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NEW PERSPECTIVES ON RAY'S THEOREM FOR THE LOCAL TIMES OF DIFFUSIONS[1]

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A new global isomorphism theorem is obtained that expresses the local times of transient regular diffusions under $P^{x,y}$, in terms of related Gaussian processes. This theorem immediately gives an explicit description of the local times of diffusions in terms of 0th order squared Bessel processes similar to that of Eisenbaum and Ray's classical description in terms of certain randomized fourth order squared Bessel processes. The proofs given are very simple. They depend on a new version of Kac's lemma for h-transformed Markov processes and employ little more than standard linear algebra. The global isomorphism theorem leads to an elementary proof of the Markov property of the local times of diffusions and to other recent results about the local times of general strongly symmetric Markov processes. The new version of Kac's lemma gives simple, short proofs of Dynkin's isomorphism theorem and an unconditioned isomorphism theorem due to Eisenbaum.

1. Introduction. In a classical paper [8], published in 1963, Ray describes the total local time $\{L^r_\infty; r \in I\}$ of a transient regular diffusion on some interval $I \subseteq \mathbb{R}$, which may be infinite, starting at $x \in I$ and conditioned to die at a fixed point $y \in I$. Ray's theorem has been the subject of many investigations, reformulations and new proofs. See, for example, the works of Williams [15], Sheppard [13], Biane and Yor [1] and Eisenbaum [2]. In some of these works the description of $\{L^r_\infty; r \in I\}$ looks quite different from the one given by Ray.

In all of these papers the law of $\{L^r_\infty; r \in I\}$ is described piecewise, in three separate regions: $r \le x$, $x \le r \le y$ and $r \ge y$, conditioned to agree at the endpoints. In Theorem 1.1 we present a single global description of $\{L^r_\infty; r \in I\}$. It is inspired by our work on the Dynkin isomorphism theorem and our generalization of the second Ray–Knight theorem [4], which was done with Eisenbaum, Kaspi and Shi. However, our proofs do not depend on this work. They are elementary, using little beyond basic linear algebra. One of the most remarkable consequences of Ray's theorem is that the local time process $\{L^r_\infty; r \in I\}$ is Markovian in r. The proof of this result is particularly simple in our formulation.

Let X be a transient regular diffusion. It is known that we can always find a measure m on the state space I, called the speed measure, so that the 0-potential density of X with respect to m is symmetric. We denote this symmetric 0-potential density by $v(r,s)$. Throughout this paper we normalize the

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local times with respect to the speed measure, so that

$$E^r(L^s_\infty) = v(r,s). \tag{1.1}$$

It is easy to show, using symmetry, that $v(r,s)$ is positive definite (see, e.g., [6], Theorem 3.3). Therefore, we can find a mean zero Gaussian process $G = \{G_r, r \in I\}$ with

$$E(G_r G_s) = v(r,s). \tag{1.2}$$

Let $G_{r,z}$ denote the projection of G_r on the orthogonal complement of G_z, that is,

$$G_{r,z} = G_r - \frac{E(G_r G_z)}{E(G_z G_z)} G_z. \tag{1.3}$$

For $h_y(r) = v(r,y)$, let $P^{x,y}$ denote the h_y-transform of P^x, where P^x is the law of X starting at x. Under $P^{x,y}$, the process X starts at x and is conditioned to die at y. (Sometimes $P^{x,y}$ is denoted by P^{x/h_y}.)

THEOREM 1.1. *Let $L^{\cdot}_\infty$ denote the total accumulated local time of X. Let $\bar{G}$ be an independent copy of G. Let $x \le y$. Then under $P^{x,y} \times P_{G,\bar{G}}$ or $P^{y,x} \times P_{G,\bar{G}}$,*

$$\begin{aligned}
&\left\{L^r_\infty + \left(\frac{G^2_{r,x}}{2} + \frac{\bar{G}^2_{r,x}}{2}\right)1_{\{r \le x\}} + \left(\frac{G^2_{r,y}}{2} + \frac{\bar{G}^2_{r,y}}{2}\right)1_{\{r \ge y\}} : r \in I\right\} \\
&\qquad \overset{law}{=} \left\{\frac{G^2_r}{2} + \frac{\bar{G}^2_r}{2} : r \in I\right\}
\end{aligned} \tag{1.4}$$

where $I \subset \mathbb{R}$.

Of all the references given above, Theorem 1.1 is most closely related to Eisenbaum's results in [2]. She gives an explicit description of the law of $\{L^r_\infty; r \in I\}$ separately, in the three regions: $r \le x$, $x \le r \le y$, and $r \ge y$. It is easy to derive a similar description from Theorem 1.1. Let $Z = \{Z_t(x); (x,t) \in \mathbb{R} \times \mathbb{R}^+\}$ be a zero-dimensional squared Bessel process starting at x. That is, $Z_t(x)$ is a measurable processes satisfying

$$Z_t(x) = x + 2\int_0^t \sqrt{Z_s(x)}\, dW_s \tag{1.5}$$

where W_t is a linear Brownian motion. Let $\bar{Z} = \{\bar{Z}_t(x); (x,t) \in \mathbb{R} \times \mathbb{R}^+\}$ be an independent copy of Z. In addition, let B_t be a standard two-dimensional Brownian motion independent of Z and $\bar{Z}$. Recall that we can write the 0-potential $v(r,s)$ of a diffusion as $v(r,s) = p(r)q(s)$, $r \le s$, for some increasing positive continuous function p and decreasing positive continuous function q. Set $\tau(r) = p(r)/q(r)$ and $\phi(r) = q(r)/p(r)$.

THEOREM 1.2. *Let* $\{L^r_\infty; r \in I\}$ *be as in Theorem* 1.1. *Then*

$$\{L^r_\infty; r \in I\} \stackrel{law}{=} \{\Psi_r; r \in I\} \tag{1.6}$$

where

$$\Psi_r = \tfrac{1}{2} q^2(r) |B_{\tau(r)}|^2, \qquad x \le r \le y, \tag{1.7}$$

$$\Psi_r = \tfrac{1}{2} q^2(r) Z_{\tau(r)-\tau(y)}(|B_{\tau(y)}|^2), \qquad r \ge y, \tag{1.8}$$

$$\Psi_r = \tfrac{1}{2} p^2(r) \bar{Z}_{\phi(r)-\phi(x)}(\phi^2(x)|B_{\tau(x)}|^2), \qquad r \le x. \tag{1.9}$$

Note that $|B_t|^2$ is a two-dimensional squared Bessel process starting at 0.
The next corollary follows easily from Theorem 1.2.

COROLLARY 1.1. $\{L^r_\infty; r \in I\}$ *is a Markov process under* $P^{x,y}$.

Using (1.6) we can easily find the generator of $\{L^r_\infty; r \in I\}$ in the three regions $r \le x$, $x \le r \le y$ and $r \ge y$, as in [2]. Recall that the δ-dimensional squared Bessel process Y_t has generator

$$2x \frac{d^2}{d^2x} + \delta \frac{d}{dx}.$$

It is then easy to check that the nonhomogeneous diffusion $f(t)Y_{g(t)}$ has generator

$$2xg'(t)f(t)\frac{d^2}{d^2x} + \left(\delta g'(t)f(t) + \frac{f'(t)}{f(t)}x\right)\frac{d}{dx}.$$

Using this we can check that $\{L^r_\infty; r \in I\}$ has generator

$$q^2(r)\tau'(r)x\frac{d^2}{d^2x} + \frac{2q'(r)}{q(r)}x\frac{d}{dx}, \qquad r \le x, \tag{1.10}$$

$$q^2(r)\tau'(r)x\frac{d^2}{d^2x} + \left(q^2(r)\tau'(r) + \frac{2q'(r)}{q(r)}x\right)\frac{d}{dx}, \qquad x \le r \le y, \tag{1.11}$$

$$-q^2(r)\tau'(r)x\frac{d^2}{d^2x} + \frac{2p'(r)}{p(r)}x\frac{d}{dx}, \qquad r \ge y. \tag{1.12}$$

We also give a relatively simple derivation of Ray's theorem using the ideas that go into the proof of Theorem 1.1.

THEOREM 1.3 (Ray's theorem). *Let* $L^{\cdot}_\infty$ *denote the total accumulated local time of* X. *Let* $G = \{G_r, r \in \mathbb{R}\}$ *be as defined in* (1.2) *and let* $\{G^{(i)}_r, r \in \mathbb{R}\}$,

$i = 1, \dots, 4$, be four independent copies of G. Let $J = \inf X$ and $S = \sup X$ and let $x < y$. Then under the measure $P^{x,y} \times P_{G^{(1)},G^{(2)},G^{(3)},G^{(4)}}$,

$$\{J, L^r_\infty; r \le x\} \overset{law}{=} \{J, \Lambda_r; r \le x\}, \tag{1.13}$$

$$\{L^r_\infty; x \le r \le y\} \overset{law}{=} \{(G^{(1)}_r)^2 + (G^{(2)}_r)^2/2; x \le r \le y\}, \tag{1.14}$$

$$\{S, L^r_\infty; r \ge y\} \overset{law}{=} \{S, \Gamma_r; r \ge y\}, \tag{1.15}$$

where

$$\Lambda_r = \tfrac{1}{2}\mathbb{1}_{\{r>J\}} \sum_{i=1}^4 (G^{(i)}_{r,J})^2, \tag{1.16}$$

$$\Gamma_r = \tfrac{1}{2}\mathbb{1}_{\{r<S\}} \sum_{i=1}^4 (G^{(i)}_{r,S})^2. \tag{1.17}$$

Here $G^{(i)}_{r,J}$ and $G^{(i)}_{r,S}$ are defined as in (1.3), with G replaced by $G^{(i)}$.

Theorem 1.1 is a simple consequence of a new version of Kac's lemma, applied to h-transformed processes, which we give in Lemma 2.2. Theorem 1.2 is a immediate consequence of Theorem 1.1. It is proved in Section 6, where we also provide a proof of Corollary 1.1. The proof of Theorem 1.3, which is given in Section 5, uses Theorem 1.1 and some interesting equalities for the moment generating functions of Gaussian Markov processes. These equalities are given in Section 4.

Lemma 2.2 can also be used to obtain isomorphisms for symmetric Markov processes which are not necessarily diffusions. In particular we note that an expression of the form of (1.4) is valid for any transient Markov process with symmetric Green's function. In this case we get that under $P^{x,y} \times P_{G,\bar{G}}$,

$$\left\{L^r_\infty + \frac{G^2_{r,y}}{2} + \frac{\bar{G}^2_{r,y}}{2} : r \in I\right\} \overset{\text{law}}{=} \left\{L^r_{T_y} + \frac{G^2_r}{2} + \frac{\bar{G}^2_r}{2} : r \in I\right\} \tag{1.18}$$

where T_y is the first hitting time of y. Obviously, when $x = y$, $L^r_{T_y} \equiv 0$ and (1.18) is more useful. In this case, using an elementary equality for the moment generating function of squares of Gaussian processes and recognizing that L^y_∞, the total accumulated local time at y is an exponential random variable with mean $v(y, y)$, (1.18) states that under $P^{y,y} \times P_G$,

$$\left\{L^r_\infty + \frac{G^2_{r,y}}{2} : r \in I\right\} \overset{\text{law}}{=} \left\{\frac{(G_{r,y} + (v(r,y))/(v(y,y))\sqrt{2L^y_\infty})^2}{2} : r \in I\right\}. \tag{1.19}$$

Equation (1.19) is equivalent to Theorem 1.2 in [4]. Using Lemma 2.2 we give a simple proof of this theorem in Section 9, and also justify all the other assertions in this paragraph.

Consider now the local time of standard Brownian motion starting at $x > 0$ and killed the first time it hits 0. [The 0-potential of this process is $v(r,s) = 2(r \wedge s)$, for $r, s > 0$.] Using Lemma 2.2 we easily obtain the following well-known isomorphism; see [10], Volume 2, Section 52.

THEOREM 1.4 (First Ray–Knight theorem). *Let $L^r_{T_0}$ denote the local time of standard Brownian motion starting at $x > 0$ and killed the first time it hits* 0. *Let $\{W_r, r \in \mathbb{R}^+\}$ and $\{\bar{W}_r, r \in \mathbb{R}^+\}$ be independent standard Brownian motions starting at* 0. *Then under $P^x \times P_{W,\bar{W}}$,*

$$\{L^r_{T_0} : r \in \mathbb{R}^+\} \overset{law}{=} \{H_r : r \in \mathbb{R}^+\} \tag{1.20}$$

where H_r is a second order squared Bessel process starting at 0, *between* 0 *and x, and then proceeds as a* 0*th order squared Bessel process from x. Equivalently,*

$$\bigl\{L^r_{T_0} + (W^2_{r-x} + \bar{W}^2_{r-x})\mathbb{1}_{\{r \ge x\}} : r \in \mathbb{R}^+\bigr\} \overset{law}{=} \{W^2_r + \bar{W}^2_r : r \in \mathbb{R}^+\}. \tag{1.21}$$

As an application of Theorem 1.1 we give an interesting modification of Theorem 1.4. We again consider standard Brownian motion starting at $x > 0$ and killed the first time it hits 0. But now we use the h-transform $h(r) = v(r, y)$ to condition this process to hit $y > x$ and die at y, so that the process never does hit 0. The total accumulated local time of this process satisfies the following isomorphism:

$$\begin{aligned} \Biggl\{L^r_\infty + \biggl(\Bigl(W_r - \frac{r}{x}W_x\Bigr)^2 + \Bigl(\bar{W}_r - \frac{r}{x}\bar{W}_x\Bigr)^2\biggr)\mathbb{1}_{\{r \le x\}} \\ + (W^2_{r-y} + \bar{W}^2_{r-y})\mathbb{1}_{\{r \ge y\}} : r \in \mathbb{R}^+\Biggr\} \\ \overset{\text{law}}{=} \{W^2_r + \bar{W}^2_r : r \in \mathbb{R}^+\} \end{aligned} \tag{1.22}$$

under $P^{x,y} \times P_{W,\bar{W}}$. Note the two independent Brownian bridges between 0 and x.

As we mentioned above, Corollary 1.1, which states that the total accumulated local time of a transient diffusion under $P^{x,y}$ is a Markov process in the spatial variable, is an immediate consequence of Theorem 1.2. In [8] and [13] this property is proved by computing the explicit conditional expectations which define the Markov property. In Theorem 8.1 we use Theorem 1.1 to simplify the computations in these papers. This gives us an alternate, more elementary, direct proof of Corollary 1.1, which has the advantage that it is self contained. The proof of Corollary 1.1 given in Section 6 uses results about Bessel processes. See [14] for a proof of Corollary 1.1 using excursion theory.

Our work is inspired by Dynkin's isomorphism theorem but in this paper we found it much simpler to work from Kac's lemma directly. In fact, using this approach we can give a simple short proof of Dynkin's theorem. The only proofs we know of in the literature, including our own, are long and difficult. We give this proof in the Appendix in Section A.1. Using the same ideas we also give a simple short proof of an "unconditioned" isomorphism theorem due to Eisenbaum and some results relating the two isomorphism theorems.

2. Variations of Kac's formula. Let $\mathbf{1}^t$ denote the transpose of the n-dimensional vector $(1 \dots 1)$. In what follows we use the notation $A^{(l)}$ for the matrix obtained by replacing the lth column of the $n \times n$ matrix A by $\mathbf{1}^t$. We use $\{Y\}_l$ to denote the lth element of the vector Y.

The next lemma is given in [4] for symmetric processes. Because it is used to prove Lemma 2.2, which is the main tool is many of our proofs, we include a sketch of its proof.

LEMMA 2.1. *Let X be a Markov process with finite 0-potential density $u(x, y)$. Assume that a local time L_t^y exists for each y, normalized so that $E^x(L_\infty^y) = u(x, y)$. Let Θ be the matrix with elements $\Theta_{i,j} = u(x_i, x_j)$, $i, j = 1, \dots, n$. Let Λ be the matrix with elements $\{\Lambda\}_{i,j} = \lambda_i \delta_{i,j}$. For all $\lambda_1, \dots, \lambda_n$ sufficiently small and $1 \le l \le n$,*

$$E^{x_l} \exp\left(\sum_{i=1}^n \lambda_i L_\infty^{x_i}\right) = \frac{\det((I - \Theta\Lambda)^{(l)})}{\det(I - \Theta\Lambda)}. \tag{2.1}$$

PROOF. By Kac's moment formula (see, e.g., [5], Section 3.6),

$$E^x\left(\prod_{i=1}^n L_\infty^{y_i}\right) = \sum_\pi u(x, y_{\pi(1)}) u(y_{\pi(1)}, y_{\pi(2)}) \cdots u(y_{\pi(n-1)}, y_{\pi(n)}) \tag{2.2}$$

where the sum goes over all permutations π of $\{1, \dots, n\}$. Hence

$$\begin{aligned} E^{x_l}&\left(\left(\sum_{i=1}^n \lambda_i L_\infty^{x_i}\right)^k\right) \\ &= k! \sum_{j_1, \dots, j_k = 1}^n u(x_l, x_{j_1}) \lambda_{j_1} u(x_{j_1}, x_{j_2}) \lambda_{j_2} u(x_{j_2}, x_{j_3}) \\ &\qquad \cdots u(x_{j_{k-2}}, x_{j_{k-1}}) \lambda_{j_{k-1}} u(x_{j_{k-1}}, x_{j_k}) \lambda_{j_k} \\ &= k! \sum_{j_k=1}^n \{(\Theta\Lambda)^k\}_{l, j_k} \\ &= k! \{(\Theta\Lambda)^k \mathbf{1}^t\}_l \end{aligned} \tag{2.3}$$

for all k.

It follows from this that

$$E^{x_l}\exp\left(\sum_{i=1}^{n}\lambda_i L_\infty^{x_i}\right)=\sum_{k=0}^{\infty}\{(\Theta\Lambda)^k\mathbf{1}^t\}_l=\{(I-\Theta\Lambda)^{-1}\mathbf{1}^t\}_l. \tag{2.4}$$

Consequently,

$$(I-\Theta\Lambda)Y=\mathbf{1}^t \tag{2.5}$$

where Y is an n-dimensional vector with components $E^{x_l}\exp(\sum_{i=1}^n\lambda_i L_\infty^{x_i})$, $l=1,\dots,n$. Equation (2.1) now follows from Cramér's theorem. (Note that (2.5) can also be found in [5], Section 6.) $\square$

With the goal of studying h-transforms of a Markov process with 0-potential density $v(x,y)$, suppose that the Markov process X in Lemma 2.1 has 0-potential density $u(x,y)$ of the form

$$u(x,y)=\frac{1}{h(x)}v(x,y)h(y) \tag{2.6}$$

where $h(x)\neq 0$, for all x. In the next lemma we modify the right-hand side of (2.1) so that it is written in terms of the matrix Σ, with elements $\Sigma_{i,j}=v(x_i,x_j)$, rather than in terms of Θ, which has elements $u(x_i,x_j)$.

LEMMA 2.2. *Let X be a Markov process with continuous* 0*-potential density $u(x,y)$ as given in* (2.6). *Assume that a local time L_t^y exists for each y, normalized so that $E^x(L_\infty^y)=u(x,y)$. Let Σ be the matrix with elements $\Sigma_{i,j}=v(x_i,x_j)$, $i,j=1,\dots,n$. Let Λ be the matrix with elements $\{\Lambda\}_{i,j}=\lambda_i\delta_{i,j}$. For all $\lambda_1,\dots,\lambda_n$ sufficiently small and $1\le l\le n$,*

$$E^{x_l}\exp\left(\sum_{i=1}^{n}\lambda_i L_\infty^{x_i}\right)=\frac{\det(I-\widehat{\Sigma}\Lambda)}{\det(I-\Sigma\Lambda)} \tag{2.7}$$

where

$$\widehat{\Sigma}_{j,k}=\left(\Sigma_{j,k}-\frac{h(x_j)\Sigma_{l,k}}{h(x_l)}\right),\qquad j,k=1,\dots,n. \tag{2.8}$$

Clearly $\widehat{\Sigma}$ depends on the starting point of the Markov process and on the function h. Rather than introduce additional cumbersome notation we leave this dependency unstated. In our applications of Lemma 2.2 it is always clear what these quantities are.

PROOF OF LEMMA 2.2. By (2.6), $\Theta=H^{-1}\Sigma H$, where $H_{j,k}=h(x_j)\delta_{j,k}$. Therefore $(I-\Theta\Lambda)=H^{-1}(I-\Sigma\Lambda)H$. It now follows from (2.5) that $H^{-1}(I-\Sigma\Lambda)HY=\mathbf{1}^t$ or, equivalently,

$$(I-\Sigma\Lambda)HY=\mathbf{h} \tag{2.9}$$

where $\mathbf{h} = (h(x_1), \dots, h(x_n))^t$. Consequently by Cramér's theorem,

$$h(x_l) E^{x_l} \exp\left(\sum_{i=1}^n \lambda_i L_\infty^{x_i}\right) = (HY)_l = \frac{\det((I - \Sigma\Lambda)^{(l,\mathbf{h})})}{\det((I - \Sigma\Lambda))} \tag{2.10}$$

where $(I - \Sigma\Lambda)^{(l,\mathbf{h})}$ is the matrix obtained by replacing the lth column of $(I - \Sigma\Lambda)$ by $\mathbf{h}$. Thus

$$E^{x_l} \exp\left(\sum_{i=1}^n \lambda_i L_\infty^{x_i}\right) = \frac{\det((I - \Sigma\Lambda)^{(l,\widehat{\mathbf{h}})})}{\det((I - \Sigma\Lambda))} \tag{2.11}$$

where $\widehat{\mathbf{h}} = \frac{1}{h(x_l)}\mathbf{h}$. Note that $\widehat{\mathbf{h}}_l = 1$.

Let B be the matrix obtained by subtracting the $h(x_j)/h(x_l)$ times the lth row of $(I - \Sigma\Lambda)^{(l,\widehat{\mathbf{h}})}$ from the jth row for each $j \neq l$. We see that

$$\begin{aligned} B_{l,l} &= 1, && \\ B_{j,l} &= 0, && j \neq l, \\ B_{j,k} &= (I - \widehat{\Sigma}\Lambda)_{j,k}, && j,k \neq l. \end{aligned} \tag{2.12}$$

Thus

$$\det\big((I - \Sigma\Lambda)^{(l,\widehat{\mathbf{h}})}\big) = \det B = \det(M_{l,l}) \tag{2.13}$$

where $M_{l,l}$ is the (l,l)th minor of $(I - \widehat{\Sigma}\Lambda)$. Since by (2.8), $\widehat{\Sigma}_{l,k} = 0$ for all k, we also have that $\det(I - \widehat{\Sigma}\Lambda) = \det(M_{l,l})$. Thus we get (2.7) from (2.11). $\square$

REMARK 2.1. Refer to Lemma 2.2 and note that $(\det(I - \Sigma\Lambda))^{-1/2} = E\exp(\sum_{i=1}^n \lambda_i G_{x_i}^2/2)$ (see, e.g., Lemma 4.1), where G is a mean zero Gaussian process with covariance $EG_{x_i}G_{x_j} = \sum_{i,j}$. Suppose that $\widehat{\Sigma}$ in (2.8) is symmetric and positive definite. Then there is a mean zero Gaussian process $\mathcal{G}$ with covariance $E\mathcal{G}_{x_i}\mathcal{G}_{x_j} = \widehat{\Sigma}_{i,j}$. Since finite joint distributions determine stochastic processes, we get the isomorphism

$$\left\{L_\infty^r + \frac{\mathcal{G}^2}{2} + \frac{\bar{\mathcal{G}}^2}{2} : r \in I\right\} \stackrel{\text{law}}{=} \left\{\frac{G_r^2}{2} + \frac{\bar{G}_r^2}{2} : r \in I\right\} \tag{2.14}$$

under $P^{x_l} \times P_{G,\bar{G}} \times P_{\mathcal{G},\bar{\mathcal{G}}}$ where $\bar{G}$ and $\bar{\mathcal{G}}$ are independent copies of G and $\mathcal{G}$. Thus we have reduced the question of obtaining isomorphism theorems of the form of (1.4) to checking whether $\widehat{\Sigma}$ is symmetric and positive definite.

In particular let $h(s) \stackrel{\text{def}}{=} v(s, y)$ and set $x_l = x$ and assume that v is symmetric. Then

$$\widehat{\Sigma}_{j,k} = v(x_j, x_k) - \frac{v(x, x_k)v(x_j, y)}{v(x, y)}. \tag{2.15}$$

When $x = y$ this is the covariance of $G_{r,y} = G_r - \frac{v(r,y)}{v(y,y)} G_y$ and we can write (2.7) as

$$E^{y,y} \exp\left(\sum_{i=1}^n \lambda_i L_\infty^{x_i}\right) = \frac{\det(I - \Sigma^{(y)}\Lambda)}{\det(I - \Sigma\Lambda)} \tag{2.16}$$

where $\Sigma^{(y)}$ denotes the covariance of $G_{r,y}$. This immediately gives the isomorphism that under $P^{y,y} \times P_{G,\bar{G}}$,

$$\left\{L_\infty^r + \frac{G_{r,y}^2}{2} + \frac{\bar{G}_{r,y}^2}{2} : r \in I\right\} \stackrel{\text{law}}{=} \left\{\frac{G_r^2}{2} + \frac{\bar{G}_r^2}{2} : r \in I\right\}. \tag{2.17}$$

(See the discussion immediately below for an explanation of the notation $E^{y,y}$ and $P^{y,y}$.)

3. Diffusions. Let X be a Markov process with 0-potential density $v(r,s)$. Let h be an excessive function for X. We use X^h to denote the h-transform of X, which is a Markov process with 0-potential densities v^h given by

$$v^h(r,s) = \frac{1}{h(r)} v(r,s) h(s). \tag{3.1}$$

In general, one uses $P^{x/h}$ to denote the probability of the process X^h starting at x and $E^{x/h}$ to denote expectation with respect to X^h starting at x.

In all the isomorphism theorems obtained in this paper we use the excessive function $h(s) := h_y(s) = v(s,y)$. X^{h_y} can be thought of as the Markov process X conditioned to die at y, (but not necessarily at the first time it hits y). In this case we use $P^{x,y}$ instead of $P^{x/h}$ and $E^{x,y}$ instead of $E^{x/h}$ for the process X^{h_y} starting at x.

In general, for a Markov process Y, with 0-potential density $u(r,s)$, $Y_{(y)}$, the process obtained by killing Y at T_y, the first time the process hits y, has 0-potential density

$$u_{(y)}(r,s) = u(r,s) - \frac{u(r,y)u(y,s)}{u(y,y)}. \tag{3.2}$$

See (A.4). Applying this to X^{h_y} we see that $(X^{h_y})_{(y)}$ has 0-potential density

$$\begin{aligned}(v^{h_y})_{(y)}(r,s) &= v^{h_y}(r,s) - \frac{v^{h_y}(r,y)v^{h_y}(y,s)}{v^{h_y}(y,y)} \\ &= \frac{1}{h_y(r)}\left(v(r,s) - \frac{v(r,y)v(y,s)}{v(y,y)}\right) h_y(s).\end{aligned} \tag{3.3}$$

Let

$$x_1 < x_2 < \cdots < x_l = x < x_{l+1} < \cdots < x_m = y < x_{m+1} < \cdots < x_n. \tag{3.4}$$

Let Σ be the matrix with elements $\Sigma_{i,j} = v(x_i, x_j)$, Σ^z the matrix with elements

$$\Sigma^z_{i,j} = \{v(x_i, x_j) - v(x_i, z)v(z, x_j)/v(z, z)\}\mathbb{1}_{\{x_i, x_j < z\}}, \tag{3.5}$$

Σ_z be the matrix with elements

$$\{\Sigma_z\}_{i,j} = \{v(x_i, x_j) - v(x_i, z)v(z, x_j)/v(z, z)\}\mathbb{1}_{\{x_i, x_j > z\}} \tag{3.6}$$

and

$$\Sigma^{(z)} = \Sigma^z + \Sigma_z. \tag{3.7}$$

In the rest of this section we take X to be a regular diffusion with 0-potential density $v(r, s)$. In this case there exist two positive continuous functions p and q, with p increasing and q decreasing such that

$$v(r, s) = p(r)q(s) \qquad \text{for } r \le s. \tag{3.8}$$

See, for example, [8], Section 1.

THEOREM 3.1. *Let X be a regular diffusion with* 0*-potential density $v(r, s)$. Let Λ be a matrix with elements $\{\Lambda\}_{i,j} = \lambda_i \delta_{i,j}$. For all $\lambda_1, \dots, \lambda_n$ sufficiently small*,

$$E^{x,y} \exp\left(\sum_{i=1}^n \lambda_i L^{x_i}_{T_y}\right) = \frac{\det(I - \Sigma^x \Lambda)}{\det(I - \Sigma^y \Lambda)} \tag{3.9}$$

and

$$E^{y,x} \exp\left(\sum_{i=1}^n \lambda_i L^{x_i}_{T_x}\right) = \frac{\det(I - \Sigma_y \Lambda)}{\det(I - \Sigma_x \Lambda)}. \tag{3.10}$$

PROOF. It suffices to prove (3.9) with $m = n$ since $L^{x_i}_{T_y} = 0$ for all $x_i > y$. We use Lemma 2.2 with $x_l = x$. Comparing (3.3) and (2.6) we see that in the case we are considering here the matrix Σ in Lemma 2.2 is now Σ^y, $h(z) = h_y(z) = v(z, y)$ and therefore

$$\begin{aligned} \{\widehat{\Sigma^y}\}_{j,k} &= \Sigma^y_{j,k} - \frac{\Sigma^y_{l,k} v(x_j, y)}{v(x_l, y)} \\ &= v(x_j, x_k) - \frac{v(x, x_k)v(x_j, y)}{v(x, y)}. \end{aligned} \tag{3.11}$$

We claim that

$$\begin{aligned} &\{\widehat{\Sigma^y}\}_{j,k} = \Sigma^x_{j,k}, && j, k \le l, \\ &\{\widehat{\Sigma^y}\}_{j,k} = 0, && j \le l < k, \\ &\{\widehat{\Sigma^y}\}_{j,k} = 0, && l < j \le k. \end{aligned} \tag{3.12}$$

This shows us that $\det(I - \widehat{\Sigma^y}\Lambda) = \det(I - \Sigma^x \Lambda)$, and (3.9) follows immediately from Lemma 2.2.

It is simple to prove the claim. We just use the decomposition $v(r,s) = p(r)q(s)$ for $r \le s$ to see that

$$x_j, x_k \le x \quad \Longrightarrow \quad \frac{v(x_j, y)}{v(x, y)} = \frac{v(x_j, x)}{v(x, x)} \tag{3.13}$$

whereas

$$x_j \le x < x_k \quad \text{or} \quad x < x_j \le x_k \quad \Longrightarrow \quad \frac{v(x, x_k)v(x_j, y)}{v(x, y)} = v(x_j, x_k). \tag{3.14}$$

The proof of (3.10) is similar. $\square$

THEOREM 3.2. *Let X to be a regular diffusion with 0-potential density $v(r,s)$. Let $L^{\cdot}_{\infty}$ denote the accumulated local time of X^{h_y}. Let Σ be the matrix with elements $\Sigma_{i,j} = v(x_i, x_j)$, $i, j = 1, \dots, n$, and $x < y$. For all $\lambda_1, \dots, \lambda_n$ suficiently small, we have*

$$E^{x,y} \exp\left(\sum_{i=1}^{n} \lambda_i L^{x_i}_{\infty}\right) = \frac{\det(I - \Sigma^x \Lambda)\det(I - \Sigma_y \Lambda)}{\det(I - \Sigma\Lambda)} \tag{3.15}$$

where Λ is a matrix with elements $\{\Lambda\}_{i,j} = \lambda_i \delta_{i,j}$.

PROOF. Using the Markov property and the additivity of local times we have that

$$\begin{aligned} &E^{x,y} \exp\left(\sum_{i=1}^{n} \lambda_i L^{x_i}_{\infty}\right) \\ &\qquad = E^{x,y} \exp\left(\sum_{i=1}^{n} \lambda_i L^{x_i}_{T_y}\right) E^{y,y} \exp\left(\sum_{i=1}^{n} \lambda_i L^{x_i}_{\infty}\right). \end{aligned} \tag{3.16}$$

It now follows from Theorem 3.1 and (2.16) that

$$E^{x,y} \exp\left(\sum_{i=1}^{n} \lambda_i L^{x_i}_{\infty}\right) = \frac{\det(I - \Sigma^x \Lambda)}{\det(I - \Sigma^y \Lambda)} \frac{\det(I - \Sigma^{(y)} \Lambda)}{\det(I - \Sigma\Lambda)}. \tag{3.17}$$

Using the simple facts that $\Sigma^{(y)} = \Sigma_y + \Sigma^y$ and $\Sigma_y \Sigma^y = 0$ which imply that $(I - \Sigma^{(y)}\Lambda) = (I - \Sigma_y \Lambda)(I - \Sigma^y \Lambda)$, we get (3.15). $\square$

PROOF OF THEOREM 1.1. It follows immediately from (3.15). $\square$

The next lemma is used in the proof of Theorem 1.3.

LEMMA 3.1. *Let X be a diffusion. For $t \le r_n < \cdots < r_1 \le x \le y$,*

$$E^{t,x} \exp\left(\sum_{i=1}^{n} \lambda_i L_\infty^{r_i}\right) = E^{t,y} \exp\left(\sum_{i=1}^{n} \lambda_i L_\infty^{r_i}\right). \tag{3.18}$$

PROOF. Refer to Lemma 2.2 with $x_l = t$. Since $\frac{h_x(x_i)}{h_x(t)} = \frac{h_y(x_i)}{h_y(t)}$ for all $t \le x_i \le x \le y$ we see that the total accumulated local times of X^{h_x} and X^{h_y} are the same. $\square$

4. Moment generating functions of Gaussian Markov processes. We begin with a general relationship about the moment generating function of squares of Gaussian processes which we provide for the convenience of the reader. A proof is given in [4].

LEMMA 4.1. *Let $\zeta = (\zeta_1, \ldots, \zeta_n)$ be a mean zero, n-dimensional Gaussian random variable with covariance matrix Σ. Assume that Σ is invertible. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be an n-dimensional vector and Λ an $n \times n$ diagonal matrix with λ_j as its jth diagonal entry. Let $u = (u_1, \ldots, u_n)$ be an n-dimensional vector. We can choose λ_i, $i = 1, \ldots, n$, sufficiently small so that $(\Sigma^{-1} - \Lambda)$ is invertible and*

$$\begin{aligned} &E \exp\left(\sum_{i=1}^{n} \lambda_i (\zeta_i + u_i)^2/2\right) \\ &\qquad = \frac{1}{(\det(I - \Sigma\Lambda))^{1/2}} \exp\left(\frac{u\Lambda u^t}{2} + \frac{u\Lambda\widetilde{\Sigma}\Lambda u^t}{2}\right) \end{aligned} \tag{4.1}$$

where $\widetilde{\Sigma} \overset{def}{=} (\Sigma^{-1} - \Lambda)^{-1}$ and $u = (u_1, \ldots, u_n)$.

REMARK 4.1. It follows from (4.1) that

$$E \exp\left(\sum_{i=1}^{n} v_i \zeta_i + \frac{\lambda_i \zeta_i^2}{2}\right) = \frac{1}{(\det(I - \Sigma\Lambda))^{1/2}} \exp\left(\frac{v\widetilde{\Sigma} v^t}{2}\right). \tag{4.2}$$

Note that $E \exp(\sum_{i=1}^{n} \lambda_i \zeta_i^2/2) = (\det(I - \Sigma\Lambda))^{-1/2}$.

We define a probabiltiy measure $\widetilde{P}$ on $\mathbb{R}^n$, in terms of its expectation operator $\widetilde{E}$, by

$$\widetilde{E}\big(g(\zeta_1, \ldots, \zeta_n)\big) = \frac{E(g(\zeta_1, \ldots, \zeta_n) \exp(\sum_{i=1}^{n} \lambda_i \zeta_i^2/2))}{E(\exp(\sum_{i=1}^{n} \lambda_i \zeta_i^2/2))} \tag{4.3}$$

for all measurable functions g on $\mathbb{R}^n$. Under $\widetilde{P}$, $\zeta = (\zeta_1, \ldots, \zeta_n)$ is a mean zero, n-dimensional Gaussian random variable with covariance matrix $\widetilde{\Sigma}$. This follows from (4.2) which shows that $\widetilde{E} \exp(i(v, \zeta)) = \exp(-(v\widetilde{\Sigma} v^t)/2)$.

The next lemma is used in the proof of Theorem A.2.

LEMMA 4.2. *Let (ζ_1,ζ_2) be an $\mathbb{R}^2$ valued Gaussian random variable with mean zero. Then for all $s\neq 0$,*

$$\frac{E(\zeta_1\exp(s\zeta_2))}{sE(\exp(s\zeta_2))}=E(\zeta_1\zeta_2). \tag{4.4}$$

PROOF. $t\zeta_1+s\zeta_2$ is a mean zero Gaussian random variable with variance $t^2E(\zeta_1^2)+2tsE(\zeta_1\zeta_2)+s^2E(\zeta_2^2)$. Take its moment generating function, evaluated at one to get

$$E\big(\exp(t\zeta_1+s\zeta_2)\big)=\exp\big(t^2E(\zeta_1^2)/2+tsE(\zeta_1\zeta_2)+s^2E(\zeta_2^2)/2\big). \tag{4.5}$$

Differentiating this with respect to t and then setting $t=0$ we get

$$E\big(\zeta_1\exp(s\zeta_2)\big)=sE(\zeta_1\zeta_2)\exp\big(s^2E(\zeta_2^2)/2\big) \tag{4.6}$$

which gives (4.4). $\square$

We have the following immediate corollary of Lemma 4.1.

COROLLARY 4.1. *Let $\eta=\{\eta_x; x\in S\}$ be a mean zero Gaussian process and f_x a real valued function on S. It follows from Lemma 4.1 that for $a^2+b^2=c^2+d^2$,*

$$\begin{aligned}&\big\{(\eta_x+f_xa)^2+(\widetilde{\eta}_x+f_xb)^2; x\in S\big\}\\&\quad\stackrel{law}{=}\big\{(\eta_x+f_xc)^2+(\widetilde{\eta}_x+f_xd)^2; x\in S\big\},\end{aligned} \tag{4.7}$$

where $\widetilde{\eta}$ is an independent copy of η.

Let $G=\{G_y, y\in\mathbb{R}\}$ be a mean zero Gaussian Markov process with covariance $E_GG_xG_y=v(x,y)$. In the remainder of this section we assume that G is such that

$$v(x,y)=p(x)q(y),\qquad x\le y, \tag{4.8}$$

where p is increasing and q is decreasing. As in Section 1, set

$$G_{y,z}=G_y-\frac{v(y,z)}{v(z,z)}G_z. \tag{4.9}$$

$G_{y,z}$ is the projection of G_y on the orthogonal complement of G_z.

LEMMA 4.3. *Let $t<s\le r_j\le\cdots\le r_2\le r_1$. Consider $\{G_{r_i,s}, 1\le i\le j\}$ and let Σ_s denote its covariance matrix and $\widetilde{\Sigma}_s:=(\Sigma_s^{-1}-\Lambda)^{-1}$. Similarly for $\{G_{r_i,t}, 1\le i\le j\}$. Let $\tau(s)=p(s)/q(s)$.Then*

$$\frac{\det(I-\Sigma_s\Lambda)}{\det(I-\Sigma_t\Lambda)}=\frac{1}{1-2K(q,\Lambda,\widetilde{\Sigma}_s)(\tau(s)-\tau(t))} \tag{4.10}$$

where $K(q,\Lambda,\widetilde{\Sigma}_s)=(1/2)(q\Lambda q^t+q\Lambda\widetilde{\Sigma}_s\Lambda q^t)$ *for* Λ, *as given in Lemma* 4.1, *with* $n=j$ *and* $q=(q(r_1),\dots,q(r_j))$.

Furthermore, consider $\{G_{r_i},1\le i\le j\}$ *and let* Σ *denote its covariance matrix. Then*

$$\tau(s)\det(I-\Sigma_t\Lambda)-\tau(t)\det(I-\Sigma_s\Lambda) = \big(\tau(s)-\tau(t)\big)\det(I-\Sigma\Lambda). \tag{4.11}$$

PROOF. For each $i\le j$ we can write

$$\begin{aligned} G_{r_i,t} &= G_{r_i,s}+\left(\frac{v(s,r_i)}{v(s,s)}G_s-\frac{v(t,r_i)}{v(t,t)}G_t\right)\\ &= G_{r_i,s}+q(r_i)\left(\frac{1}{q(s)}G_s-\frac{1}{q(t)}G_t\right)\\ &:= G_{r_i,s}+\rho_{r_i,s,t}. \end{aligned}\tag{4.12}$$

Using (4.8), we see that the two processes $\{G_{r_i,s},i=1,\dots,j\}$ and $\{\rho_{r_i,s,t},i=1,\dots,j\}$ are independent. Also, note that

$$\begin{aligned} E_G\left(\left(\frac{1}{q(s)}G_s-\frac{1}{q(t)}G_t\right)^2\right) &= \frac{v(s,s)}{q^2(s)}+\frac{v(t,t)}{q^2(t)}-2\frac{v(t,s)}{q(t)q(s)}\\ &= \tau(s)-\tau(t). \end{aligned}\tag{4.13}$$

It follows from (4.1) that

$$\begin{aligned} &\big(\det(I-\Sigma_t\Lambda)\big)^{-1/2}\\ &\quad = \big(\det(I-\Sigma_s\Lambda)\big)^{-1/2}E_G\exp\left(K(q,\Lambda,\widetilde{\Sigma}_s)\left(\frac{1}{q(s)}G_s-\frac{1}{q(t)}G_t\right)^2\right). \end{aligned}\tag{4.14}$$

Equation (4.10) now follows from (4.13) and (4.14).

To obtain (4.11), we first consider $\det(I-\Sigma_s\Lambda)$. The entries of $I-\Sigma_s\Lambda$ are of the form

$$\delta_{l,m}-\big(v(r_l,r_m)-\tau(s)q(r_l)q(r_m)\big)\lambda_m,\qquad 1\le l,m\le j. \tag{4.15}$$

For each $l=2,\dots,m$, multiply the first row of $\det(I-\Sigma_s\Lambda)$ by $q(r_l)/q(r_1)$ and subtract it from the lth row. The resulting matrix has no terms in $\tau(s)$ in rows 2 through j. This implies that $\det(I-\Sigma_s\Lambda)$ is of the form $A+\tau(s)B$, where neither A nor B contain terms in $\tau(s)$. It is also clear that $A=\det(I-\Sigma\Lambda)$. [To see this think of what we get if $\tau(s)=0$.] Thus we have shown that

$$\det(I-\Sigma_s\Lambda)=\det(I-\Sigma\Lambda)+\tau(s)B \tag{4.16}$$

where B is not a function of $\tau(s)$. Exactly the same argument shows that

$$\det(I-\Sigma_t\Lambda)=\det(I-\Sigma\Lambda)+\tau(t)B \tag{4.17}$$

since the only difference in the entries of $\det(I-\Sigma_s\Lambda)$ and $\det(I-\Sigma_t\Lambda)$ are the terms $\tau(s)$ and $\tau(t)$. Equation (4.11) follows from (4.16) and (4.17). $\square$

Note that $\tau(r)$ is an increasing function of r. We consider the distribution function $F(r)=(\tau(r)/\tau(x))\wedge 1$. The next lemma is used in the proof of Ray's theorem.

LEMMA 4.4. *Let $r_{j+1}\le t<s\le r_j\le x$. Then*

$$\int_t^s \left(E\exp\left(\frac{1}{2}\sum_{i=1}^{j}\lambda_i G_{r_i,r}^2\right)\right)^4 dF(r)=\frac{F(s)-F(t)}{\det(I-\Sigma_s\Lambda)\det(I-\Sigma_t\Lambda)}. \tag{4.18}$$

PROOF. Let $K=K(q,\Lambda,\widetilde{\Sigma}_s)$. Setting $t=r$ in (4.10) and then integrating, it follows that

$$\begin{aligned}\big(\det(I-\Sigma_s\Lambda)\big)^2\int_t^s\left(\frac{1}{\det(I-\Sigma_r\Lambda)}\right)^2 dF(r)\\ =\int_t^s\left(\frac{1}{1-2K(\tau(s)-\tau(r))}\right)^2 dF(r).\end{aligned}\tag{4.19}$$

Set $v=\tau(r)/\tau(s)$. The right-hand side of (4.19) is equal to

$$\begin{aligned}&\frac{\tau(s)}{\tau(x)}\int_{\tau(t)/\tau(s)}^{I}\left(\frac{1}{1-2K\tau(s)(1-v)}\right)^2 dv\\ &\quad=\frac{(\tau(s)-\tau(t))/\tau(x)}{1-2K(\tau(s)-\tau(t))}=\big(F(s)-F(t)\big)\frac{\det(I-\Sigma_s\Lambda)}{\det(I-\Sigma_t\Lambda)}\end{aligned}\tag{4.20}$$

where for the last equality we use (4.10). Combining (4.19) and (4.20), we get

$$\int_t^s\left(\frac{1}{\det(I-\Sigma_r\Lambda)}\right)^2 dF(r)=\frac{F(s)-F(t)}{\det(I-\Sigma_s\Lambda)\det(I-\Sigma_t\Lambda)} \tag{4.21}$$

which is (4.18). $\square$

The case $r_1\le t<s\le x$ is a degenerate form of (4.18), in which there are no squares of Gaussian processes present. In this case both sides of (4.18) are equal to $F(s)-F(t)$.

5. Ray's Theorem. We give a proof of Theorem 1.3. Since (1.14) follows immediately from (1.4), we proceed to the proof of (1.13). Let $r_n\le\cdots\le r_{j+1}\le t<s\le r_j\le\cdots\le r_2\le r_1\le x$. Note that

$$P^{x,y}(J\le z)=\frac{\tau(z)}{\tau(x)}\wedge 1=F(z) \tag{5.1}$$

where F as defined just before Lemma 4.4. To verify this see (A.1). Using (5.1), we see that we can write (4.18) as

$$
\begin{aligned}
&E^{x,y}\biggl(\biggl(E_G \exp\biggl(\sum_{i=1}^{n}\lambda_i \mathbb{1}_{\{r_i>J\}}\biggl(G_{r_i}-\frac{v(r_i,J)}{v(J,J)}G_J\biggr)^2\Big/2\biggr)\biggr)^4, t\le J<r_j\biggr)\\
&\quad=\frac{F(r_j)-F(t)}{\det(I-\Sigma_{r_j}\Lambda)\det(I-\Sigma_t\Lambda)}.
\end{aligned}
\tag{5.2}
$$

We now consider the local time process of X. It follows from the Markov property, Lemma 3.1 and Lemma A.1 that

$$
\begin{aligned}
&E^{x,y}\biggl(\exp\biggl(\sum_{i=1}^{j}\lambda_i L_\infty^{r_i}\biggr), T_t<\infty\biggr)\\
&\quad=E^{x,y}\biggl(\exp\biggl(\sum_{i=1}^{j}\lambda_i L_{T_t}^{r_i}\biggr), T_t<\infty\biggr)E^{t,y}\exp\biggl(\sum_{i=1}^{j}\lambda_i L_\infty^{r_i}\biggr)\\
&\quad=E^{x,t}\exp\biggl(\sum_{i=1}^{j}\lambda_i L_{T_t}^{r_i}\biggr)P^{x,y}(T_t<\infty)E^{t,y}\exp\biggl(\sum_{i=1}^{j}\lambda_i L_\infty^{r_i}\biggr)\\
&\quad=\frac{1}{\det(I-\Sigma_t\Lambda)}F(t)\frac{1}{\det(I-\Sigma\Lambda)}
\end{aligned}
\tag{5.3}
$$

where for the last line we use Theorem 1.1, (3.10) with y,x replaced by x,t, respectively, and the fact that $r_1,\dots,r_j$ are between t and x. It follows from (5.3) and (4.11) that

$$
\begin{aligned}
&E^{x,y}\biggl\{\exp\biggl(\sum_{i=1}^{n}\lambda_i L_\infty^{r_i}\biggr), t\le J<r_j\biggr\}\\
&\quad=E^{x,y}\biggl\{\exp\biggl(\sum_{i=1}^{j}\lambda_i L_\infty^{r_i}\biggr), t\le J<r_j\biggr\}\\
&\quad=\frac{1}{\det(I-\Sigma\Lambda)}\biggl(\frac{F(r_j)}{\det(I-\Sigma_{r_j}\Lambda)}-\frac{F(t)}{\det(I-\Sigma_t\Lambda)}\biggr)\\
&\quad=\frac{F(r_j)-F(t)}{\det(I-\Sigma_{r_j}\Lambda)\det(I-\Sigma_t\Lambda)}.
\end{aligned}
\tag{5.4}
$$

Equation (1.13) follows from (5.2) and (5.4).

The proof of (1.15) is an exact replication of the proof of (1.13). We consider $y \le r_1 \le \cdots \le r_j \le s < t$ and set

$$\begin{aligned} G_{r_i,t} &= G_{r_i,s} + \left(\frac{v(r_i,s)}{v(s,s)} G_s - \frac{v(r_i,t)}{v(t,t)} G_t\right) \\ &= G_{r_i,s} + p(r_i)\left(\frac{1}{p(s)} G_s - \frac{1}{p(t)} G_t\right). \end{aligned} \tag{5.5}$$

We also note that $P^{x,y}(S \le z) = I(z)$, where $I(z) = (1 - \tau(y)/\tau(z)) \vee 0$.

We proceed to obtain analogies of all the results in Section 4 and in the proof of (1.13). Essentially, $p(\cdot)$ replaces $q(\cdot)$, $1/\tau(\cdot)$ replaces $\tau(\cdot)$, and $I(\cdot)$ replaces $F(\cdot)$.

6. Expressing local times in terms of Bessel processes. To begin, we motivate the choice of $\Psi(r)$ in (1.7)–(1.9). Let W_t denote a linear Brownian motion. By checking covariances it is easy to verify that

$$\{G_r; r \in I\} \overset{\text{law}}{=} \{q(r) W_{\tau(r)} : r \in I\} \overset{\text{law}}{=} \{p(r) W_{\phi(r)} : r \in I\} \tag{6.1}$$

and therefore

$$\{G_{r,y}; r \ge y\} \overset{\text{law}}{=} \{q(r)(W_{\tau(r)} - W_{\tau(y)}) : r \ge y\}, \tag{6.2}$$

$$\{G_{r,x}; r \le x\} \overset{\text{law}}{=} \{p(r)(W_{\phi(r)} - W_{\phi(x)}) : r \le x\}. \tag{6.3}$$

Also, since $v(r,s)$ is the 0-potential of a diffusion, $\{G_{r,x}; r \le x\}$ and $\{G_{r,y}; r \ge y\}$ are independent.

Let $\bar{B} = \{\bar{B}_t, t \in \mathbb{R}^+\}$ and $\widehat{B} = \{\widehat{B}_t, t \in \mathbb{R}^+\}$ be two planar Brownian motions independent of each other. It follows from (6.1)–(6.3) that (1.4) is equivalent to

$$\begin{aligned} &\bigl\{L^r_\infty + \tfrac{1}{2} p^2(r) |\bar{B}_{\phi(r)-\phi(x)}|^2 \mathbb{1}_{\{r \le x\}} + \tfrac{1}{2} q^2(r) |\widehat{B}_{\tau(r)-\tau(y)}|^2 \mathbb{1}_{\{r \ge y\}} : r \in I\bigr\} \\ &\quad \overset{\text{law}}{=} \bigl\{\tfrac{1}{2} q^2(r) |\widehat{B}_{\tau(r)}|^2 : r \in I\bigr\} \\ &\quad \overset{\text{law}}{=} \bigl\{\tfrac{1}{2} p^2(r) |\bar{B}_{\phi(r)}|^2 : r \in I\bigr\} \end{aligned} \tag{6.4}$$

where $L = \{L^r_\infty, r \in I\}$ is independent of $\bar{B}$ and $\widehat{B}$. Using this, we see that

$$L^r_\infty \overset{\text{law}}{=} \tfrac{1}{2} q^2(r) |\widehat{B}_{\tau(r)}|^2, \qquad x \le r \le y, \tag{6.5}$$

$$L^r_\infty + \tfrac{1}{2} q^2(r) |\widehat{B}_{\tau(r)-\tau(y)}|^2 \mathbb{1}_{\{r \ge y\}} \overset{\text{law}}{=} \tfrac{1}{2} q^2(r) |\widehat{B}_{\tau(r)}|^2, \qquad r \ge y, \tag{6.6}$$

$$L^r_\infty + \tfrac{1}{2} p^2(r) |\bar{B}_{\phi(r)-\phi(x)}|^2 \mathbb{1}_{\{r \le x\}} \overset{\text{law}}{=} \tfrac{1}{2} p^2(r) |\bar{B}_{\phi(r)}|^2, \qquad r \le x. \tag{6.7}$$

The last two equalities can be written as

$$
\begin{aligned}
&L^r_\infty + \tfrac{1}{2}q^2(r)|\widehat{B}_{\tau(r)-\tau(y)}|^2 \mathbb{1}_{\{r\geq y\}} \\
&\qquad \stackrel{\text{law}}{=} \tfrac{1}{2}q^2(r)|\widehat{B}_{\tau(y)} + \widehat{B}_{\tau(r)-\tau(y)}|^2, \qquad r \geq y,
\end{aligned}
\tag{6.8}
$$

$$
\begin{aligned}
&L^r_\infty + \tfrac{1}{2}p^2(r)|\bar{B}_{\phi(r)-\phi(x)}|^2 \mathbb{1}_{\{r\leq x\}} \\
&\qquad \stackrel{\text{law}}{=} \tfrac{1}{2}p^2(r)|\bar{B}_{\phi(x)} + \bar{B}_{\phi(r)-\phi(x)}|^2, \qquad r \leq x.
\end{aligned}
\tag{6.9}
$$

By the additivity property of squared Bessel processes (see, e.g., [9], Chapter XI, Theorem 1.2) and using the fact that $|B_{\phi(x)}|^2 \stackrel{\text{law}}{=} \phi^2(x)|B_{\tau(x)}|^2$ in (6.9), we see that L^r_∞ is equal in law to Ψ_r, separately, in each of the three regions (1.7)–(1.9).

PROOF OF THEOREM 1.2. It suffices to show that (6.4) holds with L^r_∞ replaced by Ψ_r, since, for example, by taking Laplace transforms, (1.6) determines the finite-dimensional distributions of L^r_∞. Let $B = \{B_t, t \in \mathbb{R}^+\}$ be a planar Brownian motion independent of $\bar{B}$ and $\widehat{B}$. Using (1.7)–(1.9) and the additivity property of squared Bessel processes, we see that

$$
\begin{aligned}
&\{\Psi_r + \tfrac{1}{2}p^2(r)|\bar{B}_{\phi(r)-\phi(x)}|^2 \mathbb{1}_{\{r\leq x\}} + \tfrac{1}{2}q^2(r)|\widehat{B}_{\tau(r)-\tau(y)}|^2 \mathbb{1}_{\{r\geq y\}} : r \in I\} \\
&\quad \stackrel{\text{law}}{=} \{\tfrac{1}{2}p^2(r)|\bar{B}_{\phi(r)-\phi(x)} + \phi(x)B_{\tau(x)}|^2 \mathbb{1}_{\{r\leq x\}} \\
&\qquad + \tfrac{1}{2}q^2(r)|B_{\tau(r)}|^2 \mathbb{1}_{\{x<r<y\}} \\
&\qquad + \tfrac{1}{2}q^2(r)|\widehat{B}_{\tau(r)-\tau(y)} + B_{\tau(y)}|^2 \mathbb{1}_{\{r\geq y\}} : r \in I\}.
\end{aligned}
\tag{6.10}
$$

Since this last process is clearly Markovian, it suffices to show that it agrees in law with $\frac{1}{2}q^2(r)|B_{\tau(r)}|^2$ separately in the regions $\{r \leq x\}$, $\{x \leq r \leq y\}$ and $\{r \geq y\}$. This is obvious for the latter two regions. As for the region $\{r \leq x\}$, we again use the fact that $q(r)B_{\tau(r)} \stackrel{\text{law}}{=} p(r)B_{\phi(r)}$ to see that

$$
\begin{aligned}
&\{\tfrac{1}{2}p^2(r)|\bar{B}_{\phi(r)-\phi(x)} + \phi(x)B_{\tau(x)}|^2 : r \leq x\} \\
&\qquad \stackrel{\text{law}}{=} \{\tfrac{1}{2}p^2(r)|B_{\phi(r)}|^2 : r \leq x\} \\
&\qquad \stackrel{\text{law}}{=} \{\tfrac{1}{2}q^2(r)|B_{\tau(r)}|^2 : r \leq x\}.
\end{aligned}
\tag{6.11}
$$

□

PROOF OF COROLLARY 1.1. To establish Corollary 1.1 it suffices to show that

$$
E(H(\Psi_s)|\mathcal{F}_r^\Psi) = E(H(\Psi_s)|\Psi_r)
\tag{6.12}
$$

for each $r < s$ and any measurable function H, where $\mathcal{F}_t^\Psi$ denotes the σ-algebra generated by the stochastic process $\Psi_u, u \leq t$.

This is not difficult; for example, take $y \le r < s$. For each $a \in \mathbb{R}^1$, the continuous process $Z = \{Z_t(a); t \in \mathbb{R}^+\}$ induces a probability measure $P_Z^a(\cdot)$ on $C(\mathbb{R}, \mathbb{R}^+)$. Let X_t denote the canonical coordinate process on $C(\mathbb{R}, \mathbb{R}^+)$. Then, under $\{P_Z^a(\cdot); a \in \mathbb{R}\}$, X_t is a time-homogeneous strong Markov process. Using independence and the Markov property just described we have

$$E\big(H(\Psi_s)\big|\mathcal{F}_r^{\Psi}\big) = E\big(H\big(\tfrac{1}{2}q^2(s)Z_{\tau(s)-\tau(y)}(|B_{\tau(y)}|^2)\big)\big|\mathcal{F}_r^{\Psi}\big) \tag{6.13}$$
$$= E_Z^{Z_{\tau(r)-\tau(y)}(|B_{\tau(y)}|^2)}\big(H\big(\tfrac{1}{2}q^2(s)X_{\tau(s)-\tau(r)}\big)\big)$$

which implies (6.12) when $y \le r < s$. In a similar manner we can check that (6.12) holds for all $r < s$. □

7. Proof of the first Ray–Knight theorem.

PROOF OF THEOREM 1.4. We first obtain (1.21). We use Lemma 2.2 with $0 < x_1 < \cdots < x_l = x < x_{l+1} < \cdots < x_n$ and $h \equiv 1$. Here $\Sigma_{j,k} = 2(x_j \wedge x_k)$, which implies that

$$\frac{1}{\det(I - \Sigma\Lambda)} = \left(E \exp\left(\sum_{i=1}^{n} \lambda_i W_{x_i}^2\right)\right)^2 \tag{7.1}$$

where $\{W_x, x \in \mathbb{R}^+\}$ is standard Brownian motion starting at 0. Furthermore,

$$\widehat{\Sigma}_{j,k} = \Sigma_{j,k} - \Sigma_{l,k} = 0, \qquad k \le j \wedge l, \tag{7.2}$$

and

$$\widehat{\Sigma}_{j,k} = \Sigma_{j,k} - \Sigma_{l,k} = 2\big((x_j - x_l) \wedge (x_k - x_l)\big), \qquad l < j \wedge k. \tag{7.3}$$

This defines enough elements of $\widehat{\Sigma}_{j,k}$ for us to see that

$$\frac{1}{\det(I - \widehat{\Sigma}\Lambda)} = \left(E\left(\exp\left(\sum_{i=l+1}^{n} \lambda_i W_{x_i - x_l}^2\right)\right)\right)^2. \tag{7.4}$$

Equation (1.21) now follows from Lemma 2.2. Equation (1.20) follows from the additivity property of squared Bessel processes (see, e.g., [9], Chapter XI, Theorem 1.2).

As for (1.22), the 0-potential of standard Brownian motion starting at $x > 0$ and killed the first time it hits 0 is $v(r, s) = 2(r \wedge s)$. Thus the Gaussian process on the right-hand side of Theorem 1.1 is simply the square root of 2 times Brownian motion starting at zero. The rest follows from Theorem 1.1. □

8. A computational proof of the Markov property of local times of diffusions. In this section we give a simple, direct proof of Corollary 1.1. It is a direct consequence of the explicit computations given in Theorem 8.1. We first state Theorem 8.1, then prove Corollary 1.1 and finally return to the proof of Theorem 8.1.

Let

$$
\begin{aligned}
c &= \frac{v(w,z)}{v(z,z)} = \frac{q(w)}{q(z)},\\
\sigma^2 &= v(w,w) - \frac{v^2(w,z)}{v(z,z)} = EG_{w,z}^2, \qquad (8.1)\\
K = K_\lambda &= \frac{1}{1-\lambda\sigma^2} = E\left(\exp\left(\frac{\lambda}{2}G_{w,z}^2\right)\right).
\end{aligned}
$$

THEOREM 8.1. *Let $x_1 < \cdots < x_n = z \le w$ and let H be a bounded continuous function on $\mathbb{R}^n$. Let $H_n(L_\infty^{\cdot}) := H(L_\infty^{x_1}, \dots, L_\infty^{x_n})$.*

For all $\lambda \in \mathbb{C}$ sufficiently small the following hold:

(i) *If $x \le y \le z \le w$,*

$$
E^{x,y}\big(\exp(\lambda L_\infty^w)H_n(L_\infty^{\cdot})\big) = E^{x,y}\big(\exp(c^2\lambda K_\lambda L_\infty^z)H_n(L_\infty^{\cdot})\big). \tag{8.2}
$$

(ii) *If $x \le z \le w \le y$,*

$$
E^{x,y}\big(\exp(\lambda L_\infty^w)H_n(L_\infty^{\cdot})\big) = K_\lambda E^{x,y}\big(\exp(c^2\lambda K_\lambda L_\infty^z)H_n(L_\infty^{\cdot})\big). \tag{8.3}
$$

(iii) *If $z \le w \le x \le y$,*

$$
\begin{aligned}
&E^{x,y}\big(\exp(\lambda L_\infty^w)H_n(L_\infty^{\cdot});\, L_\infty^{x_n} \ne 0\big)\\
&\quad = K_\lambda^2 E^{x,y}\big(\exp(c^2\lambda K_\lambda L_\infty^z)H_n(L_\infty^{\cdot});\, L_\infty^{x_n} \ne 0\big).
\end{aligned} \tag{8.4}
$$

(iv) *If $z \le w \le x \le y$,*

$$
\begin{aligned}
&E^{x,y}\big(\exp(\lambda L_\infty^w)H_n(L_\infty^{\cdot});\, L_\infty^{x_n} = 0\big)\\
&\quad = E^{x,y}\left(\left(\frac{\tau(x)-\tau(w)+(\tau(w)-\tau(z))K_\lambda}{\tau(x)-\tau(z)}\right)H_n(L_\infty^{\cdot});\, L_\infty^{x_n} = 0\right).
\end{aligned} \tag{8.5}
$$

ALTERNATIVE PROOF OF COROLLARY 1.1. Fix $x < y$ and let $z \le w$. To establish the Markov property we show that for every $f \in \mathcal{S}$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^+$, we can find a bounded continuous function $\bar{f}$, possibly depending on z and w, such that for any finite sequence of points $x_1 < \cdots < x_n = z \le w$,

$$
E^{x,y}\big(f(L_\infty^w)H_n(L_\infty^{\cdot})\big) = E^{x,y}\big(\bar{f}(L_\infty^z)H_n(L_\infty^{\cdot})\big). \tag{8.6}
$$

This implies that

$$
E^{x,y}\big(f(L_\infty^w)|\sigma(L_\infty^r; r \le z)\big) = E^{x,y}\big(f(L_\infty^w)|\sigma(L_\infty^z)\big) \tag{8.7}
$$

for every $f \in \mathcal{S}$, and hence for all bounded measurable f by the conditional dominated convergence theorem, thus proving the Markov property.

Refer to Theorem 8.1. It is easy to see that both sides of equations (8.2)–(8.5) are analytic in λ in the region $\mathrm{Re}\,\lambda < \delta$ for some $\delta > 0$. Therefore, they hold for λ purely imaginary. This gives us (8.6) for $f(x) = e^{ipx}$ for any real p, and also an explicit formula for $\bar{f}$. This leads easily to (8.6) for any $f \in \mathcal{S}$ and completes the alternative proof of Corollary 1.1. $\square$

In preparation for the proof of Theorem 8.1 we make the following definitions. Let $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and

$$B_G \stackrel{\text{def}}{=} \exp\left(\frac{1}{2}\sum_{i=1}^n \lambda_i G_{x_i}^2\right), \tag{8.8}$$

$$B_{G,x} \stackrel{\text{def}}{=} \exp\left(\frac{1}{2}\sum_{i=1}^n \lambda_i G_{x_i,x}^2 \mathbb{1}_{\{x_i \le x\}}\right), \tag{8.9}$$

$$B_{G,y} \stackrel{\text{def}}{=} \exp\left(\frac{1}{2}\sum_{i=1}^n \lambda_i G_{x_i,y}^2 \mathbb{1}_{\{x_i \ge y\}}\right), \tag{8.10}$$

$$\Pi_L = \Pi_L(\bar{\lambda}) \stackrel{\text{def}}{=} \exp\left(\sum_{i=1}^n \lambda_i L_\infty^{x_i}\right). \tag{8.11}$$

We next note the following simple lemma.

LEMMA 8.1. *Let $\{G_r, r \in \mathbb{R}\}$ be a mean zero Gaussian process with covariance as given in* (3.8). *Then for $z \le w$,*

$$E\left(B_G \exp\left(\frac{\lambda}{2} G_w^2\right)\right) = K^{1/2} E\left(B_G \exp\left(\frac{\lambda}{2} c^2 K G^2(z)\right)\right). \tag{8.12}$$

PROOF. We write

$$G_w = G_{w,z} + cG_z. \tag{8.13}$$

Since $G_{w,z}$ is independent of G_{x_i}, $i = 1, \ldots, n$, and $EG_{w,z}^2 = \sigma^2$, we see from Lemma 4.1 that

$$E\left(B_G \exp\left(\frac{\lambda}{2} G_w^2\right)\right) = K^{1/2} E\left(B_G \exp\left(\frac{\lambda c^2 G_z^2}{2} + \frac{\lambda^2 c^2 \sigma^2 G_z^2}{2(1-\lambda\sigma^2)}\right)\right) \tag{8.14}$$

from which we get (8.12). $\square$

PROOF OF THEOREM 8.1. Using the same analyticity argument as in the proof of Corollary 1.1, we see that in order to prove (8.3), it suffices to show that for all $\lambda, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$ sufficiently small,

$$E^{x,y}\big(\Pi_L(\bar{\lambda}) \exp(\lambda L_\infty^w)\big) = K_\lambda E^{x,y}\big(\Pi_L(\bar{\lambda}) \exp(c^2 \lambda K_\lambda L_\infty^z)\big). \tag{8.15}$$

The results used below to obtain (8.15) are proved earlier in this paper for $\lambda, \lambda_1, \dots, \lambda_n$ real. However, it is easy to check that the proofs of these results remain valid when $\lambda, \lambda_1, \dots, \lambda_n$ are complex. This is also the case in the rest of the proof of this theorem.

Since $x \le z \le w \le y$, $\mathbb{1}_{\{x_i \ge y\}} = 0$. Using this, Theorem 1.1, Lemma 8.1 and Theorem 1.1, again we get

$$\begin{aligned} &E^{x,y} E\big(\Pi_L B_{G,x} B_{\bar{G},x} \exp(\lambda L^w_\infty)\big) \\ &\quad = E\left(B_G B_{\bar{G}} \exp\left(\lambda\left(\frac{G_w^2}{2} + \frac{\bar{G}_w^2}{2}\right)\right)\right) \\ &\quad = K E\left(B_G B_{\bar{G}} \exp\left(\lambda c^2 K\left(\frac{G_z^2}{2} + \frac{\bar{G}_z^2}{2}\right)\right)\right) \\ &\quad = K E^{x,y} E\big(\Pi_L B_{G,x} B_{\bar{G},x} \exp(\lambda c^2 K L^z_\infty)\big) \end{aligned} \tag{8.16}$$

from which we get (8.15) and consequently (8.3).

We now prove (8.2). It follows from Theorem 1.1, Lemma 8.1 and the fact that $B_{G,x}$ and $B_{G,y}$ are independent, that

$$\begin{aligned} &E^{x,y}\left(\Pi_L B_{G,x} B_{\bar{G},x} B_{G,y} B_{\bar{G},y} \exp\left(\lambda\left(L^w_\infty + \frac{G_{w,y}^2}{2} + \frac{\bar{G}_{w,y}^2}{2}\right)\right)\right) \\ &\quad = E\left(B_G B_{\bar{G}} \exp\left(\lambda\left(\frac{G_w^2}{2} + \frac{\bar{G}_w^2}{2}\right)\right)\right) \\ &\quad = K E\left(B_G B_{\bar{G}} \exp\left(\lambda c^2 K\left(\frac{G_z^2}{2} + \frac{\bar{G}_z^2}{2}\right)\right)\right) \\ &\quad = K E^{x,y}\big(\Pi_L \exp(\lambda c^2 K L^z_\infty)\big) \\ &\qquad \times \left(E\left(\exp\left(\lambda c^2 K \frac{G_{z,y}^2}{2}\right) B_{G,y}\right)\right)^2 (E B_{G,x})^2 \end{aligned} \tag{8.17}$$

where, at the last step we use Theorem 1.1 again. Also, obviously,

$$\begin{aligned} &E^{x,y}\left(\Pi_L B_{G,x} B_{\bar{G},x} B_{G,y} B_{\bar{G},y} \exp\left(\lambda\left(L^w_\infty + \frac{G_{w,y}^2}{2} + \frac{\bar{G}_{w,y}^2}{2}\right)\right)\right) \\ &\quad = E^{x,y}\big(\Pi_L \exp(\lambda L^w_\infty)\big)(E B_{G,x})^2 \left(E\left(\exp\left(\lambda \frac{G_{w,y}^2}{2}\right) B_{G,y}\right)\right)^2. \end{aligned} \tag{8.18}$$

We claim that

$$E\left(B_{G,y} \exp\left(\lambda \frac{G_{w,y}^2}{2}\right)\right) = K^{1/2} E\left(B_{G,y} \exp\left(\lambda c^2 K \frac{G_{z,y}^2}{2}\right)\right). \tag{8.19}$$

Substituting this in the right-hand side of (8.18) and comparing it to the last line of (8.17) gives (8.2).

To establish (8.19), one can check that

$$G_{w,y} = G_{w,z} + cG_{z,y} \tag{8.20}$$

and $G_{w,z}$ is independent of both $G_{z,y}$ and $B_{G,y}$. Using (8.20) in place of (8.13) and continuing with the proof of Lemma 8.1, we get (8.19).

To obtain (8.4), we note that

$$\begin{aligned} &E^{x,y}\big(\Pi_L \exp(\lambda L^w_\infty), T_z < \infty\big) \\ &\qquad = E^{x,y}\big(\Pi_L \exp(\lambda L^w_{T_z}), T_z < \infty\big) E^{z,y}\big(\exp(\lambda L^w_\infty)\big) \end{aligned} \tag{8.21}$$

since, in the first passage from x to z, $\Pi_L = 1$. It follows from Lemma A.1 that

$$E^{x,y}\big(\exp(\lambda L^w_{T_z}), T_z < \infty\big) = E^{x,z}\big(\exp(\lambda L^w_{T_z})\big) P^{x,y}(T_z < \infty) \tag{8.22}$$

and from (3.10) that

$$E^{x,z}\big(\exp(\lambda L^w_{T_z})\big) = K. \tag{8.23}$$

Furthermore, since $z \le w < y$, we can use (8.3) to get

$$E^{z,y}\big(\Pi_L \exp(\lambda L^w_\infty)\big) = K E^{z,y}\big(\Pi_L \exp(c^2 \lambda K L^z_\infty)\big). \tag{8.24}$$

Using (8.21)–(8.24) we get

$$\begin{aligned} &E^{x,y}\big(\Pi_L \exp(\lambda L^w_\infty), T_z < \infty\big) \\ &\qquad = K^2 P^{x,y}(T_z < \infty) E^{z,y}\big(\Pi_L \exp(c^2 \lambda K L^z_\infty)\big). \end{aligned} \tag{8.25}$$

Also, by (8.21) and (8.22) with w replaced by z, we see that

$$\begin{aligned} &E^{x,y}\big(\Pi_L \exp(c^2 \lambda K L^z_\infty), T_z < \infty\big) \\ &\qquad = E^{x,z}\big(\exp(c^2 \lambda K L^z_{T_z})\big) P^{x,y}(T_z < \infty) \\ &\qquad\quad \times E^{z,y}\big(\Pi_L \exp(c^2 \lambda K L^z_\infty)\big). \end{aligned} \tag{8.26}$$

Recognizing that the first expectation to the right of the equality sign in (8.26) is equal to one, we can substitute (8.26) into (8.25) to get (8.4).

To obtain (8.5), we note that

$$\begin{aligned} &E^{x,y}\big(\exp(\lambda L^w_\infty) \big| L^{x_i}_\infty, i = 1, \ldots, n; L^{x_n}_\infty = 0\big) \\ &\qquad = E^{x,y}\big(\exp(\lambda L^w_\infty) \big| z \le J\big) \\ &\qquad = \frac{P^{x,y}(w \le J) + E^{x,y}(\exp(\lambda L^w_\infty), z \le J \le w)}{P^{x,y}(z \le J)}. \end{aligned} \tag{8.27}$$

Using (5.1) and (5.4), we get (8.5). [To see that this is the same as (2.19), [13] note that $q^2(x)(\tau(x) - \tau(z)) = u(x,x) - (u^2(x,z)/u(z,z))$.] $\square$

9. Strongly symmetric Markov processes. Recall (2.16) which holds for all Markov processes with symmetric 0-potential densities. We get (1.18) by adding

$$\left(E\exp\left(\frac{1}{2}\sum_{i=1}^{n}\lambda_i G^2_{x_i,y}\right)\right)^2$$

to each side of (3.16), which holds for all Markov processes not just diffusions, and using (2.16).

Since $G_r = G_{r,y} + \frac{v(r,y)}{v(y,y)} G_y$ it follows from Corollary 4.1 that the right-hand side of (2.17) is equal, in law, to

$$\frac{G^2_{r,y}}{2} + \frac{(\bar{G}_{r,y} + (v(r,y)/v(y,y))\sqrt{G^2_y + \bar{G}^2_y}\,)^2}{2}. \tag{9.1}$$

Using this in (2.17) and cancelling $G^2_{r,y}/2$ from each side we get that under $P^{y,y} \times P_{G,\bar{G}}$,

$$\begin{aligned}\left\{L^r_\infty + \frac{G^2_{r,y}}{2} : r \in I\right\}& \\ \stackrel{\text{law}}{=} \left\{\frac{(G_{r,y} + (v(r,y)/v(y,y))\sqrt{G^2_y + \bar{G}^2_y}\,)^2}{2} : r \in I\right\}&.\end{aligned} \tag{9.2}$$

Note that $(G^2_y + \bar{G}^2_y)/2$ is an exponential random variable with mean $v(y,y)$ independent of $G_{r,y}$. By Lemma 2.2, L^y_∞ is also an exponential random variable with mean $v(y,y)$. Thus we get (1.19).

We now explain why (1.19) is equivalent to Theorem 1.2 in [4]. Consider a transient symmetric Markov process X with symmetric 0-potential density $v(r,s)$. Under both P^y and $P^{y,y}$, L^y_∞ the total accumulated local time of X at y, is an exponential random variable with mean $v(y,y)$. (See Lemmas 2.1 and 2.2.) Let T be an exponential random variable independent of X. Let V be the Markov process X killed when it's local time at y is equal to T. We show that under $P^{y,y} \times P_G$,

$$\begin{aligned}\left\{L^r_{\tau(L^y_\infty \wedge t^-)} + \frac{G^2_{r,y}}{2} : r \in I\right\}& \\ \stackrel{\text{law}}{=} \left\{\frac{(G_{r,y} + (v(r,y))/(v(y,y))\sqrt{2(L^y_\infty \wedge t)}\,)^2}{2} : r \in I\right\}&\end{aligned} \tag{9.3}$$

where $L^r_{\tau(L^y_\infty \wedge T^-)}$ is the total accumulated local time of V^{h_y} and $\tau(t) := \inf\{s : L^y_s > t\}$ is the right continuous inverse of $s \mapsto L^y_s$. This is Theorem 1.2 in [4]. [Note that since τ is right continuous, increasing with only a countable number of jumps, $\tau(L^y_\infty \wedge t^-) = \tau^-(L^y_\infty \wedge t)$, where $\tau^-(t) := \sup\{s : L^y_s < t\}$ is the left continuous inverse of $s \mapsto L^y_s$.]

We get (9.3) by using (1.19) on V. However, to do this we need to know the 0-potential density of V. This is given in [4], Lemma 5.2. It is

$$\bar{v}(s,t) = v(s,t) - \frac{v(s,y)v(t,y)}{v(y,y)} + \frac{v(s,y)v(t,y)}{v^2(y,y)} E(L_\infty^y \wedge T). \tag{9.4}$$

Let $\mathcal{G}_r$ be a mean zero Gaussian process with covariance $\bar{v}$ and G_r be a mean zero Gaussian process with covariance v. One can check that

$$\mathcal{G}_{r,y} \stackrel{\text{law}}{=} G_{r,y} \tag{9.5}$$

and

$$\frac{\bar{v}(r,y)}{\bar{v}(y,y)} = \frac{v(r,y)}{v(y,y)}. \tag{9.6}$$

Using (1.19) on V and (9.5) and (9.6) we see that under $P^{y,y} \times P_G$,

$$\begin{aligned} &\left\{ L^r_{\tau(L^y_\infty \wedge T^-)} + \frac{G^2_{r,y}}{2} : r \in I \right\} \\ &\quad \stackrel{\text{law}}{=} \left\{ \frac{\left(G_{r,y} + (v(r,y))/(v(y,y))\sqrt{2(L^y_\infty \wedge T)}\right)^2}{2} : r \in I \right\}. \end{aligned} \tag{9.7}$$

The mean of T is arbitrary; consequently we get (9.3) since (9.7) is the Laplace transform of (9.3).

Now let X be a recurrent symmetric Markov process. Let T be an exponential random variable independent of X. Let V be the Markov process X killed when it's local time at y is equal to T. It follows from [4], Lemma 5.1, that the 0-potential density of V can be written as

$$v(r,s) = w(r,s) + C \tag{9.8}$$

where $C = E(T)$ and $w(r,s)$ is the 0-potential of the process X killed when it first hits y. [So, in particular, $w(r,y) = w(y,s) = 0$.] In this case $h_y(r) = C$, so $v^h(r,s) = v(r,s)$ [see (3.1)]. Repeating the argument that led to (9.3), we get that under $P^y \times P_G$, for all $t > 0$,

$$\left\{ L^r_{\tau(t)} + \frac{G^2_{r,y}}{2} : r \in I \right\} \stackrel{\text{law}}{=} \left\{ \frac{(G_{r,y} + \sqrt{2t})^2}{2} : r \in I \right\} \tag{9.9}$$

where $\tau(t) := \inf\{s : L^y_s > t\}$ and G_r is a mean zero Gaussian process with covariance $w(r,s)$. This is Theorem 1.1 in [4].

In both Theorems 1.1 and 1.2 of [4] there is a constant b, which we take to be zero. That the constant can be added in both (9.3) and (9.9) is a consequence of Corollary 4.1 on the moment generating function of squares of Gaussian process and has nothing to do with the local times.

APPENDIX

Let X and $P^{x,y}$ be as given in the third paragraph of Section 1. Let T_r denote the first hitting time of r by X. Then

$$P^{x,y}(T_r < \infty) = \frac{v(x,r)v(r,y)}{v(x,y)v(r,r)} \tag{A.1}$$

and

$$P^{x}(T_r < \infty) = \frac{v(x,r)}{v(r,r)}. \tag{A.2}$$

To obtain (A.1), we just use the Markov property

$$E^{x,y}(L^r_\infty) = P^{x,y}(T_r < \infty)E^{r,y}(L^r_\infty) \tag{A.3}$$

together with the fact that for all l, $E^{l,y}(L^r_\infty) = v(l,r)v(r,y)/v(l,y)$. This last equality is just the simplest case of Kac's moment formula (2.2) applied to the h-transform of X, that is, to the Markov process with 0-potential $v(s,t)h(t)/h(s)$ where $h(s) = v(s,y)$. (A.2) is obtained similarly using the simple equality $E^x(L^r_\infty) = P^x(T_r < \infty)E^r(L^r_\infty)$ and (2.2) again.

Using the Markov property and (A.2), we see that for any measurable function f,

$$\begin{aligned}
&\int v(x,y)f(y)\,dy \\
&\quad = E^x\biggl(\int_0^\infty f(X_t)\,dt\biggr) \\
&\quad = E^x\biggl(\int_0^{T_r} f(X_t)\,dt\biggr) + E^x\biggl(\int_{T_r}^\infty f(X_t)\,dt\biggr) \\
&\quad = E^x\biggl(\int_0^{T_r} f(X_t)\,dt\biggr) + P^x(T_r < \infty)E^r\biggl(\int_0^\infty f(X_t)\,dt\biggr) \\
&\quad = E^x\biggl(\int_0^{T_r} f(X_t)\,dt\biggr) + \frac{v(x,r)}{v(r,r)}\int v(r,y)f(y)\,dy
\end{aligned} \tag{A.4}$$

which immediately gives (3.2).

We provide the following lemma for the convenience of the reader.

LEMMA A.1. *Let $H \in \mathcal{F}_{T_r}$. Then*

$$E^{x,y}(H 1\!\!1_{\{T_r<\infty\}}) = E^{x,r}(H)P^{x,y}(T_r < \infty), \qquad H \in \mathcal{F}_{T_r}. \tag{A.5}$$

PROOF. For any h-transform with $0 < h(x) < \infty$ and stopping time T we have

$$E^{x/h}(H 1\!\!1_{\{T<\zeta\}}) = \frac{1}{h(x)}E^x(Hh(X_T)), \qquad H \in \mathcal{F}_T. \tag{A.6}$$

See (62.20) of [12]. Applying this with $T = T_r$ and $h_y(s) = v(y,s)$ shows that

$$E^{x,y}\big(H\mathbb{1}_{\{T_r<\infty\}}\big) = \frac{v(r,y)}{v(x,y)}E^x\big(H\mathbb{1}_{\{T_r<\infty\}}\big), \qquad H \in \mathcal{F}_{T_r}, \tag{A.7}$$

and with $h_r(s) = v(r,s)$

$$E^{x,r}\big(H\mathbb{1}_{\{T_r<\infty\}}\big) = \frac{v(r,r)}{v(r,x)}E^x\big(H\mathbb{1}_{\{T_r<\infty\}}\big), \qquad H \in \mathcal{F}_{T_r}. \tag{A.8}$$

Therefore, by (A.1),

$$E^{x,y}\big(H\mathbb{1}_{\{T_r<\infty\}}\big) = E^{x,r}\big(H\mathbb{1}_{\{T_r<\infty\}}\big)P^{x,y}(T_r<\infty), \qquad H \in \mathcal{F}_{T_r}. \tag{A.9}$$

Next, we take $H = 1$ in (A.8) and use (A.2) to see that

$$P^{x,r}\big(\mathbb{1}_{\{T_r<\infty\}}\big) = \frac{v(r,r)}{v(x,r)}P^x\big(\mathbb{1}_{\{T_r<\infty\}}\big) = 1. \tag{A.10}$$

Using (A.9) and (A.10), we get (A.5). □

Note that (A.1) and hence (A.10) requires that the 0-potential of X exists, that is, that X is transient.

A.1. *Other isomorphism theorems.* We begin with short simple proofs of two well-known isomorphisms between local times and Gaussian processes. They are not as neat as the ones presented in the body of this paper but they hold in greater generality.

Let X be a strongly symmetric Markov process with state space S and 0-potential density $v(r,s)$. Let W be the h transform of X where $h(s) = v(s,y)$, that is, W has 0-potential density

$$u(r,s) = \frac{v(r,s)v(s,y)}{v(r,x)}. \tag{A.11}$$

Let G be a mean zero Gaussian process with covariance $\Sigma_{r,s} = v(r,s)$.

THEOREM A.1 (Dynkin's isomorphism theorem). *Let $\{L^s_\infty, s \in S\}$ denote the total accumulated local time of W, which we assume exists. For all x, y in S and measurable functions F on $\mathbb{R}^n$, for all n,*

$$E^{x,y}E\left(F\left(L^{\cdot}_\infty + \frac{G^2_{\cdot}}{2}\right)\right) = \frac{1}{v(x,y)}E\left(G_xG_yF\left(\frac{G^2_{\cdot}}{2}\right)\right). \tag{A.12}$$

PROOF. To prove this theorem it suffices to show that

$$v(x,y)E^{x,y}\exp\left(\sum_{i=1}^n \lambda_i L^{x_i}_\infty\right) = \frac{E(G_xG_y\exp(\sum_{i=1}^n \lambda_i G^2_{x_i}/2))}{E\exp(\sum_{i=1}^n \lambda_i G^2_{x_i}/2)} \tag{A.13}$$

for all $\lambda_1,\dots,\lambda_n$ sufficiently small where $x = x_1$ and $y = x_n$. It follows from

Remark 4.1 that

$$
\begin{aligned}
\frac{E(G_{x_1}G_{x_n}\exp(\sum_{i=1}^n \lambda_i G_{x_i}^2/2))}{E\exp(\sum_{i=1}^n \lambda_i G_{x_i}^2/2)} &= \{\widetilde{\Sigma}\}_{1,n} \\
&= \{(I-\Sigma\Lambda)^{-1}\Sigma\}_{1,n} \\
&= \sum_{j=1}^n \{(I-\Sigma\Lambda)^{-1}\}_{1,j} v(x_j,y).
\end{aligned} \tag{A.14}
$$

To handle the left-hand side of (A.13) we use (2.9) with $h(z) = v(z,y)$ and $x_1 = x$. Clearly,

$$\{HY\}_1 = \{(I-\Sigma\Lambda)^{-1}\mathbf{h}\}_1. \tag{A.15}$$

Thus we see that

$$v(x,y)E^{x,y}\exp\left(\sum_{i=1}^n \lambda_i L_\infty^{x_i}\right) = \sum_{j=1}^n \{(I-\Sigma\Lambda)^{-1}\}_{1,j} v(x_j,y). \tag{A.16}$$

Comparing this with (A.14) gives (A.13). □

There is some similarity between this proof and the one in [10], Vol. 1, Section 27, of what the authors refer to as a "caricature" of Dynkin's isomorphism theorem.

The isomorphism in Theorem A.1 is for processes conditioned to die at a fixed point in their state space. It is a simple consequence of Lemma 2.2 which deals with this situation. In the next isomorphism theorem, which is due to Eisenbaum (it is stated below Théorème 1.3 in [3]), we consider processes with no further condition imposed on their lifetimes. It is a simple consequence of Lemma 2.1 which deals with this situation.

THEOREM A.2. *Let $\{L_\infty^s, s \in S\}$ denote the total accumulated local time of X, which we assume exists. For all x in S and measurable functions F on $\mathbb{R}^n$, for all n,*

$$
\begin{aligned}
&E^x E\left(F\left(L_\infty^{\cdot} + \frac{(G(\cdot)+s)^2}{2}\right)\right) \\
&\qquad = E\left(\left(1+\frac{G(x)}{s}\right)F\left(\frac{(G(\cdot)+s)^2}{2}\right)\right)
\end{aligned} \tag{A.17}
$$

for all $s \neq 0$.

PROOF. To prove this theorem it suffices to show that

$$E^x \exp\left(\sum_{i=1}^n \lambda_i L_\infty^{x_i}\right) = 1 + \frac{E(G_x \exp(\sum_{i=1}^n \lambda_i (G_{x_i}+s)^2/2))}{sE\exp(\sum_{i=1}^n \lambda_i (G_{x_i}+s)^2/2)} \tag{A.18}$$

for all $\lambda_1, \dots, \lambda_n$ sufficiently small where $x = x_1$. Expanding the squares $(G_{x_i} + s)^2$ and cancelling the terms in s^2, we see that the term to the right of the plus sign in (A.18) is equal to

$$\text{(A.19)} \qquad \frac{E(G_{x_1} \exp(s \sum_{i=1}^n \lambda_i G_{x_i}) \exp(\sum_{i=1}^n \lambda_i G_{x_i}^2/2))}{sE(\exp(s \sum_{i=1}^n \lambda_i G_{x_i}) \exp(\sum_{i=1}^n \lambda_i G_{x_i}^2/2))}.$$

By Remark 4.1, we can write (A.19) in the form

$$\text{(A.20)} \qquad \frac{\widetilde{E}(G_{x_1} \exp(s \sum_{i=1}^n \lambda_i G_{x_i}))}{s\widetilde{E}(\exp(s \sum_{i=1}^n \lambda_i G_{x_i}))},$$

where $\widetilde{E}$ signifies that, in this formulation, $\{G_{x_i}, i = 1, \dots, n\}$ is a mean zero Gaussian process with covariance $\widetilde{\Sigma} = (I - \Sigma\Lambda)^{-1}\Sigma$. By Lemma 4.2, (A.20) is equal to

$$\text{(A.21)} \qquad \sum_{i=1}^n \widetilde{\Sigma}_{1,i}\lambda_i = \{\widetilde{\Sigma}\Lambda\mathbf{1}^t\}_1.$$

Therefore to prove this theorem it suffices to show that

$$\text{(A.22)} \qquad E^x \exp\left(\sum_{i=1}^n \lambda_i L_\infty^{x_i}\right) = 1 + \{\widetilde{\Sigma}\Lambda\mathbf{1}^t\}_1.$$

This is indeed the case since, by (2.4) with $x = x_1$,

$$\text{(A.23)} \qquad \begin{aligned} E^{x_1} \exp\left(\sum_{i=1}^n \lambda_i L_\infty^{x_i}\right) &= \{I + (I - \Sigma\Lambda)^{-1}\Sigma\Lambda\mathbf{1}^t\}_1 \\ &= 1 + \{(I - \Sigma\Lambda)^{-1}\Sigma\Lambda\mathbf{1}^t\}_1 \\ &= 1 + \{\widetilde{\Sigma}\Lambda\mathbf{1}^t\}_1. \end{aligned}$$

□

We now present some equalities which combine Theorems A.1 and A.2. To simplify the expressions we make the following change of notation. Let m be a finite discrete measure on S of the form $\sum_{i=1}^n \lambda_i \delta_{x_i}(\cdot)$, where λ_i are real or complex and are such that the relationships given exist. In this notation we write (A.13) as

$$\text{(A.24)} \qquad v(x, y)E^{x,y} \exp\left(\int L_\infty^r \, dm(r)\right) = \frac{E(G_x G_y \exp(\int G_r^2 dm(r)/2))}{E \exp(\int G_r^2 dm(r)/2)}$$

and (A.22) as

$$\text{(A.25)} \qquad E^x \exp\left(\int L_\infty^r \, dm(r)\right) = 1 + \int \widetilde{\Sigma}_{1,y} m(dy)$$

where $x = x_1$. [We use $\{L_\infty^s, s \in S\}$ to indicate the total accumulated local time of the Markov process under consideration. Which process that is is indicated by the probability measure. Thus in (A.24) we are considering the total accumulated local time of W whereas in (A.25) we are considering the total accumulated local time of X.]

The next result gives a relation between the conditioned (Theorem A.1) and the unconditioned (Theorem A.2) isomorphism theorems.

THEOREM A.3.

$$\begin{aligned} &E^x \exp\left(\int L^r_\infty\, dm(r)\right) \\ &\quad = 1 + \int \left(E^{x,y} \exp\left(\int L^r_\infty\, dm(r)\right)\right) v(x,y)\, dm(y). \end{aligned} \tag{A.26}$$

PROOF. This follows from (A.25) and (A.14). □

We can use the above isomorphism theorems to obtain interesting formulas for local times. Let X be a transient symmetric Markov process with 0-potential density $v(r,s)$. Let Y be the process X killed at T_0 and denote its zero potential by $w(r,s)$. Thus $w(r,s) = v(r,s) - v(r,0)v(s,0)/v(0,0)$. Let $f_x = v(x,0)/v(0,0)$.

THEOREM A.4. *Under the above conditions*,

$$\begin{aligned} &E^0 \exp\left(\int L^r_{\tau(L^0_\infty \wedge t^-)}\, dm(r)\right) \\ &\quad = e^{tA} e^{-t/v(0,0)} + \frac{1}{1 - v(0,0)A}\left(1 - e^{-t/v(0,0)}\right) \end{aligned} \tag{A.27}$$

where $L^r_{\tau(L^0_\infty \wedge t^-)}$ *is defined in* (9.3) *and*

$$\begin{aligned} A = &\int f_x^2\, dm(x) \\ &+ \iint E^{x,y} \exp\left(\int L^r_{T_0}\, dm(r)\right) w(x,y) f_x f_y\, dm(x)\, dm(y). \end{aligned} \tag{A.28}$$

Now let X be a recurrent symmetric Markov process. Let Y be the process X killed at T_0 and denote it's zero potential by $w(r,s)$.

THEOREM A.5. *Under the above conditions*,

$$E^0 \exp\left(\int L^r_{\tau(t)}\, dm(r)\right) = e^{tB}, \tag{A.29}$$

where $L^r_{\tau(t)}$ *is defined in* (9.9) *and*

$$\begin{aligned} B &= \int dm(r) + \iint E^{x,y} \exp\left(\int L^r_{T_0}\, dm(r)\right) w(x,y)\, dm(x)\, dm(y) \\ &= \int E^x \exp\left(\int L^r_{T_0}\, dm(r)\right) dm(x). \end{aligned} \tag{A.30}$$

When X is transient we have a simple formula for $w(r,s)$ in terms of $v(r,s)$. When X is recurrent the situation isn't so simple. In Section 6 of [4], $w(r,s)$ is obtained for recurrent symmetric Lévy processes.

PROOF OF THEOREM A.4. Under P^0, L^0_∞, the total accumulated local time of X at zero, is an exponential random variable with mean $v(0,0)$. Let ρ be an independent random variable equal in law to L^0_∞. It follows from (9.3) that

$$\begin{aligned}E^0\exp\biggl(\int L^r_{\tau(L^0_\infty\wedge t^-)}\,dm(r)\biggr)\\ = \frac{E\exp(\int(\eta_r+f_r\sqrt{2(\rho\wedge t)})^2\,dm(r)/2)}{E\exp(\int\eta_r^2\,dm(r)/2)}\end{aligned}\tag{A.31}$$

where η_r is a mean zero Gaussian process with covariance $w(r,s)$. Furthermore, by Lemma 4.1, we see that the second line of (A.31) equals

$$E\exp\biggl((\rho\wedge t)\biggl(\int f_x^2\,dm(x)+\int\int\widetilde{w}(x,y)f_xf_y\,dm(x)\,dm(y)\biggr)\biggr).\tag{A.32}$$

To understand what $\widetilde{w}$ is, recall that the support of m is a finite set $x_1,\dots,x_n$. Thus we are really considering the matrix $\widetilde{W}$ with elements $\{\widetilde{w}(x_i,x_j)\}_{i,j=1}^n$. We see from Lemma 4.1 that $\widetilde{W}=(W^{-1}-\Lambda)^{-1}$ where W is the matrix with elements $\{w(x_i,x_j)\}_{i,j=1}^n$ and Λ is a diagonal matrix with elements $\Lambda_{j,j}=m(\{x_j\})$. We now see by (A.14) that

$$\{\widetilde{W}\}_{j,k}=\frac{E(\eta_{x_j}\eta_{x_k}\exp(\sum_{i=1}^n\lambda_i\eta_{x_i}^2/2))}{E\exp(\sum_{i=1}^n\lambda_i\eta_{x_i}^2/2)}.\tag{A.33}$$

Therefore, using (A.13), we get

$$E^0\exp\biggl(\int L^r_{\tau(L^0_\infty\wedge t^-)}\,dm(r)\biggr)=E\exp\bigl((\rho\wedge t)A\bigr)\tag{A.34}$$

since the covariance of η is the 0-potential of Y. Equation (A.27) follows easily from (A.34). $\square$

PROOF OF THEOREM A.5. It follows from (9.9) that

$$E^0\exp\biggl(\int L^r_{\tau(t)}\,dm(r)\biggr)=\frac{E\exp(\int(\eta_r+\sqrt{2t})^2\,dm(r)/2)}{E\exp(\int\eta_r^2\,dm(r)/2)}\tag{A.35}$$

where η_r is a mean zero Gaussian process with covariance $w(r,s)$. Thus (A.32) holds in this case with $f_x\equiv 1$ and $\rho=\infty$. Mimicking the rest of the proof of Theorem A.4, we get the first equation in (A.30). To get the second, we note that by Theorem A.3,

$$
\begin{aligned}
&E^x \exp\left(\int L_{T_0}^r \, dm(r)\right) \\
&\qquad = 1 + \int \left(E^{x,y} \exp\left(\int L_{T_0}^r \, dm(r)\right)\right) w(x,y) \, dm(y).
\end{aligned}
\tag{A.36}
$$

Using this in the first equation in (A.30) we get the second. □

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