cap \{m_1, \ldots, m_j\}^c$, by the inclusion–exclusion formula for the union of events the right-hand side of (A.4) is

$$1 + \sum_{j'=1}^{\ell} (-1)^{j'} \sum_{0<m'_1<\cdots<m'_{j'}\le \ell} B_{m'_1} \ldots B_{m'_{j'}} = \prod_{m=1}^{\ell} (1 - B_m),$$

which equals the left-hand side of (A.4). Equation (4.5) follows as $A_0 \otimes A_1 \otimes \cdots \otimes A_k = \varnothing$. $\square$

PROOF OF LEMMA 4.4. Let $\mathrm{med}(X)$ be the median of X. With $X^s = X_1 - X_2$ as in (4.7), we observe $P\{|X^s| > x\} \ge 1/2P\{|X - \mathrm{med}(X)| > x\}$, which implies $2E(|X^s| \wedge c)^2 \ge E(|X - \mathrm{med}(X)| \wedge c)^2$. Let $c'_n = c_2(tn)$. It follows from (4.7) that

$$\sup_a P\left\{|S_n - a| \le \frac{c'_n}{4}\right\} \le \frac{\sqrt{2}C(c'_n/2)}{[nE(|X - \mathrm{med}(X)| \wedge c'_n)^2]^{1/2}} = (1 + o(1))C\sqrt{\frac{t}{2}}.$$

This gives (i). The proof of Lemma 2.3 of Klass and Zhang (1994) gives (ii).

Let us prove (iii). Set $\mu_n = \mu(c'_n)$ and $p = P\{|X| > c'_n\}$. Let P^* and E^* be the conditional probability and expectation given $|X_i| \le c'_n,\ 1 \le i \le n$. Then $(1 - p)E^* X_1^2 = (c'_n)^2(1/(nt) - p)$. For $C_0 > 0$ we have

$$\begin{aligned} P\{|S_n - n\mu_n| \le C_0 c'_n\} &\ge (1 - p)^n P^*\{|S_n - n\mu_n| \le C_0 c'_n\} \\ &\ge (1 - p)^n \{1 - nE^* X_1^2/(C_0 c'_n)^2\} \\ &= (1 - p)^{n-1}\{1 - p - (t^{-1} - np)/C_0^2\}. \end{aligned}$$

Since the right-hand side is log-concave in p, its minimum is reached either at $p = 0$ or $p = 1/(nt)$, so that $P\{|S_n - n\mu_n| \le C_0 c'_n\} \ge \min(1 - 1/(tC_0^2), (1 - 1/(nt))^n)$. This gives the first inequality of (iii) with $C_0 = M - 1/t$ and the second one for $\delta tn|\mu_n| \ge c'_n/2$ with $C_0 = (1 - \delta t)/(2\delta t)$. Let $X'_j = \min\{\max(X_j, -c'_n), c'_n\},\ j = 1, 2$. For $\delta tn|\mu_n| \le c'_n/2$,

$$\frac{E(X'_1 - X'_2)^2}{2} = \mathrm{Var}(X'_1) \ge E(X'_1)^2 - \{|\mu_n| + pc'_n\}^2 \ge \frac{(c'_n)^2}{tn} - \left(\frac{c'_n}{n}\right)^2 \left(\frac{1}{2\delta t} + \frac{1}{t}\right)^2.$$

Since $(X'_1 - X'_2) \le \min\{|X^s|^2, (2c'_n)^2\}$, by (4.7),

$$P\left\{|S_n| \le \frac{c'_n}{2}\right\} \le Cc'_n[nE\{|X^s| \wedge (2c'_n)\}^2]^{-1/2} \le C\left[\frac{2}{t} - \frac{2}{n}\left(\frac{1}{2\delta t} + \frac{1}{t}\right)^2\right]^{-1/2}. \ \square$$

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