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# THE FIRST EXIT TIME OF A BROWNIAN MOTION FROM AN UNBOUNDED CONVEX DOMAIN[1]

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Consider the first exit time $\tau_D$ of a $(d+1)$-dimensional Brownian motion from an unbounded open domain $D = \{(x, y) \in \mathbb{R}^{d+1} : y > f(x), x \in \mathbb{R}^d\}$ starting at $(x_0, f(x_0) + 1) \in \mathbb{R}^{d+1}$ for some $x_0 \in \mathbb{R}^d$, where the function $f(x)$ on $\mathbb{R}^d$ is convex and $f(x) \to \infty$ as the Euclidean norm $|x| \to \infty$. Very general estimates for the asymptotics of $\log \mathbb{P}(\tau_D > t)$ are given by using Gaussian techniques. In particular, for $f(x) = \exp\{|x|^p\}$, $p > 0$,

$$\lim_{t\to\infty} t^{-1}(\log t)^{2/p} \log \mathbb{P}(\tau_D > t) = -j_\nu^2/2,$$

where $\nu = (d-2)/2$ and $j_\nu$ is the smallest positive zero of the Bessel function $J_\nu$.

## 1. Introduction.

Let $B(t) = (B_1(t), \dots, B_d(t)) \in \mathbb{R}^d$, $t \ge 0$, be a standard $d$-dimensional Brownian motion, where $B_i(t)$, $1 \le i \le d$, are independent Brownian motions starting at 0. Consider the first exit time $\tau_D$ of a $(d+1)$-dimensional Brownian motion from the unbounded open domain

$$D = \{(x, y) \in \mathbb{R}^{d+1} : y > f(x), x \in \mathbb{R}^d\} \tag{1.1}$$

starting at the point $(x_0, f(x_0) + 1) \in \mathbb{R}^{d+1}$ for some $x_0 \in \mathbb{R}^d$, where the function $f(x)$ on $\mathbb{R}^d$ is convex and $f(x) \to \infty$ as the Euclidean norm $|x| \to \infty$. That is, the exit time or the stopping time

$$\tau_D = \inf\{t : (x_0 + B(t), f(x_0) + 1 + W(t)) \notin D\}, \tag{1.2}$$

which plays a key role in the probabilistic solution to the Dirichlet problem. Here and throughout the paper, $W(t)$ is a standard one-dimensional Brownian motion starting at 0, independent of $B(t)$.

It is well known that, for a bounded smooth open (connected) domain $\widetilde{D}$,

$$\lim_{t\to\infty} t^{-1} \log \mathbb{P}(\tau_{\widetilde{D}} > t) = -\lambda_1(\widetilde{D}), \tag{1.3}$$

where $\lambda_1(\widetilde{D}) > 0$ is the principal eigenvalue of $-\Delta/2$ in $\widetilde{D}$ with Dirichlet boundary condition. Is it natural to consider what happens for an unbounded domain such as $D$ in (1.1)?

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When $f(x) = A \cdot |x|$, $A > 0$, the unbounded domain in (1.1), called $D(\theta)$, is the right circular cone of angle $\theta = \arccos(A/\sqrt{1+A^2})$. Burkholder (1977) proved that $\mathbb{E}(\tau_{D(\theta)}^p) < \infty$ if and only if $p < p(\theta, d)$, where $p(\theta, d)$ is a number defined in terms of the smallest 0 of a certain hypergeometric function. There is also a close connection between moments of the exit time and the least harmonic majorant of $|x|^p$ in the domain. In DeBlassie (1987, 1988), Burkholder's result, together with techniques from partial differential equations, is used to find an exact formula for $\mathbb{P}\{\tau_{D(\theta)} > t\}$ as an infinite series involving confluent hypergeometric functions. Using this formula, the exact asymptotics in $t$ for $\mathbb{P}\{\tau_{D(\theta)} > t\}$ follow and the result also holds for more general cones. We should mention here that in $\mathbb{R}^2$ formulas for $\mathbb{P}\{\tau_{D(\theta)} > t\}$ have existed for many years. Indeed, Spitzer (1958) derived an expression for $\mathbb{P}\{\tau_{D(\theta)} > t\}$ in his study of the winding of two-dimensional Brownian motion. Recently, a uniform treatment that covers all the above results was presented in Bañuelos and Smits (1997) for very general cones. In particular, for exit time $\tau_C$ from a cone $C$,

$$\mathbb{P}\{\tau_C > t\} \sim ct^{-\gamma_C} \qquad \text{as } t \to \infty, \tag{1.4}$$

where $c >$ is a known constant and $\gamma_C$ is determined by the first eigenvalue of the Dirichlet problem for the Laplace–Beltrami operator on a subset of the unit sphere determined by the cone $C$. The special geometric structure of the cone (scale invariance) is essential for these results.

When $f(x) = A \cdot |x|^2$, $A > 0$, the unbounded domain in (1.1), called $D_2$, is parabolic in shape. Denote the exit time from $D_2$ by $\tau_2$. Bañuelos, DeBlassie and Smits (2001) proved that, for $d = 1$, there are two positive constants $A_1$ and $A_2$ such that

$$-A_1 \le \liminf_{t\to\infty} t^{-1/3} \log \mathbb{P}(\tau_2 > t) \le \limsup_{t\to\infty} t^{-1/3} \log \mathbb{P}(\tau_2 > t) \le -A_2. \tag{1.5}$$

Their techniques, based on conformal transformation and elementary principles of large deviations, are completely different from those used in the study of cones. Indeed, they change the problem to the study of the exit time of a degenerate diffusion from an infinite strip on $\mathbb{R}^2$, with singular generator

$$L = \frac{1}{8}\frac{1}{u^2 + v^2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right). \tag{1.6}$$

Their bounds are derived in terms of infinite series involving certain Feynman–Kac functionals. It is their work that motivates the present study and we thank them for providing a copy of the paper and useful discussions.

In this paper, we provide general upper and lower estimates for the asymptotic tail distribution of the exit time $\tau_D$ on $\mathbb{R}^{d+1}$. Our techniques, separate for upper and lower bounds, are completely different from those used before. The following simple fact is the key step for our upper bound estimates. It is based on a powerful Gaussian technique, Slepian's inequality, after putting pieces into context.

PROPOSITION 1.1. *Let $f$ be a convex function on $\mathbb{R}^d$. Then*

$$\mathbb{P}(\tau_D > t) = \mathbb{P}\big(f\big(x_0 + B(s)\big) \le W(s) + f(x_0) + 1,\ 0 \le s \le t\big)$$
$$\le \mathbb{P}\big(f\big(x_0 + B(s)\big) \le \sqrt{s}\xi + f(x_0) + 1,\ 0 \le s \le t\big),$$

*where $\xi$ is a standard normal random variable independent of* $\{B(t) \in \mathbb{R}^d, t \ge 0\}$.

Slepian's inequality (lemma) and its variations provide a very useful tool in the theory of Gaussian processes and probability in Banach spaces. Very nice discussions with various applications can be found in Ledoux and Talagrand (1991) and Lifshits (1995). The simplest form of Slepian's lemma for centered Gaussian processes $X_t$ and $Y_t$ with index $t \in T$ states that if $\mathbb{E}X_t^2 = \mathbb{E}Y_t^2$ and $\mathbb{E}X_t X_s \le \mathbb{E}Y_t Y_s$ for any $t, s \in T$, then, for any $x$,

$$\mathbb{P}\Big(\sup_{t\in T} X_t \le x\Big) \le \mathbb{P}\Big(\sup_{t\in T} Y_t \le x\Big). \tag{1.7}$$

An interesting and useful extension of Slepian's inequality, including reversed direction under certain conditions, can be found in Li and Shao (2002b) with applications. In particular, a conjecture of Kesten (1992) on the random pursuit problem for Brownian particles is confirmed there, which can be viewed as the exit probability from a cone.

Returning to our exit time problem, we need further restrictions on $f$ and notation to state our main results. Assume the convex $f(x)$ is symmetric with respect to the set of all orthogonal transformations of $\mathbb{R}^d$. Then, as given in Rockafellar [(1970), page 110] $f(x) = h(|x|)$ and $h(x)$ is a nondecreasing lower semicontinuous convex function on $[0, \infty)$ with $h(0)$ finite. It is easy to see that the asymptotic rate of $\mathbb{P}(\tau_D > t)$ depends neither on the fixed initial starting point $(x_0, f(x_0) + 1) = (x_0, h(|x_0|) + 1) \in \mathbb{R}^{d+1}$ nor any bounded part of the domain, so we will assume for simplicity $x_0 = 0$ and the conditions on $h(x)$ are only for $x \ge K > 0$. Here and throughout this paper, we use the letter $K$ and its modifications, $K_1$, $K_\delta$, and so on, for various positive constants that may be different from one line to another, and we use $\log x$ for the natural logarithm. Also, let $j_\nu$ be the smallest positive zero of the Bessel function $J_\nu$, $\nu = (d-2)/2$ and $j_{-1/2} = \pi/2$.

Define

$$H(x) = \int_K^x \big(h^{-1}\big(\sqrt{s} + h(0) + 1\big)\big)^{-2}\, ds. \tag{1.8}$$

Assume that, for any $0 < \delta < 1$ and $0 < q = q(\delta) < 1$, there exists $u_0 = u_0(\delta, q)$ such that, for all $x \ge u_0$,

$$H(x) \ge \delta^{-1} H(x^q). \tag{1.9}$$

THEOREM 1.1. *If condition* (1.9) *holds for any* $0 < \delta < 1$ *and for some* $\delta_0 > 0$ *small*,

$$\inf_{x \ge u_0} \left( \frac{j_\nu^2}{2} \cdot \frac{tH(x)}{x} + \frac{x}{2t} \right) \ge t^{\delta_0} \tag{1.10}$$

*for* $t$ *large*, *then*

$$\log \mathbb{P}(\tau_D > t) \le -(1-\delta) \inf_{x \ge u_0} \left( \frac{j_\nu^2}{2} \cdot \frac{tH(x)}{x} + \frac{x}{2t} \right). \tag{1.11}$$

*In particular*, *for* $h^{-1}(x) = Ax^\alpha (\log x)^\beta$, $x > 1$,

$$\log \mathbb{P}(\tau_D > t) \le \begin{cases} -(1-\delta)^2 (1+\alpha)(2\alpha)^{-\alpha/(1+\alpha)} \\ \quad \times \left(2^{-1}(1+\alpha)\right)^{2\beta/(1+\alpha)} C^{1/(1+\alpha)} t^{(1-\alpha)/(1+\alpha)} (\log t)^{-2\beta/(1+\alpha)}, \\ \qquad\qquad\qquad\qquad\qquad \text{if } 0 < \alpha < 1,\ \beta \in \mathbb{R}, \\ -(1-\delta)^2 2^{-1} A^{-2} j_\nu^2 t (\log t)^{-2\beta}, \qquad \text{if } \alpha = 0,\ \beta > 0, \\ -K^{-1} (\log t)^{-\beta}, \qquad\qquad \text{if } \alpha = 1,\ \beta \le 0, \end{cases}$$

*for* $t$ *sufficiently large*, *where* $C = (1-\alpha)^{-1} 2^{2\beta - 1} A^{-2} j_\nu^2$.

To see that our upper estimates are reasonably sharp, we have the following general lower bound in the case $f = h(|x|)$. In fact, we conjecture that our lower bounds in Theorem 1.2 are sharp in general. To state them, let us first define a class of functions $\mathcal{G}$ with

$$\mathcal{G} = \{ g \in C^2[0, \infty) : g(0) < h(0) + 1,\ g' \ge 0,\ g'' \le 0 \\ \text{for } t \text{ large and } \lim_{t \to \infty} \sqrt{t} g'(t) = \infty \}. \tag{1.12}$$

THEOREM 1.2. *Assume that* $h' > 0$ *and* $h'' \ge 0$ *for* $x \ge K$. *For any* $\delta > 0$ *fixed and* $t > 0$ *sufficiently large*,

$$\log \mathbb{P}(\tau_D > t) \\ \ge -(1+\delta) \inf_{g \in \mathcal{G}} \left( \frac{j_\nu^2}{2} \int_K^t \frac{1}{(h^{-1}(g(s)))^2} \, ds + \frac{1}{2} \int_K^t (g'(s))^2 \, ds \right), \tag{1.13}$$

*where $K > 1$ is a constant. In particular, for $h^{-1}(x) = Ax^{\alpha}(\log x)^{\beta}$, $x > 1$,*

$$\log \mathbb{P}(\tau_D > t) \geq \begin{cases} -(1+\delta)^2 2^{-1}(1-\alpha)^{-1} \\ \quad \times \left(\alpha^{-\alpha}(1+\alpha)^{2\beta+2}A^{-2}j_\nu^2\right)^{1/(1+\alpha)} t^{(1-\alpha)/(1+\alpha)}(\log t)^{-2\beta/(1+\alpha)}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } 0 < \alpha < 1, \beta \in \mathbb{R}, \\ -(1+\delta)^2 2^{-1} A^{-2} j_\nu^2\, t(\log t)^{-2\beta}, \qquad \text{if } \alpha = 0, \beta > 0, \\ -(1+\delta)^2 2^{\beta-1}(1-\beta)^{-1}A^{-1} j_\nu (\log t)^{1-\beta}, \qquad \text{if } \alpha = 1, \beta \leq 0, \end{cases}$$

*for $t$ sufficiently large.*

Note that from Theorems 1.1 and 1.2 we have, for the constants in (1.5),

$$A_1 = 3^{1/3}(3/2)(j_\nu)^{4/3}, \qquad A_2 = (3/2)(j_\nu)^{4/3},$$

which are not too far apart. Furthermore, for $f(x) = |x|^{\gamma}$, $\gamma > 1$,

$$\begin{aligned} &\liminf_{t\to\infty} t^{-(\gamma-1)/(\gamma+1)} \log \mathbb{P}(\tau_D > t) \\ &\qquad \geq -2^{-1}(1+\gamma)^{2\gamma/(\gamma+1)}\gamma^{(2-\gamma)/(\gamma+1)}(\gamma-1)^{-1} j_\nu^{2\gamma/(\gamma+1)}, \\ &\limsup_{t\to\infty} t^{-(\gamma-1)/(\gamma+1)} \log \mathbb{P}(\tau_D > t) \\ &\qquad \leq -2^{-1}(1+\gamma)(\gamma-1)^{-\gamma/(\gamma+1)} j_\nu^{2\gamma/(\gamma+1)} \end{aligned} \tag{1.14}$$

and, for $f(x) = \exp\{|x|^p\}$, $p > 0$,

$$\lim_{t\to\infty} \frac{(\log t)^{2/p}}{t} \log \mathbb{P}(\tau_D > t) = \frac{-j_\nu^2}{2}. \tag{1.15}$$

Thus, as seen in (1.15), both our upper and lower estimates are sharp for fast exponential growing $f$. In the case of $f$ growing slightly faster than linear, say $f = |x|(\log|x|)^p$, $p > 0$, our upper bound does not provide the correct rate and more work needs to be done in this direction. It seems a challenging problem to find the limiting constant in (1.14), assuming it exists. Also, the case $f(x) = \langle x, Qx\rangle$ is of special interest where $Q$ is a positive-definite matrix.

It is also worth pointing out a general framework for this type of problem, which leads us to this work from the point of view of the theory of Gaussian processes. There are two types of probability estimates that have attracted a lot of attention recently for the general Gaussian process $X_t$, $t \in T$. One is lower tail probabilities which concern $\mathbb{P}(\sup_{t\in S}(X_t - X_{t_0}) \leq x)$ as $x \to 0$, with $t_0 \in S$ fixed; see Li and Shao (2002a) for details. The other is small ball (small deviation) probabilities; see (2.3) and the full discussion there. Both types of probability estimates can also be viewed as exit time problems if the Gaussian process has scaling property. In fact, our approach in this paper is based on this point of view together with related techniques.

Although our upper bound estimate is based on a purely Gaussian technique, namely, Slepian's inequality, it is possible to extend the approach to some non-Gaussian settings. For example, one could use the extension of Slepian's inequality for the commuting semigroups given in Dudley and Stroock (1987). The stable processes are also of great interest.

Finally, we want to point out the following connections with the heat equation. Let

$$v(x,t) = \mathbb{P}_x\{\tau_D \ge t\}, \qquad x \in \mathbb{R}^{d+1}.$$

Then $v$ solves

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{1}{2}\Delta v \qquad \text{in } D, \\ v(x,0) &= 1, \qquad x \in D. \end{aligned} \tag{1.16}$$

So our results can be viewed as the long-time behavior of $\log v(x,t)$. Furthermore, a closely related and useful technique in studying certain asymptotic problems is the logarithmic transformation $V = -\log v(x,t)$, which changes (1.16) into a nonlinear evolution equation for $V$. This can then be viewed as a stochastic control problem; see, for example, Fleming and Soner (1993). For other connections and the interplay between the exit probability and principal eigenvalues, see Chen and Li (2003) for discussions on a concrete and relatively simple example of a degenerate (nonuniform elliptic) generator.

The rest of the paper is organized as follows. In Section 2, we present several estimates on exit probabilities with moving boundary. They are necessary for the proofs of Theorems 1.1 and 1.2 and important in their own right. Detailed discussions on the connection with small ball estimates are given with examples. In Section 3, we give the proofs of Proposition 1.1 and Theorem 1.1, which require detailed asymptotic analysis. The proof of Theorem 1.2 is presented in Section 4.

## 2. Exit probabilities with moving boundary.

First, we mention the following useful results due to Novikov (1979) for one-dimensional Brownian motion. Consider the following stopping times:

$$\sigma_1 = \inf\{t \ge 0 : W(t) \le g_1(t)\}$$

and

$$\sigma_2 = \inf\{t \ge 0 : |W(t)| \ge g_2(t)\}.$$

LEMMA 2.1. *Let $g_1(t)$ be a continuous function, $g_1(0) < 0$, and let $K > 0$ be such that when $t \ge K$ the function $g_1''(t)$ is continuous and $g_1'(t) \ge 0$, $g_1''(t) \le 0$. Then, if $\sqrt{t}\, g_1'(t) \to \infty$ as $t \to \infty$,*

$$\log \mathbb{P}(\sigma_1 > t) = -\tfrac{1}{2}\int_K^t (g_1'(s))^2\, ds\big(1 + o(1)\big) \qquad \text{as } t \to \infty, \tag{2.1}$$

*where* $\mathbb{P}(\sigma_1 > t) = \mathbb{P}(W(s) \ge g_1(s), 0 \le s \le t)$.

*Let* $g_2(t)$ *be a continuous positive function and let* $K > 0$ *be such that for* $t \ge K$ *the function* $g_2''(t)$ *is continuous and either* $g_2'(t) \ge 0$, $g_2''(t) \le 0$, $g_2(t) \to \infty$, $\sqrt{t} g_2'(t) \to 0$ *as* $t \to \infty$ *or* $g_2'(t) \le 0$, $g_2''(t) \ge 0$, $g_2(t) \to 0$ *as* $t \to \infty$. *Then*

$$\log \mathbb{P}(\sigma_2 > t) = -\frac{\pi^2}{8} \int_K^t \frac{1}{g_2^2(s)}\, ds + \frac{1}{2} \log g_2(t)\bigl(1 + o(1)\bigr) \tag{2.2}$$

*as* $t \to \infty$, *where* $\mathbb{P}(\sigma_1 > t) = \mathbb{P}(|W(s)| \le g_2(s), 0 \le s \le t)$.

Indeed, sharper two-sided inequalities are also given in Novikov (1979) for $\mathbb{P}(\sigma_1 > t)$ and $\mathbb{P}(\sigma_2 > t)$. The arguments used combine the Girsanov transformation with Laplace transform techniques. Early work with various applications can be found in Mogulskii (1974), Lai (1977), Portnoy (1978) and Lai and Wijsman (1979). Some approximations related to these boundary crossing probabilities for fixed $t$ can be found in Novikov, Frishling and Kordzakhia (1999) along with references to other approaches and applications. Related results for processes with independent increments are given in Greenwood and Novikov (1986).

However, conditions for (2.2) can be significantly weakened for the upper bounds as given in Theorem 2.1 for the radial part of the $d$-dimensional Brownian motion by using piecewise approximation arguments. Similar arguments have been used in Shi (1996), Li (1999, 2001), Berthet and Shi (2000) and Chen, Kuelbs and Li (2000), and all deal with small ball probabilities for various processes under weighted norms. In general, the small ball probability (or small deviation) studies the behavior of

$$\log \mathbb{P}(\|X\| \le \varepsilon) \tag{2.3}$$

for a random process under various norms $\|\cdot\|$ as $\varepsilon \to 0$. In the last few years, there has been considerable progress on the small ball estimate for Gaussian processes. As was established in Kuelbs and Li (1993) [see also Li and Linde (1999) for further developments], the behavior of (2.3) for a Gaussian random element $X$ is determined up to a constant by the metric entropy of the unit ball of the reproducing kernel Hilbert space associated with $X$ and vice versa. For other connections and applications of small ball probabilities, we refer readers to a recent survey paper by Li and Shao (2001).

To see a connection between small ball probability under weighted sup-norm and exit probability with moving boundary, the following simple example is instructive. Recall that $B(t)$ denotes a $d$-dimensional Brownian motion starting from 0, and $|B(t)| = (\sum_{i=1}^d B_i^2(t))^{1/2}$ its radial part. It is well known now [see, e.g., Li (1999)] that, for $T > 0$ fixed,

$$\begin{aligned} &\log \mathbb{P}\Big(\sup_{0\le s\le T} |B(s)|/g(s) \le \varepsilon\Big) \\ &\quad = \log \mathbb{P}\big(|B(s)| \le \varepsilon g(s),\ 0 \le s \le T\big) \\ &\quad \sim -\frac{j_\nu^2}{2}\int_0^T \frac{1}{g^2(s)}\,ds\,\frac{1}{\varepsilon^2} \qquad \text{as } \varepsilon \to 0 \end{aligned} \tag{2.4}$$

under regularity conditions on the weight function $g$. On the other hand, consider the stopping time $\tau = \inf\{t > 0 : |B(t)| > t^\alpha\}$ with $\alpha < 1/2$. By using the Brownian scaling property,

$$\begin{aligned} \mathbb{P}(\tau > t) &= \mathbb{P}\big(|B(s)| \le s^\alpha, 0 \le s \le t\big) \\ &= \mathbb{P}\big(|B(st)| \le s^\alpha t^\alpha, 0 \le s \le 1\big) \\ &= \mathbb{P}\big(|B(s)| \le s^\alpha t^{\alpha-1/2}, 0 \le s \le 1\big) \\ &= \mathbb{P}\Big(\sup_{0\le s\le 1} \frac{|B(s)|}{s^\alpha} < t^{\alpha-1/2}\Big) \end{aligned}$$

and thus the small ball estimate (2.4) implies, as $t \to \infty$,

$$\log \mathbb{P}(\tau > t) \sim -2^{-1} j_\nu^2 (1-2\alpha)^{-1} t^{1-2\alpha}. \tag{2.5}$$

As can be seen, the simple argument above does not work for the general moving boundary $g$. However, the basic ideas used in the small ball estimates for weighted norms can be used and indeed that is what we do in Theorem 2.1. Note that the time interval is not fixed in contrast with (2.4). We present detailed upper bounds since we need a uniform estimate for the proof of Theorem 1.1. These results are not available in the literature for the Bessel process as far as we know, although Novikov's argument could be applied under much stronger smoothness conditions as can be seen in the proof of Theorem 2.2.

To get started, we need the following well-known exact distribution function given in Ciesielski and Taylor (1962):

$$\mathbb{P}\Big(\sup_{0\le s\le 1} |B(s)| \le x\Big) = \frac{2^{1-\nu}}{\Gamma(\nu+1)} \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^{1-\nu} J_{\nu+1}(j_{\nu,n})} \exp\Big\{-\frac{j_{\nu,n}^2}{2x^2}\Big\},$$

where $\nu = (d-2)/2$ and $0 < j_{\nu,1} < j_{\nu,2} < \cdots$ are the positive zeros of the Bessel function $J_\nu$ (and, of course, $J_{\nu+1}$ denotes the Bessel function of index $\nu+1$). This implies the small ball estimate

$$\mathbb{P}\Big(\sup_{0\le s\le 1} |B(s)| \le x\Big) \sim \frac{2^{1-\nu}}{\Gamma(\nu+1) j_\nu^{1-\nu} J_{\nu+1}(j_\nu)} \exp\Big\{-\frac{j_\nu^2}{2x^2}\Big\} \qquad \text{as } x \to 0,$$

where $j_\nu \equiv j_{\nu,1}$ is the smallest positive zero of the Bessel function $J_\nu$, $\nu = (d-2)/2$. Furthermore, we have, for any $x > 0$,

$$K^{-1}\exp\left\{-\frac{j_\nu^2}{2x^2}\right\} \le \mathbb{P}\left(\sup_{0\le s\le 1}|B(s)| \le x\right) \le K\exp\left\{-\frac{j_\nu^2}{2x^2}\right\} \tag{2.6}$$

for some universal constant $K > 0$.

Now we can state the following general upper bound estimates.

THEOREM 2.1. *For any positive integer $n$ and finite partition $P = \{t_i, 0 \le i \le n\}$ of $[0,t]$ such that*

$$0 = t_0 < t_1 < \cdots < t_n = t,$$

*it follows that*

$$\begin{aligned} &\mathbb{P}\big(|B(s)| \le g(s),\ 0 \le s \le t\big) \\ &\quad \le K^n \exp\left\{-\frac{j_\nu^2}{2}\sum_{i=1}^n (t_i - t_{i-1})\left(\sup_{t_{i-1}\le s\le t_i} g(s)\right)^{-2}\right\}, \end{aligned} \tag{2.7}$$

*where the absolute constant $K > 0$ is the one given in* (2.6). *In particular, for a nondecreasing function $g$ and the uniform partition $t_i = in^{-1}t$, $0 \le i \le n$, we have*

$$\mathbb{P}\big(|B(s)| \le g(s),\ 0 \le s \le t\big) \le K^n \exp\left\{-\frac{j_\nu^2}{2}\int_{t/n}^t g^{-2}(s)\,ds\right\}. \tag{2.8}$$

PROOF. Since $B(t) = (B_1(t), \ldots, B_d(t)) \in \mathbb{R}^d$ has independent increments, we have

$$\begin{aligned} &\mathbb{P}\big(|B(s)| \le g(s),\ 0 \le s \le t\big) \\ &\quad = \mathbb{P}\big(|B(s)| \le g(s),\ t_{i-1} \le s \le t_i,\ 1 \le i \le n\big) \\ &\quad = \mathbb{P}\left(\bigcap_{i=1}^{n-1} A_i,\ |B(s)| \le g(s),\ t_{n-1} \le s \le t_n\right) \\ &\quad = \mathbb{E}\Big(\mathbb{1}_{\{\bigcap_{i=1}^{n-1} A_i\}}\mathbb{P}\big(|B(s) - B(t_{n-1}) + x| \le g(s), t_{n-1} \le s \le t_n\big)\Big| \\ &\qquad\qquad B(t_{n-1}) = x\Big), \end{aligned} \tag{2.9}$$

where

$$A_i = \big\{|B(s)| \le g(s),\ t_{i-1} \le s \le t_i\big\}, \qquad 1 \le i \le n.$$

By Anderson's inequality for Gaussian measure and the scaling property of $B(t)$,

$$\begin{aligned}
&\mathbb{P}\big(|B(s) - B(t_{n-1}) + x| \le g(s),\ t_{n-1} \le s \le t_n\big)\\
&\quad \le \mathbb{P}\big(|B(s) - B(t_{n-1})| \le g(s),\ t_{n-1} \le s \le t_n\big)\\
&\quad \le \mathbb{P}\Big(|B(s)| \le \sup_{t_{n-1} \le s \le t_n} g(s),\ 0 \le s \le t_n - t_{n-1}\Big)\\
&\quad = \mathbb{P}\Big(\sup_{0 \le s \le 1} |B(s)| \le (t_n - t_{n-1})^{-1/2} \sup_{t_{n-1} \le s \le t_n} g(s)\Big).
\end{aligned}$$

Putting the above estimates into (2.9) and iterating, we obtain

$$\begin{aligned}
&\mathbb{P}\big(|B(s)| \le g(s),\ 0 \le s \le t\big)\\
&\quad \le \prod_{i=1}^{n} \mathbb{P}\Big(\sup_{0 \le s \le 1} |B(s)| \le (t_i - t_{i-1})^{-1/2} \sup_{t_{i-1} \le s \le t_i} g(s)\Big).
\end{aligned} \tag{2.10}$$

Next note that, by (2.6),

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{0 \le s \le 1} |B(s)| \le (t_i - t_{i-1})^{-1/2} \sup_{t_{i-1} \le s \le t_i} g(s)\Big)\\
&\quad \le K \exp\Big\{-\frac{j_\nu^2}{2}(t_i - t_{i-1})\Big(\sup_{t_{i-1} \le s \le t_i} g(s)\Big)^{-2}\Big\}
\end{aligned} \tag{2.11}$$

and we have the upper bound (2.7).

If $g(s)$ is nondecreasing and the partition $t_i = in^{-1}t$, $0 \le i \le n$, then

$$\sum_{i=1}^{n} \frac{t_i - t_{i-1}}{g^2(t_i)} \ge \sum_{i=1}^{n-1} \frac{t_{i+1} - t_i}{g^2(t_i)} \ge \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} g^{-2}(s)\, ds = \int_{t/n}^{t} g^{-2}(s)\, ds.$$

and (2.8) follows. □

Next, we turn to the lower bounds for the Bessel process. We use Novikov's argument this time due to its simplicity although a different one based on approximations can be given.

THEOREM 2.2. *Let $g(t)$ be a continuous strictly positive function such that $g''(t) \le 0$ is continuous and $g'(t) \ge 0$. Then*

$$\begin{aligned}
&\mathbb{P}\big(|B(s)| \le g(s),\ 0 \le s \le t\big)\\
&\quad \ge K^{-1} \exp\Big\{\frac{d}{2} \log \frac{g(t)}{g(0)} - \frac{1}{2}\int_0^t (g'(s))^2\, ds - \frac{j_\nu^2}{2}\int_0^t g^{-2}(s)\, ds\Big\}.
\end{aligned} \tag{2.12}$$

*In particular, under the additional condition $\sqrt{t}g'(t) \to 0$ as $t \to \infty$,*

$$\log \mathbb{P}\big(|B(s)| \le g(s),\ 0 \le s \le t\big) \ge -(1+\delta)\frac{j_\nu^2}{2}\int_0^t g^{-2}(s)\, ds \tag{2.13}$$

*for and* $\delta > 0$ *and* $t$ *large*.

PROOF. We introduce the transformed process

$$\Xi(t) = g(t) \int_0^t G(s)\, dB(s), \tag{2.14}$$

where $G(s)$ is a diagonal matrix with diagonal elements equal to $1/g(s)$. By Itô's formula, the stochastic differential of the process $\Xi(t)$

$$d\Xi(t) = \frac{f'(t)}{f(t)} \Xi(t)\, dt + dB(t). \tag{2.15}$$

Since the matrix of the Gaussian diffusion process $\Xi(t)$ is the identity matrix, and the drift is square integrable, we see from Liptser and Shiryaev (1977) that the measures of $\Xi(s)$ and $B(s)$ are equivalent on $[0, t]$. In particular, we have

$$\mathbb{P}\big(|\Xi(s)| \le g(s),\ 0 \le s \le t\big) = \mathbb{E} Z(t) I\{|B(s)| \le g(s), 0 \le s \le t\}, \tag{2.16}$$

where $I\{\cdot\}$ is the indicator function and

$$Z(t) = \exp\left\{ \int_0^t \frac{g'(s)}{g(s)} \langle B(s), dB(s)\rangle - \frac{1}{2} \int_0^t \left| \frac{g'(s)}{g(s)} B(s) \right|^2 ds \right\}. \tag{2.17}$$

Using representation (2.14) for $\Xi(t)$ on the left-hand side of (2.16) and making the time change $u = \int_0^t 1/g^2(s)\, ds$ in the components of the stochastic integral $\int_0^t G(s)\, dB(s)$, we obtain

$$\begin{aligned} \mathbb{P}\big(|\Xi(s)| \le g(s), 0 \le s \le t\big) &= \mathbb{P}\left( |B(s)| \le 1, 0 \le s \le \int_0^t \frac{1}{g^2(s)}\, ds \right) \\ &= \mathbb{P}\left( \sup_{0 \le s \le 1} |B(s)| \le \left( \int_0^t \frac{1}{g^2(s)}\, ds \right)^{-1/2} \right) \\ &\ge K^{-1} \exp\left\{ -\frac{j_\nu^2}{2} \int_0^t \frac{1}{g^2(s)}\, ds \right\} \end{aligned} \tag{2.18}$$

by Brownian scaling and (2.6). On the other hand, by Itô's formula and integration by parts,

$$\begin{aligned} &\int_0^t \frac{g'(s)}{g(s)} \langle B(s), dB(s) \rangle \\ &\quad = \frac{1}{2} \int_0^t \frac{g'(s)}{g(s)}\, d\big(|B(s)|^2 - (2\nu + 2)s\big) \\ &\quad = -(\nu + 1) \log \frac{g(t)}{g(0)} + \frac{1}{2} \frac{g'(t)}{g(t)} |B(t)|^2 - \frac{1}{2} \int_0^t |B(s)|^2\, d\left( \frac{g'(s)}{g(s)} \right), \end{aligned}$$

where $\nu = (d-2)/2$ as before in order to avoid confusion. Therefore, from the representation for $Z(t)$ in (2.17), it follows, on the set $\{|B(s)| \le g(s), 0 \le s \le t\}$,

$$
\begin{aligned}
Z(t) &= \exp\left\{-\frac{d}{2}\log\frac{g(t)}{g(0)} + \frac{1}{2}\frac{g'(t)}{g(t)}|B(t)|^2 - \frac{1}{2}\int_0^t |B(s)|^2\frac{g''(s)}{g(s)}\,ds\right\} \\
&\le \exp\left\{-\frac{d}{2}\log\frac{g(t)}{g(0)} + \frac{1}{2}g'(t)g(t) - \frac{1}{2}\int_0^t g''(s)g(s)\,ds\right\} \qquad (2.19)\\
&= \exp\left\{-\frac{d}{2}\log\frac{g(t)}{g(0)} + \frac{1}{2}g'(0)g(0) + \frac{1}{2}\int_0^t (g'(s))^2\,ds\right\}
\end{aligned}
$$

for $g' \ge 0$ and $g'' \le 0$. Thus, (2.12) follows from (2.16), (2.18) and (2.19). The estimate (2.13) is clear under the condition $\sqrt{t}g'(t) \to 0$ as $t \to \infty$ and we finish the proof. □

Combining Theorems 2.1 and 2.2 with appropriate modifications to the proofs, the following results follow easily and we omit the details.

COROLLARY 2.1. *Let $g(t)$ be a continuous strictly positive function such that $g''(t) \le 0$ is continuous, $g'(t) \ge 0$ and $\sqrt{t}g'(t) \to 0$ as $t \to \infty$. Then*

$$\log \mathbb{P}\big(|B(s)| \le g(s),\ 0 \le s \le t\big) \sim -\frac{j_\nu^2}{2}\int_0^t g^{-2}(s)\,ds \qquad \text{as } t \to \infty.$$

*In fact, if $g(t) = At^p(\log t)^q$ for $t \ge t_0 \ge 2$ and $g(t) = g(t_0) > 0$ for $0 \le t \le t_0$, then, as $t \to \infty$,*

$$
\log \mathbb{P}\big(|B(s)| \le g(s),\ 0 \le s \le t\big)
\sim \begin{cases}
-2^{-1}A^{-2}j_\nu^2(1-2p)^{-1}t^{1-2p}(\log t)^{-2q}, & \text{if } 0 < p < 1/2,\ q \in \mathbb{R},\\
-2^{-1}A^{-2}j_\nu^2 t(\log t)^{-2q}, & \text{if } p = 0,\ q \ge 0,\\
-2^{-1}A^{-2}j_\nu^2(1-2q)^{-1}(\log t)^{1-2q}, & \text{if } p = 1/2,\ q < 0.
\end{cases}
$$

We conjecture that the above corollary also holds for the noninteger-dimensional Bessel process. Finally, some related exit probabilities investigated in the literature can be found in Uchiyama (1980), Bass and Cranston (1983), Durbin (1985, 1992), Lerche (1986) and Anderson and Pitt (1997). Our approach can also be used to obtain related estimates.

**3. Upper bound estimates.** First, we need to prove Proposition 1.1. Let $\tau_D$ be defined as in (1.2). Then

$$
\begin{aligned}
\mathbb{P}(\tau_D > t) &= \mathbb{P}\big(f(x_0 + B(s)) < W(s) + f(x_0) + 1, 0 \le s \le t\big) \\
&= \mathbb{P}\Big(\sup_{0 \le s \le t} f(x_0 + B(s)) - W(s) < f(x_0) + 1\Big) \\
&\le \mathbb{P}\Big(\sup_{0 \le s \le t} f(x_0 + B(s)) - \sqrt{s}\xi < f(x_0) + 1\Big),
\end{aligned}
$$

where the inequality follows from Slepian's lemma by conditioning on $B(s), 0 \le s \le t$. To justify it, we simply note that $\mathrm{Var}(-W(s)) = \mathrm{Var}(-\sqrt{s}\xi)$ and

$$\mathbb{E}W(s)W(s') = \min(s, s') \le \sqrt{ss'} = \mathbb{E}(\sqrt{s}\xi\sqrt{s'}\xi).$$

Thus, Proposition 1.1 follows from Slepian's lemma; see (1.7) or Slepian (1961). Note that in most papers and books Slepian's lemma is proved and used for mean-zero Gaussian random vectors. Here we, in fact, use a form that the same mean depends on the index parameter. The standard proof can be modified to cover this case. A direct proof, together with other applications of Slepian's lemma, can be found in an elegant paper by Dudley and Stroock (1987).

Next, we turn to the proof of Theroem 1.1. It follows from Proposition 1.1 that

$$\begin{aligned}
\mathbb{P}(\tau_D > t) &= \mathbb{P}\big(h(|B(s)|) \le W(s) + h(0) + 1, 0 \le s \le t\big) \\
&\le \mathbb{P}\big(h(|B(s)|) \le \sqrt{s}\xi + h(0) + 1, 0 \le s \le t\big) \\
&= \mathbb{P}\big(h(|B(s)|) \le \sqrt{s}\xi + h(0) + 1, \xi \ge (u_0/t)^{1/2}, 0 \le s \le t\big) \\
&\quad + \mathbb{P}\big(h(|B(s)|) \le \sqrt{s}\xi + h(0) + 1, \xi < (u_0/t)^{1/2}, 0 \le s \le t\big) \\
&= I + II \qquad \text{(say)}.
\end{aligned}$$

Note that the second term $II$ above can be easily bounded by

$$\begin{aligned}
II &= \mathbb{P}\big(h(|B(s)|) \le \sqrt{s}\xi + h(0) + 1, \xi < (u_0/t)^{1/2}, 0 \le s \le t\big) \\
&\le \mathbb{P}\big(h(|B(s)|) \le u_0^{1/2} + h(0) + 1, \xi < (u_0/t)^{1/2}, 0 \le s \le t\big) \\
&= \mathbb{P}\big(\xi < (u_0/t)^{1/2}\big)\,\mathbb{P}\Big(\sup_{0 \le s \le t} |B(s)| \le h^{-1}\big(u_0^{1/2} + h(0) + 1\big)\Big) \\
&\le \exp\{-Kt\}
\end{aligned}$$

for $t$ large, where the last inequality follows from the scaling property of the Bessel process and (2.6).

It is harder to handle the first term $I$. We have

$$\begin{aligned}
I &= \mathbb{P}\big(h(|B(s)|) \le \sqrt{s}\xi + h(0) + 1, \xi \ge (u_0/t)^{1/2}, 0 \le s \le t\big) \\
&= \frac{1}{2\pi}\int_{(u_0/t)^{1/2}}^{\infty} \mathbb{P}\big(|B(s)| \le h^{-1}\big(\sqrt{s}u + h(0) + 1\big), 0 \le s \le t\big) \qquad (3.1) \\
&\quad \times \exp\left\{-\frac{u^2}{2}\right\} du.
\end{aligned}$$

Now, using the scaling property of the Bessel process and (2.6), we see that

$$\begin{aligned}
&\mathbb{P}\big(|B(s)| \le h^{-1}\big(\sqrt{s}u + h(0) + 1\big),\ 0 \le s \le t\big) \\
&\quad = \mathbb{P}\big(|B(s)| \le uh^{-1}\big(\sqrt{s} + h(0) + 1\big),\ 0 \le s \le u^2 t\big) \qquad (3.2) \\
&\quad \le \exp\{(u^2t)^{1-q}\log K\}\exp\{-2^{-1}j_v^2 u^{-2}\big(H(u^2t) - H\big((u^2t)^q\big)\big)\}
\end{aligned}$$

by choosing $n = (u^2 t)^{1-q}$ in the estimate (2.8).

For $u^2 t \ge u_0$, it is clear from (1.9),

$$H(u^2t) - H\big((u^2t)^q\big) \ge (1-\delta)H(u^2t). \tag{3.3}$$

Hence, by combining (3.1)–(3.3), we obtain, for $t$ large,

$$\begin{aligned}
I &\le \int_{(u_0/t)^{1/2}}^{\infty} \exp\{(u^2t)^{1-q}\log K\}\exp\{-(1-\delta)2^{-1}j_\nu^2 u^{-2}H(u^2t)\}\exp\left\{-\frac{u^2}{2}\right\}du \\
&\le \exp\left\{-(1-\delta)2^{-1}\inf_{u\ge(u_0/t)^{1/2}}\big(j_\nu^2u^{-2}H(u^2t)+u^2\big)\right\} \\
&\quad\times \int_{(u_0/t)^{1/2}}^{\infty}\exp\left\{(u^2t)^{1-q}\log K - \delta\frac{u^2}{2}\right\}du \\
&\le \exp\{K_\delta t^{(1-q)/q}\}\exp\left\{-(1-\delta)2^{-1}\inf_{x\ge u_0}\big(j_\nu^2 tH(x)/x + x/t\big)\right\},
\end{aligned}$$

where we used the estimate

$$\begin{aligned}
&\int_{(u_0/t)^{1/2}}^{\infty}\exp\left\{(u^2t)^{1-q}\log K - \delta\frac{u^2}{2}\right\}du \\
&\qquad\le \int_{(u_0/t)^{1/2}}^{Q}\exp\{(u^2t)^{1-q}\log K\}du + \int_Q^\infty \exp\left\{-\delta\frac{u^2}{4}\right\}du \\
&\qquad\le \exp\{K_\delta t^{(1-q)/q}\}
\end{aligned}$$

for some $K_\delta > 0$ and $t$ large, where $Q = (4\delta^{-1}\log K)^{1/(2q)}t^{(1-q)/(2q)}$. We thus finished the general estimate (1.11).

Next, we apply the estimate to the special case $h^{-1}(x) = Ax^\alpha(\log x)^\beta$ with $0 \le \alpha < 1$. It is easy to see from (1.8), as $x \to \infty$,

$$\begin{aligned}
H(x) &\sim \int_K^x A^{-2}s^{-\alpha}(\log\sqrt{s})^{-2\beta}\,ds \\
&\sim (1-\alpha)^{-1}2^{2\beta}A^{-2}x^{1-\alpha}(\log x)^{-2\beta}
\end{aligned}$$

and thus condition (1.9) is satisfied. Hence, in the case $0 < \alpha < 1$ and $\beta \in \mathbb{R}$, we have, for $t$ large,

$$\begin{aligned}
&\inf_{x\ge u_0}\left(\frac{j_\nu^2}{2}\frac{tH(x)}{x} + \frac{x}{2t}\right) \\
&\qquad\ge (1-\delta)(1+\alpha)(2\alpha)^{-\alpha/(1+\alpha)}\big(2^{-1}(1+\alpha)\big)^{2\beta/(1+\alpha)} \\
&\qquad\quad\times C^{1/(1+\alpha)}t^{(1-\alpha)/(1+\alpha)}(\log t)^{-2\beta/(1+\alpha)},
\end{aligned} \tag{3.4}$$

where $C = (1-\alpha)^{-1}2^{2\beta-1}A^{-2}j_\nu^2$. Note that the inf in (3.4) is attained around

$$x = (2\alpha C)^{1/(\alpha+1)}\big(2^{-1}(1+\alpha)\big)^{2\beta/(1+\alpha)}t^{2/(1+\alpha)}(\log t)^{-2\beta/(1+\alpha)}$$

for $t$ large.

In the case $\alpha = 0$ and $\beta > 0$, we have, for $t$ large,

$$\inf_{x \geq u_0} \left( \frac{j_\nu^2}{2} \frac{tH(x)}{x} + \frac{x}{2t} \right) \geq (1-\delta) 2^{-1} A^{-2} j_\nu^2 t (\log t)^{-2\beta}, \tag{3.5}$$

similarly to the estimate in (3.4).

Finally, in the case $\alpha = 1$ and $\beta \leq 0$, our condition (1.9) is not satisfied. But the basic idea of our argument works and we can obtain, for $t$ large,

$$\log \mathbb{P}(\tau_D > t) \leq -K^{-1} (\log t)^{-\beta} \tag{3.6}$$

for some constant $K > 0$. Since the correct rate for $\beta = 0$ is $\log t$, the above result is not sharp and we omit the details of its proof. Note that the rate $(\log t)^{-\beta}$ is the best we can do via the method we used. In other words, we lost the correct rate in the key estimate, Proposition 1.1, as can be seen from the case $\alpha = 1$ and $\beta = 0$, where the upper bound goes to a constant as $t \to \infty$.

**4. Proof of Theorem 1.2.** For any $g \in \mathcal{G}$, we have

$$\begin{aligned} \mathbb{P}(\tau_D > t) &= \mathbb{P}\big(h(|B(s)|) \leq W(s) + h(0) + 1, 0 \leq s \leq t\big) \\ &\geq \mathbb{P}\big(h(|B(s)|) \leq g(s) \leq W(s) + h(0) + 1, 0 \leq s \leq t\big) \\ &= \mathbb{P}\big(|B(s)| \leq h^{-1}(g(s)), 0 \leq s \leq t\big) \\ &\quad \times \mathbb{P}\big(W(s) \geq g(s) - h(0) - 1, 0 \leq s \leq t\big). \end{aligned}$$

Thus, for $t > 0$ sufficiently large, we have, from Theorem 2.2,

$$\log \mathbb{P}\big(|B(s)| \leq h^{-1}(g(s)), 0 \leq s \leq t\big) \geq -(1+\delta) \frac{j_\nu^2}{2} \int_K^t \frac{1}{(h^{-1}(g(s)))^2}\, ds. \tag{4.1}$$

For the second term, we have, from (2.1),

$$\log \mathbb{P}\big(W(s) \geq g(s) - h(0) - 1, 0 \leq s \leq t\big) \geq -\frac{1+\delta}{2} \int_K^t (g'(s))^2\, ds. \tag{4.2}$$

Combining the above estimates, we have (1.13).

For the remaining part of the proof, we need to find a "good" function $g$ corresponding to $h^{-1}(x) = Ax^\alpha (\log x)^\beta$. It is natural to require that integrals in (1.13) grow at about the same rate, that is, $g' \cdot h^{-1}(g) \approx 1$. This relationship forces

$$g(x) \approx x^{1/(1+\alpha)} (\log x)^{-\beta/(1+\alpha)}.$$

To determine the best constant $H > 0$ in front of the above function, let

$$g(x) = Hx^{1/(1+\alpha)} (\log x)^{-\beta/(1+\alpha)},$$

which is in $\mathcal{G}$. Note that the condition $\beta \leq 0$ fits naturally for the case $\alpha = 1$.

Next, we need some detailed estimates for the integrals in (1.13). In the case $0 \le \alpha < 1$, it is easy to see, as $t \to \infty$,

$$\begin{aligned}\frac{1}{2}\int_K^t (g'(s))^2\,ds &\sim \frac{H^2}{2(1+\alpha)^2}\int_K^t s^{-2\alpha/(1+\alpha)}(\log s)^{-2\beta/(1+\alpha)}\,ds \\ &\sim \frac{H^2}{2(1+\alpha)(1-\alpha)}t^{(1-\alpha)/(1+\alpha)}(\log t)^{-2\beta/(1+\alpha)}\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\frac{j_\nu^2}{2}\int_K^t \frac{1}{(h^{-1}(g(s)))^2}\,ds &= \frac{j_\nu^2}{2A^2}\int_K^t g^{-2\alpha}(s)\big(\log g(s)\big)^{-2\beta}\,ds \\ &\sim \frac{(1+\alpha)^{2\beta} j_\nu^2}{2H^{2\alpha}A^2}\int_K^t s^{-2\alpha/(1+\alpha)}(\log s)^{-2\beta/(1+\alpha)}\,ds \\ &\sim \frac{(1+\alpha)^{2\beta+1} j_\nu^2}{2(1-\alpha)H^{2\alpha}A^2}t^{(1-\alpha)/(1+\alpha)}(\log t)^{-2\beta/(1+\alpha)}.\end{aligned} \tag{4.4}$$

In the case $\alpha = 1$ and $\beta < 0$, we have, as $t \to \infty$,

$$\frac{1}{2}\int_K^t (g'(s))^2\,ds \sim \frac{H^2}{8}\int_K^t s^{-1}(\log s)^{-\beta}\,ds \sim \frac{H^2}{8(1-\beta)}(\log t)^{1-\beta} \tag{4.5}$$

and

$$\begin{aligned}\frac{j_\nu^2}{2}\int_K^t \frac{1}{(h^{-1}(g(s)))^2}\,ds &= \frac{j_\nu^2}{2A^2}\int_K^t g^{-2}(s)\big(\log g(s)\big)^{-2\beta}\,ds \\ &\sim \frac{2^{2\beta-1} j_\nu^2}{H^2A^2}\int_K^t s^{-1}(\log s)^{-\beta}\,ds \\ &\sim \frac{2^{2\beta-1} j_\nu^2}{(1-\beta)H^2A^2}(\log t)^{1-\beta}.\end{aligned} \tag{4.6}$$

Hence, in the case $0 < \alpha < 1$ and $\beta \in \mathbb{R}$, we have, by combining estimates (1.13), (4.3) and (4.4) for $t$ large,

$$\begin{aligned}&\log \mathbb{P}(\tau_D > t) \\ &\quad \ge -(1+\delta)^2 \inf_{H>0}\left(\frac{H^2}{2(1+\alpha)(1-\alpha)} + \frac{(1+\alpha)^{2\beta+1} j_\nu^2}{2(1-\alpha)H^{2\alpha}A^2}\right) \\ &\qquad \times t^{(1-\alpha)/(1+\alpha)}(\log t)^{-2\beta/(1+\alpha)} \\ &\quad = -(1+\delta)^2 2^{-1}(1-\alpha)^{-1}\big(\alpha^{-\alpha}(1+\alpha)^{2\beta+2}A^{-2}j_\nu^2\big)^{1/(1+\alpha)} \\ &\qquad \times t^{(1-\alpha)/(1+\alpha)}(\log t)^{-2\beta/(1+\alpha)},\end{aligned} \tag{4.7}$$

where we used the fact that $\inf_{x>0}(x + Cx^{-q}) = (1+q)q^{-1}(Cq)^{1/(1+q)}$ for $C > 0$ and $q > 0$.

In the case $\alpha = 0$ and $\beta > 0$, we have, by combining estimates (1.13), (4.3) and (4.4) as in (4.7) and by taking $H > 0$ sufficiently small,

$$\log \mathbb{P}(\tau_D > t) \geq -(1+\delta)^2 \frac{j_\nu^2}{2A^2} t (\log t)^{-2\beta} \tag{4.8}$$

for $t$ large.

Finally, in the case $\alpha = 1$ and $\beta \leq 0$, we see, for $t$ large,

$$\begin{aligned} \log \mathbb{P}(\tau_D > t) &\geq -(1+\delta)^2 \inf_{H>0} \left( \frac{H^2}{8(1-\beta)} + \frac{2^{2\beta-1} j_\nu^2}{(1-\beta)H^2A^2} \right) (\log t)^{1-\beta} \\ &= -(1+\delta)^2 \frac{2^{\beta-1} j_\nu}{(1-\beta)A} (\log t)^{1-\beta}. \end{aligned} \tag{4.9}$$

This completes the proof of the lower bound.

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