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INTEGRATION BY PARTS ON δ-BESSEL BRIDGES, $\delta > 3$ AND RELATED SPDEs

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We study a white-noise driven semilinear partial differential equation on the spatial interval [0, 1] with Dirichlet boundary condition and with a singular drift of the form cu^{-3}, $c > 0$. We prove existence and uniqueness of a non-negative continuous adapted solution u on $[0, \infty) \times [0, 1]$ for every nonnegative continuous initial datum x, satisfying $x(0) = x(1) = 0$. We prove that the law π_δ of the Bessel bridge on [0, 1] of dimension $\delta > 3$ is the unique invariant probability measure of the process $x \mapsto u$, with $c = (\delta - 1)(\delta - 3)/8$ and, if $\delta \in \mathbb{N}$, that u is the radial part in the sense of Dirichlet forms of the $\mathbb{R}^\delta$-valued solution of a linear stochastic heat equation. An explicit integration by parts formula w.r.t. π_δ is given for all $\delta > 3$.

1. Introduction. We are concerned with the following white-noise driven stochastic partial differential equation (SPDE) on the spatial interval [0, 1]:

$$
\begin{cases}
\dfrac{\partial u_\delta}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u_\delta}{\partial \theta^2} + \dfrac{(\delta-1)(\delta-3)}{8(u_\delta)^3} + \dfrac{\partial^2 W}{\partial t\,\partial\theta}, \\
u_\delta(t,0) = u_\delta(t,1) = 0, \qquad t \geq 0, \\
u_\delta(0,\theta) = x(\theta), \qquad \theta \in [0,1],
\end{cases} \tag{1}
$$

where $x : [0, 1] \mapsto [0, \infty)$ is continuous and satisfies $x(0) = x(1) = 0$, W is a Brownian sheet and $\delta > 3$.

In this paper we prove first that, for all $\delta > 3$, there exists a unique continuous nonnegative solution u_δ of (1) on $[0, \infty) \times [0, 1]$ such that $(u_\delta)^{-3} \in L^1_{\mathrm{loc}}([0, \infty) \times (0, 1))$, and that u_δ is adapted. Notice that the nonlinearity in (1) is singular enough to make the standard techniques noneffective.

Secondly, we study the ergodicity of the solution of (1): we prove that the process $x \mapsto u_\delta$ is symmetric w.r.t. the law π_δ of the δ-dimensional Bessel bridge on [0, 1] and that π_δ is the unique invariant probability measure of $x \mapsto u_\delta$.

One of the main tools is the following integration by parts formula w.r.t. the probability measure π_δ, $\delta > 3$:

$$
\int_{K_0} \partial_h \varphi \, d\pi_\delta = -\int_{K_0} \varphi(x)\left(\langle x, h''\rangle + \frac{(\delta-1)(\delta-3)}{4}\langle x^{-3}, h\rangle\right)\pi_\delta(dx), \tag{2}
$$

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where $\varphi : L^2(0,1) \mapsto \mathbb{R}$ is Fréchet differentiable with bounded gradient, $h : [0,1] \mapsto \mathbb{R}$ is twice continuously differentiable with compact support in $(0,1)$ and h'' is the second derivative of h, $\partial_h \varphi$ is the directional derivative of φ along $h \in L^2(0,1)$ and $\langle \cdot, \cdot \rangle$ is the canonical scalar product in $L^2(0,1)$. This result allows us to prove that $x \mapsto u_\delta$ is a gradient system, that is, it is the diffusion associated with the symmetric Dirichlet form with state space $K_0 := \{x \in L^2(0,1), x \geq 0\}$:

$$W^{1,2}(\pi_\delta) \ni \varphi, \psi \mapsto \mathcal{D}^\delta(\varphi, \psi) := \frac{1}{2} \int_{K_0} \langle \nabla \varphi, \nabla \psi \rangle \, d\pi_\delta$$

where ∇ denotes the gradient in the Hilbert space $H := L^2(0,1)$.

Finally, if $\delta \in \mathbb{N} \cap [4, \infty)$, we prove that the process $x \mapsto u_\delta$ is the radial part in the sense of Dirichlet forms of the Gaussian process Z_δ, solution of the $\mathbb{R}^\delta$-valued linear SPDE:

$$\begin{cases} \dfrac{\partial Z_\delta}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 Z_\delta}{\partial \theta^2} + \dfrac{\partial^2 \overline{W}}{\partial t \, \partial \theta}, \\ Z_\delta(t, \overline{x})(0) = Z_\delta(t, \overline{x})(1) = 0, \qquad t \geq 0, \\ Z_\delta(0, \overline{x}) = \overline{x}, \end{cases} \tag{3}$$

where $\overline{x} \in L^2(0,1; \mathbb{R}^\delta)$, $\overline{W} := (W_1, W_2, \dots, W_\delta) \mapsto \mathbb{R}^\delta$, and $\{W_i\}_{i=1,\dots,\delta}$ are independent copies of W. By this we mean the following: it is well known that Z_δ is associated with the Dirichlet form $(\Lambda^\delta, W^{1,2}(\mu_\delta))$ on $H^\delta = L^2(0,1; \mathbb{R}^\delta)$:

$$W^{1,2}(\mu_\delta) \ni F, G \mapsto \Lambda^\delta(F, G) := \frac{1}{2} \int_{H^\delta} \langle \overline{\nabla} F, \overline{\nabla} G \rangle_{H^\delta} \, d\mu_\delta$$

where μ_δ is the law on $L^2(0,1)$ of a Brownian bridge of dimension δ over $[0,1]$, $F, G : H^\delta \mapsto \mathbb{R}$ and $\overline{\nabla} F : H^\delta \mapsto H^\delta$ is the gradient of F in H^δ. We set

$$\Phi_\delta : H^\delta \mapsto K_0, \qquad \Phi_\delta(y)(\tau) := |y(\tau)|_{\mathbb{R}^\delta}, \qquad \tau \in [0,1].$$

Then we prove that $\mathcal{D}^\delta$ is the image of Λ^δ under the map Φ_δ, that is, π_δ is the image of μ_δ under Φ_δ and

$$\begin{aligned} W^{1,2}(\pi_\delta) &= \{\varphi \in L^2(\pi_\delta) : \varphi \circ \Phi_\delta \in W^{1,2}(\mu_\delta)\}, \\ \mathcal{D}^\delta(\varphi, \psi) &= \Lambda^\delta(\varphi \circ \Phi_\delta, \psi \circ \Phi_\delta) \qquad \forall\, \varphi, \psi \in W^{1,2}(\pi_\delta). \end{aligned}$$

In [12], Nualart and Pardoux proved existence and uniqueness of a pair (u_3, η), where u_3 is a continuous function of $(t, \theta) \in \mathcal{O} := [0, +\infty) \times [0,1]$ and η is a measure on $\mathcal{O}$, solving the SPDE with reflection:

$$\begin{cases} \dfrac{\partial u_3}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 u_3}{\partial \theta^2} + \dfrac{\partial^2 W}{\partial t \, \partial \theta} + \eta, \\ u_3(0, \cdot) = x, \quad u_3(t, 0) = u_3(t, 1) = 0, \\ u_3 \geq 0, \ d\eta \geq 0, \ \displaystyle\int_{\mathcal{O}} u_3 \, d\eta = 0. \end{cases} \tag{4}$$

[See Section 3.] In [16] and [17], we proved that the process $x \mapsto u_3$ is symmetric w.r.t. the law π_3 of the three-dimensional Bessel bridge on [0, 1], π_3 is the unique invariant probability measure of $x \mapsto u_3$, and $x \mapsto u_3$ is the diffusion associated with the Dirichlet form $(\mathcal{D}^3, W^{1,2}(\pi_3))$,

$$W^{1,2}(\pi_3) \ni \varphi, \psi \mapsto \mathcal{D}^3(\varphi, \psi) := \tfrac{1}{2} \int_{K_0} \langle \nabla\varphi, \nabla\psi\rangle \, d\pi_3,$$

where ∇ denotes the gradient in H. One of the key tools was the following integration by parts formula w.r.t. the probability measure π_3 on $L^2(0, 1)$:

$$\int_{K_0} \partial_h \varphi \, d\pi_3 = -\int_{K_0} \varphi(x)\langle x, h''\rangle \, d\pi_3 - \int_0^1 dr\, h(r) \int_{K_0} \varphi(x)\sigma_0(r, dx), \tag{5}$$

where the measure $\sigma_0(r, \cdot)$ is explicitly defined in terms of two independent three-dimensional Bessel bridges, respectively on $[0, r]$ and on $[0, 1-r]$, glued at $r \in (0, 1)$; see (15) below. The last term of (5) was interpreted as a boundary term and applied to characterize η as a family of additive functionals of u_3. Finally, we proved that $x \mapsto u_\delta$ is the radial part in the sense of Dirichlet forms of the Gaussian process Z_3, solution of the $\mathbb{R}^3$-valued SPDE (3) above with $\delta = 3$.

Mueller in [10] and Mueller and Pardoux in [11] considered the following SPDE with periodic boundary condition:

$$\begin{cases} \dfrac{1}{2}\dfrac{\partial \hat{u}}{\partial t} = \dfrac{\partial^2 \hat{u}}{\partial \theta^2} + \hat{u}^{-\alpha} + g(\hat{u})\dfrac{\partial^2 W}{\partial t\, \partial\theta}, & t \geq 0,\ \theta \in \mathbb{S}^1 := \mathbb{R}/\mathbb{Z}, \\ \hat{u}(0, \cdot) = \hat{x}, \end{cases}$$

where $\alpha > 0$, $\hat{x} : \mathbb{S}^1 \mapsto \mathbb{R}$ is continuous, $\inf \hat{x} > 0$ and g satisfies suitable growth conditions, and proved that $\alpha = 3$ is the critical exponent for $\hat{u}$ to hit zero in finite time. More precisely, the following was proved:

1. If $\alpha > 3$, then a.s. $\hat{u}(t, \theta) > 0$ for all $t \geq 0$, $\theta \in \mathbb{S}^1$.
2. If $\alpha < 3$, then with positive probability, there exist $t > 0$, $\theta \in \mathbb{S}^1$, such that $\hat{u}(t, \theta) = 0$.

It seems that the critical case $\alpha = 3$ is treated here for the first time. Our result says that the solution of (1) can possibly hit 0 in (0, 1) in a finite time, but in a way that the nonlinearity u^{-3} does not blow up in $L^1_{\mathrm{loc}}([0, \infty) \times (0, 1))$, so that we have existence for all times. Notice that we do not require any strict positivity of the initial datum: our result cover even the case of $x \equiv 0$.

The results presented above allow also to prove that for all continuous $x : [0, 1] \mapsto [0, \infty)$ with $x(0) = x(1) = 0$, for all $\alpha \geq 3$ and $C > 0$ the following

SPDE admits a unique continuous nonnegative adapted solution $\hat{u}_\alpha$, being well defined for all $t \geq 0$:

$$\begin{cases} \dfrac{\partial \hat{u}_\alpha}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 \hat{u}_\alpha}{\partial \theta^2} + \dfrac{C}{(\hat{u}_\alpha)^\alpha} + \dfrac{\partial^2 W}{\partial t\, \partial \theta}, \\ \hat{u}_\alpha(t,0) = \hat{u}_\alpha(t,1) = 0, \qquad t \geq 0, \\ \hat{u}_\alpha(0,\cdot) = x, \end{cases} \tag{6}$$

while for all $0 \leq \alpha < 3$ and $C \geq 0$ the following SPDE of Nualart–Pardoux type admits a unique solution $(\hat{u}_\alpha, \hat{\eta}_\alpha)$:

$$\begin{cases} \dfrac{\partial \hat{u}_\alpha}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 \hat{u}_\alpha}{\partial \theta^2} + \dfrac{C}{(\hat{u}_\alpha)^\alpha} + \dfrac{\partial^2 W}{\partial t\, \partial \theta} + \hat{\eta}_\alpha, \\ \hat{u}_\alpha(0,\cdot) = x, \quad \hat{u}_\alpha(t,0) = \hat{u}_\alpha(t,1) = 0, \qquad t \geq 0, \\ \hat{u}_\alpha \geq 0,\ d\hat{\eta}_\alpha \geq 0,\ \displaystyle\int_{\mathcal{O}} \hat{u}_\alpha\, d\hat{\eta}_\alpha = 0, \end{cases} \tag{7}$$

and $\hat{\eta}_\alpha \neq 0$.

The family $(u_\delta)_{\delta \geq 3}$, defined by (1) and (4), reveal several analogies with the family of Bessel processes $(\rho_\delta)_{\delta \geq 1}$. Indeed, recall that:

1. If $(B_t)_{t \geq 0}$ is a linear BM and $x \geq 0$, then, for all $\delta > 1$, there exists a unique continuous nonnegative solution $(\rho_\delta(t,x))_{t \geq 0}$ of the SDE:

$$d\rho_\delta = \frac{\delta - 1}{2\rho_\delta}\, dt + dB, \qquad t \geq 0,\ \rho_\delta(0,x) = x, \tag{8}$$

   and, for $\delta = 1$, there exists a unique pair (ρ_1, L), with $t \mapsto \rho_1(t,x)$ continuous and nonnegative and $t \mapsto L(t,x)$ continuous and monotone nondecreasing, satisfying

$$d\rho_1 = dL + dB, \qquad \rho_1(0,x) = x,\ L(0,x) = 0,\ \int \rho_1\, d_t L = 0. \tag{9}$$

   For all $\delta \geq 1$, $\rho_\delta = (\rho_\delta(t,x))_{t \geq 0, x \geq 0}$ is called the δ-dimensional Bessel process.
2. The process ρ_δ is the diffusion associated with the Dirichlet form:

$$W^{1,2}\big([0,\infty), x^{\delta-1}\, dx\big) \ni f, g \mapsto \gamma^\delta(f,g) := \frac{\omega_\delta}{2}\int_0^\infty f'(x) g'(x) x^{\delta-1}\, dx,$$

   where $\omega_\delta := \pi^{\delta/2}/\,\Gamma(1 + \delta/2)$.
3. If $\delta \in \mathbb{N} \cap [1,\infty)$ and $(B_\delta(t))_{t \geq 0}$ is a δ-dimensional Brownian motion, then ρ_δ is characterized as the radial part in the sense of Dirichlet forms of B_δ; that is, the Dirichlet form γ^δ, generating ρ_δ, is the image of

$$W^{1,2}(\mathbb{R}^\delta) \ni F, G \mapsto \tfrac{1}{2}\int_{\mathbb{R}^\delta} \langle \nabla F, \nabla G \rangle\, dx$$

   under the map $\mathbb{R}^\delta \ni y \mapsto |y| \in [0,\infty)$. Notice that, in this case, it is even true that the law of ρ_δ is equal to the law of $|B_\delta|$.

4. For all $\alpha \geq 1$ and $c > 0$ there exists a unique continuous nonnegative solution ρ of the following SDE:

$$d\rho = \frac{c}{(\rho)^\alpha}\, dt + dB, \qquad t \geq 0,\ \rho(0) \geq 0, \tag{10}$$

while for all $0 < \alpha < 1$ and $c \geq 0$ there exists a unique pair (ρ, L) such that: $\rho(\cdot)$ is continuous nonnegative, $L(\cdot)$ is continuous and monotone nondecreasing,

$$d\rho = \frac{c}{(\rho)^\alpha}\, dt + dL + dB, \qquad \rho(0) = x,\ L(0) = 0,\ \int \rho\, d_t L = 0 \tag{11}$$

and moreover, $L \neq 0$.

5. The following integration by parts formulae hold for the invariant measure $\mathbb{1}_{[0,\infty)}(x)x^{\delta-1}\, dx$ of ρ_δ:

$$\int_0^\infty f'(x)x^{\delta-1}\, dx = -\int_0^\infty f(x)\frac{\delta-1}{x}x^{\delta-1}\, dx, \qquad \delta > 1, \tag{12}$$

$$\int_0^\infty f'(x)\, dx = -f(0) \qquad \forall\, f \in C_0^\infty([0,\infty)). \tag{13}$$

In particular, in the critical case $\delta = 1$ a boundary term appears, while for $\delta > 1$ only a logarithmic-derivative term appears.

Notice that the exponent in the nonlinear term of (8) is equal to -1, that is, to minus the critical dimension for (10)–(11) and (12)–(13): the same happens for the exponent in the nonlinear term of (1), which is equal to -3, that is, to minus the critical dimension for (6)–(7) and (2)–(5). Moreover, the maps

$$(1,\infty) \ni \delta \mapsto \frac{\delta-1}{2} \in (0,\infty), \qquad (3,\infty) \ni \delta \mapsto \frac{(\delta-1)(\delta-3)}{8} \in (0,\infty)$$

are both increasing and bijective.

For a general theory of integration by parts formulae in infinite dimension, see [9]. For integration by parts formulae and infinite-dimensional Dirichlet forms on stationary Bessel processes, see [7]. Part of the results of this paper has been announced in [18].

The paper is organized as follows. In Section 2 we prove the integration by parts formula (2). Section 3 is devoted to the study of equation (1). In Section 4 we study equations (6) and (7).

We fix some notation: We set $H := L^2(0,1)$ and we denote the canonical scalar product in H by $\langle\cdot,\cdot\rangle$ and the associated norm by $\|\cdot\|$. We set $K_0 := \{x \in H, x \geq 0\}$, $\mathcal{O} := [0,+\infty) \times [0,1]$ and

$$C_a := C_a(0,1) := \{c : [0,1] \mapsto \mathbb{R} \text{ continuous},\ c(0) = c(1) = a\}, \qquad a \geq 0,$$

$$A : D(A) \subset H \mapsto H, \qquad D(A) := W^{2,2} \cap W_0^{1,2}(0,1), \qquad A := \frac{1}{2}\frac{\partial^2}{\partial\theta^2}.$$

We also denote by $C_c^2(0,1)$ the set of all $h : [0,1] \mapsto \mathbb{R}$, being twice continuously differentiable and with compact support in $(0,1)$. By $W = \{W(t,\theta) : (t,\theta) \in \mathcal{O}\}$ we denote a two-parameter Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$; that is, W is a Gaussian process with zero mean and covariance function

$$\mathbb{E}[W(t,\theta)W(t',\theta')] = (t \wedge t')(\theta \wedge \theta'), \qquad (t,\theta), (t',\theta') \in \mathcal{O}.$$

We denote by $\mathcal{F}_t$ the σ-field generated by the random variables $\{W(s,\theta) : (s,\theta) \in [0,t] \times [0,1]\}$. Moreover we set:

- $x^{a,b}_{\delta,r}$, for $a, b \geq 0$, $\delta \geq 2$ and $r \in]0,1]$, is a δ-Bessel bridge between a and b over $[0,r]$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and independent of W: see Chapter XI of [13].
- π^a_δ, $\delta \geq 2$, $a \geq 0$, is the law on $L^2(0,1)$ of $x^a_\delta := x^{a,a}_{\delta,1}$. Moreover, $\pi_\delta := \pi^0_\delta$.
- Let $r \in (0,1)$. For $y \in C([0,r])$ and $z \in C([0,1-r])$ we set

$$y \underset{r}{\oplus} z \in H, \qquad \Big[y \underset{r}{\oplus} z\Big](\tau) := y(\tau)\mathbb{1}_{[0,r]}(\tau) + z(\tau - r)\mathbb{1}_{(r,1]}(\tau).$$

Then we define for all $\varphi \in C_{\mathrm{b}}(H)$, $a > 0$, $r \in (0,1)$:

$$\int \varphi(x)\sigma_a(r,dx) := \frac{\sqrt{2}a^2 e^{-a^2/(2r(1-r))}}{\sqrt{\pi r^3(1-r)^3}(1-e^{-2a^2})}\mathbb{E}\Big[\varphi\Big(x^{a,0}_{3,r} \underset{r}{\oplus} \hat{x}^{0,a}_{3,1-r}\Big)\Big], \tag{14}$$

$$\int \varphi(x)\sigma_0(r,dx) := \frac{1}{\sqrt{2\pi r^3(1-r)^3}}\mathbb{E}\Big[\varphi\Big(x^{0,0}_{3,r} \underset{r}{\oplus} \hat{x}^{0,0}_{3,1-r}\Big)\Big], \tag{15}$$

where $\{x^{b,c}_{3,r}, \hat{x}^{c,b}_{3,1-r}\}$ are independent, and $\{x^{b,c}_{3,r}, \hat{x}^{b,c}_{3,r}\}$ are identically distributed, $r \in (0,1)$, $b, c \geq 0$. We introduce the following function spaces:

- $C_{\mathrm{b}}(H)$ is the space of all $\varphi : H \mapsto \mathbb{R}$ being bounded and uniformly continuous in the norm of H. The space $C^1_{\mathrm{b}}(H)$ is that of Fréchet differentiable $\varphi \in C_{\mathrm{b}}(H)$ with bounded and continuous gradient $\nabla\varphi : H \mapsto H$.
- $\mathrm{Exp}(H)$ is the linear span of $\{1, \cos(\langle\cdot, h\rangle), \sin(\langle\cdot, h\rangle) : h \in C^2_c(0,1)\}$.
- $\mathrm{Lip}(H)$ is the space of all $\varphi : H \mapsto \mathbb{R}$ such that

$$\|\varphi\|_{\mathrm{Lip}} := \sup_x |\varphi(x)| + \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|} < \infty.$$

  $\mathrm{Lip}(K_0)$ is the set of $\varphi : K_0 \mapsto \mathbb{R}$ such that $H \ni x \mapsto \varphi(x^+)$ is in $\mathrm{Lip}(H)$, where $x^+(\tau) := \sup\{x(\tau), 0\}$, $\tau \in [0,1]$.
- $C^1_{\mathrm{b}}(K_0)$ is the set of all $\varphi \in \mathrm{Lip}(K_0)$ such that there exists a bounded continuous vector field $\nabla\varphi : K_0 \mapsto H$, which we call the gradient of φ, satisfying

$$\lim_{t \downarrow 0} \frac{1}{t}\big(\varphi(x + th) - \varphi(x)\big) = \langle\nabla\varphi(x), h\rangle \qquad \forall\, x, h \in K_0.$$

2. Integration by parts on the δ-Bessel bridge. The aim of this section is to prove the following.

THEOREM 1. *For all $\delta > 3$, $a \ge 0$, $\varphi \in C^1_{\mathrm{b}}(H)$ and $h \in C^2_c(0,1)$, we have*

$$\int_{K_0} \partial_h \varphi \, d\pi^a_\delta = -\int_{K_0} \varphi(x)\left(\langle x, h''\rangle + \frac{(\delta-1)(\delta-3)}{4}\langle x^{-3}, h\rangle\right)\pi^a_\delta(dx). \tag{16}$$

We set

$$\kappa(\delta) := \frac{(\delta-1)(\delta-3)}{4}.$$

We recall the following result, proved in (1)–(2) and Remark 2 of [17]:

THEOREM 2. *For all $\varphi \in C^1_{\mathrm{b}}(K_0)$, $a \ge 0$ and $h \in C^2_c(0,1)$, we have*

$$\int_{K_0} \partial_h \varphi \, d\pi^a_3 = -\int_{K_0} \varphi(x)\langle x, h''\rangle \, d\pi^a_3 - \int_0^1 dr \, h(r) \int \varphi(x)\sigma_a(r, dx). \tag{17}$$

LEMMA 1. *Let $(B(t))_{t\in[0,1]}$ a Brownian motion. For all $a \ge 0$ and $\delta \ge 2$ there exists a unique continuous $(x^a_\delta(t))_{t\in[0,1]}$, adapted to the filtration of B, such that for all $t \in (0,1)$: $x^a_\delta(t) > 0$ and*

$$x^a_\delta(t) = a + B(t) + \int_0^t \left[\frac{\delta-1}{2x^a_\delta(s)} - \frac{x^a_\delta(s)}{1-s} + \gamma(1-s, x^a_\delta(s), a)\right] ds, \tag{18}$$

where for $t > 0$, $y, b \ge 0$,

$$\gamma(t, y, b) := \frac{\partial}{\partial y} \log \int_0^{\frac{\pi}{2}} (\sin\phi)^{\delta-2} \cosh\left(\frac{yb}{t}\cos\phi\right) d\phi.$$

Moreover, x^a_δ is a Bessel bridge of dimension δ between a and a over $[0,1]$,

$$0 \le a \le a' \implies x^a_\delta(t) \le x^{a'}_\delta(t) \qquad \forall\, t \in [0,1], \text{ a.s.} \tag{19}$$

and $a \mapsto x^a_\delta$ is continuous in the sup-norm topology. If $\delta \in \mathbb{N}$, then x^0_δ is equal in law to the modulus of a Brownian bridge of dimension δ between 0 and 0 over $[0,1]$.

PROOF. Recall that the transition semigroup $(p_\delta(t,a,b))_{t\ge 0, a, b > 0}$ of the Bessel process of dimension $\delta \ge 2$ is

$$p_\delta(t,a,b) := \frac{1}{t}\left(\frac{b}{a}\right)^{\nu} b \exp\left(-\frac{a^2+b^2}{2t}\right) I_{\delta/2-1}\left(\frac{ab}{t}\right), \tag{20}$$

where I_ν is the modified Bessel function of index $\nu \ge 0$:

$$I_\nu(z) = \frac{2(\frac{1}{2}z)^\nu}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})} \int_0^{\frac{\pi}{2}} (\sin\phi)^{2\nu} \cosh(z\cos\phi)\, d\phi, \qquad z \ge 0;$$

see Chapter XI of [13]. By Girsanov's theorem, a Bessel bridge of dimension $\delta \ge 2$ between a and a over $[0,1]$ is a weak solution of (18); see XI.3.11 in [13]. Suppose that $a > 0$ and that $(x, \hat{B})$ is a weak solution of (18), where $\hat{B}$ is a Brownian motion. By Theorem IX.3.5 of [13], there exists a unique continuous process $(q(t))_{t\in[0,1]}$, adapted to the filtration of $\hat{B}$, such that

$$q(t) = [\delta]t + \int_0^t 2\sqrt{|q(s)|}\, d\hat{B}_s - \int_0^t \frac{2q(s)}{1-s}\, ds, \qquad t \in [0,1], \tag{21}$$

where $[\delta] \in \mathbb{N}$ and $[\delta] \le \delta < [\delta] + 1$. By Itô's formula, the square of the modulus of a Brownian bridge of dimension $[\delta]$ between 0 and 0 over $[0,1]$ is a weak solution of (21). By pathwise uniqueness we have uniqueness in law, so that q is equal in law to the square of the modulus of a Brownian bridge of dimension $[\delta]$ between 0 and 0 over $[0,1]$. In particular, $q(t) > 0$ for all $t \in (0,1)$; see Chapter XI of [13]. Then, setting $\hat{x} := \sqrt{q}$, we have $\hat{x} > 0$ for all $t \in (0,1)$ and by Itô's formula,

$$\hat{x}(t) = \hat{B}(t) + \int_0^t \left(\frac{[\delta]-1}{2\hat{x}(s)} - \frac{\hat{x}(s)}{1-s} \right) ds, \qquad t \in [0,1].$$

Since $x(0) = a > 0$, by continuity $T_0 := \inf\{t \in (0,1] : x(t) = 0\} > 0$ almost surely. Then for all $t \in [0, T_0)$, since $\gamma \ge 0$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big[\big(\hat{x}(t) - x(t)\big)^+\big]^2 \\
&\quad = \left(\frac{\delta-1}{2}\left(\frac{1}{\hat{x}(t)} - \frac{1}{x(t)} \right) - \frac{\delta - [\delta]}{2\hat{x}(t)} - \frac{\hat{x}(t) - x(t)}{1-t} - \gamma\big(1-t, x(t), a\big) \right) \\
&\qquad \times \big(\hat{x}(t) - x(t)\big)^+ \\
&\quad \le 0.
\end{aligned}$$

Since $x(0) \ge 0 = \hat{x}(0)$, we obtain $x(t) \ge \hat{x}(t)$ for all $t \in [0, T_0)$; since $\hat{x} > 0$ on $(0,1)$, then $x(t) > 0$ on $[0,1]$. Then, we have proved that every weak solution $(x, \hat{B})$ of (18) with $a > 0$, satisfies $x(t) > 0$ for all $t \in [0,1]$.

Therefore, we can prove pathwise uniqueness for (18) if $a > 0$. Indeed, let $(x^1, \hat{B})$ and $(x^2, \hat{B})$ be two weak solutions of (18) with the same driving Brownian motion $\hat{B}$. An explicit computation yields

$$\left| \frac{\partial \gamma(t,y,b)}{\partial y} \right| \le \frac{b^2}{t^2}, \qquad \frac{\partial \gamma(t,y,b)}{\partial b} \ge 0 \qquad \forall\, t > 0,\ b \ge 0,\ y \ge 0. \tag{22}$$

Then, since $x^1 > 0$ and $x^2 > 0$ on $(0,1)$, we have, for all $t \in (0,1)$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(x^1(t) - x^2(t)\big)^2 \\
&\quad = \left(\frac{\delta-1}{2}\left(\frac{1}{x^1(t)} - \frac{1}{x^2(t)} \right) - \frac{x^1(t) - x^2(t)}{1-t} \right.
\end{aligned}$$

$$+\gamma\big(1-t,x^1(t),a\big)-\gamma\big(1-t,x^2(t),a\big)\Big)\big(x^1(t)-x^2(t)\big)$$
$$\le \frac{a^2}{(1-t)^2}\big(x^1(t)-x^2(t)\big)^2$$

so that $x^1 \equiv x^2$. By Yamada–Watanabe's theorem, every weak solution is a strong solution, and for all $a>0$, $\delta \ge 2$, we have existence of a solution x_δ^a of (18), adapted to the filtration of the fixed Brownian motion $(B(t))_{t\in[0,1]}$; see [15]. Consider now $a' \ge a > 0$. By (22), arguing as in (23), we can prove that a.s. for all $t \in (0,1)$,

$$\frac{1}{2}\frac{d}{dt}\big[\big(x_\delta^a(t)-x_\delta^{a'}(t)\big)^+\big]^2 \le \frac{a^2}{(1-t)^2}\big[\big(x_\delta^a(t)-x_\delta^{a'}(t)\big)^+\big]^2$$

and since $(a-a')^+ = 0$, then $x_\delta^a \le x_\delta^{a'}$. If now $a_n \downarrow 0$, then we set $x_\delta^0 := \lim_n x_\delta^{a_n} \ge 0$. By the above considerations, x_δ^0 is a strong solution of (18) with $a=0$ and $x_\delta^0 > 0$ on $(0,1)$. In particular, x_δ^0 is continuous and by Dini's theorem $x_\delta^{a_n} \downarrow x_\delta^0$ uniformly on $[0,1]$. Arguing like in (23) we obtain pathwise uniqueness for $a=0$ in the class of continuous $(x(t))_{t\in[0,1]}$ such that $x>0$ on $(0,1)$. Continuity of $a \mapsto x_\delta^a$ follows analogously. If $\delta \in \mathbb{N}$, by Itô's formula $(x_\delta^0)^2$ is a weak solution of (21). Since uniqueness in law holds for (21), then x_δ^0 is equal in law to the modulus of a Brownian bridge of dimension δ between 0 and 0 over $[0,1]$. $\square$

REMARK 1. In the proof of Lemma 1, we proved pathwise uniqueness for (18) with $a=0$ in the class of continuous nonnegative $(x(t))_{t\in[0,1]}$. Notice that if we omit the requirement of the nonnegativity, then pathwise uniqueness does not hold for (18) with $a=0$. Indeed, notice that $\gamma(\cdot,\cdot,0)\equiv 0$. Let $\hat{B} := -B$ and call $\hat{x}_\delta^0$ the nonnegative strong solution of

$$\hat{x}_\delta^0(t) = \hat{B}(t) + \int_0^t \left(\frac{\delta-1}{2\hat{x}_\delta^0(s)} - \frac{\hat{x}_\delta^0(s)}{1-s}\right) ds, \qquad t\in[0,1],$$

obtained by Lemma 1. Then, (x_δ^0, B) and $(-\hat{x}_\delta^0, B)$ are different solutions of (18), since $x_\delta^0 > 0$ and $-\hat{x}_\delta^0 < 0$ on $(0,1)$. This shows that also the uniqueness in law fails for (18) if $a=0$. Nevertheless, every weak solution of (18) is a strong solution. Indeed, if (x,B) is a solution of (18) with $a=0$, then either $x\equiv x_\delta^0$ or $x \equiv -\hat{x}_\delta^0$.

PROOF OF THEOREM 1. We fix $\delta > 3$ and we let $\nu > 0$ such that $\delta = 2(\nu+1)$. Fix $a>0$: then π_δ^a is absolutely continuous w.r.t. π_2^a,

$$\pi_\delta^a(dx) = \frac{p_2(1,a,a)}{p_\delta(1,a,a)} \exp\left(-\frac{\nu^2}{2}\int_0^1 \frac{d\tau}{(x(\tau))^2}\right)\pi_2^a(dx),$$

where $(p_d(t,a,b))_{t,a,b\geq 0}$ is the transition semigroup of the Bessel process of dimension $d \geq 2$ defined in (20); see XI.1.22 in [13]. Then we have

$$\pi_\delta^a(dx) = \frac{p_3(1,a,a)}{p_\delta(1,a,a)} \exp\left(-\frac{\nu^2 - 1/4}{2}\int_0^1 \frac{d\tau}{(x(\tau))^2}\right)\pi_3^a(dx). \tag{23}$$

Notice that $\nu^2 - 1/4 = \kappa(\delta)$. We define

$$\gamma_\varepsilon(x) := \frac{p_3(1,a,a)}{p_\delta(1,a,a)} \exp\left(-\frac{\kappa(\delta)}{2}\int_0^1 \frac{d\tau}{(\varepsilon + x(\tau))^2}\right), \qquad x \in K_0. \tag{24}$$

Then γ_ε is in $C_{\mathrm{b}}^1(K_0)$ and for all $x, h \in K_0$,

$$\begin{aligned}\langle \nabla \log \gamma_\varepsilon(x), h\rangle &:= \lim_{t\downarrow 0} \frac{1}{t}\big(\log \gamma_\varepsilon(x+th) - \log \gamma_\varepsilon(x)\big)\\ &= \kappa(\delta)\int_0^1 \frac{1}{(\varepsilon + x(\tau))^3} h(\tau)\,d\tau.\end{aligned}$$

Let $h \in C_c^2(0,1) \cap K_0$. By (17) in Theorem 2, we obtain

$$\begin{aligned}\int_{K_0} \partial_h \varphi \gamma_\varepsilon \, d\pi_3^a = &-\int_{K_0} \varphi(x)\big[\langle x, h''\rangle + \langle \nabla \log \gamma_\varepsilon(x), h\rangle\big]\gamma_\varepsilon(x)\pi_3^a(dx)\\ &- \int_0^1 dr\ h(r) \int \varphi(x)\gamma_\varepsilon(x)\sigma_a(r, dx).\end{aligned}$$

For all $x \in K_0$ and $\varepsilon > 0$, we have

$$\gamma_\varepsilon(x)\langle \nabla \log \gamma_\varepsilon(x), h\rangle \leq \kappa(\delta)\frac{p_3(1,a,a)}{p_\delta(1,a,a)} \exp\left(-\frac{\kappa(\delta)}{2}\int_0^1 \frac{d\tau}{(x(\tau))^2}\right)\langle x^{-3}, h\rangle$$

and by (19),

$$\begin{aligned}&\int_{K_0} \frac{p_3(1,a,a)}{p_\delta(1,a,a)} \exp\left(-\frac{\kappa(\delta)}{2}\int_0^1 \frac{d\tau}{(x(\tau))^2}\right)\langle x^{-3}, h\rangle \pi_3^a(dx)\\ &\quad = \int_0^1 \mathbb{E}\big[\big(x_\delta^a(\tau)\big)^{-3}\big]h(\tau)d\tau \leq \int_0^1 \mathbb{E}\big[\big(x_\delta^0(\tau)\big)^{-3}\big]h(\tau)\,d\tau\\ &\quad = \int_0^1 d\tau\, \frac{h(\tau)}{[\tau(1-\tau)]^{\delta/2}} \int_0^\infty dy\, \frac{C_\delta y^{\delta-1}}{y^3} \exp\left\{-\frac{y^2}{2\tau(1-\tau)}\right\} < \infty\end{aligned}$$

since $\delta > 3$ and h has compact support in $(0,1)$. By the dominated convergence theorem, we obtain, for $a > 0$,

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0} \int_{K_0} \varphi(x)\big[\langle x, h''\rangle + \langle \nabla \log \gamma_\varepsilon(x), h\rangle\big]\gamma_\varepsilon(x)\pi_3^a(dx)\\ &\quad = \int_{K_0} \varphi(x)\big(\langle x, h''\rangle + \kappa(\delta)\langle x^{-3}, h\rangle\big)\pi_\delta^a(dx).\end{aligned}$$

Now we turn to the last term in (25). Notice that by (14),

$$\left|\int \varphi(x)\gamma_\varepsilon(x)\sigma_a(r,dx)\right| \le \frac{C(a)}{\sqrt{r^3(1-r)^3}}\|\varphi\|_\infty \psi^{a,0}_{\varepsilon,r}\psi^{0,a}_{\varepsilon,1-r} \tag{25}$$

where

$$\psi^{b,c}_{\varepsilon,r} := \mathbb{E}\left[\exp\left(-\frac{\kappa(\delta)}{2}\int_0^r \frac{d\tau}{(\varepsilon + x^{b,c}_{3,r}(\tau))^2}\right)\right] \le 1, \qquad b,c \ge 0,$$

since $\kappa(\delta) > 0$. By monotone convergence, for all $r \in (0,1)$,

$$\lim_{\varepsilon\downarrow 0}\psi^{0,a}_{\varepsilon,1-r} = \mathbb{E}\left[\exp\left(-\frac{\kappa(\delta)}{2}\int_0^{1-r} \frac{d\tau}{(x^{0,a}_{3,1-r}(\tau))^2}\right)\right] = 0, \tag{26}$$

since by the law of the iterated logarithm, a.s.,

$$\int_0^{r'} \frac{d\tau}{(x^{0,a}_{3,1-r}(\tau))^2} = +\infty \qquad \forall\, r' \in (0, 1-r],$$

and by the dominated convergence theorem we have that the last term in (25) tends to 0 as $\varepsilon \downarrow 0$. Then, (16) is proved for $a > 0$. Since

$$\mathbb{E}\big[\big\langle (x^0_\delta)^{-3}, h\big\rangle\big] = \int_0^1 \mathbb{E}\big[\big(x^0_\delta(\tau)\big)^{-3}\big]h(\tau)\,d\tau < \infty, \tag{27}$$

and since (16) can be written in the form

$$\mathbb{E}\big[\partial_h\varphi(x^a_\delta)\big] = -\mathbb{E}\big[\varphi(x^a_\delta)\big(\langle h'', x^a_\delta\rangle + \kappa(\delta)\big\langle h, (x^a_\delta)^{-3}\big\rangle\big)\big], \tag{28}$$

by (19) and by the dominated convergence theorem we can let $a \downarrow 0$ in (28) and obtain (16) for all $a \ge 0$. $\square$

COROLLARY 1. *Let $a \ge 0$. For all $\psi \in \mathrm{Lip}(H)$ there exists a field $\nabla\psi \in L^\infty(K_0, \pi^a_\delta; H)$ such that for all $h \in C^2_c(0,1)$,*

$$\lim_{t\downarrow 0}\frac{1}{t}\big(\psi(\cdot + th) - \psi\big) =: \partial_h\psi = \langle\nabla\psi, h\rangle \qquad \textit{weakly in } L^2(\pi^a_\delta).$$

We call $\nabla\psi$ the gradient of ψ. Then, (16) *holds for all $\varphi \in \mathrm{Lip}(H)$. Moreover, for all $\psi \in \mathrm{Lip}(H)$ and $\varphi \in \mathrm{Exp}(H)$, we have*

$$\tfrac{1}{2}\int_{K_0}\langle\nabla\psi, \nabla\varphi\rangle\,d\pi^a_\delta = -\int_{K_0}\psi L^a_\delta\varphi\,d\pi^a_\delta$$

where $L^a_\delta\varphi \in L^1(\pi^a_\delta)$ is defined as

$$L^a_\delta\varphi(x) := \frac{1}{2}\mathrm{Tr}\big[D^2\varphi(x)\big] + \langle x, A\nabla\varphi(x)\rangle + \frac{\kappa(\delta)}{2}\big\langle x^{-3}, \nabla\varphi(x)\big\rangle,$$

$$\pi^a_\delta\text{-a.e.}, x \in K_0.$$

PROOF. The family $\{(\psi(\cdot + th) - \psi))/t\}_{t>0}$ is bounded in $L^2(\pi_\delta^a)$. For all $\varphi \in \mathrm{Exp}(H)$:

$$\begin{aligned}\lim_{t\downarrow 0}\int_{K_0}\frac{1}{t}\big(\psi(\cdot + th) - \psi\big)\varphi\, d\pi_\delta^a \\ = -\int_{K_0}\psi\langle\nabla\varphi, h\rangle\, d\pi_\delta^a - \int_{K_0}\varphi\psi(x)\big(\langle x, h''\rangle + \kappa(\delta)\langle x^{-3}, h\rangle\big)\pi_\delta^a(dx).\end{aligned}\tag{29}$$

Indeed, by (16), (29) holds for all $\psi \in C_{\mathrm{b}}^1(H)$; moreover, the family of functionals

$$C_{\mathrm{b}}^1(H) \ni \psi \mapsto \int_{K_0}\frac{1}{t}\big(\psi(\cdot + th) - \psi\big)\varphi\, d\pi_\delta^a, \qquad t > 0,$$

is uniformly bounded in the sup-norm, by (16). By the density of $C_{\mathrm{b}}^1(H)$ in $C_{\mathrm{b}}(H)$ in the sup-norm, we obtain (29) for all $\psi \in C_{\mathrm{b}}(H)$. Then, (29) allows us to identify all limit points of $(\psi(\cdot + th) - \psi))/t$ in the weak topology of $L^2(\pi_\delta^a)$ as $t \downarrow 0$. The last formula follows from (16). $\square$

3. SPDE generated by the δ-Bessel bridge. This section is devoted to the proof of the following:

THEOREM 3. *Let $\delta > 3$ and $a \geq 0$.*

(i) *For all $x \in K_0 \cap C_a$, there exists a unique random continuous nonnegative $u_\delta^a : [0,\infty) \times [0,1] \mapsto [0,\infty)$, such that $(u_\delta^a)^{-3} \in L_{\mathrm{loc}}^1([0,\infty) \times (0,1))$, solving the SPDE*

$$\begin{cases}\dfrac{\partial u_\delta^a}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u_\delta^a}{\partial\theta^2} + \dfrac{(\delta-1)(\delta-3)}{8(u_\delta^a)^3} + \dfrac{\partial^2 W}{\partial t\,\partial\theta}, \\ u_\delta^a(t,0) = u_\delta^a(t,1) = a, \qquad t \geq 0, \\ u_\delta^a(0,\cdot) = x.\end{cases}\tag{30}$$

Moreover, u_δ^a is $(\mathcal{F}_t)$-adapted. We set $X_\delta^a(t,x) := u_\delta^a(t,\cdot) \in K_0 \cap C_a$, $t \geq 0$.

(ii) *The process X_δ^a is symmetric with respect to its unique invariant probability measure π_δ^a, law of the δ-dimensional Bessel bridge between a and a over* $[0,1]$. *Moreover, X_δ^a is strong Feller: indeed, for all bounded and Borel $\varphi : H \mapsto \mathbb{R}$ we have, for all $x, x' \in K_0 \cap C_a$, $t > 0$,*

$$\big|\mathbb{E}\big[\varphi\big(X_\delta^a(t,x)\big)\big] - \mathbb{E}\big[\varphi\big(X_\delta^a(t,x')\big)\big]\big| \leq \|\varphi\|_\infty (1 \wedge t)^{-1/2}\|x - x'\|.\tag{31}$$

(iii) *X_δ^a is the diffusion associated with the Dirichlet form $(\mathcal{D}^{\delta,a}, W^{1,2}(\pi_\delta^a))$, closure in $L^2(\pi_\delta^a)$ of the symmetric bilinear form $(D^{\delta,a}, \mathrm{Lip}(H))$*

$$\mathrm{Lip}(H) \ni \varphi, \psi \mapsto D^{\delta,a}(\varphi,\psi) := \tfrac{1}{2}\int_{K_0}\langle\nabla\varphi, \nabla\psi\rangle\, d\pi_\delta^a.$$

(iv) *Let* $\delta \in \mathbb{N} \cap [4, \infty)$ *and* $a = 0$. *We set*: $\Phi_\delta : H^\delta \mapsto K_0$, $\Phi_\delta(y)(\tau) := |y(\tau)|_{\mathbb{R}^\delta}$, $\tau \in [0, 1]$. *Then* $\mathcal{D}^\delta := \mathcal{D}^{\delta,0}$ *is the image of* Λ^δ *under the map* Φ_δ; *that is*, π_δ *is the image of* μ_δ *under* Φ_δ *and*

$$W^{1,2}(\pi_\delta) = \{\varphi \in L^2(\pi_\delta) : \varphi \circ \Phi_\delta \in W^{1,2}(\mu_\delta)\},$$
$$\mathcal{D}^\delta(\varphi, \psi) = \Lambda^\delta(\varphi \circ \Phi_\delta, \psi \circ \Phi_\delta) \qquad \forall\, \varphi, \psi \in W^{1,2}(\pi_\delta).$$

In (iv), for all $\delta \in \mathbb{N}$, $\delta \geq 4$, we denote by $(\Lambda^\delta, W^{1,2}(\mu_\delta))$ the Dirichlet form with state space $H^\delta = L^2(0, 1; \mathbb{R}^\delta)$:

$$W^{1,2}(\mu_\delta) \ni F, G \mapsto \Lambda^\delta(F, G) := \frac{1}{2} \int_{H^\delta} \langle \overline{\nabla} F, \overline{\nabla} G\rangle_{H^\delta}\, d\mu_\delta$$

where μ_δ is the law on $L^2(0, 1; \mathbb{R}^\delta)$ of a δ-dimensional Brownian bridge between 0 and 0 over $[0, 1]$, $F, G : H^\delta \mapsto \mathbb{R}$ and $\overline{\nabla} F : H^\delta \mapsto H^\delta$ is the gradient of F in H^δ. It is well known that $(\Lambda^\delta, W^{1,2}(\mu_\delta))$ generates the process Z_δ, solution of the $\mathbb{R}^\delta$-valued linear SPDE (3); see [16] and Chapter 8 of [5].

REMARK 2. A solution of (30) is defined as a continuous process $u : [0, \infty) \times [0, 1] \mapsto [0, \infty)$, such that for all $h \in C_c^2(0, 1)$ and $t \geq 0$,

$$\begin{aligned} \langle u(t, \cdot), h\rangle &= \langle x, h\rangle + \frac{1}{2} \int_0^t \langle u(s, \cdot), h''\rangle\, ds - \langle W(t, \cdot), h'\rangle \\ &\quad + \frac{\kappa(\delta)}{2} \int_0^t \int_0^1 \frac{1}{(u(s, \theta))^3} h(\theta)\, d\theta\, ds, \end{aligned} \tag{32}$$

so that the requirement $(u_\delta^a)^{-3} \in L^1_{\mathrm{loc}}([0, \infty) \times (0, 1))$ is necessary for (32) to be meaningful. Let now $h_0(\theta) := \theta(1 - \theta)$, $\theta \in [0, 1]$. We can find a sequence (h_n) in $C_c^2(0, 1)$ such that $h_n(\theta) \uparrow h_0(\theta)$ for all $\theta \in [0, 1]$ and $h_n''(\theta)\, d\theta \rightharpoonup h_0''(\theta)\, d\theta + \delta_0(d\theta) - \delta_1(d\theta)$ in the dual space of $C([0, 1])$ as $n \to \infty$. Then by (32) and the continuity we obtain the a priori estimate

$$\int_0^T \int_0^1 \frac{\theta(1 - \theta)}{(u_\delta^a(s, \theta))^3}\, d\theta\, ds < \infty \qquad \forall\, T \geq 0,\ \forall\, a \geq 0, \tag{33}$$

$$\int_0^T \int_0^1 \frac{1}{(u_\delta^a(s, \theta))^3}\, d\theta\, ds < \infty \qquad \forall\, T \geq 0,\ \forall\, a > 0. \tag{34}$$

REMARK 3. By Theorem 3(iv), we can say that u_δ^0 is the radial part of Z_δ. This result can not be extended to the case $a > 0$, since a Bessel bridge of integer dimension $\delta \geq 2$ between a_1 and a_2 has the law of the modulus of a Brownian bridge of dimension δ between $\overline{a}_1$ and $\overline{a}_2$, for some $\overline{a}_i \in \mathbb{R}^\delta$, $|\overline{a}_i| = a_i$, $i = 1, 2$, if and only if $a_1 a_2 = 0$.

We recall now the definition given by Nualart and Pardoux in [12] of a solution of the SPDE with reflection:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \theta^2} - f\big(\theta, u(t,\theta)\big) + \dfrac{\partial^2 W}{\partial t\,\partial\theta} + \eta(t,\theta), \\ u(0,\cdot) = x, \quad u(t,0) = u(t,1) = a, \qquad t \geq 0, \\ u \geq 0,\ d\eta \geq 0,\ \displaystyle\int_{\mathcal{O}} u\,d\eta = 0, \end{cases} \tag{35}$$

with $x:[0,1] \mapsto [0,+\infty)$ continuous, $a \geq 0$, $x(0) = x(1) = a$ and $f:[0,1] \times [0,\infty) \mapsto \mathbb{R}$ measurable. We suppose that:

(H1) $f(\theta,\cdot)$ is continuously differentiable for all $\theta \in [0,1]$ and for some $c > 0$,

$$|f| \leq c, \qquad |\partial_y f(\theta,y)| \leq c, \qquad \forall\, \theta \in [0,1],\ y \in [0,\infty).$$

(H2) There exists $C \geq 0$ such that for all $\theta \in [0,1]$:

$$\left|\int_0^t f(\theta,u)\,du\right| \leq C \qquad \forall\, t \geq 0.$$

Following [12], we set:

DEFINITION 1. A pair (u,η) is said to be a solution of the SPDE with reflection (35), also called the Nualart–Pardoux equation, if:

- $\{u(t,\theta):(t,\theta) \in \mathcal{O}\}$ is a continuous and adapted process; that is, $u(t,\theta)$ is $\mathcal{F}_t$-measurable for all $(t,\theta) \in \mathcal{O}$, and a.s. $u(\cdot,\cdot)$ is continuous on $\mathcal{O}$, $u(t,\cdot) \in K_0 \cap C_a(0,1)$ for all $t \geq 0$, $a \geq 0$ and $u(0,\cdot) = x$.
- η is a random positive measure on $[0,\infty) \times (0,1)$ such that $\eta([0,T] \times [\delta, 1-\delta]) < +\infty$ for all $T, \delta > 0$, and η is adapted, that is, $\eta(B)$ is $\mathcal{F}_t$-measurable for every Borel set $B \subset [0,t] \times (0,1)$.
- For all $t \geq 0$ and $h \in C_c^2(0,1)$,

$$\begin{aligned} \langle u(t,\cdot),h\rangle &- \tfrac{1}{2}\int_0^t \langle u(s,\cdot),h''\rangle\,ds + \int_0^t \langle f(\cdot,u(s,\cdot)),h\rangle\,ds \\ &= \langle x,h\rangle - \langle W(t,\cdot),h'\rangle + \int_0^t\int_0^1 h(\theta)\eta(ds,d\theta). \end{aligned}$$

- $\int_{\mathcal{O}} u\,d\eta = 0$.

In [12], the following theorem is proved.

THEOREM 4. *Assume that f satisfies* (H1) *and* (H2). *Then for all $x \in K_0 \cap C_a(0,1)$, there exists a unique solution (u,η) of* (35).

We set

$$F : K_0 \mapsto \mathbb{R}, \qquad F(x) := \int_0^1 d\theta \int_0^{x(\theta)} f(\theta, s)\, ds,$$

$$\pi^F_{3,a}(dx) := \frac{1}{\pi^a_3(e^{-2F})} \exp(-2F(x))\pi^a_3(dx).$$

The following theorem has been proved in [16] and [17].

THEOREM 5. *If u is the solution of the Nualart–Pardoux SPDE* (35), *then the process $x \mapsto u$ is the diffusion associated with the symmetric Dirichlet form $(\mathcal{E}, W^{1,2}(\pi^F_{3,a}))$, closure in $L^2(\pi^F_{3,a})$ of the symmetric bilinear form*

$$\mathrm{Exp}(H) \ni \varphi, \psi \mapsto \tfrac{1}{2} \int_{K_0} \langle \nabla\varphi, \nabla\psi \rangle \, d\pi^F_{3,a}.$$

In particular, $x \mapsto u$ is symmetric with respect to $\pi^F_{3,a}$, and moreover, $\pi^F_{3,a}$ is the unique invariant probability measure of $x \mapsto u$. Finally, $\mathrm{Lip}(K_0) \subset W^{1,2}(\pi^F_{3,a})$.

REMARK 4. Let u_δ be the unique solution to (30), for all $\delta > 3$. For all $(t, \theta) \in \mathcal{O}$, $(3, \infty) \ni \delta \mapsto u_\delta(t, \theta)$ is nondecreasing, and as $\delta \downarrow 3$: $u_\delta \downarrow u$ uniformly on $[0, T] \times [0, 1]$, $T \geq 0$, and

$$\frac{\delta - 3}{4(u_\delta)^3}\, dt\, d\theta \to \eta(dt, d\theta) \qquad \text{distributionally on } \mathcal{O},$$

where (u, η) is the solution of the SPDE with reflection (35), with $f \equiv 0$.

In the proof of Theorem 3 we consider solutions to SPDEs of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 u}{\partial \theta^2} - g(u) + \dfrac{\partial^2 W}{\partial t\, \partial\theta}, \\ u(t, 0) = u(t, 1) = b, \qquad t \geq 0, \\ u(0, \cdot) = x(\cdot) + b \in L^2(0, 1), \end{cases} \tag{36}$$

where $g : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$ is measurable, $g(u) := g(\cdot, u(\cdot, \cdot))$, $b \in \mathbb{R}$ and $x \in C_0$.

LEMMA 2. *Let $g_\rho : [0, 1] \times \mathbb{R} \mapsto \mathbb{R}$ be measurable, $\rho > 0$, such that $\mathbb{R} \ni y \mapsto g_\rho(\theta, y)$ is monotone nondecreasing, Lipschitz-continuous uniformly in $\theta \in [0, 1]$ and satisfies*

$$|g_\rho(\theta, y)| \leq c(1 + |y|) \qquad \forall\, y \in \mathbb{R}, \ \rho > 0,$$

for some $c \geq 0$. For all $b \in \mathbb{R}$, let u^b_ρ be the unique solution of the SPDE (36) *with $g := g_\rho$. Then, a.s. we have:*

(a) *for all* $\rho_1, \rho_2 > 0$, *we have a.s. for all* $t \geq 0$,

$$\|(u^b_{\rho_1}(t,\cdot) - u^b_{\rho_2}(t,\cdot))^+\|^2 \leq -2\int_0^t \langle g_{\rho_1}(u^b_{\rho_2}) - g_{\rho_2}(u^b_{\rho_2}), (u^b_{\rho_1} - u^b_{\rho_2})^+\rangle\, ds.$$

In particular, if $\rho \mapsto g_\rho(\cdot,\cdot)$ *is monotone nondecreasing (nonincreasing), then* $\rho \mapsto u^b_\rho(\cdot,\cdot)$ *is monotone nonincreasing (nondecreasing) for all* $b \in \mathbb{R}$.

(b) $b \mapsto u^b_\rho(\cdot,\cdot)$ *is monotone nondecreasing for all* $\rho > 0$.

PROOF. We prove the first assertion; the second one follows analogously. Let $\rho_1 \geq \rho_2 > 0$ and set $\phi := (u^b_{\rho_1} - u^b_{\rho_2})^+$. Then by Lemma 6.1, page 147 in [1],

$$\begin{aligned}
\frac{d}{dt}\|\phi\|^2 &= 2\langle \phi, A(u^b_{\rho_1} - u^b_{\rho_2})\rangle - 2\langle \phi, g_{\rho_1}(u^b_{\rho_1}) - g_{\rho_2}(u^b_{\rho_2})\rangle \\
&= -\left\|\frac{\partial \phi}{\partial \theta}\right\|^2 - 2\langle \phi, g_{\rho_1}(u^b_{\rho_1}) - g_{\rho_1}(u^b_{\rho_2})\rangle - 2\langle \phi, g_{\rho_1}(u^b_{\rho_2}) - g_{\rho_2}(u^b_{\rho_2})\rangle \\
&\leq -2\langle \phi, g_{\rho_1}(u^b_{\rho_2}) - g_{\rho_2}(u^b_{\rho_2})\rangle. \qquad \square
\end{aligned}$$

PROOF OF THEOREM 3. We divide the proof into several steps. In steps 1–4 we prove (i) and (ii). The idea is to approximate u^a_δ from below, by means of solutions v_ε, $\varepsilon > 0$, of Nualart–Pardoux-type equations. We choose v_ε so that its invariant measure converges to π^a_δ as $\varepsilon \downarrow 0$. In step 5 we prove (iii) and in step 6 we prove (iv). We choose the realization of x^a_δ given in Lemma 1 with a Brownian motion B on $(\Omega, \mathcal{F}, \mathbb{P})$ independent of W.

Step 1. Uniqueness of solutions of (30) follows from the dissipativity of the coefficients: indeed, let u^1 and u^2 be two nonnegative continuous solutions of (30), and set for $\varepsilon > 0$, $h_\varepsilon(\theta) := [\theta(1-\theta)/\varepsilon] \wedge 1$, $\theta \in [0,1]$ and $\phi := u^1 - u^2$. By (33), $h_\varepsilon u_i^{-3} \in L^1([0,T]\times[0,1])$, for all $T \geq 0$, $i = 1,2$, and by Theorem 6.4, page 131 in [1], $\phi(t,\cdot) \in C^1([0,1])$ for all $t > 0$ and $\partial\phi/\partial\theta$ is in $L^\infty_{\mathrm{loc}}(\mathcal{O})$. Then

$$\begin{aligned}
\|h_\varepsilon \phi\|^2(t) &= \int_0^t \biggl[-\left\langle h_\varepsilon\left(\frac{\partial \phi}{\partial \theta}\right)^2\right\rangle - \left\langle h'_\varepsilon \phi, \frac{\partial \phi}{\partial \theta}\right\rangle \\
&\qquad + \kappa(\delta)\left\langle h_\varepsilon(u^1 - u^2), \frac{1}{(u^1)^3} - \frac{1}{(u^2)^3}\right\rangle\biggr]\, ds \\
&\leq -\int_0^t \left\langle h'_\varepsilon \phi, \frac{\partial \phi}{\partial \theta}\right\rangle ds.
\end{aligned}$$

As $\varepsilon \downarrow 0$, $\langle h'_\varepsilon \phi, \frac{\partial\phi}{\partial\theta}\rangle \to 0$ since $\phi(t,0) = \phi(t,1) = 0$, $t \geq 0$, so that $\phi \equiv 0$.

Step 2. Notice that $[0,\infty) \ni y \mapsto -\kappa(\delta)/2(\varepsilon + y)^3$ satisfy (H1) and (H2) above. Let $x \in K_0 \cap C_a$. We define for all $\varepsilon > 0$ and $c > 0$:

- $v_{\varepsilon,c}$ as the solution of the SPDE (36) with $b = 0$ and

$$g(\theta, y) = -\frac{\kappa(\delta)}{2} \frac{1}{(\varepsilon + y^+)^3} - \frac{y^-}{c}, \qquad (\theta, y) \in [0, 1] \times \mathbb{R},$$

- $(v_\varepsilon, \eta^\varepsilon)$ as the solution of the SPDE with reflection (35) with

$$f(\theta, y) = -\frac{\kappa(\delta)}{2} \frac{1}{(\varepsilon + y)^3}, \qquad (\theta, y) \in [0, 1] \times [0, \infty).$$

By the proof of Theorem 4 given in [12], we obtain in particular that for all $\varepsilon > 0$, $v_{\varepsilon,c} \uparrow v_\varepsilon$ uniformly on bounded sets of $\mathcal{O}$, as $c \downarrow 0$. By Lemma 2(a) we obtain that $\varepsilon \mapsto v_{\varepsilon,c}$ is nonincreasing for all $c > 0$ and, letting $c \downarrow 0$, that $\varepsilon \mapsto v_\varepsilon$ is nonincreasing. Notice that $w_{\varepsilon,c} := \varepsilon + v_{\varepsilon,c}$ is solution of

$$\begin{cases} \dfrac{\partial w_{\varepsilon,c}}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 w_{\varepsilon,c}}{\partial \theta^2} + \dfrac{\kappa(\delta)}{2(\varepsilon \vee w_{\varepsilon,c})^3} + \dfrac{(w_{\varepsilon,c} - \varepsilon)^-}{c} + \dfrac{\partial^2 W}{\partial t\, \partial \theta}, \\ w_{\varepsilon,c}(t, 0) = w_{\varepsilon,c}(t, 1) = a + \varepsilon, \qquad t \geq 0, \\ w_{\varepsilon,c}(0, \cdot) = x + \varepsilon. \end{cases}$$

By Lemma 2, we have, for all $\varepsilon_2 \geq \varepsilon_1 > 0$,

$$\begin{aligned} &\|(w_{\varepsilon_1,c} - w_{\varepsilon_2,c})^+(t, \cdot)\|^2 \\ &\quad \leq \kappa(\delta) \int_0^t \int_0^1 \left(\frac{1}{(\varepsilon_1 \vee w_{\varepsilon_2,c})^3} - \frac{1}{(\varepsilon_2 \vee w_{\varepsilon_2,c})^3} \right) (w_{\varepsilon_1,c} - w_{\varepsilon_2,c})^+ \, d\theta\, ds. \end{aligned}$$

Letting $c \downarrow 0$, we obtain

$$\begin{aligned} &\big\| \big(\varepsilon_1 + v_{\varepsilon_1} - (\varepsilon_2 + v_{\varepsilon_2})\big)^+(t, \cdot) \big\|^2 \\ &\quad \leq \kappa(\delta) \int_0^t \int_0^1 \left(\frac{1}{(\varepsilon_2 + v_{\varepsilon_2})^3} - \frac{1}{(\varepsilon_2 + v_{\varepsilon_2})^3} \right) (\varepsilon_1 + v_{\varepsilon_1} - \varepsilon_2 - v_{\varepsilon_2})^+ \, d\theta\, ds = 0. \end{aligned}$$

We obtain that $\varepsilon \mapsto \varepsilon + v_\varepsilon$ is nondecreasing and therefore v_ε converges uniformly on $\mathcal{O}$ as $\varepsilon \downarrow 0$ to a continuous function which we denote by u_δ^a. We set for all $\varepsilon, c > 0$: $Y_{\varepsilon,c}(t, x) := v_{\varepsilon,c}(t, \cdot)$, $Y_\varepsilon(t, x) := v_\varepsilon(t, \cdot)$, $X_\delta^a(t, x) := u_\delta^a(t, \cdot)$, $t \geq 0$.

We shall prove that the process X_δ^a enjoys the desired properties. The proof will be based only on monotonicity arguments, on the integration by parts formula w.r.t. π_δ^a (16) and on the explicit knowledge of the invariant measure of Y_ε, given by Theorem 5.

Step 3. We have for all $t \geq 0$, $x, x' \in K_0 \cap C_a$:

$$\|Y_{\varepsilon,c}(t, x) - Y_{\varepsilon,c}(t, x')\|^2 \leq e^{-\pi^2 t} \|x - x'\|^2,$$

and, letting $c \to 0$ and then $\varepsilon \to 0$, we obtain

$$\|X_\delta^a(t, x) - X_\delta^a(t, x')\|^2 \leq e^{-\pi^2 t} \|x - x'\|^2. \tag{37}$$

Since $Y_\varepsilon(t,\cdot)$ and $X_\delta^a(t,\cdot)$ are a.s. 1-Lipschitz continuous in the norm of H, they can be continuously extended to processes in K_0. We still denote the extensions, respectively, by Y_ε and X_δ^a.

Let $a>0$. By Theorem 5, the process Y_ε is symmetric with respect to the probability measure $\gamma_\varepsilon\, d\pi_3^a/Z_\varepsilon$; that is,

$$\int \psi(x)\mathbb{E}[\varphi(Y_\varepsilon(t,x))]\frac{1}{Z_\varepsilon}\gamma_\varepsilon(x)\pi_3^a(dx)=\int \mathbb{E}\left[\psi(Y_\varepsilon(t,x))\right]\varphi(x)\frac{1}{Z_\varepsilon}\gamma_\varepsilon(x)\pi_3^a(dx),$$

where γ_ε is defined in (24) and $Z_\varepsilon>0$ is a normalization constant. By the dominated convergence theorem and (23) we obtain

$$\mathbb{E}[\psi(x_\delta^a)\varphi(X_\delta^a(t,x_\delta^a))]=\mathbb{E}[\psi(X_\delta^a(t,x_\delta^a))\varphi(x_\delta^a)], \tag{38}$$

that is, X_δ^a is symmetric w.r.t. π_δ^a for $a>0$. By Lemma 2(b), $a\mapsto X_\delta^a$ is monotone, and by the uniqueness of solutions of (30), a.s. $X_\delta^a(t,x)\downarrow X_\delta^0(t,x)$ uniformly as $a\downarrow 0$. By (37) and the continuity of $a\mapsto x_\delta^a$, we can let $a\downarrow 0$ in (38) and obtain that X_δ^0 is symmetric w.r.t. π_δ^0.

Now let $a\ge 0$ and m_1 and m_2 be invariant probability measures for X_δ^a. If q_1 and q_2 are random variable with law, respectively, m_1 and m_2, and independent of W, by (37) we have, for all $\varphi\in C_{\mathrm{b}}(H)$,

$$|m_1(\varphi)-m_2(\varphi)|=\left|\mathbb{E}[\varphi(X_\delta^a(t,q_1))-\varphi(X_\delta^a(t,q_2))]\right|\to 0 \qquad \text{as } t\to\infty.$$

Therefore, π_δ^a is the unique invariant probability measure of X_δ^a. Finally, we notice that $v_{\varepsilon,c}$ satisfies a white-noise driven SPDE with dissipative nonlinearity of Nemytskii type. By Proposition 4.4.4 of [2], we have, for all bounded and Borel $\varphi: H\mapsto\mathbb{R}$, $x,y\in K_0\cap C_a$, $t>0$,

$$\left|\mathbb{E}[\varphi(Y_{\varepsilon,c}(t,x))]-\mathbb{E}[\varphi(Y_{\varepsilon,c}(t,y))]\right|d\le \|\varphi\|_\infty(1\wedge t)^{-1/2}\|x-y\|,$$

and (31) follows letting $c,\varepsilon\downarrow 0$.

Step 4. Fix $t\ge 0$. By dominated convergence, we obtain, for $h\in C_c^2(0,1)\cap K_0$,

$$\begin{aligned}&\langle Y_\varepsilon(t,x)-x,h\rangle-\frac{1}{2}\int_0^t\langle h'',Y_\varepsilon(s,x)\rangle\,ds+\langle h',W(t,\cdot)\rangle\\&\qquad\ge\int_0^t\int_0^1\frac{\kappa(\delta)h}{2(\varepsilon+Y_\varepsilon)^3}\,d\theta\,ds.\end{aligned}$$

Since, by step 2, $\varepsilon\mapsto\varepsilon+v_\varepsilon$ is nondecreasing, we can let $\varepsilon\downarrow 0$, and obtain by Beppo–Levi's theorem:

$$\begin{aligned}&\langle X_\delta^a(t,x)-x,h\rangle-\frac{1}{2}\int_0^t\langle h'',X_\delta^a(s,x)\rangle\,ds+\langle h',W(t,\cdot)\rangle\\&\qquad\ge\frac{\kappa(\delta)}{2}\int_0^t\int_0^1\frac{h}{(X_\delta^a)^3}\,d\theta\,ds.\end{aligned}$$

Since π_δ^a is invariant for X_δ^a, we obtain

$$\mathbb{E}\biggl[\langle X_\delta^a(t,x_\delta^a)-x_\delta^a,h\rangle-\frac{1}{2}\int_0^t\langle h'',X_\delta^a(s,x_\delta^a)\rangle\,ds+\langle h',W(t,\cdot)\rangle\biggr]=-\frac{t}{2}\mathbb{E}[\langle h'',x_\delta^a\rangle],$$

$$\frac{\kappa(\delta)}{2}\mathbb{E}\biggl[\int_0^t\int_0^1\frac{h}{(X_\delta^a(s,x_\delta^a))^3}\,d\theta\,ds\biggr]=t\frac{\kappa(\delta)}{2}\mathbb{E}\biggl[\int_0^1\frac{h}{(x_\delta^a)^3}\,d\theta\biggr].$$

By (16) with $\varphi\equiv 1$, we obtain that for all $t\geq 0$ there exists a measurable set $G_t\subseteq C_a\times\Omega$, with $[\pi_\delta^a\otimes\mathbb{P}](G_t)=1$, such that, for all $(x,\omega)\in G_t$,

$$\begin{aligned}\langle X_\delta^a(t,x),h\rangle&=\langle x,h\rangle+\frac{1}{2}\int_0^t\langle h'',X_\delta^a(s,x)\rangle\,ds\\&\quad-\langle h',W(t,\cdot)\rangle+\frac{\kappa(\delta)}{2}\int_0^t\int_0^1\frac{h}{(X_\delta^a)^3}\,d\theta\,ds.\end{aligned}\tag{39}$$

By continuity and by the Fubini–Tonelli theorem, there exists a set $G\subset C_a$ with $\pi_\delta^a(G)=1$, such that for all $x\in G$, a.s. (39) holds for all $t\geq 0$.

By (31), the law of $X_\delta^a(t,x)$ is absolutely continuous w.r.t. π_δ^a for all $t>0$ and $x\in K_0\cap C_a$. Therefore, for all $n\in\mathbb{N}$, $X_\delta^a(1/n,x)\in G$ almost surely. Since $(W(\cdot+1/n,\cdot)-W(1/n,\cdot))$ is a Brownian sheet independent of $\mathcal{F}_{1/n}$, we obtain that a.s.,

$$\begin{aligned}\langle X_\delta^a(t+1/n,x),h\rangle&=\langle X_\delta^a(1/n,x),h\rangle+\frac{1}{2}\int_{1/n}^{t+1/n}\langle h'',X_\delta^a(s,x)\rangle\,ds\\&\quad-\langle W(t+1/n,\cdot)-W(1/n,\cdot),h'\rangle\\&\quad+\frac{\kappa(\delta)}{2}\int_{1/n}^{t+1/n}\int_0^1\frac{h}{(X_\delta^a(s,x))^3}\,d\theta\,ds.\end{aligned}$$

By continuity, we obtain that for all $x\in K_0\cap C_a$, u_δ^a solves a.s. (30).

Step 5. We prove (iii). Let $\delta>3$, $a\geq 0$. We set, for all $\psi\in\mathrm{Lip}(H)$, $\lambda>0$, $\varepsilon>0$,

$$\begin{aligned}R_\varepsilon(\lambda)\psi(x)&:=\int_0^\infty e^{-\lambda t}\mathbb{E}[\psi(Y_\varepsilon(t,x))]\,dt,&&x\in K_0,\\R_\delta^a(\lambda)\psi(x)&:=\int_0^\infty e^{-\lambda t}\mathbb{E}\bigl[\psi\bigl(X_\delta^a(t,x)\bigr)\bigr]\,dt=\lim_{\varepsilon\downarrow 0}R_\varepsilon(\lambda)\psi(x),&&x\in K_0.\end{aligned}$$

By (37) and Theorem 5, $\{R_\varepsilon(\lambda)\psi,R_\delta^a(\lambda)\psi\}\subset\mathrm{Lip}(K_0)\subset W^{1,2}(\gamma_\varepsilon d\pi_3^a)$. By Theorem 5 and by (17) we have, for all $\varphi\in\mathrm{Exp}(H)$,

$$\begin{aligned}
&\int_{K_0} (\psi - \lambda R_\varepsilon(\lambda)\psi)\varphi\gamma_\varepsilon \, d\pi_3^a \\
&\qquad = \tfrac{1}{2}\int_{K_0} \langle \nabla R_\varepsilon(\lambda)\psi, \nabla\varphi\rangle \gamma_\varepsilon \, d\pi_3^a \\
&\qquad = -\int_{K_0} R_\varepsilon(\lambda)\psi\big(\tfrac{1}{2}\mathrm{Tr}[D^2\varphi] + \langle x, A\nabla\varphi\rangle + \tfrac{1}{2}\langle \nabla \log \gamma_\varepsilon, \nabla\varphi\rangle\big)\gamma_\varepsilon \, d\pi_3^a \\
&\qquad\quad - \tfrac{1}{2}\int_0^1 dr \int_{K_0} R_\varepsilon(\lambda)\psi \langle \nabla\varphi, \delta_r\rangle \gamma_\varepsilon \sigma_a(r, dx),
\end{aligned}$$

where δ_r is the Dirac mass at $r \in (0, 1)$ and $\langle \nabla\varphi(x), \delta_r\rangle$ is well defined since $\nabla\varphi(x) \in D(A)$ for all $x \in K_0$. For $a > 0$, letting $\varepsilon \downarrow 0$, by (25)–(26) and Corollary 1,

$$\begin{aligned}
\int_{K_0} (\psi - \lambda R_\delta^a(\lambda)\psi)\varphi \, d\pi_\delta^a &= -\int_{K_0} R_\delta^a(\lambda)\psi L_\delta^a \varphi \, d\pi_\delta^a \\
&= \tfrac{1}{2}\int_{K_0} \langle \nabla R_\delta^a(\lambda)\psi, \nabla\varphi\rangle \, d\pi_\delta^a.
\end{aligned}$$

Then, by (19), (27) and (37), we can let $a \downarrow 0$ and obtain, by Corollary 1,

$$\begin{aligned}
\int_{K_0} (\psi - \lambda R_\delta^0(\lambda)\psi)\varphi \, d\pi_\delta^0 &= -\int_{K_0} R_\delta^0(\lambda)\psi L_\delta^0 \varphi \, d\pi_\delta^0 \\
&= \tfrac{1}{2}\int_{K_0} \langle \nabla R_\delta^0(\lambda)\psi, \nabla\varphi\rangle \, d\pi_\delta^0.
\end{aligned}$$

By a standard approximation procedure, for all $\psi \in \mathrm{Lip}(H)$ there exists a sequence $(\varphi_i)_{i\in\mathbb{N}} \subset \mathrm{Exp}(H)$ such that

$$\sup_i \|\varphi_i\|_{\mathrm{Lip}} < \infty, \qquad \lim_{i\to\infty} \varphi_i(x) = \psi(x) \qquad \forall\, x \in H.$$

By Corollary 1, ψ admits a generalized gradient $\nabla\psi \in L^\infty(K_0, \pi_\delta^a; H)$. We claim that $(\nabla\varphi_i)_i$ converges to $\nabla\psi$ weakly in $L^2(K_0, \pi_\delta^a; H)$. Indeed, let $\mathcal{K}$ be a weak limit of $(\nabla\varphi_i)_i$. By Corollary 1 we have, for all $\varphi \in \mathrm{Exp}(H)$ and $h \in C_c^2(0, 1)$,

$$\begin{aligned}
&\int_{K_0} \langle \mathcal{K}, h\rangle \varphi \, d\pi_\delta^a \\
&\qquad = -\int_{K_0} \psi \langle \nabla\varphi, h\rangle \, d\pi_\delta^a - \int_{K_0} \psi\varphi(x)(\langle x, h''\rangle + \kappa(\delta)\langle x^{-3}, h\rangle)\pi_\delta^a(dx) \\
&\qquad = \int_{K_0} \langle \nabla\psi, h\rangle \varphi \, d\pi_\delta^a
\end{aligned}$$

and this proves the claim. We obtain, for all $\psi_1, \psi_2 \in \mathrm{Lip}(H)$,

$$\int_{K_0} \lambda R_\delta^a(\lambda)\psi_1\psi_2 \, d\pi_\delta^a + \tfrac{1}{2}\int_{K_0} \langle \nabla R_\delta^a(\lambda)\psi_1, \nabla\psi_2\rangle \, d\pi_\delta^a = \int_{K_0} \psi_1\psi_2 \, d\pi_\delta^a. \tag{40}$$

Therefore, $(D^{\delta,a}, \mathrm{Lip}(H))$ is closable in $L^2(\pi_\delta^a)$, and the unique continuous extension of $(R_\delta^a(\lambda))_{\lambda>0}$ to $L^2(\pi_\delta^a)$ is the strongly continuous resolvent associated with the closure $(\mathcal{D}^{\delta,a}, W^{1,2}(\pi_\delta^a))$: see the proof of Theorem 5 in [17] for details.

Step 6. We prove (iv). Let $\delta \in \mathbb{N} \cap [4,\infty)$ and $a = 0$. By the last assertion in Lemma 1, the image measure of μ_δ under Φ_δ is π_δ. Therefore there exists a measurable set $\Omega_0 \subseteq H^\delta$ with $\mu_\delta(\Omega_0) = 1$, such that for all $y \in \Omega_0$, $|y| > 0$ on $(0,1)$, so that for all $h \in C_0(0,1)$ the following map is well defined:

$$\Omega_0 \ni y \mapsto h\frac{y}{|y|} \in C\big([0,1];\mathbb{R}^\delta\big).$$

Since Z_δ is a strong Feller Gaussian process (see [5]), for all $G \in \mathrm{Lip}(H^\delta)$ there exists a sequence $\{G_n\} \subset C_{\mathrm{b}}^1(H^\delta)$, such that

$$\|G_n\|_{\mathrm{Lip}(H^\delta)} \le \|G\|_{\mathrm{Lip}(H^\delta)}, \qquad G_n \to G \text{ in } W^{1,2}(\mu_\delta).$$

Then by a density argument, for all $G \in \mathrm{Lip}(H^\delta)$,

$$\lim_{t\downarrow 0}\frac{1}{t}\left[G\left(y + th\frac{y}{|y|}\right) - G(y)\right] = \left\langle \overline{\nabla} G(y), h\frac{y}{|y|}\right\rangle_{H^\delta} \qquad \text{in } L^2(\mu_\delta).$$

Then, for $h \in C_0(0,1)$ and $G := \varphi \circ \Phi_\delta$ with $\varphi \in \mathrm{Lip}(H)$,

$$\begin{aligned}\langle \nabla\varphi(|y|), h\rangle &:= \lim_{t\downarrow 0}\frac{1}{t}\big(\varphi(|y| + th) - \varphi(|y|)\big) \\ &= \lim_{t\downarrow 0}\frac{1}{t}\left[[\varphi\circ\Phi_\delta]\left(y + th\frac{y}{|y|}\right) - [\varphi\circ\Phi_\delta](y)\right] \\ &= \left\langle \overline{\nabla}[\varphi\circ\Phi_\delta](y), h\frac{y}{|y|}\right\rangle_{H^\delta} \qquad \text{in } L^2(\mu_\delta).\end{aligned}$$

For all $\varphi, \psi \in \mathrm{Lip}(H)$, it follows that

$$\mathcal{D}^\delta(\varphi,\psi) = \Lambda^\delta\left(\varphi\circ\Phi_\delta, \psi\circ\Phi_\delta\right), \tag{41}$$

and by the density of $\mathrm{Lip}(H)$ in $W^{1,2}(\pi_\delta)$, we have that for every $\varphi \in W^{1,2}(\pi_\delta)$, $\varphi\circ\Phi_\delta \in W^{1,2}(\mu_\delta)$ and (41) holds for all $\varphi, \psi \in W^{1,2}(\pi_\delta)$. It remains to prove that if $\varphi \in L^2(\pi_\delta)$ satisfies $\varphi\circ\Phi_\delta \in W^{1,2}(\mu_\delta)$, then $\varphi \in W^{1,2}(\pi_\delta)$. It is enough to prove that $\{[R_\delta^0(1)\psi]\circ\Phi_\delta : \psi \in \mathrm{Lip}(H)\}$ is dense in $\{\varphi\circ\Phi_\delta : \varphi \in L^2(\pi_\delta)\} \cap W^{1,2}(\mu_\delta)$ w.r.t. Λ_1^δ.

By (41), $\mathcal{Y}_\delta := \{\varphi\circ\Phi_\delta : \varphi \in W^{1,2}(\pi_\delta)\}$ is a closed subspace of $W^{1,2}(\mu_\delta)$. Therefore, setting $\Lambda_1^\delta := (\cdot,\cdot)_{L^2(\mu_\delta)} + \Lambda^\delta$, for all $G \in W^{1,2}(\mu_\delta)$ there exists a unique $\Gamma_\delta G \in W^{1,2}(\pi_\delta)$, such that for all $\varphi \in W^{1,2}(\pi_\delta)$,

$$\Lambda_1^\delta(G, \varphi\circ\Phi_\delta) = \Lambda_1^\delta([\Gamma_\delta G]\circ\Phi_\delta, \varphi\circ\Phi_\delta) = \mathcal{D}_1^\delta(\Gamma_\delta G, \varphi),$$

where $\mathcal{D}_1^\delta := (\cdot,\cdot)_{L^2(\pi_\delta)} + \mathcal{D}^\delta$. Moreover, Γ_δ is Markovian; that is, $G \ge 0$ implies

$\Gamma_\delta G \ge 0$ and $\Gamma_\delta 1 = 1$. Therefore

$$\|\Gamma_\delta G\|_{L^\infty(\pi_\delta)} \le \|G\|_{L^\infty(\mu_\delta)}, \qquad \forall\, G \in W^{1,2}(\mu_\delta) \cap L^\infty(\mu_\delta).$$

By (i)–(iii), $\mathcal{D}^\delta$ is a quasiregular symmetric Dirichlet form; see [8]. Then, for all $h \in C_c^2(0,1)$, $\varphi \in W^{1,2}(\pi_\delta) \cap L^\infty(\pi_\delta)$,

$$\int_{K_0} \langle \nabla\varphi, h\rangle \, d\pi_\delta = -\int_{K_0} \varphi^*(x)\big(\langle x, h''\rangle + \kappa(\delta)\langle x^{-3}, h\rangle\big)\pi_\delta(dx),$$

where φ^* is a $\mathcal{D}^\delta$-quasicontinuous π_δ-version of φ. For all $\psi \in \mathrm{Lip}(H)$ we have, by (40),

$$\Lambda_1^\delta\big(G, [R_\delta^0(1)\psi] \circ \Phi_\delta\big) = \mathcal{D}_1^\delta\big(\Gamma_\delta G, [R_\delta^0(1)\psi]\big) = \int_{K_0} (\Gamma_\delta G)^* \psi \, d\pi_\delta$$

for all $G \in W^{1,2}(\mu_\delta) \cap L^\infty(\mu_\delta)$. Then there exists $C_\psi \ge 0$ such that

$$\big|\Lambda_1^\delta\big(G, [R_\delta^0(1)\psi] \circ \Phi_\delta\big)\big| \le C_\psi \|G\|_\infty \qquad \forall\, G \in W^{1,2}(\mu_\delta) \cap L^\infty(\mu_\delta),$$

and by Theorem 4.2 in [6], there exists a finite signed measure Σ_ψ on H^δ, charging no Λ^δ-exceptional set, such that, for all $G \in W^{1,2}(\mu_\delta) \cap L^\infty(\mu_\delta)$,

$$\Lambda^\delta\big(G, [R_\delta^0(1)\psi] \circ \Phi_\delta\big) = -\int_{H^\delta} G^* \, d\Sigma_\psi,$$

where G^* is a Λ^δ-quasicontinuous μ_δ-version of G, and for all $\varphi \in W^{1,2}(\pi_\delta)$,

$$\int_{H^\delta} [\varphi \circ \Phi_\delta]^* \, d\Sigma_\psi = \int_{H^\delta} \varphi \circ \Phi_\delta \cdot \psi \circ \Phi_\delta \, d\mu_\delta.$$

Suppose now that $\varphi \in L^2(\pi_\delta)$, $\varphi \circ \Phi_\delta \in W^{1,2}(\mu_\delta)$ and

$$\Lambda_1^\delta\big(\varphi \circ \Phi_\delta, [R_\delta^0(1)\psi] \circ \Phi_\delta\big) = 0 \qquad \forall\, \psi \in \mathrm{Lip}(H).$$

We set $G_m := ([\varphi \circ \Phi_\delta]^* \wedge m) \vee (-m)$, $m \in \mathbb{N}$ and

$$G_{n,m}(y) := \mathbb{E}\big[G_m \circ \Phi_\delta(Z_\delta(1/n, y))\big], \qquad y \in H^\delta,$$

where Z_δ is the solution of (3). By the strong Feller property of Z_δ, $(G_{n,m}) \subset \mathrm{Lip}(H^\delta)$, $|G_{n,m}| \le m$, $G_{n,m} \to \varphi_m \circ \Phi_\delta$ Λ^δ-quasi everywhere as $n \to \infty$ and in $W^{1,2}(\mu_\delta)$. Moreover,

$$\Lambda_1^\delta\big(G_{n,m}, [R_\delta^0(1)\psi] \circ \Phi_\delta\big) = -\int G_{n,m} \, d\Sigma_\psi,$$

and passing to the limit in $n \to \infty$ and $m \to \infty$, we obtain, for all $\psi \in \mathrm{Lip}(H)$,

$$\begin{aligned} 0 &= \Lambda_1^\delta\big(\varphi \circ \Phi_\delta, [R_\delta^0(1)\psi] \circ \Phi_\delta\big) \\ &= -\int [\varphi \circ \Phi_\delta]^* \, d\Sigma_\psi = -\int_{K_0} [\varphi \circ \Phi_\delta]^* \cdot \psi \circ \Phi_\delta \, d\mu_\delta, \end{aligned}$$

which implies $\varphi \equiv 0$ in $L^2(\pi_\delta)$. $\square$

COROLLARY 2. *For all* $\delta > 3$, $a \geq 0$, $(t, \theta) \in (0, \infty) \times (0, 1)$ *and* $x \in K_0 \cap C_a$, *the law of* $u_\delta^a(t, \theta)$ *is absolutely continuous w.r.t. the Lebesgue measure* dy *on* $[0, \infty)$.

PROOF. The proof follows from Theorem 3(ii). $\square$

COROLLARY 3. *For all* $\delta > 3$, *the log-Sobolev and the Poincaré inequalities hold for* (1); *that is, for all* $\varphi \in W^{1,2}(\pi_\delta^a)$,

$$\int_{K_0} |\varphi - \pi_\delta^a(\varphi)|^2 \, d\pi_\delta^a \leq \frac{1}{2\pi^2} \int_{K_0} \|\nabla \varphi\|^2 \, d\pi_\delta^a,$$

$$\int_{K_0} \varphi^2 \log(\varphi^2) \, d\pi_\delta^a \leq \frac{1}{2\pi^2} \int_{K_0} \|\nabla \varphi\|^2 \, d\pi_\delta^a + \|\varphi\|^2_{L^2(\pi_\delta^a)} \log\bigl(\|\varphi\|^2_{L^2(\pi_\delta^a)}\bigr).$$

For the proof see, for example, [14], [4] and [3].

REMARK 5. The construction of solutions of (30) in the proof of Theorem 3 uses pathwise methods, and the identification of X_δ^a as the Markov process associated with the Dirichlet form $\mathcal{D}^{\delta,a}$ is obtained a posteriori. One can follow another approach, constructing a Markov process properly associated with $\mathcal{D}^{\delta,a}$, and then proving, by the integration by parts formula (16) and by Fukushima's decomposition, that the process solves (30). However this approach gives only weak solutions and requires the proof of quasiregularity of $\mathcal{D}^{\delta,a}$; see [8]. On the other hand, the pathwise approach followed here gives existence of strong solutions of (30), that is, adapted to the filtration of the driving noise, and gives also the quasiregularity of $\mathcal{D}^{\delta,a}$ by Theorem IV.5.1 in [8].

4. SPDEs with positive unbounded drift. In this section we apply the results of the previous sections, to prove the following.

THEOREM 6. *Let* $a \geq 0$.

(i) *Let* $\alpha \geq 3$, $C > 0$. *For all* $x \in K_0 \cap C_a$, *there exists a unique nonnegative continuous adapted* $\hat{u}$ *on* $\mathcal{O}$, *such that* $(\hat{u})^{-\alpha} \in L^1_{\mathrm{loc}}([0, \infty) \times (0, 1))$, *solution of*

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 \hat{u}}{\partial \theta^2} + \dfrac{C}{(\hat{u})^\alpha} + \dfrac{\partial^2 W}{\partial t \, \partial \theta}, \\ \hat{u}(t, 0) = \hat{u}(t, 1) = 0, \qquad t \geq 0, \\ \hat{u}(0, \cdot) = x, \end{cases} \tag{42}$$

(ii) *Let* $0 < \alpha < 3$, $C \geq 0$. *Then for all* $x \in K_0 \cap C_a$, *there exists a unique* $(\hat{u}, \hat{\eta})$, *such that* $(\hat{u})^{-\alpha} \in L^1_{\mathrm{loc}}([0, \infty) \times (0, 1))$, *solution of the following SPDE of* *the Nualart–Pardoux type*:

$$
\begin{cases}
\dfrac{\partial \hat{u}}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 \hat{u}}{\partial \theta^2} + \dfrac{C}{(\hat{u})^\alpha} + \dfrac{\partial^2 W}{\partial t\,\partial\theta} + \hat{\eta}, \\
\hat{u}(0,\cdot) = x, \quad \hat{u}(t,0) = \hat{u}(t,1) = 0, \qquad t \ge 0, \\
\hat{u} \ge 0,\ d\hat{\eta} \ge 0,\ \displaystyle\int_{\mathcal{O}} \hat{u}\,d\hat{\eta} = 0.
\end{cases}
\tag{43}
$$

Moreover, $(\hat{\eta})_{x\in K_0\cap C_a}$ *is not identically equal to* 0.

PROOF. Let $\hat{f}:(0,\infty)\mapsto\mathbb{R}$, smooth and monotone nondecreasing, possibly unbounded in a neighbourhood of 0. We claim that there exists a unique pair $(\hat{u},\hat{\eta})$, solution of the Nualart–Pardoux equation (35) with $f=\hat{f}$, such that $h\hat{f}(\hat{u})\in L^1([0,T]\times[0,1])$ for all $h\in C_c^2(0,1)$ and $T\ge 0$. Indeed, if we set, for $\varepsilon>0$, $(\hat{u}^\varepsilon,\hat{\eta}^\varepsilon)$ as the solution of the Nualart–Pardoux (35) with $f=\hat{f}(\cdot+\varepsilon)$, then, arguing as in step 2 of Proof of Theorem 3, we have that $\varepsilon\mapsto\hat{u}^\varepsilon$ is monotone nonincreasing and $\varepsilon\mapsto\varepsilon+\hat{u}^\varepsilon$ is monotone nondecreasing. Moreover, $\varepsilon\mapsto\hat{\eta}^\varepsilon$ is monotone nondecreasing. Therefore, $\hat{u}^\varepsilon$ converges uniformly on bounded subsets of $\mathcal{O}$ to a continuous function $\hat{u}$ and $\hat{\eta}^\varepsilon$ converges distributionally to a measure $\hat{\eta}$, and by Beppo–Levi's theorem, $(\hat{u},\hat{\eta})$ is the wanted solution. Uniqueness follows from Proof of Theorem 4, given in [12].

Therefore, for all $\alpha\ge 0$ and $C\ge 0$, there exists a unique pair $(\hat{u},\hat{\eta})$, solution of the SPDE with reflection (35) with $f(\theta,y)=-Cy^{-\alpha}$, $y>0$. If $\alpha=3$ and $C>0$, then we proved in Theorem 3 that $\hat{\eta}\equiv 0$.

Let $\alpha>3$, $C>0$ and $x\in K_0\cap C_a$. Notice that we can write

$$
\frac{1}{y^\alpha} = \frac{1}{y^\alpha}\vee 1 + \frac{1}{y^\alpha}\wedge 1 - 1, \qquad y>0.
$$

Consider for all $\varepsilon>0$, the solution $(\hat{v}^\varepsilon,\hat{\zeta}^\varepsilon)$ of the SPDE with reflection (35) with

$$
f(\theta,y) = -C\biggl(\frac{1}{(\varepsilon+y)^3}\vee 1 - 1\biggr), \qquad (\theta,y)\in[0,1]\times[0,\infty).
$$

By Lemma 2(a), $\hat{u}^\varepsilon\ge\hat{v}^\varepsilon$ and $\hat{\eta}^\varepsilon\le\hat{\zeta}^\varepsilon$, $\varepsilon>0$. Arguing as in steps 2–4 Proof of Theorem 3, we can prove that, letting $\varepsilon\downarrow 0$, $\hat{v}^\varepsilon$ converges, uniformly on bounded sets of $\mathcal{O}$, to a continuous $\hat{v}$, such that, for all $h\in C_c^2(0,1)$, $t\ge 0$,

$$
\begin{aligned}
\langle\hat{v}(t,\cdot),h\rangle = \langle x,h\rangle + \frac{1}{2}\int_0^t \langle h'',\hat{v}(s,\cdot)\rangle\,ds - \langle h',W(t,\cdot)\rangle \\
+ C\int_0^t\int_0^1 h\biggl(\frac{1}{(\hat{v})^3}\vee 1 - 1\biggr),
\end{aligned}
$$

$$
\lim_{\varepsilon\downarrow 0}\int_0^t\int_0^1 h\,d\hat{\zeta}^\varepsilon = 0 \qquad \text{so that } \int_0^t\int_0^1 h\,d\hat{\eta} = \lim_{\varepsilon\downarrow 0}\int_0^t\int_0^1 h\,d\hat{\eta}^\varepsilon = 0.
$$

Therefore, $\hat{\eta} = 0$ and $\hat{u}$ satisfies (6).

Let $\alpha \in (0, 3)$. By Theorem 10 in [17] we have, for all $h \in C_c^2(0, 1)$ and $\varphi \in C_{\mathrm{b}}(H)$,

$$\int_{K_0} \mathbb{E}\left[\int_0^1 h(\theta) \int_0^\infty e^{-t} \hat{\eta}(dt, d\theta)\right] \varphi \exp(-2F_\alpha)\, d\pi_3^a$$
$$= \tfrac{1}{2} \int_0^1 dr\, h(r) \int_{K_0} \varphi e^{-2F_\alpha} d\sigma_a(r, \cdot),$$

where

$$F_\alpha(x) = \begin{cases} \dfrac{C}{\alpha - 1} \displaystyle\int_0^1 \frac{1}{[x(\theta)]^{\alpha-1}}\, d\theta, & 1 < \alpha < 3, \\ C \displaystyle\int_0^1 \log\left[\frac{1}{x(\theta)}\right] d\theta, & \alpha = 1, \\ -\dfrac{C}{1 - \alpha} \displaystyle\int_0^1 [x(\theta)]^{1-\alpha}\, d\theta, & 0 < \alpha < 1. \end{cases}$$

For all $\alpha \in (0, 3)$, e^{-2F_α} is in $L^1(\pi_3^a)$ and not identically equal to 0. Therefore $(\hat{\eta})_{x \in K_0 \cap C_a}$ is not identically 0. $\square$

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