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ON DRIFT AND ENTROPY GROWTH FOR RANDOM WALKS ON GROUPS

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In this paper we consider drift and entropy growth for symmetric finitary random walks on finitely generated groups. We construct examples of various intermediate asymptotics of the drift for such random walks. We establish general inequalities which connect drift, entropy and exponential growth rate of groups. Then we apply these inequalities to get estimates for entropy in particular examples.

1. Introduction. Let G be a group and μ a symmetric measure on it. We consider random walks on G induced by this measure. In this paper we assume that the support of μ is finite and generates the group G. We consider two functions

$$H(n) = H^{G,\mu}(n) = -\sum_{g\in G} \big(\mu^{*n}(g)\big) \ln\big(\mu^{*n}(g)\big)$$

and

$$L(n) = L^{G,\mu}(n) = \mathbf{E}_{\mu^{*n}} l(g).$$

Here l denotes the word length, corresponding to the fixed set of generators, and μ^{*n} is the nth convolution of μ. The function $H(n)$ is called the *entropy* and $L(n)$ is called the *drift* (or escape) of the random walk. $L(n)$ shows how fast (on average) the random walk is moving away from the origin.

It is known that $H(n)$ is asymptotically linear if and only if $L(n)$ is asymptotically linear [14] and if and only if the Poisson boundary of the random walk is nontrivial [11]. In particular, it is so for any nonamenable group. On the other hand, for many examples (e.g., for any Abelian group) $L(n)$ is asymptotically $\sqrt{n}$.

Until recently it was unknown whether there exist simple random walks on groups with intermediate growth rate of $L(n)$. The question of finding such random walks appears in [9]. I learned about this question from A. M. Vershik in 1999. The problem of finding asymptotics for $H(n)$ is discussed in [1].

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First examples of random walks on groups with intermediate growth rate of $L(n)$ were found by the author in [3, 6]. In these examples $L(n) \asymp n^{1-1/2^k}$ (for any positive integer k) and $L(n) \asymp n/\ln(n)$. Related questions about the drift (escape) for these groups were also considered in [12].

In this paper we find new possibilities for the rate of $L(n)$. In particular, we show that $L(n)$ can be asymptotically equal to

$$\frac{n}{\ln(\ln(\cdots \ln(n)\cdots))}.$$

We also estimate the growth of the entropy for random walks on these groups.

The structure of the paper is as follows. In Section 2 we state that certain functions are concave. We use this auxiliary lemma in the next section.

In Section 3 we consider a two-dimensional simple random walk. We find asymptotics of some class of functionals depending on local times of this two-dimensional random walk.

In Section 4 we construct examples of groups and we apply the results of the previous section to find asymptotics of the drift in these groups.

In Section 5 we give some general estimates for the entropy $H(n)$. We apply these estimates to the examples considered in Section 4. These examples show that there are infinitely many possibilities for asymptotics of the entropy. They also show that the growth of the entropy (as well as of the drift) can be very close to linear and yet sublinear.

The main results of this paper were announced in [7].

2. An auxiliary lemma.

LEMMA 1. *Take* $0 < \alpha \leq 1$.

(i) *Let*

$$\tilde{L}_{k,\alpha}(x) = \frac{x}{\underbrace{(\ln(\ln(\cdots \ln(x))\cdots))}_{k}{}^{\alpha}}.$$

Then $\tilde{L}_{k,\alpha}(x)$ *is concave on the segment* $[T_k, \infty)$, *where*

$$T_k = \underbrace{\exp\bigl(\exp\cdots\exp(4k)\cdots\bigr)\cdots}_{k}.$$

(ii) *There exist a continuous increasing function* $L_{k,\alpha}:[0,\infty) \to [0,\infty)$ *and* $X_{k,\alpha} > 0$ *such that* $L_{k,\alpha}(x)$ *is concave*, $L_{k,\alpha}(0) = 0$ *and* $L_{k,\alpha}(x) = \tilde{L}_{k,\alpha}(x)$ *for any* $x > X_{k,\alpha}$.

We prove this lemma in the Appendix.

3. Some functionals of two-dimensional random walk. We say that the random walk is simple if μ is equidistributed. In this section we consider a simple random walk on $\mathbb{Z}^2$. Let $b_z^{(n)}$ be the number of times the random walk has visited the element z, $z \in \mathbb{Z}^2$, from the moment 0 up to the moment n. Let $R^{(n)}$ be the number of different elements visited until the moment n ($R^{(n)}$ is called *range* of the random walk). First we formulate a simple property of the range.

LEMMA 2. *There exist $q_1, q_2 > 0$ such that, for any $n > 1$,*

$$\Pr\left[R^{(n)} \geq q_1 \frac{n}{\ln(n)}\right] \geq q_2.$$

PROOF. In fact, $R(n)/(n/\ln(n)) \to \pi$ almost surely [2]. $\square$

LEMMA 3. *Let f be a concave strictly increasing function such that $f(0) = 0$. Consider a simple random walk on $\mathbb{Z}^2$. There exists $K > 0$ such that*

$$\mathbf{E}\left(\sum_{z\in\mathbb{Z}^2} f\big(b_z^{(n)}\big)\right) \leq K f(\ln(n)) \frac{n}{\ln(n)}.$$

PROOF. Note that

$$\sum_{z\in\mathbb{Z}^2} f\big(b_z^{(n)}\big) \leq f\big(n/R^{(n)}\big) R^{(n)},$$

since f is concave. Hence

$$\mathbf{E}\left[\sum_{z\in\mathbb{Z}^2} f\big(b_z^{(n)}\big)\right] \leq \mathbf{E}\big[f\big(n/R^{(n)}\big)R^{(n)}\big].$$

Note that $xf(n/x)$ is concave on $(0, \infty)$, since $f(x)$ is concave on $(0, \infty)$. For a smooth f we know that $(xf(n/x))'' = n^2 f''(n/x)/x^3$. To prove this in the general case it is sufficient to consider an approximation of f by twice differentiable concave functions.

Consequently,

$$\mathbf{E}\left[f\left(\frac{n}{R^{(n)}}\right)R^{(n)}\right] \leq E\big[R^{(n)}\big] f\left(\frac{n}{\mathbf{E}[R^{(n)}]}\right) \asymp f(\ln(n)) \frac{n}{\ln(n)}.$$

Then there exists $K > 0$ such that

$$\mathbf{E}\left(\sum_{z\in\mathbb{Z}^2} f\big(b_z^{(n)}\big)\right) \leq K f(\ln(n)) \frac{n}{\ln(n)}.$$

This completes the proof of the lemma. $\square$

The following lemma gives an estimate from the other side.

LEMMA 4. *Let f be a strictly increasing function on $[0,\infty)$ such that $f(0)=0$ and $f(Cx)\le Cf(x)$ for any $C>1$. Then for n large enough and for some positive ε we have*

$$\mathbf{E}\left[\sum_{z\in\mathbb{Z}^2} f\bigl(b_z^{(n)}\bigr)\right]\ge \varepsilon f(\ln(n))\frac{n}{\ln(n)}.$$

PROOF. Note that for any $\varepsilon_1>0$ there exists $K>0$ such that for n large enough

$$\Pr\bigl[b_0^{(n)}\ge K\ln(n)\bigr]\ge 1-\varepsilon_1.$$

(This follows from [5], Theorem 1.) Let $n\ge 4$ and let $m=[n/2]$. Since $m>1$, Lemma 2 implies that

$$\Pr\left[R^{(m)}\ge q_1\frac{m}{\ln(m)}\right]\ge q_2.$$

Let $x_1^{(n)},\dots,x_s^{(n)}$ be different points visited by the random walk up to the moment n, enumerated in the order of visiting.

Let $\beta_i^n=b^{(n)}_{x_i^{(n)}}$.

Take ε_1 such that $\varepsilon_1\le 1/2$.

Note that, for any $0\le i\le q_1\frac{m}{\ln(m)}$,

$$\begin{aligned}\Pr\bigl[\beta_i^{(n)}\ge K\ln(m)\bigr]&\ge \Pr\bigl[\beta_i^{(n)}\ge K\ln(m),\,R^{(m)}\ge i\bigr]\\&=\Pr\bigl[\beta_i^{(n)}\ge K\ln(m)|R^{(m)}\ge i\bigr]\Pr\bigl[R^{(m)}\ge i\bigr]\\&\ge \Pr\bigl[\beta_i^{(n)}\ge K\ln(m)|R^{(m)}\ge i\bigr]q_2\\&\ge \Pr\bigl[\beta_0^{(n)}\ge K\ln(m)\bigr]q_2\ge(1-\varepsilon_1)q_2\ge\frac{q_2}{2}.\end{aligned}$$

Hence for any n large enough,

$$\mathbf{E}\left[\sum_{z\in\mathbb{Z}^2} f\bigl(b_z^{(n)}\bigr)\right]\ge\frac{q_2}{2}f\left(K\ln\left(\left[\frac{n}{2}\right]\right)\right)q_1\frac{n/2}{\ln(n/2)}\asymp f(\ln(n))\frac{n}{\ln(n)}.\qquad\square$$

COROLLARY 1. (i) *Let $L_{k,\alpha}(x)$ be the function defined in Lemma 1 ($0<\alpha\le 1$). Then*

$$\mathbf{E}\left[\sum_{z\in\mathbb{Z}^2} L_{k,\alpha}\bigl(b_z^{(n)}\bigr)\right]\asymp L_{k+1,\alpha}(n).$$

(ii) *Let* $f(x) = x^\alpha$ $(0 < \alpha \le 1)$. *Then*

$$\mathbf{E}\left[\sum_{z\in\mathbb{Z}^2} f\left(b_z^{(n)}\right)\right] \asymp n/\ln(n)^{(1-\alpha)}.$$

PROOF. This corollary follows from Lemmas 1, 3 and 4 since, for $n > N$,

$$\frac{n}{\ln(n)} L_{k,\alpha}(\ln(n)) \asymp \frac{n}{\ln(n)} \frac{\ln(n)}{(\ln(\ln\cdots\ln(n)\cdots))^\alpha} \asymp L_{k+1,\alpha}(n)$$

and

$$\frac{n}{\ln(n)} \ln(n)^{1-\alpha} = \frac{n}{\ln(n)^\alpha}. \qquad \square$$

4. Main result. First we recall the definition of the wreath product.

DEFINITION 1. Let C, D be groups and denote by $\sum_C D$ the space of functions $f : C \to D$ with finite support. The wreath product of C and D is a semidirect product of C and $\sum_C D$, where C acts on $\sum_C D$ by shifts: if $c \in C$, $f : C \to D$, $f \in \sum_C D$, then ${}^c f(x) = f(xc^{-1})$, $x \in C$. Let $C \wr D$ denote the wreath product.

Note that if C is infinite, then the group of finite D-valued configurations $\sum_C D$ is infinitely generated. However, the wreath product $C \wr D$ is finitely generated whenever C and D are finitely generated.

LEMMA 5. *Let* $a_1, a_2, \dots, a_k$ *generate* A *and let* μ *be the measure equidistributed on these generators and their inverses.*

Then for some finite symmetric measure ν *on* $B = \mathbb{Z}^2 \wr A$ *the drift of the corresponding simple random walk satisfies*

$$L^{B,\nu}(n) \asymp \mathbf{E} \sum_{z\in\mathbb{Z}^2} L^{A,\mu}\left(b_z^{(n)}\right).$$

PROOF. The proof of this lemma is similar to that of Lemma 3 in [6].

For any $a \in A$, $\tilde{a}^e$ denotes the function from $\mathbb{Z}^2$ to A such that $\tilde{a}^e(0) = a$ and $\tilde{a}^e(x) = e$ for any $x \neq 0$. Let $a^e = (0, \tilde{a}^e)$. Let w_1', w_2' be the standard generators of $\mathbb{Z}^2$ and $w_1 = (w_1', e)$, $w_2 = (w_2', e)$.

Consider the following set of generators of B:

$$(a_j^e)^p w_s (a_n^e)^q,$$

$p, q = 0, 1$ or -1, $s = 1$ or 2 and $1 \le j, n \le k$.

Consider the simple random walk on B, corresponding to this set of generators.

Let μ_2 be the measure on A such that $\mu_2(e) = 1/2$ and $\mu_2(a_j) = \mu_2(a_j^{-1}) = 1/(4k)$ for any $1 \le j \le k$. The random walk on $B = \mathbb{Z}^2 \wr A$ is realized as follows: Let X_n, $n \geq 0$, be a simple symmetric random walk on $\mathbb{Z}^2$ and denote by $\{b_z^{(n)} : z \in \mathbb{Z}^2\}$ and $R^{(n)}$ its occupation process and its range, as introduced in the previous section.

Let $\xi_n^{(z)}$ be independent and identically distributed symmetric random walks on the group A, with steps distributed with respect to μ_2. The random walk on $B = \mathbb{Z}^2 \wr A$ is

$$U_n = (X_n, \phi_n)$$

with

$$\phi_n(z) = \xi^{(z)}_{2b_z^{(n)} - \delta_{(z,0)} - \delta_{(z,X_n)}}.$$

Let $c_z^{(n)} = l_A(\phi_n(z))$.

Then

$$\tfrac{1}{2} \sum_{z \in \mathbb{Z}^2} c_z^{(n)} \leq l_B(U_n) \leq 2\left(\sum_{z \in Z^2} c_z^{(n)} + R \right).$$

Hence

$$\tfrac{1}{2}\mathbf{E}\left[\sum_{z \in \mathbb{Z}^2} c_z^{(n)} \right] = \tfrac{1}{2} \sum_{z \in \mathbb{Z}^2} \mathbf{E}[c_z^{(n)}] \leq \mathbf{E}[l_B(U_n)] \leq 2\left(\sum_{z \in \mathbb{Z}^2} \mathbf{E}[c_z^{(n)}] + \mathbf{E}[R] \right).$$

It is clear that

$$L^{A,\mu_2}(n) \asymp L^{A,\mu}(n).$$

Note that

$$\mathbf{E}[c_z^{(n)} | X_k,\ k = 0, 1, 2, \dots, n] = L^{A,\mu_2}(2b_z^{(n)} - \delta_{(z,0)} - \delta_{(z,X_n)}).$$

Hence

$$\begin{aligned} &\mathbf{E}[\min(L^{A,\mu_2}(2b), L^{A,\mu_2}(2b-1), L^{A,\mu_2}(2b-2))] \\ &\quad \leq \mathbf{E}[c_z^{(n)} | b_z^{(n)} = b] \\ &\quad \leq \mathbf{E}[\max(L^{A,\mu_2}(2b), L^{A,\mu_2}(2b-1), L^{A,\mu_2}(2b-2))]. \end{aligned}$$

There exist $C_2, C_3 > 0$ such that

$$C_2 L^{A,\mu}(n) \leq L^{A,\mu_2}(2n-2), L^{A,\mu_2}(2n-1), L^{A,\mu_2}(2n) \leq C_3 L^{A,\mu}(n).$$

Hence

$$C_2 \mathbf{E}\left[\sum_{z \in Z^2} L^{A,\mu}(b_z^{(n)}) \right] \leq \sum_{z \in \mathbb{Z}^2} \mathbf{E}[c_z^{(n)}] \leq C_3 \mathbf{E}\left[\sum_{z \in \mathbb{Z}^2} L^{A,\mu}(b_z^{(n)}) \right].$$

This completes the proof of the lemma. $\square$

THEOREM 1. (i) *Let F be a finite group. Consider the following groups that are defined recurrently*:

$$G_1 = \mathbb{Z}^2 \wr F; \qquad G_{i+1} = \mathbb{Z}^2 \wr G_i.$$

Then for some simple random walk on G_i and for any n large enough, *we have*

$$L^{G_i}(n) \asymp \frac{n}{\underbrace{\ln(\ln \cdots \ln n)\cdots)}_{k}}.$$

(ii) *Consider the following groups that are defined recurrently*:

$$F_1 = \mathbb{Z}; \qquad F_{i+1} = \mathbb{Z} \wr F_i;$$

let

$$H_{1,i} = \mathbb{Z}^2 \wr F_i, \qquad H_{j+1,i} = \mathbb{Z}^2 \wr H_{j,i}.$$

Then for some simple random walk on $H_{j,i}$ and for any n large enough, *we have*

$$L_{H_{j,i}}(n) \asymp \frac{n}{\underbrace{\sqrt[2^i]{\ln(\ln(\cdots \ln}}_{j}(n)\cdots))}.$$

PROOF. (i) We prove the theorem by induction on i. Base $i = 1$. In this case $G_i = \mathbb{Z}^2 \wr F$ and $L(n)$ is asymptotically equal to $n/\ln(n)$ [3]. The induction step follows from the previous lemma and Corollary 1.

(ii) We prove the statement by induction on j. For $H_{1,i} = F_i$ the asymptotics of the drift is found in [6]. It is proven there that, for some random walk on F_i,

$$L_{F_i}(n) \asymp n^{1-1/2^k}.$$

The induction step follows from the previous lemma and Corollary 1. □

5. Estimates of the entropy. In this section we give estimates for the entropy of a random walk. It is known (see [1]) that for a wide class of measures on nilpotent groups $H(n) \asymp \ln(n)$. As mentioned before, $H(n)$ is asymptotically linear for any nonamenable group. In this section we study intermediate examples.

Let $v(n)$ be the growth function of the group [i.e., $v(n) = \#\{g \in G : l(g) \le n\}$]. Let

$$l = \lim_{n\to\infty} L(n)/n,$$

$$h = \lim_{n\to\infty} H(n)/n,$$

$$v = \lim_{n\to\infty} \ln(v(n))/n.$$

(It is a well-known fact that these limits exist, since $L(n)$, $H(n)$ and $v(n)$ are subadditive. See, e.g., [15].) It is known (see [9, 15]) that

$$h \le vl.$$

The following lemma generalizes this fact.

LEMMA 6. *For any* $\varepsilon > 0$ *there exists* $C > 0$ *such that*

$$H(n) \le (v+\varepsilon)L(n) + \ln(n) + C.$$

PROOF. Let $a_i^{(n)} = \Pr_{\mu^{*n}}[l(g) = i]$. Then by definition

$$L(n) = \sum_{i=0}^{n} i a_i^{(n)}.$$

Comparing μ^{*n} with the measure which is equidistributed on every sphere in the group we get

$$\begin{aligned} H(n) &\le \sum_{i=1}^{n} a_i^{(n)} \ln\bigl(v(i)/a_i^{(n)}\bigr) \\ &= \sum_{i=1}^{n} a_i^{(n)} \ln(v(i)) + \sum_{i=1}^{n} a_i^{(n)}\bigl(-\ln\bigl(a_i^{(n)}\bigr)\bigr) \le \sum_{i=1}^{n} a_i^{(n)} \ln(v(i)) + \ln(n). \end{aligned}$$

For any $\varepsilon > 0$ there exists $K > 0$ such that $v(i) \le K(v+\varepsilon)^i$. Hence

$$\begin{aligned} H(n) &\le \sum_{i=1}^{n} a_i^{(n)}\bigl(i(v+\varepsilon) + \ln(K)\bigr) + \ln(n) \\ &= (v+\varepsilon)\sum_{i=1}^{n} a_i^{(n)} i + \ln(K) + \ln(n) \\ &= (v+\varepsilon)L(n) + \ln(K) + \ln(n). \qquad \square \end{aligned}$$

Another lemma estimates the entropy from the other side:

LEMMA 7. (i) *There exists* $C > 0$ *such that*

$$H(n) \ge C\mathbf{E}_{\mu^{*n}} l^2(g)/n - \ln(n) \ge CL^2(n)/n - \ln(n).$$

(ii) *There exists* $K > 0$ *such that*

$$L(n) \le K\sqrt{n\bigl(\ln(v(n)) + \ln(n)\bigr)}.$$

PROOF. (i) Let $p_n(x)$ be the probability of hitting x after n steps. In [14] it is shown that there exist $K_1, K_2 > 0$ such that, for any x and n,

$$p_n(x) \le K_1 n^{3/4} \exp\bigl(-K_2 l(x)^2/n\bigr) \le K_1 n \exp\bigl(-K_2 l(x)^2/n\bigr).$$

Then note that

$$-\sum_{g\in G\,:\,l(g)=i} \mu^{*n}(g)\ln\bigl(\mu^{*n}(g)\bigr) \ge \bigl(-\ln(K_1) - \ln(n) + K_2 i^2/n\bigr)a_i^{(n)}.$$

Hence

$$\begin{aligned} H(n) &\ge -\ln(K_1) - \ln(n) + K_2 \sum_{i=0}^{n} i^2/n a_i^{(n)} \\ &= -\ln(K_1) - \ln(n) + K_2/n\mathbf{E}_{\mu^{*n}} l^2(g) \ge C/n\mathbf{E}_{\mu^{*n}} l^2(g) - \ln(n). \end{aligned}$$

The last inequality follows from the fact that, for some $C_2 > 0$, $H(n) \ge C_2$.

(ii) This follows from the first part of the lemma, since $H(n) \le \ln(v(n))$. □

As a corollary from the two previous lemmas we get the following theorem.

THEOREM 2. *Let G_i be the groups defined in Theorem* 1. *Then for some random walk on G_i, we have*

$$K_1 n/\underbrace{\ln\bigl(\ln\cdots\ln}_{i}(n)\cdots\bigr)^2 \le H_{G_i}(n) \le K_2 n/\underbrace{\ln\bigl(\ln\cdots\ln}_{i}(n)\cdots\bigr)$$

for some positive constants K_1 and K_2. In particular, all G_i have different asymptotics of the entropy.

Moreover, using arguments from [8] one can show that $H_{G_i}(n)$ in Theorem 2 is asymptotically equivalent to the right-hand term of the second inequality, that is, for some $K_3 > 0$,

$$H_{G_i}(n) \ge K_3 n/\underbrace{\ln\bigl(\ln\cdots\ln}_{i}(n)\ \cdots\bigr).$$

APPENDIX

PROOF OF THE AUXILIARY LEMMA. In this Appendix we give the proof of Lemma 1.

(i) First we consider the case $\alpha = 1$. Let $m_k(x) = (\underbrace{\ln(\ln\cdots\ln(x)\cdots)}_{k})$. We want to prove that $x/m_k(x)$ is concave on $[T_k, \infty)$. Note that

$$\left(\frac{x}{m_k}\right)''(x) = \frac{-xm_k''(x)m_k^2(x) - 2m_k'(x)m_k^2(x) + 2xm_k(x)(m_k'(x))^2}{m_k^4(x)}.$$

Since $m_k(x) > 0$ on $[T_k, \infty)$, it suffices to prove that

$$2x\big(m_k'(x)\big)^2 - x m_k(x) m_k''(x) \le 2 m_k'(x) m_k(x).$$

To prove this we will show that

$$2x\big(m_k'(x)\big)^2 < 1/2 m_k(x) m_k'(x)$$

and that

$$m_k(x)\big(-m_k''(x)\big)x \le \tfrac{3}{2} m_k'(x) m_k(x).$$

Note that

$$m_k'(x) = \frac{1}{x \underbrace{\ln(x)\ln(\ln(x))\cdots\ln(\ln(\cdots\ln(x)\cdots))}_{k-1}}.$$

We see that $0 < m_k'(x) < 1/x$ and we know that $m_k(x) > 4$, $x \in [T_k, \infty)$. This proves the first inequality. Now let

$$r_k(x) = \frac{1}{m_k'(x)} = x \underbrace{\ln(x)\ln(\ln(x))\cdots\ln\big(\ln(\cdots\ln(x)\cdots)\big)}_{k-1}.$$

Note that

$$r_k'(x) = m_1(x) m_2(x) \cdots m_{k-1}(x) + \sum_{i=1}^{k-1} r_k(x) \frac{m_i'(x)}{m_i(x)}.$$

Note that for $1 \le i \le k$ it holds that $m_i'(x) \le 1/x$. Since $m_k(x) \ge 2k$ and, for $1 \le i \le k-1$, $m_i(x) \ge 2k$, we get

$$r_k'(x) x \le 1.5 r_k(x).$$

This implies that

$$\big(-m_k''(x)\big)x = \frac{x r_k'(x)}{r_k^2(x)} \le \frac{3}{2 r_k(x)} = \frac{3}{2} m_k'.$$

So we have proven the second inequality.

Now consider $0 < \alpha \le 1$. We have already proven that

$$h_k(x) = \frac{x}{\underbrace{(\ln(\ln\cdots\ln(x)\cdots))}_{k}}$$

is concave on $[T_k, \infty)$.

Let

$$g_k(x) = x h_k\left(\frac{1}{x}\right).$$

Since $g_k''(x) = x^{-3}h_k''(1/x)$ we get that $g_k(x)$ is concave on $(0, 1/T_k]$.

Note that if f is an increasing function and both f and g are concave, then $f(g)$ is concave. In fact, in this case, for any $0 \le t \le 1$, it holds that

$$f\big(g(tx + (1-t)y)\big) \le f\big(tg(x) + (1-t)g(y)\big) \le tf(g(x)) + (1-t)f(g(y)).$$

Note that $f(x) = x^\alpha$ is concave and increasing on $[0, \infty)$ for $0 < \alpha \le 1$ and that $g_k(x)$ is positive on $(0, 1/T_k]$. Therefore $f(g_k(x))$ is concave on $(0, 1/T_k]$. But this implies that

$$\frac{x}{\underbrace{(\ln(\ln \cdots \ln(x)\cdots))}_{k}{}^{\alpha}} = xf\left(g_k\left(\frac{1}{x}\right)\right)$$

is concave on $[T_k, \infty)$.

(ii) Let $\beta_{k,\alpha} = \tilde{L}_{k,\alpha}(T_k)/(2T_k)$. Consider the function $y_{k,\alpha}(x) = \beta_{k,\alpha}x$. Note that $y_{k,\alpha}(T_k) < \tilde{L}_{k,\alpha}(T_k)$ and that there exists $N_{k,\alpha} > 0$ such that $y_{k,\alpha}(x) > \tilde{L}_{k,\alpha}(x)$ for any $x > N_{k,\alpha}$. Take maximal z such that $y_{k,\alpha}(z) = \tilde{L}_{k,\alpha}(z)$. Let $L_{k,\alpha}(x) = y_{k,\alpha}(x)$ if $0 \le x \le z$ and $L_{k,\alpha}(x) = \tilde{L}_{k,\alpha}(x)$ if $x \ge z$. Note that $y_{k,\alpha}(x)$ is concave for $x \in [0, z]$, that $\tilde{L}_{k,\alpha}(x)$ is concave if $x \ge z$ and that $\tilde{L}'_{k,\alpha}(z) \le y'_{k,\alpha}(z)$. This implies that $L_{k,\alpha}$ is concave. □

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