The Annals of Probability
1996, Vol. 24, No. 1, 293–319

POTENTIAL THEORY FOR ELLIPTIC SYSTEMS

BY Z. Q. CHEN AND Z. ZHAO

Cornell University and University of Missouri

The existence and uniqueness theorem is proved for solutions of the Dirichlet boundary value problems for weakly coupled elliptic systems on bounded domains. The elliptic systems are only assumed to have measurable coefficients and have singular coefficients for the lower-order terms. A probabilistic representation theorem for solutions of the Dirichlet boundary value problems is obtained by using the switched diffusion process associated with the system. A strong positivity result for solutions of the Dirichlet boundary value problems is proved. Formulas expressing resolvents and kernel functions for the system by those of the component elliptic operators are also obtained.

1. Introduction. Let D be a d-dimensional Euclidean domain and N a positive integer. For

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} : D \to \mathbb{R}^N,$$

consider the following weakly coupled elliptic operator:

$$Su = \begin{pmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_N \end{pmatrix} u + Qu. \tag{1.1}$$

Here, for each k, $L_k = \frac{1}{2}\nabla \cdot (a^k \nabla) + b^k \cdot \nabla$ is a strictly elliptic operator with measurable coefficients, that is,

$$L_k = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}^k \frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{d} b_i^k \frac{\partial}{\partial x_i}, \tag{1.2}$$

with $a_{ij}^k = a_{ji}^k$, and there exists a constant $\lambda > 1$ such that, for almost every x in D,

$$\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \le \lambda|\xi|^2, \qquad \xi \in \mathbb{R}^d, \tag{1.3}$$

Received June 1994; revised November 1994.

AMS 1991 *subject classifications*. Primary 60H30, 35J45; secondary 60J60.

Key words and phrases. Weakly coupled elliptic system, weak solution, Dirichlet boundary value problem, Dirichlet space, switched diffusion, irreducibility, resolvent, kernel function.

and $Q = (q_{kl})$ is an $N \times N$ matrix-valued measurable function on D such that

$$q_{kl} \geq 0 \quad \text{a.e. on } D \text{ for } k \neq l \tag{1.4}$$

and, for each fixed k,

$$\sum_{l=1}^{N} q_{kl} \leq 0 \quad \text{a.e. on } D. \tag{1.5}$$

The assumptions on the coefficients b^k and Q are quite general and can be singular:

$$1_D |b^k|^2 \in K_d \quad \text{and} \quad 1_D q_{kl} \in K_d \tag{1.6}$$

for $k, l = 1, 2, \ldots, N$. Here a function f defined on $\mathbb{R}^d$ is said to be in class K_d if

$$\lim_{\alpha \downarrow 0} \left[\sup_x \int_{|y-x| \leq \alpha} \frac{|f(y)|}{|y-x|^{d-2}}\, dy \right] = 0 \quad \text{when } d \geq 3,$$

$$\lim_{\alpha \downarrow 0} \left[\sup_x \int_{|y-x| \leq \alpha} |f(y)| \ln\{|y-x|^{-1}\}\, dy \right] = 0 \quad \text{when } d = 2$$

and

$$\sup_x \int_{|y-x| \leq 1} |f(y)|\, dy < \infty \quad \text{when } d = 1.$$

In [2], we showed by using Dirichlet space theory that there exists a strong Markov process $Y = (X, \Lambda)$ on $D \times \{1, 2, \ldots, N\}$, which is called the switched diffusion process, whose infinitesimal generator is S with Dirichlet boundary condition. This process Y is defined for all starting points in $D \times \{1, 2, \ldots, N\}$ except possibly for a set of zero capacity. This exceptional set can be taken of the form $\mathscr{N} \times \{1, 2, \ldots, N\}$, where $\mathscr{N} \subset D$ has zero logarithmic or Newtonian capacity depending on whether $d = 2$ or $d \geq 3$ (cf. [2]). Recall that a set of zero capacity will not be hit by the process Y with starting point in its complement. In order to study the sample path behavior of Y by using the nonsymmetric Dirichlet space theory (especially its additive functional theory), we assumed in [2] the following regularity condition on coefficients a^k, b^k and q_{kl}:

(1.7) *$a^k \in W^{1,2}(D)^d$, $\sum_{i=1}^{d} (\partial b_i^k / \partial x_i)$ is bounded from below for each k, and $\sum_{k=1}^{N} q_{kl}$ is bounded from above for each l.*

The above condition amounts to saying that the dual operator of the α-resolvent of Y is sub-Markovian when α is sufficiently large (see [2]). Condition (1.7) is dropped in this paper. Therefore, the approach in [2] is not applicable in this paper and a new approach will be developed.

In this paper, we will use the switched diffusion process Y to study the potential theory for the elliptic system S, including weak solutions for the

Dirichlet boundary value problems for the system

$$(1.8)\qquad \begin{aligned} Su &= 0 \quad \text{in } D,\\ u &= \phi \quad \text{on } \partial D \times \{1, 2, \dots, N\} \end{aligned}$$

and their probabilistic representations, where ϕ is a continuous function defined on $\partial D \times \{1, 2, \dots, N\}$.

DEFINITION 1.1. A function u defined on $D \times \{1, 2, \dots, N\}$ such that, for each k, $u(\cdot, k)$ and its distributional derivatives $(\partial u/\partial x_i)(\cdot, k)$, $1 \le i \le d$, are locally square integrable on D is said to be a *weak solution* of $Su = 0$ if

$$(1.9)\qquad \begin{aligned} \frac{1}{2}\sum_{i,j=1}^{d} \int_D a_{ij}^k \frac{\partial u}{\partial x_i}(\cdot, k)\frac{\partial \phi}{\partial x_j}\,dx - \sum_{i=1}^{d}\int_D b_i^k \frac{\partial u}{\partial x_i}(\cdot, k)\,\phi\,dx \\ - \sum_{l=1}^{N}\int_D q_{kl}\, u(\cdot, l)\,\phi\,dx = 0 \end{aligned}$$

for $\phi \in C_c^\infty(D)$ and $k \in \{1, 2, \dots, N\}$, where $C_c^\infty(D)$ is the space of smooth functions with compact support in D.

Protter and Weinberger [14] proved a maximum principle for solutions of the Dirichlet boundary value problems for a class of weakly coupled elliptic operator S whose L_k in (1.1) is of the form $\frac{1}{2}(a^k\nabla)\cdot\nabla + b^k\cdot\nabla$ and whose coefficients a^k, b^k and q_{kl} are all assumed to be bounded. Therefore, the solution of the Dirichlet boundary value problem for S of this type is unique. Eizenberg and Freidlin [5] studied the Dirichlet boundary value problems for elliptic operators S with $L_k = \frac{1}{2}(a^k\nabla)\cdot\nabla + b^k\cdot\nabla$ and with strict inequality in (1.4) and equality in (1.5). Under the smoothness assumption that D is a bounded C^1 smooth domain, $a^k \in C^2(\overline{D})$ and $b^k, q_{kl} \in C^1(\overline{D})$, they proved a probabilistic representation theorem for solutions of the Dirichlet problem as well as an existence and uniqueness theorem in the Appendix of [5].

Let

$$(1.10)\qquad \tau = \inf\{t > 0\colon \Lambda_t \neq \Lambda_0\}$$

be the first switching time of Y. In this paper, we first identify the preswitching process $\{Y_t, 0 \le t < \tau\}$ of Y and derive the switching distribution of Y at its first switching time τ. Clearly, for $0 \le t < \tau$, $Y_t = (X_t, \Lambda_0)$. Let $X_t^0 = X_t$ for $0 \le t < \tau$ and $X_t^0 = \partial$ for $t \ge \tau$, where ∂ is a cemetery point added to D as a one-point compactification. We show in Proposition 2.1 that there exists a subset $\mathscr{N}$ of D having zero capacity such that for $x \in D \setminus \mathscr{N}$ (in the sequel we abbreviate it as "for q.e. $x \in D$," where q.e. stands for quasi-every) and $k \in \{1, 2, \dots, N\}$, $(X^0, P^{(x,k)})$ is a strong Markov process on D starting from x whose infinitesimal generator is $L_k + q_{kk}$ with zero Dirichlet boundary condition. Let $\mathscr{F}_{\tau-}$ denote the σ-field of events (of Y) strictly prior to the first switching time τ; that is, $\mathscr{F}_{\tau-}$ is the σ-field generated by $\mathscr{F}_0$ and the sets $A \cap [\tau > t]$ for $A \in \mathscr{F}_t$ and $t > 0$, where $\{\mathscr{F}_t\}_{t \geq 0}$ is the minimum completed admissible σ-field generated by Y. We show in Theorem 2.5 below that, for q.e. $x \in D$, $k \in \{1, 2, \dots, N\}$ and $l \in \{1, 2, \dots, N\} \setminus \{k\}$,

$$E^{(x,k)}[\phi(X_\tau), \Lambda_\tau = l \mid \mathscr{F}_{\tau-}] = E^{(x,k)}[\phi(X_\tau), \Lambda_\tau = l \mid X_{\tau-}] = \phi(X_{\tau-})\left(-\frac{q_{kl}}{q_{kk}}\right)(X_{\tau-})$$

for any bounded continuous functions ϕ on D. Therefore, we can identify the switched diffusion process $Y = (X, \Lambda)$ with the following strong Markov process obtained through a patching procedure of Ikeda, Nagasawa Watanabe [9] from diffusion processes associated with $L_k + q_{kk}$, $k = 1, 2, \dots, N$. Suppose that we start from the point (x, k). Let X^k be the strong Markov process in D which is continuous up to its lifetime η whose infinitesimal generator is $L_k + q_{kk}$ with zero Dirichlet boundary condition and denote by ∂_k the cemetery point for X^k added to D as a one-point compactification. Let

$$Y_t = (X_t, \Lambda_t) = (X_t^k, k) \quad \text{for } t < \eta. \tag{1.11}$$

If $\lim_{t \uparrow \eta} X_t^k = \partial_k$, set $Y_t = \Delta$ for $t \geq \eta$, where Δ is a cemetery point added to $D \times \{1, 2, \dots, N\}$ as a one-point compactification. If $\lim_{t \uparrow \eta} X_t^k = X_{\eta-}^k \in D$, while setting $Y_t = \Delta$ for $t \geq \eta$ with probability $(\sum_{l=1}^N q_{kl}/q_{kk})(X_{\eta-})$, put $X_\eta = X_{\eta-}$ with probability $(1 - (\sum_{l=1}^N q_{kl}/q_{kk}))(X_{\eta-})$ and let Λ_η jump to $l \in \{1, 2, \dots, N\} \setminus \{k\}$ with probability $(-q_{kl}/q_{kk})(X_{\eta-})$. Then iterating the above procedure with the starting point (X_τ, Λ_τ), we get a strong Markov process $Y = (X, \Lambda)$ on $D \times \{1, 2, \dots, N\}$ (cf. [9] and [13]). This process Y is the switched diffusion process associated with the operator S in (1.1). It follows from this patching-together construction that the diffusion component X of $Y = (X, \Lambda)$ has continuous sample path up to the lifetime ζ of Y. Since the diffusion process X^k of $L_k + q_{kk}$ can be chosen to start from any point in D (see [3]), by the above patching-together construction, the switched diffusion process $Y = (X, \Lambda)$ can be modified to start from any point in $D \times \{1, 2, \dots, N\}$.

Using the switched diffusion process Y described above, we will prove in this paper the existence and uniqueness theorem for weak solutions of (1.8) for the elliptic system S of (1.1) on a bounded domain D under our general condition (1.6). For the existence part, we show directly that the function u given by

$$u(x, k) = E^{(x,k)}[\phi(X_{\tau(D)}, \Lambda_{\tau(D)})] \tag{1.12}$$

is a weak solution for (1.8), where $\tau(D) = \inf\{t > 0\colon (X_t, \Lambda_t) \notin D \times \{1, 2, \dots, N\}\}$. Here, by extending $a^k = I$, $b^k = 0$ and $q_{kl} = 0$ off D, we may assume the switched diffusion process $Y=(X,\Lambda)$ is on $\mathbb{R}^d\times\{1,2,\dots,N\}$. Denote by $u_0(\cdot,k)$ the unique weak solution of

$$(1.13)\qquad \begin{aligned}(L_k+q_{kk})u_0(\cdot,k)&=0 &&\text{in } D,\\ u_0(\cdot,k)&=\phi(\cdot,k) &&\text{on } \partial D.\end{aligned}$$

We show that the weak solution u of (1.8) satisfies the equation

$$(1.14)\qquad u(x,k)=u_0(x,k)+\sum_{\substack{l=1\\ l\neq k}}^{N} G^k(q_{kl}u(\cdot,l))(x)$$

for $k=1,2,\dots,N$, where G^k is the Green operator of the operator L_k+q_{kk} on D with zero Dirichlet boundary condition. A consequence of this is that any locally bounded weak solution of $Su=0$ has a continuous version. We mention here that a probabilistic representation theorem for solutions (not their existence) of (1.8) was obtained in [2] using a Dirichlet space approach under the extra condition (1.7).

Under the mild condition that the system S is irreducible (see Definition 4.1), we prove the following strong positivity result for solution u of (1.8). Suppose that $\phi\geq 0$ and $\sum_{k=1}^N\phi(\cdot,k)\not\equiv 0$ on ∂D. Then $u(\cdot,k)>0$ in D for $k=1,2,\dots,N$. We remark here that any weakly coupled elliptic system can be decomposed into several independent irreducible elliptic subsystems. Let G_α be the α-resolvent for S and G_α^k be the α-resolvent for L_k+q_{kk} on D with zero Dirichlet boundary condition. We show that, for large $\alpha>0$ and L^2-integrable function f on $D\times\{1,2,\dots,N\}$,

$$(1.15)\qquad G_\alpha f(\cdot,k)=G_\alpha^k(f(\cdot,k))+\sum_{\substack{l=1\\ l\neq k}}^{N} G_\alpha^k(q_{kl}G_\alpha f(\cdot,l)).$$

Using this, we can show that G_α maps bounded Borel measurable functions into bounded continuous functions on $D\times\{1,2,\dots,N\}$. Let

$$G_\alpha^0=\begin{pmatrix}G_\alpha^1 & 0 & \cdots & 0\\ 0 & G_\alpha^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & G_\alpha^N\end{pmatrix}\quad\text{and}\quad Q^0=\begin{pmatrix}0 & q_{12} & \cdots & q_{1N}\\ q_{21} & 0 & \cdots & q_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ q_{N1} & 0 & \cdots & 0\end{pmatrix}.$$

Equation (1.15) can be rewritten as

$$(1.16)\qquad G_\alpha=G_\alpha^0+G_\alpha^0\big(Q^0G_\alpha\big).$$

We prove that there exists a constant $\alpha_0>0$ such that, for $\alpha>\alpha_0$,

$$(1.17)\qquad G_\alpha=\sum_{n=0}^{\infty}G_\alpha^0\big(Q^0G_\alpha^0\big)^n,$$

where the series is convergent in $W_0^{1,2}(D)^d$. If we denote by $p(t,(x,k),(y,r))$ the fundamental solution for S with zero Dirichlet boundary condition [or, equivalently, the transition density function of $Y = (X, \Lambda)$] and by $p_k(t, x, y)$ the fundamental solution for $L_k + q_{kk}$ on D with zero Dirichlet boundary condition, then

$$\begin{aligned} p(t,(x,k),(y,r)) &= \delta_{kr} p_k(t,x,y) \\ &\quad + \sum_{n=1}^{\infty} \int \cdots \int_{0<t_1<t_2<\cdots<t_n<t} \sum_{\substack{1\le l_1,l_2,\dots,l_n\le N \\ l_i\ne l_{i+1} \\ l_0=k,\, l_n=r}} \int_D \cdots \int_D p_k(t_1,x,y_1) q_{kl_1}(y_1) \\ &\qquad \times p_{l_1}(t_2-t_1,y_1,y_2)\cdots q_{l_{n-1}r}(y_n) \\ &\qquad \times p_r(t-t_n,y_n,y)\, dt_1\, dt_2 \cdots dt_n\, dy_1\, dy_2 \cdots dy_n. \end{aligned} \tag{1.18}$$

Here δ_{kr} is the Kronecker symbol in k, r, which equals 1 if $k = r$ and is 0 if $k \neq r$.

As we indicated above, this paper is quite independent of [2], but it uses several probabilistic potential theory results proved in [3] for operators $L_k + q_{kk}$, $k = 1, 2, \dots, N$.

The rest of this paper is organized as follows. In Section 2, we study the switching distribution of the switched diffusion process $Y = (X, \Lambda)$ associated with S, and we derive the resolvent identity (1.15). We also show that $Y = (X, \Lambda)$ can only switch finitely many times within a finite time interval. In Section 3, existence, uniqueness and probabilistic representation theorems for solutions of (1.8) are proved. The strong positivity result mentioned above is proved in Section 4. Finally, resolvent and kernel identities (1.17) and (1.18) are established in Section 5.

2. Switched diffusion processes and their resolvents. Let D be a d-dimensional Euclidean domain, let dx be the Lebesgue measure in D and let $W_0^{1,2}(D)$ be the Sobolev space on D of order $(1, 2)$ with zero boundary condition; that is,

$$W_0^{1,2}(D) = \overline{C_c^\infty(D)}^{\|\cdot\|_{1,2}}, \tag{2.1}$$

where

$$\|f\|_{1,2} = \left(\int_D |\nabla f|^2\, dx + \int_D |f|^2\, dx \right)^{1/2}. \tag{2.2}$$

Let $W = D \times \{1, 2, \dots, N\}$ and $dm(x, k) = dx\, dn(k)$, where dn is the counting measure on $\{1, 2, \dots, N\}$. A cemetery point Δ is added to W as a one-point compactification, and a function f defined on W is extended to $W_\Delta = W \cup \{\Delta\}$ by setting $f(\Delta) = 0$ unless otherwise specified. We associate the elliptic system S of (1.1) with a bilinear form $(\mathscr{F}, \mathscr{E})$ on $L^2(W, m)$, where

$$\mathscr{F} = \left\{ u \in L^2(W, m) \colon u(\cdot, k) \in W_0^{1,2}(D) \text{ for } k = 1, 2, \dots, N \right\} \tag{2.3}$$

and, for $u, v \in \mathscr{F}$,

$$\begin{aligned}\mathscr{E}(u,v) = {} & \frac{1}{2}\sum_{k=1}^{N}\int_D \sum_{i,j=1}^{d} a_{ij}^k(x)\frac{\partial u}{\partial x_i}(x,k)\frac{\partial v}{\partial x_j}(x,k)\,dx \\ & -\frac{1}{2}\sum_{k=1}^{N}\int_D \sum_{i=1}^{d} b_i^k(x)\frac{\partial u}{\partial x_i}(x,k)v(x,k)\,dx \\ & -\sum_{k,l=1}^{N}\int_D q_{kl}(x)u(x,k)v(x,l)\,dx.\end{aligned} \tag{2.4}$$

It is known (cf. [4] and [10]) that if $g \in K_d$, then for any $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that, for $f \in W_0^{1,2}(D)$,

$$\int_D |g| f^2\,dx \le \varepsilon \int_D |\nabla f|^2\,dx + C_\varepsilon \int_D |f|^2\,dx. \tag{2.5}$$

Since $1_D|b^k|^2 \in K_d$, $1_D q_{kl} \in K_d$ and (a_{ij}^k) is uniformly elliptic and bounded, it is easy to see that there exist constants $\alpha_0 > 0$ and $A > 1$ such that

$$A^{-1}\sum_{k=1}^{N}\|u(\cdot,k)\|_{1,2}^2 \le \mathscr{E}_{\alpha_0}(u,u) \le A\sum_{k=1}^{N}\|u(\cdot,k)\|_{1,2}^2 \quad \text{for } u \in \mathscr{F} \tag{2.6}$$

and

$$|\mathscr{E}(u,v)| \le A\sqrt{\mathscr{E}_{\alpha_0}(u,u)}\,\sqrt{\mathscr{E}_{\alpha_0}(v,v)} \quad \text{for } u, v \in \mathscr{F}, \tag{2.7}$$

where $\mathscr{E}_{\alpha_0} = \mathscr{E} + \alpha_0(\ ,\ )_{L^2(W,m)}$. For $u \in \mathscr{F}$, $u^+ \wedge 1 \in \mathscr{F}$ and

$$\mathscr{E}(u^+ \wedge 1, u - u^+ \wedge 1) \ge 0 \tag{2.8}$$

[see [2] for detailed computations for (2.6) to (2.8)]. Thus $(\mathscr{F}, \mathscr{E})$ is a nonsymmetric regular Dirichlet space on $L^2(W, m)$. Therefore, by a fundamental theorem from [1], there exists a Hunt process $Y = (X, \Lambda)$ on $D \times \{1, 2, \dots, N\}$ associated with $(\mathscr{F}, \mathscr{E})$ such that the infinitesimal generator of Y is S whose domain of definition is

$$\mathscr{D}(S) = \{u \in \mathscr{F}\colon Su \in L^2(W, m)\}. \tag{2.9}$$

Here the derivatives are understood in the distributional sense. The statement $Su \in L^2(W, m)$ means that there exists $f \in L^2(W, m)$ such that, for $v \in \mathscr{F}$,

$$\mathscr{E}(u, v) = -(f, v)_{L^2(W,m)} \tag{2.10}$$

and $Su = f$. The process Y is defined for all starting points in $D \times \{1, 2, \dots, N\}$ except possibly for a set of zero capacity. Later in this section we will show that this exceptional set can be dropped.

For $\alpha > 0$, let

$$G_\alpha f(x,k) = E^{(x,k)}\left[\int_0^\infty e^{-\alpha t} f(X_t, \Lambda_t)\,dt\right] \tag{2.11}$$

for a Borel measurable function f defined on $D \times \{1, 2, \dots, N\}$ whenever the right-hand side of (2.11) makes sense, with the convention that $f(\Delta) = 0$. It is known that, for $\alpha > \alpha_0$, G_α is a bounded operator in $L^2(W, m)$ with

$$\|G_\alpha\|_2 \le \frac{1}{\alpha - \alpha_0}, \tag{2.12}$$

and, for $f \in L^2(W, m)$, $G_\alpha f \in \mathscr{F}$ with

$$\mathscr{E}_\alpha(G_\alpha f, u) = (f, u)_{L^2(W, m)}, \qquad u \in \mathscr{F}. \tag{2.13}$$

Let

$$\tau = \inf\{t > 0: \Lambda_t \neq \Lambda_0\} \tag{2.14}$$

and

$$X_t^0 = \begin{cases} X_t, & t < \tau, \\ \partial, & t \ge \tau, \end{cases} \tag{2.15}$$

where ∂ is a cemetery point added to D.

Proposition 2.1. *For q.e. $x \in D$ and $k \in \{1, 2, \dots, N\}$, $\{X^0, P^{(x,k)}\}$ is a realization of the strong Markov process that is continuous up to its lifetime and has infinitesimal generator*

$$L_k + q_{kk} = \tfrac{1}{2}\nabla \cdot (a^k \nabla) + b^k \cdot \nabla + q_{kk}, \tag{2.16}$$

whose domain of definition is

$$\mathscr{D}_k = \{f \in W_0^{1,2}(D): (L_k + q_{kk}) f \in L^2(D, dx)\}. \tag{2.17}$$

Proof. Let $\mathscr{N} \subset D$ be the exceptional set having zero capacity such that the switched diffusion process Y is well defined for each starting point in $(D \setminus \mathscr{N}) \times \{1, 2, \dots, N\}$. Let $Y_t^0 = Y_t$ when $0 \le t < \tau$ and $Y_t^0 = \Delta$ when $t \ge \tau$, where Δ is a cemetery point. That is, when starting from (x, k), Y^0 is the part process of Y on the open set $D \times \{k\}$. By Corollary 2 on page 149 of [1], $\{Y^0, P^{(x,k)}\}$ for $x \in D \setminus \mathscr{N}$ is a strong Markov process on $D \times \{k\}$ whose associated Dirichlet space is $(\mathscr{F}_k, \mathscr{E})$, where

$$\begin{aligned} \mathscr{F}_k &= \{u \in \mathscr{F}: u = 0 \text{ off } D \times \{k\}\} \\ &= \{u: u(\cdot, k) \in W_0^{1,2}(D) \text{ and } u(\cdot, l) = 0 \text{ for } l \neq k\}, \end{aligned} \tag{2.18}$$

and where $\mathscr{F}$ and $\mathscr{E}$ are defined in (2.3) and (2.4). Note that, for $u, v \in \mathscr{F}_k$,

$$\begin{aligned} \mathscr{E}(u, v) = \frac{1}{2} \sum_{i,j=1}^d \int_D a_{ij}^k \frac{\partial u(\cdot, k)}{\partial x_i} \frac{\partial v(\cdot, k)}{\partial x_j}\, dx - \sum_{i=1}^d \int_D b_i^k \frac{\partial u(\cdot, k)}{\partial x_i} v(\cdot, k)\, dx \\ - \int q_{kk} u(\cdot, k) v(\cdot, k)\, dx. \end{aligned}$$

Therefore, $(\mathscr{F}_k, \mathscr{E})$ can be identified with the Dirichlet space $(W_0^{1,2}(D), \mathscr{E}^k)$ on $L^2(D, dx)$, where

$$\begin{aligned} \mathscr{E}^k(f, g) = {} & \frac{1}{2} \sum_{i,j=1}^d \int_D a_{ij}^k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}\, dx \\ & - \sum_{i=1}^d \int_D b_i^k \frac{\partial f}{\partial x_i} g\, dx - \int q_{kk} f g\, dx \end{aligned} \tag{2.19}$$

for $f, g \in W_0^{1,2}(D)$. Note that if we identify (∂, k) with Δ, then $Y^0 = (X^0, k)$. Therefore, $\{X^0, P^{(x,k)}, x \in D \setminus \mathscr{N}\}$ is a realization of the Hunt process associated with the regular Dirichlet space $(W_0^{1,2}(D), \mathscr{E}^k)$ on $L^2(D, dx)$ that is continuous up to its lifetime τ. Or, equivalently, $\{X^0, P^{(x,k)}, x \in D \setminus \mathscr{N}\}$ is the Hunt process which is continuous up to its lifetime and has infinitesimal generator $L_k + q_{kk}$ with domain of definition $\mathscr{D}_k$. $\square$

Let G_α^k be the α-resolvent of $\{X^0, P^{(x,k)}\}$ for q.e. $x \in D$; that is,

$$G_\alpha^k f(x) = E^{(x,k)}\left[\int_0^\infty e^{-\alpha t} f(X_t^0)\, dt\right] = E^{(x,k)}\left[\int_0^\tau e^{-\alpha t} f(X_t)\, dt\right]. \tag{2.20}$$

By enlarging the constant α_0 in (2.6) and (2.7) if necessary, we may assume that

$$A^{-1}\|f\|_{1,2}^2 \le \mathscr{E}_{\alpha_0}^k(f, f) \le A\|f\|_{1,2}^2, \qquad f \in W_0^{1,2}(D). \tag{2.21}$$

Thus, for $\alpha > \alpha_0$, we have $\|G_\alpha^k\|_2 \le 1/(\alpha - \alpha_0)$ and $G_\alpha^k f \in W_0^{1,2}(D)$ for $f \in L^2(D, dx)$.

PROPOSITION 2.2. *Let $\phi \ge 0$ on D and $\alpha \ge 0$. Then*

$$E^{(x,k)}[e^{-\alpha\tau}\phi(X_{\tau-})] = G_\alpha^k(-q_{kk}\phi)(x). \tag{2.22}$$

PROOF. Let $(\Omega, Z, \zeta, Q^x, x \in D)$ be the strong Markov process having infinitesimal generator L_k with

$$\mathscr{D}(L_k) = \{f \in W_0^{1,2}(D) \colon L_k f \in L^2(D, dx)\}. \tag{2.23}$$

Let P^x be the probability measure on Ω determined by

$$E^x[f(Z_t)] = E_Q^x\left[\exp\left(\int_0^t q_{kk}(Z_s)\, ds\right) f(Z_t)\right], \tag{2.24}$$

where E_Q^x denotes the integration with respect to the probability measure Q^x. Then $(\Omega, Z, \zeta, P^x, x \in D)$ is a strong Markov process which is continuous up to its lifetime and has infinitesimal generator $L_k + q_{kk}$ whose domain of definition is $\mathscr{D}_k$ in (2.17) (cf. [3]). Thus

$$E^{(x,k)}[e^{-\alpha\tau}\phi(X_{\tau-})] = E^x[e^{-\alpha\zeta}\phi(Z_{\zeta-})],$$

which by page 286 in Sharpe [15] [putting $m_t = \exp(\int_0^t q_{kk}(Z_s)\, ds) 1_{[t<\zeta]}$ there and noting that Q^x-a.e. $Z_{\zeta-} = \partial$]

$$\begin{aligned} &= E_Q^x\left[\int_0^\infty e^{-\alpha t}\phi(Z_t)(-q_{kk}(Z_t))\exp\left(\int_0^t q_{kk}(Z_s)\, ds\right) dt\right] \\ &= E^x\left[\int_0^\infty e^{-\alpha t}(-q_{kk}\phi)(Z_t)\, dt\right] \\ &= G_\alpha^k(-q_{kk}\phi)(x). \end{aligned}$$

$\square$

THEOREM 2.3. *For* $\alpha > \alpha_0$ *and* $f \in L^2(W, m)$,

$$G_\alpha f(x,k) = G_\alpha^k(f(\cdot,k))(x) + G_\alpha^k\left(\sum_{\substack{l=1\\ l\neq k}}^N q_{kl} G_\alpha f(\cdot,l)\right)(x). \tag{2.25}$$

PROOF. For $1 \le k \le N$ and $x \in D$, let

$$\phi_k(x) = E^{(x,k)}\left[e^{-\alpha\tau} G_\alpha f(X_\tau, \Lambda_\tau)\right].$$

By the strong Markov property of $Y = (X, \Lambda)$,

$$\begin{aligned} G_\alpha f(x,k) &= E^{(x,k)}\left[\int_0^\tau e^{-\alpha t} f(X_t,k)\,dt\right] + E^{(x,k)}\left[\int_\tau^\infty e^{-\alpha t} f(X_t,\Lambda_t)\,dt\right] \\ &= G_\alpha^k(f(\cdot,k))(x) + \phi_k(x). \end{aligned} \tag{2.26}$$

Hence, $\phi_k = G_\alpha f(\cdot,k) - G_\alpha^k(f(\cdot,k))$ is in $W_0^{1,2}(D)$. Since $(\alpha - S)G_\alpha f = f$ and $(\alpha - L_k - q_{kk})G_\alpha^k(f(\cdot,k)) = f(\cdot,k)$,

$$(\alpha - L_k - q_{kk})\phi_k = \sum_{\substack{l=1\\ l\neq k}}^N q_{kl} G_\alpha f(\cdot,l). \tag{2.27}$$

Thus

$$\phi_k = G_\alpha^k\left(\sum_{\substack{l=1\\ l\neq k}}^N q_{kl} G_\alpha f(\cdot,l)\right) \tag{2.28}$$

by the following lemma. □

LEMMA 2.4. *Suppose* q *is a function defined on* D *such that* $1_D q \in K_d$ *and* $u \in W_0^{1,2}(D)$. *Then, for* $k \in \{1,2,\dots,N\}$ *and* $\alpha > \alpha_0$, $G_\alpha^k(qu) \in W_0^{1,2}(D)$ *and* $G_\alpha^k(qu)$ *is the unique weak solution of* $(\alpha - L_k - q_{kk})\phi = qu$.

PROOF. It suffices to prove the lemma for $q \ge 0$ and $u \ge 0$. Note that the resolvent of the Dirichlet space $(W_0^{1,2}, \mathscr{E}_\alpha^k)$ is $\{G_{\alpha+\beta}^k\}_{\beta>0}$. For $f, g \in L^2(D, dx)$, set

$$\mathscr{E}^{(\beta)}(f,g) = \beta\left(f - \beta G_{\alpha+\beta}^k f, g\right). \tag{2.29}$$

Let $q_n = q \wedge n$. Then $\phi_n = G_\alpha^k(q_n u) \in W_0^{1,2}(D)$. By the resolvent identity,

$$\begin{aligned} \mathscr{E}^{(\beta)}(\phi_n,\phi_n) &= \beta\left(G_{\alpha+\beta}^k(q_n u), \phi_n\right)_{L^2(D,dx)} \\ &= \left(q_n u, \beta \hat{G}_{\alpha+\beta}^k \phi_n\right), \end{aligned} \tag{2.30}$$

where $\hat{G}_{\alpha+\beta}^k$ is the adjoint of $G_{\alpha+\beta}^k$ in $L^2(D, dx)$. Since $1_D q \in K_d$, by (2.5) and (2.6) there exists a constant B which is independent of n such that

$$\left(q_n u, \beta G_{\alpha+\beta}^k \phi_n\right) \le B\sqrt{\mathscr{E}_\alpha^k(u,u)}\sqrt{\mathscr{E}_\alpha^k\left(\beta\hat{G}_{\alpha+\beta}^k\phi_n, \beta\hat{G}_{\alpha+\beta}^k\phi_n\right)},$$

which, by Lemma 3.1 in [12], is less than or equal to

$$B\sqrt{\mathscr{E}_\alpha^k(u,u)}\sqrt{\mathscr{E}^{(\beta)}(\phi_n,\phi_n)}.$$

Thus

$$\mathscr{E}^{(\beta)}(\phi_n,\phi_n) \le B^2\mathscr{E}_\alpha^k(u,u). \tag{2.31}$$

Let $n\to\infty$. By the monotone convergence theorem and (2.31),

$$\begin{aligned}\mathscr{E}^{(\beta)}\big(G_\alpha^k(qu),G_\alpha^k(qu)\big) &= \beta\big(G_{\alpha+\beta}^k(qu),G_\alpha^k(qu)\big)_{L^2(D,dx)}\\ &\le B^2\mathscr{E}_\alpha^k(u,u).\end{aligned} \tag{2.32}$$

Since $G_{\alpha+\beta}^k(qu)\le G_\alpha^k(qu)$, $G_{\alpha+\beta}^k(qu)\in L^2(D,dx)$ and therefore $G_\alpha^k(qu)=G_{\alpha+\beta}^k(qu)-\beta G_\alpha^k(G_{\alpha+\beta}^k(qu))$ is also in $L^2(D,dx)$. Thus, by Lemma 3.2 in [12], $G_\alpha^k(qu)\in W_0^{1,2}(D)$. For $f\in W_0^{1,2}(D)$,

$$\begin{aligned}\mathscr{E}_\alpha^k\big(G_\alpha^k(qu),f\big) &= \lim_{\beta\to\infty}\mathscr{E}^{(\beta)}\big(G_\alpha^k(qu),f\big)\\ &= \lim_{\beta\to\infty}\Big(qu,\beta\hat{G}_{\alpha+\beta}^k f\Big)\\ &= (qu,f),\end{aligned} \tag{2.33}$$

since $\beta\hat{G}_{\alpha+\beta}^k f$ converges to f in $(W_0^{1,2}(D),\|\cdot\|_{1,2})$. Thus $G_\alpha^k(qu)$ is the unique weak solution of $(\alpha-L_k-q_{kk})\phi=qu$. □

Recall that $\{\mathscr{F}_t\}_{t\ge0}$ is the minimum completed admissible σ-field generated by Y (cf. [1] and [6]) and τ is the first switching time for Y as defined in (2.14). Also, $\mathscr{F}_{\tau-}$ is the σ-field of events strictly prior to the stopping time τ; that is, $\mathscr{F}_{\tau-}$ is the σ-field generated by $\mathscr{F}_0$ and the sets $A\cap[\tau>t]$ for $A\in\mathscr{F}_t$ and $t>0$. Since $Y_-=\{Y_{t-},t\ge0\}$ is left-continuous, it is a predictable process with respect to the filtrations $\{\mathscr{F}_t\}_{t\ge0}$. Thus, by Corollary 3.23(2) of [8], $Y_{\tau-}1_{[\tau<\infty]}=(X_{\tau-},\Lambda_0)1_{[\tau<\infty]}$ is $\mathscr{F}_{\tau-}$-measurable. The following theorem gives the switching distribution of Y at its first switching time τ. In particular, it implies that X is continuous up to the lifetime ζ of Y.

THEOREM 2.5. *For q.e. $x\in D$, $k\in\{1,2,\dots,N\}$ and $l\in\{1,2,\dots,N\}\setminus\{k\}$, we have*

$$E^{(x,k)}\big[\phi(X_\tau),\Lambda_\tau=l|\mathscr{F}_{\tau-}\big]=\phi(X_{\tau-})\left(-\frac{q_{kl}}{q_{kk}}\right)(X_{\tau-}) \tag{2.34}$$

for any bounded continuous function ϕ on D.

PROOF. For such fixed l, let $f(x,j)=\phi(x)1_{\{l\}}(j)$, which is a bounded continuous function on $D\times\{1,2,\dots,N\}$. We see from (2.29) that

$$E^{(x,k)}\big[e^{-\alpha\tau}G_\alpha f(X_\tau,\Lambda_\tau)\big]=G_\alpha^k\left(\sum_{\substack{j=1\\ j\ne k}}^N q_{kj}G_\alpha f(\cdot,j)\right)(x),$$

which, by Proposition 2.2,

$$= E^{(x,k)}\left[e^{-\alpha\tau}\sum_{\substack{j=1\\j\neq k}}^{N} G_\alpha f(X_{\tau-}, j)\left(-\frac{q_{kj}}{q_{kk}}\right)(X_{\tau-})\right], \tag{2.35}$$

where the convention $0/0 = 0$ is used. By the resolvent identity $G_\beta f = G_\alpha f - (\beta - \alpha)G_\alpha(G_\beta f)$, it follows from (2.35) that, for $\beta > \alpha$,

$$\begin{aligned} &E^{(x,k)}\left[e^{-\alpha\tau}\beta G_\beta f(X_\tau, \Lambda_\tau)\right] \\ &\qquad = E^{(x,k)}\left[e^{-\alpha\tau}\sum_{\substack{j=1\\j\neq k}}^{N} \beta G_\beta f(X_{\tau-}, j)\left(-\frac{q_{kj}}{q_{kk}}\right)(X_{\tau-})\right]. \end{aligned} \tag{2.36}$$

Since $\beta G_\beta f$ converges to f on $D \times \{1, 2, \dots, N\}$ as $\beta \to \infty$ except possibly for a set of zero capacity, by the bounded convergence theorem it follows from (2.36) that

$$\begin{aligned} &E^{(x,k)}\left[e^{-\alpha\tau}\phi(X_\tau), \Lambda_\tau = l\right] \\ &\qquad = E^{(x,k)}\left[e^{-\alpha\tau}\phi(X_{\tau-})\left(-\frac{q_{kl}}{q_{kk}}\right)(X_{\tau-})\right]. \end{aligned} \tag{2.37}$$

For any $t \geq 0$ and $A \in \mathscr{F}_t$, by the strong Markov property of Y and (2.37), we have

$$\begin{aligned} &E^{(x,k)}\left[1_{A\cap[\tau>t]}e^{-\alpha\tau}\phi(X_\tau), \Lambda_\tau = l\right] \\ &\quad = E^{(x,k)}\left[e^{-\alpha\tau}1_{A\cap[\tau>t]}E^{(X_t,\Lambda_t)}\left[e^{-\alpha\tau}\phi(X_\tau), \Lambda_\tau = l\right]\right] \\ &\quad = E^{(x,k)}\left[e^{-\alpha\tau}1_{A\cap[\tau>t]}E^{(X_t,\Lambda_t)}\left[e^{-\alpha\tau}\phi(X_{\tau-})\left(-\frac{q_{kl}}{q_{kk}}\right)(X_{\tau-})\right]\right] \\ &\quad = E^{(x,k)}\left[1_{A\cap[\tau>t]}e^{-\alpha\tau}\phi(X_{\tau-})\left(-\frac{q_{kl}}{q_{kk}}\right)(X_{\tau-})\right]. \end{aligned}$$

Therefore, since τ is $\mathscr{F}_{\tau-}$-measurable,

$$e^{-\alpha\tau}E^{(x,k)}\left[\phi(X_\tau), \Lambda_\tau = l \mid \mathscr{F}_{\tau-}\right] = e^{-\alpha\tau}\phi(X_{\tau-})\left(-\frac{q_{kl}}{q_{kk}}\right)(X_{\tau-}). \tag{2.38}$$

This proves (2.34). □

By the strong Markov property of Y, process Y is uniquely determined by its preswitching part $\{Y_t = (X_t, \Lambda_0), 0 \leq t < \tau\}$ and its switching distribution (2.34) at the switching time τ. Recall that for quasi-every starting point (x, k) in $D \times \{1, 2, \dots, N\}$, $\{X_t, 0 \leq t < \tau\}$ is a Hunt process having infinitesimal generator $L_k + q_{kk}$ with domain of definition defined by (2.17). Therefore, the switched diffusion process $Y = (X, \Lambda)$ can also be constructed via the patching procedure from diffusion processes X^k of $L_k + q_{kk}$, $k = 1, 2, \dots, N$, as described in the Introduction. Clearly, the diffusion component X of $Y = (X, \Lambda)$ has a continuous sample path up to the lifetime ζ of Y. Since the diffusion process X^k of $L_k + q_{kk}$ can be chosen to start from any point in D (see [3]), by the patching-together construction, the switched diffusion process $Y = (X, \Lambda)$ is refined to start from any point in $D \times \{1, 2, \dots, N\}$. From now on, we always assume that Y takes this refined version.

Let

$$\begin{aligned}
\tau_1 &= \inf\{t > 0: \Lambda_t \neq \Lambda_0\},\\
\tau_2 &= \inf\{t > \tau_1: \Lambda_t \neq \Lambda_{\tau_1}\},\\
&\vdots\\
\tau_n &= \inf\{t > \tau_{n-1}: \Lambda_t \neq \Lambda_{\tau_{n-1}}\},\\
&\vdots
\end{aligned}$$

with the convention that $\inf \varnothing = +\infty$; that is, let τ_n be the nth switching time of the process $Y = (X, \Lambda)$.

THEOREM 2.6. *$P^{(x,k)}[\lim_{n\to\infty} \tau_n = \infty] = 1$ for $(x, k) \in D \times \{1, 2, \dots, N\}$.*

PROOF. Let $(X^k, \zeta_k, P_k^x, x \in D)$ be the diffusion process of $L_k + q_{kk}$ on D with zero Dirichlet boundary condition and let ζ_k be its lifetime. By Theorem 5.12 of [3] there exists a constant $\alpha > 0$ such that

$$\sup_{x\in D} E_k^x\Big[e^{-\alpha\zeta_k} 1_D\Big(X_{\zeta_k -}^k\Big)\Big] \le \tfrac{1}{2}, \qquad k = 1, 2, \dots, N. \tag{2.39}$$

Thus, for $(x, k) \in D \times \{1, 2, \dots, N\}$, by the strong Markov property of Y,

$$\begin{aligned}
&E^{(x,k)}[\exp(-\alpha\tau_n)]\\
&\quad = E^{(x,k)}\Big[E^{(X_{\tau_1}, \Lambda_{\tau_1})}\big[\exp(-\alpha(\tau_2 - \tau_1)) \cdots \\
&\qquad\qquad \times E^{(X_{\tau_{n-1}}, \Lambda_{\tau_{n-1}})}\big[\exp(-\alpha(\tau_n - \tau_{n-1}))\big] \cdots\big]\Big]\\
&\quad \le \bigg(\sup_{1\le k\le N} \sup_{x\in D} E_k^x\Big[\exp(-\alpha\zeta_k) 1_D\Big(X_{\zeta_k -}^k\Big)\Big]\bigg)^n\\
&\quad \le \frac{1}{2^n}.
\end{aligned} \tag{2.40}$$

Hence

$$E^{(x,k)}\Big[\exp\Big(-\alpha \lim_{n\to\infty} \tau_n\Big)\Big] = \lim_{n\to\infty} E^{(x,k)}\big[\exp(-\alpha\tau_n)\big] = 0 \tag{2.41}$$

and the theorem is proved. □

Theorem 2.6 tells us that the process $Y = (X, \Lambda)$ can only have finitely many switches during a finite time interval.

3. Dirichlet boundary value problems for elliptic systems. In this section, we prove an existence and uniqueness theorem for the Dirichlet boundary value problems as well as a probabilistic representation theorem for the weakly coupled elliptic system S. Recall that we only assume that S has measurable coefficients which may be singular for lower-order terms. Throughout this section, D is a bounded domain in $\mathbb{R}^d$ such that its Euclidean boundary ∂D has zero Lebesgue measure unless otherwise specified. By extending $a^k = I$, $b^k = 0$ and $q_{kl} = 0$ off D, there exists a switched diffusion process $Y = (X, \Lambda)$ on $\mathbb{R}^d \times \{1, 2, \dots, N\}$ associated with the extended system S on $\mathbb{R}^d$. Let

$$\tau(D) = \inf\{t > 0: Y_t \notin D \times \{1, 2, \dots, N\}\}. \tag{3.1}$$

Then $(Y_t,\ t < \tau(D),\ P^{(x,k)},\ (x,k) \in D \times \{1, 2, \dots, N\})$ is the switched diffusion process on $D \times \{1, 2, \dots, N\}$ with infinitesimal generator $(S, \mathscr{D}(S))$ of (2.9) on D.

For a bounded Borel measurable function ϕ defined on $\partial D \times \{1, 2, \dots, N\}$, let

$$u(x,k) = E^{(x,k)}\left[\phi(X_{\tau(D)}, \Lambda_{\tau(D)})\right], \tag{3.2}$$

$$u_0(x,k) = E^{(x,k)}\left[\phi(X_{\tau(D)}, k) 1_{[\tau(D) < \tau]}\right], \tag{3.3}$$

where $\tau = \inf\{t > 0: \Lambda_t \neq \Lambda_0\}$ is the first switching time for the process Y.

LEMMA 3.1. *For $k \in \{1, 2, \dots, N\}$ and $x \in D$,*

$$u(x,k) = u_0(x,k) + \sum_{\substack{l=1 \\ l \neq k}}^{N} G^k(q_{kl} u(\cdot, l))(x), \tag{3.4}$$

where G^k is the Green operator of $L_k + q_{kk}$ on D with zero Dirichlet boundary condition.

PROOF.

$$\begin{aligned} u(x,k) &= E^{(x,k)}\left[\phi(X_{\tau(D)}, k) 1_{[\tau(D) < \tau]}\right] \\ &\quad + E^{(x,k)}\left[\phi(X_{\tau(D)}, \Lambda_{\tau(D)}) 1_{[\tau(D) > \tau]}\right] \\ &\quad + E^{(x,k)}\left[\phi(X_{\tau(D)}, \Lambda_{\tau(D)}) 1_{[\tau(D) = \tau]}\right] \\ &= u_0(x,k) + \mathrm{I} + \mathrm{II}. \end{aligned} \tag{3.5}$$

By the strong Markov property of Y,

$$\mathrm{I} = E^{(x,k)}\left[1_{[\tau < \tau(D)]} u(X_\tau, \Lambda_\tau)\right],$$

which, by (2.34), equals

$$\sum_{\substack{l=1 \\ l \neq k}}^{N} E^{(x,k)}\left[1_{[\tau < \tau(D)]} u(X_{\tau-}, l)\left(-\frac{q_{kl}}{q_{kk}}\right)(X_{\tau-})\right],$$

which, by Proposition 2.2, is equal to

$$\sum_{\substack{l=1\\ l\neq k}}^{N} E^{(x,k)}\left[\int_0^\tau q_{kl}(X_s)u(X_s,l)\,ds\right]$$

$$= \sum_{\substack{l=1\\ l\neq k}}^{N} G^k(q_{kl}u(\cdot,l))(x).$$

Since ∂D has zero Lebesgue measure,

$$E^{(x,k)}\left[\int_0^\tau 1_{\partial D}(X_s)\,ds\right] = 0$$

(cf. [3]). Therefore,

$$\mathrm{II} \le E^{(x,k)}\left[1_{\partial D}(X_\tau)\phi(X_\tau,\Lambda_\tau)\right],$$

which, by (2.34) and Proposition 2.2, is equal to

$$\sum_{\substack{l=1\\ l\neq k}}^{N} E^{(x,k)}\left[\int_0^\tau 1_{\partial D}(X_s)\phi(X_s,l)q_{kl}(X_s)\,ds\right] = 0. \qquad \square$$

THEOREM 3.2. *Suppose ϕ is a bounded Borel measurable function on $\mathbb{R}^d \times \{1,2,\dots,N\}$ such that $\phi(\cdot,k) \in W^{1,2}(\mathbb{R}^d)$ for each k. Then*

$$u(x,k) = E^{(x,k)}\left[\phi\left(X_{\tau(D)},\Lambda_{\tau(D)}\right)\right] \tag{3.6}$$

is the unique weak solution on $D \times \{1,2,\dots,N\}$ of $Su = 0$ such that $u(\cdot,k) - \phi(\cdot,k) \in W_0^{1,2}(D)$. Furthermore, u is continuous in D.

PROOF. Without loss of generality, we may assume $\phi \ge 0$. Clearly, u is a bounded function on $D \times \{1,2,\dots,N\}$ and, by Lemma 3.1 for $k \in \{1,2,\dots,N\}$,

$$u(x,k) = u_0(x,k) + \sum_{\substack{l=1\\ l\neq k}}^{N} G^k(q_{kl}u(\cdot,l)). \tag{3.7}$$

It is known from Lemma 5.6 in [3] that, for each fixed k,

$$u_0(x,k) = E^{(x,k)}\left[\phi(X_\tau,k)\right]$$

is the unique weak solution on D of $(L_k + q_{kk})u = 0$ such that $u_0(\cdot,k) - \phi(\cdot,k) \in W_0^{1,2}(D)$. Let G_α^k be the α-resolvent of the elliptic operator $L_k + q_{kk}$ on D with Dirichlet boundary condition or, equivalently, G_α^k is the α-resolvent of the regular Dirichlet space $(W_0^{1,2}(D), \mathscr{E}^k)$ given by (2.20). Set $\xi_l = G^k(q_{kl}u(\cdot,l))$. Since D is a bounded domain and ξ_l is bounded, we have $\xi_l \in L^2(D,dx)$ and

$$\lim_{\beta\to\infty} \beta\left(\xi_l - \beta G_\beta^k \xi_l, \xi_l\right)_{L^2(D,dx)} = \lim_{\beta\to\infty} \beta\left(G_\beta^k(q_{kl}u(\cdot,l)), \xi_l\right)_{L^2(D,dx)}$$

$$= \lim_{\beta\to\infty}\left(q_{kl}u(\cdot,l), \beta\hat{G}_\beta^k \xi_l\right)_{L^2(D,dx)},$$

where $\hat{G}_\beta^k$ is the adjoint of G_β^k on $L^2(D, dx)$. Since $q_{kl} \in L^1(D, dx)$ and $\beta \hat{G}_\beta^k \xi_l$ converges to ξ_l in $L^2(D, dx)$ as $\beta \to \infty$, by the Lebesgue dominant convergence theorem

$$\lim_{\beta \to \infty} \beta \big(\xi_l - \beta G_\beta^k \xi_l, \xi_l \big)_{L^2(D, dx)} = (q_{kl} u(\cdot, l), \xi_l) < \infty. \tag{3.8}$$

Thus, by Lemma 3.2 in [12], $\xi_l = G^k(q_{kl} u(\cdot, l)) \in W_0^{1,2}(D)$ and, for $\psi \in C_c^\infty(D)$,

$$\begin{aligned} \mathscr{E}^k(\xi_l, \psi) &= \lim_{\beta \to \infty} \beta \big(\xi_l - \beta G_\beta^k \xi_l, \psi \big)_{L^2(D, dx)} \\ &= \lim_{\beta \to \infty} \Big(q_{kl} u(\cdot, l), \beta \hat{G}_\beta^k \psi \Big) \\ &= (q_{kl} u(\cdot, l), f). \end{aligned} \tag{3.9}$$

Hence ξ_l is the weak solution in $W_0^{1,2}(D)$ for $(L_k + q_{kk})\xi_l = -q_{kl} u(\cdot, l)$. Therefore,

$$u(\cdot, k) - \phi(\cdot, k) = u_0(\cdot, k) - \phi(\cdot, k) + \sum_{\substack{l=1 \\ l \neq k}}^N \xi_l(\cdot)$$

is in $W_0^{1,2}(D)$ and, by (3.7),

$$L_k u(\cdot, k) + \sum_{l=1}^N q_{kl} u(\cdot, l) = 0. \tag{3.10}$$

Hence u is the weak solution for $Su = 0$ in D with $u(\cdot, k) - \phi(\cdot, k) \in W_0^{1,2}(D)$. The uniqueness follows from Lemma 3.3.

Next we show u is continuous in D. It is known from Lemma 5.8 in [3] that $u_0(\cdot, k)$ is continuous in D. Let $(X^k, P_k^x, x \in D)$ by the diffusion of $L_k + q_{kk}$ with zero Dirichlet boundary condition on D. Then

$$\begin{aligned} \xi_l(x) &= G^k(q_{kl}(\cdot) u(\cdot, l))(x) \\ &= E_k^x\left[\int_0^t q_{kl}(X_s^k) u(X_s^k, l)\, ds \right] + E_k^x[\xi_l(X_t)]. \end{aligned} \tag{3.11}$$

Since (cf. Lemmas 5.7 and 5.8 in [3])

$$\lim_{t \downarrow 0} \sup_{x \in D} E_k^x\left[\int_0^t q(X_s^k) u(X_s^k, l)\, ds \right] = 0 \tag{3.12}$$

and $x \mapsto E_k^x[\xi_l(X_t)]$ is continuous in D, ξ_l is a continuous function on D. Therefore, u is continuous on D. □

LEMMA 3.3. *Let D be a domain in $\mathbb{R}^d$ having finite Lebesgue measure. Suppose u is a function defined on $D \times \{1, 2, \ldots, N\}$ such that $u(\cdot, k) \in W_0^{1,2}(D)$ for each k and $Su = 0$ on D. Then $u = 0$ m-a.e. on $D \times \{1, 2, \ldots, N\}$.*

REMARK. Protter and Weinberger [14] proved a maximum principle for solutions of $Su \geq 0$ for an elliptic system S whose L_k in (1.1) are of the form $\frac{1}{2}(a^k\nabla)\cdot\nabla + b^k\cdot\nabla$ and all the coefficients a^k, b^k and q_{kl} are assumed to be bounded. Since our system S has $L_k = \frac{1}{2}\nabla\cdot(a^k\nabla) + b^k\nabla$ with measurable a^k and singular coefficients b^k and q_{kl}, we cannot apply Protter and Weinberger's method using the Hopf lemma. We prove this lemma by using a technique from Trudinger [19] together with a probabilistic characterization of S-harmonic functions.

PROOF OF LEMMA 3.3. Suppose $m(\{(x,k)\colon u(x,k)\neq 0\}) > 0$. Without loss of generality, we assume

$$\theta = \operatorname*{ess\,sup}_{D\times\{1,2,\dots,N\}} u > 0. \tag{3.13}$$

Let $\beta\in(0,\theta)$ and $v=(u-\beta)^+$. Then $v(\cdot,k)$ is in $W_0^{1,2}(D)$ for each k by Theorem 7.8 in [7]. Since $Su=0$, $\mathscr{E}(u,v)=0$, where $\mathscr{E}$ is the Dirichlet form on D given by (2.4). For $y=(x,k)$, set $a(y)=a^k(x)$ and $b(y)=b^k(x)$. From $\mathscr{E}(u,v)=0$, one has

$$\begin{aligned}
&\frac{1}{2}\int_W\sum_{i,j=1}^d a_{ij}(y)\frac{\partial v}{\partial x_i}(y)\frac{\partial v}{\partial x_j}(y)m(dy)\\
&\qquad=\sum_{i=1}^d\int_W b_i(y)\frac{\partial v}{\partial x_i}(y)v(y)m(dy)\\
&\qquad\quad+\sum_{k,l=1}^N\int_D q_{kl}(x)u(\alpha,k)v(\alpha,l)\,dx,
\end{aligned}\tag{3.14}$$

where $W=D\times\{1,2,\dots,N\}$. By (2.5),

$$\begin{aligned}
&\sum_{i=1}^d\int_W b_i(y)\frac{\partial v}{\partial x_i}(y)v(y)m(dy)\\
&\qquad\leq\sqrt{\int|\nabla v|^2(y)m(dy)}\sqrt{\sum_{i=1}^d\int b_i^2(y)v^2(y)m(dy)}\\
&\qquad\leq\frac{\varepsilon}{2}\int|\nabla v|^2(y)m(dy)+\frac{\varepsilon d}{2}\int|\nabla v|^2(y)m(dy)\\
&\qquad\quad+\frac{d}{2}\frac{C_{\varepsilon^2}}{\varepsilon}\int v^2(y)m(dy)\\
&\qquad=\frac{(d+1)\varepsilon}{2}\int|\nabla v|^2(y)m(dy)+\frac{d}{2}\frac{C_{\varepsilon^2}}{\varepsilon}\int v^2(y)m(dy).
\end{aligned}\tag{3.15}$$

Note that q_{kl} satisfies (1.4) and (1.5):

$$\begin{aligned}
\sum_{k,l=1}^{N} \int_D q_{kl}(x)u(x,k)v(x,l)\,dx
&= \int \Bigg[\sum_{k,l=1}^{N} q_{kl}(x)(u(x,k) \wedge \beta)(u(x,l) - \beta)^{+} \\
&\qquad\qquad + \sum_{k,l=1}^{N} q_{kl}(x)v(x,k)v(x,l) \Bigg]\,dx \\
&\le \int \sum_{k,l=1}^{N} q_{kl}(x)v(x,k)v(x,l)\,dx \\
&\le \frac{1}{2} \sum_{k,l=1}^{N} \left[\int_D q_{kl}(x)v^2(x,k)\,dx + \int_D q_{kl}(x)v^2(x,l)\,dx \right],
\end{aligned}$$

which, by (2.5),

$$\le \varepsilon N \int |\nabla v|^2(y)m(dy) + NC_\varepsilon \int v^2(y)m(dy).$$

Thus, by (3.14) and the ellipticity (1.3),

$$\begin{aligned}
&\frac{\lambda^{-1}}{2} \int |\nabla v|^2(y)m(dy) \\
&\quad \le \left(\frac{d+1}{2} + N \right) \varepsilon \int |\nabla v|^2(y)m(dy) + \left(\frac{d}{2}\frac{C_{\varepsilon^2}}{\varepsilon} + NC_\varepsilon \right) \int v^2(y)m(dy).
\end{aligned}$$

Hence, by selecting ε sufficiently small, we have

$$\|\nabla v\|_2 \le \gamma \|v\|_2 \tag{3.16}$$

for some constant $\gamma > 0$, where $\|\cdot\|_2$ is the L^2 norm in $L^2(W,m)$. By Sobolev's inequality (see, e.g., Theorem 7.10 in [7]) for $d \ge 3$,

$$\begin{aligned}
\|v\|_{2d/(d-2)} &\le C\|\nabla v\|_2 \\
&\le C\gamma\|v\|_2 \le C\gamma\big(m(\operatorname{supp} v)\big)^{1/n}\|v\|_{2d/(d-2)},
\end{aligned} \tag{3.17}$$

where $C = C(d) > 0$, $\operatorname{supp} v$ is the smallest relatively closed subset in $D \times \{1,2,\dots,N\}$ outside of which v vanishes and the last inequality comes from Hölder's inequality. Therefore,

$$m(\operatorname{supp} v) \ge (C\gamma)^{-n}. \tag{3.18}$$

In the case of $d = 2$, the inequality of the same form with the constant C depending on d and the volume of D also follows from the Sobolev inequality by replacing $2d/(d-2)$ by any number greater than 2. Since inequality (3.18) is independent of $\beta \in (0,\theta)$, u must attain its essential supremum θ in $D \times \{1,2,\dots,N\}$ on a set of positive measure. Thus u is bounded from above and therefore u is bounded since $-u$ satisfies $S(-u) = 0$.

Let G_α be the α-resolvent of the Dirichlet space $(\mathscr{F}, \mathscr{E})$ given by (2.3) and (2.4) on $L^2(W, m)$ and let $\hat{G}_\alpha$ be the adjoint of G_α. Then, for $\alpha > \alpha_0$ and $f \in L^2(W, m)$, $\hat{G}_\alpha f \in \mathscr{F}$ and therefore

$$(3.19)\qquad \begin{aligned}(u, f)_{L^2(W,m)} &= \mathscr{E}_\alpha\big(u, \hat{G}_\alpha f\big)\\ &= \alpha\big(u, \hat{G}_\alpha f\big)_{L^2(W,m)}\\ &= (\alpha G_\alpha u, f)_{L^2(W,m)}.\end{aligned}$$

Thus $u = \alpha G_\alpha u$ for $\alpha > \alpha_0$ and therefore $u = P_t u$ m-a.e. on $D \times \{1, 2, \dots, N\}$ for $t > 0$, where P_t is the semigroup of $(\mathscr{F}, \mathscr{E})$. Since u attains its essential supremum a in $D \times \{1, 2, \dots, N\}$ on a set of positive measure, there exists a point $(x, k) \in D \times \{1, 2, \dots, N\}$ such that

$$(3.20)\qquad E^{(x,k)}\big[u(X_1, \Lambda_1); \tau(D) > 1\big] = P_1 u(x, k) = a.$$

Let $A = \{(x, l)\ u(x, l) = a\}$. By (3.20),

$$(3.21)\qquad P^{(x,k)}\big[(X_1, \Lambda_1) \in A\big] = 1.$$

We know from Remark 3 for Theorem 5.11 in [3] that the diffusion process X^k on D of $L_k + q_{kk}$ with zero Dirichlet boundary condition has transition density function $p^k(t, x, \cdot)$ which is strictly positive almost everywhere in D for each fixed $(t, x) \in \mathbb{R}^+ \times D$. Since

$$P^{(x,k)}\big[(X_1, \Lambda_1) \in (dz, k); 1 < \tau < \tau(D)\big] = p^k(1, x, z)\, dz,$$

where $\tau = \inf\{t > 0\colon \Lambda_t \neq \Lambda_0\}$, we have $m(D \times \{k\} \setminus A) = 0$. Hence $u(\cdot, k) = a > 0$ a.e. on D, which contradicts the hypothesis that $u(\cdot, k) \in W_0^{1,2}(D)$. □

PROPOSITION 3.4. *Any locally bounded weak solution of $Su = 0$ on D has a continuous version.*

PROOF. Suppose that u is a weak solution of $Su = 0$ in D which is locally bounded. For an arbitrary point $z \in D$, let $r > 0$ such that $\overline{B(z, r)} \subset D$, where $B(z, r) = \{x \in \mathbb{R}^d\colon |x - z| < r\}$. By Theorem 3.2,

$$(x, k) \mapsto E^{(x,k)}\big[u\big(X_{\tau(B(z,r))}, \Lambda_{\tau(B(z,r))}\big)\big]$$

is a continuous version of u on $B(z, r) \times \{1, 2, \dots, N\}$, where

$$\tau(B(z, r)) = \inf\{t > 0\colon X_t \notin B(z, r)\}.$$

Therefore, u has a continuous version on $D \times \{1, 2, \dots, N\}$ by using the partition of unity. □

By a similar argument as that for Lemma 4.4 in [3], we have the following result.

LEMMA 3.5. *For any weak solution u of $Su = 0$ on D, the following inequality holds*:

$$(3.22)\qquad \int_{B(z,r)} \sum_{k=1}^N |\nabla u(\cdot, k)|^2\, dx \le C \int_{B(z,R)} \sum_{k=1}^N |u(\cdot, k)|^2\, dx,$$

with $0 < r < R$ *such that* $\overline{B(z,R)} \subset D$, *where* $C > 0$ *is a constant which depends only on the ellipticity constant* λ *in* (1.3), *the coefficients in* (2.5) *for* $|b^k|^2$ *and* q_{kl} *and on the value of* $R - r$. *Here* $B(z,r)$ *denotes the Euclidean ball in* $\mathbb{R}^d$ *centered at* z *with radius* r.

THEOREM 3.6. *Let* D *be a bounded domain in* $\mathbb{R}^d$ *and* $\phi(\cdot, k) \in C(\partial D)$ *for* $k = 1, 2, \dots, N$. *Then*

$$u(x,k) = E^{(x,k)}\left[\phi\left(X_{\tau(D)}, \Lambda_{\tau(D)}\right)\right]$$

is the unique weak solution of $Su = 0$ *on* D *such that*

$$\lim_{\substack{x \to z\\ x \in D}} u(x,k) = \phi(z,k) \tag{3.23}$$

for any boundary point $z \in \partial D$ *which is regular for* $(\frac{1}{2}\Delta, D)$. *Moreover* u *is continuous on* $D \times \{1, 2, \dots, N\}$.

PROOF. Let ϕ_n be such that $\phi_n(\cdot, k) \in C^2(\partial D)$ and ϕ_n converges to ϕ uniformly on $\partial D \times \{1, 2, \dots, N\}$. Let

$$u^n(x,k) = E^{(x,k)}\left[\phi_n\left(X_{\tau(D)}, \Lambda_{\tau(D)}\right)\right] \tag{3.24}$$

and

$$u_0^n(x,k) = E^{(x,k)}\left[\phi_n\left(X_{\tau(D)}, k\right) 1_{[\tau(D) < \tau]}\right], \tag{3.25}$$

where $\tau = \inf\{t > 0\colon \Lambda_t \neq \Lambda_0\}$. By Theorem 5.11 in [3], $u_0^n(\cdot, k)$ is the unique weak solution of $(L_k + q_{kk})u_0^n(\cdot, k) = 0$ such that

$$\lim_{\substack{x \to z\\ x \in D}} u_0^n(x,k) = \phi_n(z,k) \tag{3.26}$$

for $z \in \partial D$ which is regular for $(\frac{1}{2}\Delta, D)$. It follows from (3.11), (3.12) and Lemma 5.7 of [3] that

$$\lim_{\substack{x \to z\\ x \in D}} G^k\left(q_{kl} u^n(\cdot, l)\right)(x) = 0 \tag{3.27}$$

for $z \in \partial D$ which is regular for $(\frac{1}{2}\Delta, D)$. Thus, by Lemma 3.1 and Theorem 3.2, u^n is the unique weak solution of $Su = 0$ satisfying (3.23) with ϕ_n in place of ϕ. Since u^n converges uniformly to u on $D \times \{1, 2, \dots, N\}$ as $n \to \infty$, it follows from Lemma 3.5 that u is the unique weak solution of $Su = 0$ such that (3.23) holds. $\square$

4. Strong positivity result for solutions of Dirichlet boundary value problems. In this section, we prove that if the elliptic system S is irreducible, a strong positivity result holds for solutions of the Dirichlet boundary value problem for S.

Recall that $Q = (q_{kl})_{N \times N}$ is an $N \times N$ matrix-valued, measurable function on D in (1.1) which satisfies conditions (1.4) and (1.5).

DEFINITION 4.1. The weakly coupled elliptic system S or the matrix Q is said to be *irreducible on* D if for any distinct $k, l \in \{1,2,\dots,N\}$, there exist $k_0, k_1, \dots, k_r$ in $\{1,2,\dots,N\}$ with $k_i \neq k_{i+1}$, $k_0 = k$ and $k_r = l$ such that $\{x \in D: q_{k_i k_{i+1}}(x) \neq 0\}$ has positive Lebesgue measure for $i = 0,1,\dots,r-1$.

PROPOSITION 4.1. *The matrix Q is irreducible on D if and only if Q is fully coupled on D in the sense of* [18]; *that is*, $\{1,2,\dots,N\}$ *cannot be split into two disjoint nonempty sets Γ and Σ such that $q_{kl} = 0$ a.e. on D for $k \in \Gamma$ and $l \in \Sigma$.*

PROOF. If Q is irreducible on D, then Q is fully coupled on D since otherwise $\{1,2,\dots,N\}$ can be split into two disjoint nonempty sets Γ and Σ such that $q_{kl} = 0$ a.e. on D for any $k \in \Gamma$ and $l \in \Sigma$. Let $k \in \Gamma$ and $l \in \Sigma$. By the irreducibility of Q on D, there exists $\{k_i\}_{i=0}^r \subset \{1,2,\dots,N\}$ with $k_i \neq k_{i+1}$, $k_0 = k$ and $k_r = l$ such that $\{x: q_{k_i k_{i+1}}(x) \neq 0\}$ has positive Lebesgue measure. This implies that $\{k_i\}_{i=0}^r \subset \Gamma$. In particular, $l \in \Gamma$, a contradiction.

Conversely, suppose that Q is fully coupled on D. For each fixed $k \in \{1,2,\dots,N\}$, let

$$\Gamma(k) = \{l\colon \text{there exists } \{k_i\}_{i=0}^r \subset \{1,2,\dots,N\} \text{ with } k_i \neq k_{i+1},\ k_0 = k,\ k_r = l \text{ such that } \{x\colon q_{k_i k_{i+1}}(x) \neq 0\} \text{ has positive Lebesgue measure}\}.$$

The set $\Gamma(k)$ is not empty since Q is fully coupled on D. Indeed, $\Gamma(k) = \{1,2,\dots,N\}$, since otherwise $\Sigma(k) = \{1,2,\dots,N\} \setminus \Gamma(k)$ is nonempty. Hence, there exist $r \in \Gamma(k)$ and $l \in \Sigma(k)$ such that $\{x \in D: q_{rl}(x) \neq 0\}$ has positive Lebesgue measure. This implies that $l \in \Gamma(k)$, a contradiction. □

It is clear that any weakly coupled elliptic system of (1.1) can be decomposed into several independent irreducible elliptic subsystems.

THEOREM 4.2 (Strong positivity result). *Let D be a bounded domain in $\mathbb{R}^d$ and let u be the unique weak solution of*

$$\begin{aligned} Su &= 0 \quad \text{in } D \times \{1,2,\dots,N\}, \\ u &= \phi \quad \text{on } \partial D \times \{1,2,\dots,N\} \end{aligned} \tag{4.1}$$

for a continuous function ϕ. If the elliptic system is irreducible on D and $\phi \geq 0$ is such that $\sum_{k=1}^N \phi(\cdot,k) \not\equiv 0$ on ∂D, then $u > 0$ in $D \times \{1,2,\dots,N\}$.

PROOF. Let $u_0(\cdot,k)$ be the unique solution of

$$\begin{aligned} (L_k + q_{kk}) u_0(\cdot,k) &= 0 \qquad \text{in } D, \\ u_0(\cdot,k) &= \phi(\cdot,k) \quad \text{on } \partial D. \end{aligned}$$

Since $\phi \geq 0$, $u_0(\cdot,k) \geq 0$ for each k. Without loss of generality, we may assume that $\phi(\cdot,1) \not\equiv 0$ on ∂D. Then, by Remark 2 for Theorem 5.11 in [3], $u_0(\cdot,1) > 0$ and therefore $u(\cdot,1) > 0$ in D. For any fixed $k \in \{2,3,\dots,N\}$, by the irreducibility of the system S on D, there exists $\{k_i\}_{i=0}^r$ with $k_i \neq k_{i+1}$, $k_0 = k$ and $k_r = 1$ such that $\{x \in D: q_{k_i k_{i+1}}(x) \neq 0\}$ has positive Lebesgue measure. It follows from (3.4) that

$$
\begin{aligned}
u(x,k)
&= u_0(x,k) + \sum_{\substack{l=1\\ l\neq k}}^{N} G^k\big(q_{kl}u(\cdot,l)\big)\\
&= u_0(x,k)\\
&\quad + \sum_{i=1}^{r} \sum_{\substack{l_1,\dots,l_i=1\\ l_1\neq 1,\, l_2\neq l_1,\dots,\, l_i\neq l_{i-1}}}^{N} G^k\Big(q_{kl_1}\Big(G^{l_1}q_{l_1l_2}\Big(\cdots\Big(G^{l_{i-1}}q_{l_{i-1}l_i}u_0(\cdot,l_i)\Big)\cdots\Big)\Big)\Big)\\
&\quad + \sum_{\substack{l_1,\dots,l_r=1\\ l_1\neq 1,\, l_2\neq l_1,\dots,\, l_r\neq l_{r-1}}}^{N} G^k\Big(q_{kl_1}\Big(G^{l_1}q_{l_1l_2}\Big(\cdots\Big(G^{l_{r-1}}q_{l_{r-1}l_r}u(\cdot,l_r)\Big)\cdots\Big)\Big)\Big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
u(x,k) &\geq G^k\Big(q_{kk_1}\Big(G^{k_1}q_{k_1k_2}\Big(\cdots\Big(G^{k_{r-1}}q_{k_{r-1}1}u_0(\cdot,1)\Big)\cdots\Big)\Big)\Big)\\
&> 0.
\end{aligned}
$$

Here we use the fact that $G^k\psi > 0$ in D whenever $\psi \geq 0$ in D such that $\{x \in D\colon \psi(x) > 0\}$ has positive Lebesgue measure (see Remark 3 for Theorem 5.11 in [3]). □

REMARK. Under the irreducibility condition, Sweers [18] proved a strong positivity result for solutions of

$$
\begin{aligned}
Su &= -\phi \quad \text{in } D,\\
u &= 0 \quad\;\; \text{on } \partial D,
\end{aligned}
\tag{4.2}
$$

using an analytic method. It can also be proved by using the following identity:

$$
G\phi(x,k) = G^k\phi(\cdot,k)(x) + G^k\left(\sum_{\substack{l=1\\ l\neq k}}^{N} q_{kl}G\phi(\cdot,l)\right)(x),
$$

which follows from (2.34) and the strong Markov property of the switched diffusion process $Y = (X, \Lambda)$ on D.

5. Resolvent and kernel identities. Let D be a Euclidean domain in $\mathbb{R}^d$, $W = D \times \{1, 2, \dots, N\}$ and let S be a weakly coupled elliptic system (1.1) on D. In this section, we derive formulas which express the resolvent and (transition density) kernels of S in terms of those for the component elliptic operators $L_k + q_{kk}$, $k = 1, 2, \ldots, N$. First, we present a continuity result for the α-resolvent G_α of the system.

THEOREM 5.1. *Let $\alpha > 0$ and let f be a bounded Borel measurable function on $D \times \{1, 2, \ldots, N\}$. Then $G_\alpha f$ is continuous in $D \times \{1, 2, \ldots, N\}$ and*

$$\lim_{\substack{x \to z \\ x \in D}} G_\alpha f(x, k) = 0 \tag{5.1}$$

for $z \in \partial D$ which is regular for $(\frac{1}{2}\Delta, D)$.

PROOF. By the strong Markov property of the switched diffusion process $Y = (X, \Lambda)$ associated with S on D and (2.34),

$$G_\alpha f(x, k) = G_\alpha^k f(\cdot, k)(x) + \sum_{\substack{l=1 \\ l \neq k}}^{N} G_\alpha^k (q_{kl} G_\alpha f(\cdot, l))(x), \tag{5.2}$$

where G_α^k is the α-resolvent of $L_k + q_{kk}$ on D with Dirichlet boundary condition. By Lemma 5.7 in [3], $G_\alpha^k f$ is continuous in D and

$$\lim_{\substack{x \to z \\ x \in D}} G_\alpha^k f(\cdot, k)(x) = 0 \tag{5.3}$$

for $z \in \partial D$ which is regular for $(\frac{1}{2}\Delta, D)$. Let $(X^k, P_k^x,\ x \in D)$ be the diffusion process of $L_k + q_{kk}$ on D with Dirichlet boundary condition and set $g_l = G_\alpha^k(q_{kl} G_\alpha f(\cdot, l))$. Then

$$g_l(x) = E_k^x\left[\int_0^t e^{-\alpha s} q_{kl}(X_s^k) G_\alpha f(X_s^k, l)\, ds\right] + e^{-\alpha t} E_k^x\left[g_l(X_t^k)\right]. \tag{5.4}$$

Since

$$\lim_{t \downarrow 0} \sup_{x \in D} E_k^x\left[\int_0^t e^{-\alpha s} q_{kl}(X_s^k) G_\alpha f(X_s^k, l)\, ds\right] = 0 \tag{5.5}$$

(see the proof of Lemma 5.8 in [3]) and $x \to E_k^x[g_l(X_t^k)]$ is continuous in D such that

$$\lim_{\substack{x \to z \\ x \in D}} E_k^x\left[g_l(X_t^k)\right] = 0 \tag{5.6}$$

for $z \in \partial D$ which is regular for $(\frac{1}{2}\Delta, D)$, g_l is continuous in D with the same limit behavior at the boundary as the function $x \mapsto E_k^x[g_l(X_t)]$ has. Thus, by (5.2), $G_\alpha f(\cdot, k)$ is continuous in D and (5.1) holds for each k. □

Let

$$G_\alpha^0 = \begin{pmatrix} G_\alpha^1 & 0 & \cdots & 0 \\ 0 & G_\alpha^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_\alpha^N \end{pmatrix} \quad \text{and} \quad Q^0 = Q - \begin{pmatrix} q_{11} & 0 & \cdots & 0 \\ 0 & q_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{NN} \end{pmatrix}.$$

THEOREM 5.2. *There exists a constant* $\alpha_1 > 0$ *such that, for* $\alpha > \alpha_1$ *and* $f \in L^2(W, m)$,

$$G_\alpha f = G_\alpha^0 f + \sum_{n=1}^{\infty} G_\alpha^0 \left(Q^0 G_\alpha^0\right)^n f. \tag{5.7}$$

This vector-valued series is convergent in $(W_0^{1,2}(D)^d, \|\cdot\|_{1,2})$.

PROOF. Without loss of generality, we assume $f \geq 0$. Let $(W_0^{1,2}(D), \mathscr{E}^k)$ be as in (2.20). Since $1_D|b^k|^2 \in K_d$ and $1_D q_{kk} \in K_d$, there exists a constant $\alpha_2 > \alpha_0$ (α_0 is the constant in Section 2) such that, for $\alpha > \alpha_2$, $k = 1, 2, \dots, N$ and $\phi \in W_0^{1,2}(D)$,

$$\frac{1}{4\lambda} \int_D |\nabla \phi|^2\, dx + \frac{\alpha}{2} \int_D |\phi|^2\, dx \leq \mathscr{E}_\alpha^k(\phi, \phi) \leq \lambda \int_D |\nabla \phi|^2\, dx + 2\alpha \int |\phi|^2\, dx,$$

where $\lambda > 1$ is the ellipticity constant in (1.3). If we let

$$||| \phi |||_\alpha \stackrel{\text{def}}{=} \left(\int_D |\nabla \phi|^2\, dx + 2\alpha\lambda \int_D |\phi|^2\, dx \right)^{1/2}, \tag{5.8}$$

then

$$\frac{1}{4\lambda} ||| \phi |||_\alpha^2 \leq \mathscr{E}_\alpha^k(\phi, \phi) \leq \lambda ||| \phi |||_\alpha^2 \tag{5.9}$$

for $\phi \in W_0^{1,2}(D)$ and $k = 1, 2, \dots, N$. By (2.26), for any fixed $k \in \{1, 2, \dots, N\}$,

$$\begin{aligned} G_\alpha f(x, k) &= G_\alpha^k f(\cdot, k)(x) + G_\alpha^k \left(\sum_{\substack{l=1 \\ l \neq k}}^{N} q_{kl} G_\alpha f(\cdot, l) \right)(x) \\ &= G_\alpha^k f(\cdot, k)(x) + G_\alpha^k \left(\sum_{\substack{l=1 \\ l \neq k}}^{N} q_{kl} G_\alpha^l f(\cdot, l) \right)(x) \\ &\quad + G_\alpha^k \left(\sum_{\substack{l=1 \\ l \neq k}}^{N} q_{kl} G_\alpha^l \left(\sum_{\substack{l=1 \\ l \neq k}}^{N} q_{ll_1} G_\alpha f(\cdot, l_1) \right) \right) \\ &= \sum_{i=0}^{n} \phi_i^k(x) + R_n^k(x), \end{aligned} \tag{5.10}$$

where

$$\phi_0^k = G_\alpha^k f(\cdot, k)$$

$$\phi_1^k = G_\alpha^k \left(\sum_{\substack{l=1 \\ l \neq k}}^{N} q_{kl} G_\alpha^l f(\cdot, l) \right) = G_\alpha^k \left(\sum_{\substack{l=1 \\ l \neq k}}^{N} q_{kl} \phi_0^l \right) \tag{5.11}$$

$$\vdots$$

$$\phi_i^k = G_\alpha^k \left(\sum_{\substack{l=1 \\ l \neq k}}^{N} q_{kl} \phi_{i-1}^l \right) \tag{5.12}$$

$$\vdots$$

Since $1_D q_{kl} \in K_d$ for $k, l \in \{1, 2, \ldots, N\}$, by (2.5) there exists a constant $\alpha_1 > \alpha_2$ such that, for $\alpha > \alpha_1$, $k, l \in \{1, 2, \ldots, N\}$ and $\phi, \psi \in W_0^{1,2}(D)$,

$$\int_D |q_{kl} \phi \psi| \, dx \le \delta \sqrt{\mathscr{E}_\alpha^k(\phi, \phi)} \sqrt{\mathscr{E}_\alpha^k(\psi, \psi)}, \tag{5.13}$$

where $\delta = 1/(4\lambda N)$. By the same idea employed in the proof of Lemma 2.4, it can be shown that $\phi_i^k \in W_0^{1,2}(D)$ and, for $\alpha > \alpha_1$,

$$\begin{aligned} \mathscr{E}_\alpha^k(\phi_1^k, \phi_1^k) &= \left(\sum_{\substack{l=1 \\ l \neq k}}^{N} q_{kl} \phi_0^l, \phi_1^k \right) \\ &\le \delta \left(\sum_{\substack{l=1 \\ l \neq k}}^{N} \sqrt{\mathscr{E}_\alpha^k(\phi_0^l, \phi_0^l)} \right) \sqrt{\mathscr{E}_\alpha^k(\phi_1^k, \phi_1^k)}. \end{aligned}$$

Thus

$$\mathscr{E}_\alpha^k(\phi_1^k, \phi_1^k) \le \delta^2 \sum_{\substack{l=1 \\ l \neq k}}^{N} \mathscr{E}_\alpha^k(\phi_0^l, \phi_0^l). \tag{5.14}$$

Hence, by (5.9),

$$||| \phi_1^k |||_\alpha \le 2\lambda\delta \sum_{\substack{l=1 \\ l \neq k}}^{N} ||| \phi_0^l |||_\alpha.$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{N} ||| \phi_1^k |||_\alpha &\le 2\lambda\delta(N-1) \sum_{k=1}^{N} ||| \phi_0^k |||_\alpha \\ &\le \frac{1}{2} \sum_{k=1}^{N} ||| \phi_0^k |||_\alpha. \end{aligned} \tag{5.15}$$

A similar argument yields that

$$\sum_{k=1}^{N} \| \phi_i^k \|_\alpha \le \frac{1}{2} \sum_{k=1}^{N} \| \phi_{i-1}^k \|_\alpha \le \frac{1}{2^i} \sum_{k=1}^{N} \| \phi_0^k \|_\alpha \tag{5.16}$$

and

$$\sum_{k=1}^{N} \left\| R_n^k(x) \right\|_\alpha \le \frac{1}{2^n} \sum_{k=1}^{N} \left\| G_\alpha f(\cdot, k) \right\|_\alpha. \tag{5.17}$$

Therefore, by (5.10), $G_\alpha f(\cdot, k) = \sum_{i=0}^{\infty} \phi_i^k$, which is convergent in $(W_0^{1,2}(D), \|\cdot\|_{1,2})$. The theorem is thus proved. □

Let $p_k(t, x, y)$ be the transition density function for the diffusion process of $L_k + q_{kk}$ on D with zero Dirichlet boundary condition (see Remark 5 for Theorem 5.11 in [3] for its existence).

THEOREM 5.3. *For $x, y \in D$ and $k, l \in \{1, 2, \dots, N\}$,*

$$\begin{aligned} p(t, (x,k), (y,l)) &= \delta_{kl} p_k(t, x, y) \\ &\quad + \sum_{n=1}^{\infty} \sum_{\substack{1 \le l_1, l_2, \dots, l_n < N \\ l_1 \ne k,\, l_n \ne l,\, l_i \ne l_{i+1}}} \int \cdots \int_{0 < t_1 < t_2 < \cdots < t_n < t} \int_D \cdots \int_D p_k(t_1, x, y_1) q_{kl_1}(y_1) \\ &\qquad \times p_{l_1}(t_2 - t_1, y_1, y_2) \cdots q_{l_n l}(y_n) \\ &\qquad \times p_l(t - t_n, y_n, y)\, dt_1\, dt_2 \cdots dt_n\, dy_1\, dy_2 \cdots dy_n \end{aligned} \tag{5.18}$$

converges and is the transition density function for $Y = (X, \Lambda)$ on $D \times \{1, 2, \dots, N\}$.

PROOF. This follows from Theorem 5.2 and the fact that, for $f \in L_+^2(W, m)$,

$$G_\alpha f(x, k) = \int_0^\infty e^{-\alpha t} p(t, (x,k), (y,l)) f(y, l)\, dm(y, l)$$

for $\alpha > \alpha_0$. □

Acknowledgments. The authors are grateful to Pat Fitzsimmons, Tom Kurtz and Ruth Williams for helpful discussion. Thanks are also due to the referee for a very careful reading and helpful comments.

REFERENCES

[1] CARRILLO MENENDEZ, S. (1975). Processus de Markov associé à une forme de Dirichlet non symétrique. *Z. Wahrsch. Verw. Gebiete* **33** 139–154.

[2] CHEN, Z. Q. and ZHAO, Z. (1994). Switched diffusion processes and systems of elliptic equations—a Dirichlet space approach. *Proc. Roy. Soc. Edinburgh Sect. A* **124** 673–701.

[3] CHEN, Z. Q. and ZHAO, Z. (1995). Diffusion processes and second order elliptic operators with singular coefficients for lower order terms. *Math. Ann.* **302** 323–357.

[4] CHUNG, K. L. and ZHAO, Z. (1995). *From Brownian Motion to Schrödinger Equations*. Springer, Berlin.

[5] EIZENBERG, A. and FREIDLIN, M. (1990). On the Dirichlet problem for a class of second order PDE systems with small parameter. *Stochastics Stochastics Rep.* **33** 111–148.

[6] FUKUSHIMA, M. (1980). *Dirichlet Forms and Markov Processes*. North-Holland, Amsterdam.

[7] GILBERG, D. and TRUDINGER, N. S. (1977). *Elliptic Partial Differential Equations of Second Order*. Springer, New York.

[8] HE, S.-W., WANG, J.-G. and YAN, J.-A. (1992). *Semimartingale Theory and Stochastic Calculus*. Science Press/CRC Press, Beijing.

[9] IKEDA, N., NAGASAWA, M. and WATANABE, S. (1966). A construction of Markov process by piecing out. *Proc. Japan Acad. Ser. A Math. Sci.* **42** 370–375.

[10] KATO, T. (1966). *Perturbation Theory for Linear Operators*. Springer, New York.

[11] KIFER, Y. (1992). Principal eigenvalues and equilibrium states corresponding to weakly coupled parabolic systems of PDE. Preprint.

[12] KUNITA, H. (1969). Sub-Markov semi-groups in Banach lattices. *Functional Analysis and Related Topics. Lecture Notes in Math.* **1540**. Springer, Berlin 332–343.

[13] MEYER, P. A. (1975). Renaissance, recollements, mélanges, relentissement de processus de Markov. *Ann. Inst. Fourier* (*Grenoble*) **25** 465–497.

[14] PROTTER, M. and WEINBERGER, H. (1976). *Maximum Principles in Differential Equations*. Prentice-Hall, Englewood Cliffs, NJ.

[15] SHARPE, M. (1986). *General Theory of Markov Processes*. Academic, New York.

[16] SIMON, B. (1982). Schrödinger semigroups. *Bull. Amer. Math. Soc.* **7** 447–526.

[17] SKOROKHOD, A. V. (1989). *Asymptotic Methods of the Theory of Stochastic Differential Equations*. Trans. Amer. Math. Soc., Providence, RI.

[18] SWEERS, G. (1992). Strong positivity in $C(\overline{\Omega})$ for elliptic systems. *Math. Z.* **209** 251–271.

[19] TRUDINGER, N. S. (1973). Linear elliptic operators with measurable coefficients. *Ann. Scuola Norm. Sup. Pisa Sci. Fis. Mat.* **27** 255–308.

DEPARTMENT OF MATHEMATICS
CORNELL UNIVERSITY
ITHACA, NEW YORK
E-mail: zchen@math.cornell.edu

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MISSOURI
COLUMBIA, MISSOURI 65211
E-mail: mathzz@mizzou1.missouri.edu