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NONLINEAR SUPERPROCESSES

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Nonlinear martingale problems in the McKean–Vlasov sense for superprocesses are studied. The stochastic calculus on historical trees is used in order to show that there is a unique solution of the nonlinear martingale problems under Lipschitz conditions on the coefficients.

1. Introduction. Nonlinear diffusions, also called McKean–Vlasov processes, are diffusion processes which are associated with nonlinear second order partial differential equations. Many authors have studied $\mathbb{R}^d$-valued McKean–Vlasov diffusions in detail, for example, Funaki (1984), Oelschläger (1984) and Sznitman (1984, 1991). The main issues are approximation by a sequence of weakly interacting diffusions, associated large deviations and fluctuations and finally uniqueness and existence of the nonlinear martingale problem associated with a McKean–Vlasov process.

In this paper we focus on the latter question in the setup of branching measure-valued diffusions processes, also called superprocesses. For an excellent introduction to the theory of superprocesses, we refer to Dawson (1993). In order to formulate the basic definition we need to introduce some notation. The space of finite (resp., probability) measures over a Polish space E is denoted by $M(E)$ [resp., $M_1(E)$] and is equipped with the weak topology. We fix a time $T > 0$. The space of continuous (resp., cadlag) paths from $[0, T]$ to E is denoted by C_E (resp., D_E) and $C_b(E)$ is the set of bounded continuous functions on E. The expression $\mu(f)$ with $\mu \in M_1(E)$ means $\int f\, d\mu$.

DEFINITION 1.1. (i) Let $\mathscr{L} = (L(m), \mathscr{D})_{m \in M_1(M(E))}$ be a family of linear operators with common domain $\mathscr{D} \subset C_b(E)$ where b, c are measurable functions on $M_1(M(E)) \times E$ with $c \geq 0$.

(ii) Fix $\nu \in M(E)$. A measure P on $(C_{M(E)}, \mathscr{F}, \mathscr{F}_t)$ with canonical filtration $\mathscr{F}_t$ and σ-algebra $\mathscr{F}$ generated by the coordinate process X is called a nonlinear superprocess with parameter $(\mathscr{L}, b, c)$ started from ν, if for each $f \in \mathscr{D}$, the process $M(f)$ defined by

$$\begin{aligned} M_t(f) &:= X_t(f) - \nu(f) \\ &\quad - \int_0^t X_s\big(L\big(P \circ X_s^{-1}\big) f + b\big(P \circ X_s^{-1}\big) f\big)\, ds \end{aligned} \tag{1.1}$$

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is a local martingale with increasing process

$$\int_0^t \int_E f^2(x) c\big(P \circ X_s^{-1}, x\big) X_s(dx)\, ds, \tag{1.2}$$

where $P \circ X_s^{-1} \in M_1(M(E))$ denotes the distribution of X_s under P.

In the superprocess terminology $\mathscr{L}$ describes the one-particle motion and the functions b and c determine the branching behavior. In particular, b measures the noncriticality of the branching and is sometimes viewed as an immigration function.

In terms of partial differential equations, the flow of the one-dimensional marginals $u_s := P \circ X_s^{-1}$ of a solution of the nonlinear martingale problem (1.1), (1.2) solves the (weak) nonlinear equation

$$\dot{u}_s = \mathscr{A}^*(u_s) u_s, \tag{1.3}$$

where for nice functions F on $M(E)$,

$$\mathscr{A}(m)F(\mu) = \mu\big(L(m)\nabla_{\cdot} F(\mu) + b(m)\nabla_{\cdot} F(\mu) + c(m)\nabla_{\cdot}^{(2)} F(\mu)\big) \tag{1.4}$$

with

$$\nabla_x F(\mu) := \lim_{\varepsilon \downarrow 0} \frac{F(\mu + \varepsilon\delta_z) - F(\mu)}{\varepsilon},$$

where δ_x denotes Dirac measure on $x \in E$. This is one motivation for the study of nonlinear superprocesses from the point of view of partial differential equations. Another motivation is a kind of law of large numbers for weakly interacting N-type superprocesses, which also provides a proof of the existence of a nonlinear superprocess. A weakly interacting N-type superprocesses $\mathbf{X}^N = (X^1, \dots, X^N) \in C_{M(E)^N}$ is characterized by the martingale property of the processes

$$\begin{aligned}\Bigg(& e_{\mathbf{f}}\big(\mathbf{X}_t^N\big) - e_{\mathbf{f}}\big(\mathbf{X}_0^N\big) \\ & + \int_0^t e_{\mathbf{f}}\big(\mathbf{X}_s^N\big) \sum_{j=1}^N X_s^j \left(L\left(\frac{1}{N} \sum_{i=1}^N \delta_{X_s^{i,N}} \right) f_j + b\left(\frac{1}{N} \sum_{i=1}^N \delta_{X_s^{i,N}} \right) f_j \right. \\ & \qquad \left. - c\left(\frac{1}{N} \sum_{i=1}^N \delta_{X_s^{i,N}} \right) f_j^2 \right) ds \Bigg)_{t \ge 0},\end{aligned} \tag{1.5}$$

where $e_{\mathbf{f}}(\boldsymbol{\mu}^N) := \exp(-\sum_{i=1}^N \mu_i(f_i))$ for $\boldsymbol{\mu}^N = (\mu_1, \dots, \mu_N) \in M(E)^N$ and $\mathbf{f} = (f_1, \dots, f_N) \in C_b(E)^N$. The actual proof of the approximation is based on the *propagation of chaos* technique [cf. Sznitman (1984, 1991)], which is carried out in Overbeck (1994b). The large deviations and the fluctuations associated with the approximation if the weakly interacting superprocesses are superprocesses with mean-field interaction are studied in Overbeck (1994a, b).

The main result of the present paper is the proof that there is a unique solution to (1.1), (1.2) under Lipschitz conditions on the parameter $(\mathscr{L}, b, c)$.

In order to state the result, we have to introduce for $p \geq 1$ the Wasserstein metric ρ_p on $M_1(M(E))$:

$$(1.6)\qquad \rho_p(m_1, m_2) := \left(\inf_Q \int_{M(E)\times M(E)} d^p_{(E,d)}(\mu, \nu) Q(d\mu, d\nu)\right)^{1/p},$$

where the infimum is taken over all $Q \in M_1(M(E) \times M(E))$ whose marginal distributions are m_1 and m_2 and where for a Polish space E with metric d, the metric $d_{(E,d)}$ on $M(E)$ is defined as

$$(1.7)\qquad d_{(E,d)}(\mu, \nu) := \sup\{|\mu(f) - \nu(f)|; \|f\|_{BL} \leq 1\},$$

where

$$\|f\|_{BL} = \|f\|_\infty \wedge \inf\{K; |f(x) - f(y)| \leq Kd(x, y)\ \forall x, y \in E\}.$$

Notice that if we replace $d_{(E,d)}$ by $d_{(E,d)} \wedge 2$ in the definition of ρ_1, then ρ_1 is smaller than the original ρ_1 and equivalent to the Prohorov metric and also to $d_{(M(E), d_{(E,d)})}$. Recall that by Hölder's inequality, $\rho_q \leq K_{p,q}\rho_p$ if $q \leq p$ with some constant $K_{p,q}$.

THEOREM 1. (a) *Let $L(m) = L$ be independent of m, $c = 1$ and b: $M_1(M(E)) \times E \to [-R, 0]$ for some $R > 0$. If*

$$(1.8)\qquad |b(m_1, x) + b(m_2, x)| \leq K_b \rho_2^2(m_1, m_2)$$

for some positive constant K_b, then there is a unique solution to the nonlinear martingale problem (1.1), (1.2).

(b) *Assume that $E = \mathbb{R}^q$, $q \in \mathbb{N}$, $b = 0$, c is constant and that for $f \in C_b^2(\mathbb{R}^q)$,*

$$(1.9)\quad L(m)f(x) = \nabla f(x) \cdot d(m, x) + \frac{1}{2}\sum_{i,j=1}^{q} a_{ij}(m, x)\frac{\partial^2 f}{\partial x_i\, \partial x_j}(x),$$

where $a = \sigma \cdot \sigma^*$, σ: $M_1(M(\mathbb{R}^q)) \times \mathbb{R}^q \to \mathbb{R}^q \times \mathbb{R}^q$ and d: $M_1(M(\mathbb{R}^q)) \times \mathbb{R}^q \to \mathbb{R}^q$ are bounded functions, which are Lipschitz-continuous with respect to the metric $\rho((m_1, x_1), (m_2, x_2)) = \rho_2(m_1, m_2) + |x_1 - x_2|$. Then there exists a unique solution of (1.1), (1.2).

The proof relies on the fact that for two superprocesses P^i, $i = 1, 2$, with different parameters, that is, with different one-particle motion and branching functions b and c, there exists a filtered probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}_t, \tilde{\mathscr{F}}, \mathbb{P})$ on which we can construct processes X^i with distribution P^i, $i = 1, 2$. This follows from the stochastic calculus along historical trees, recently developed by Perkins (1992, 1995) and by Evans and Perkins (1995). We specialize to the noninteractive case and, in fact, only use the existence, or more precisely the constructive part of their approach, and not the uniqueness result. Once this is established, the proof of existence and uniqueness is carried out by a Picard–Lindelöf approximation. We also emphasize that it is sufficient to prove that two solutions P^i, $i = 1, 2$, of (1.1), (1.2) have the same flow as their one-dimensional distributions; that is, $P^1 \circ X_s^{-1} = P^2 \circ X_s^{-1}$ for all $0 \leq s \leq T$.

In Theorem 1(a) the probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}_t, \tilde{\mathscr{F}}, \mathbb{P})$ is the canonical space of a marked historical process as in Evans and Perkins [(1995), Section 2]. In the proof of Theorem 1(b) we can take $(\tilde{\Omega}, \tilde{\mathscr{F}}_t, \tilde{\mathscr{F}}, \mathbb{P})$ to be the canonical space of the historical Brownian motion, as in Perkins (1992). We formulate the assertion in part (b) only for constant c and $b = 0$, but the proof in Section 3 will be carried out for nonconstant c. The more general uniqueness statement can be found in Proposition 3.3, and it requires strong conditions on c and some relations between the functions c, b and d. The existence result that covers Theorem 1(b) is proved in Corollary 3.5 under more restrictive, but also more explicit assumptions on c. These restrictions are required by the further development of the results in Perkins (1992) in his second paper [Perkins (1995)]. Despite the fact that the conditions of Perkins (1995) are simplified in our noninteractive case, they are still quite involved and give only complicated and somehow artificial examples (cf. end of Section 3).

In both cases it turns out that the historical process plays the same role for nonlinear superprocesses as the Brownian motion plays for nonlinear diffusions on $\mathbb{R}^q$, namely, as a *driving term for strong stochastic equations*. The fundamental role of the historical process also becomes apparent in several other papers, for example, in Perkins (1992, 1995) where interacting measure-valued processes are considered, in Evans and Perkins (1995), where a Clark-type formula for measure-valued processes is proved, in LeGall (1991), where the connections to Brownian excursions are investigated, and in Dynkin (1993), where the relations to quasilinear partial differential equation are explored.

In the following two sections we will prove parts (a) and (b) of Theorem 1 separately.

2. Nonlinearity in the immigration function.

In this section we consider the case in which $L(m) = L$ is a generator of a time-homogeneous Hunt process independent of m and $c = 1$. Hence the nonlinearity appears only in the immigration function b. Because we need the historical process from now on I will briefly describe it.

2.1. *Historical process*. The historical process over a one-particle motion ξ, for example, over a Hunt process with state space E, can be seen as the superprocess constructed over the *path process* of the one-particle motion. A path process is a path-valued process and evolves from a path of length s to a path of length $t > s$ by pasting on the given path ξ^s a new path of length $t - s$, which is distributed as the underlying one-particle motion started from $\xi^s(s)$. By construction this is a time-inhomogeneous Markov process with state space D_E and it has a generator $(L^h, D(L^h))$ in the sense of martingale problems [cf., e.g., Perkins (1992, 1995)]. If we superpose a critical branching mechanism onto this path process and take the usual "superprocess limit," we arrive at the historical process, which can then be viewed as the solution of the martingale problem described in (1.1), (1.2) with $c(m) = 1$, $b(m) = 0$ and $L(m) = L^h$. It is called historical because every particle carries all the information about the places it and its ancestor visited. Additionally one can reconstruct from this information the genealogy of a present particle by investigation of the overlap of the paths of two different particles. Because we only use the historical process as a tool and we will not prove theorems about it, we will omit an exact definition and refer to Dawson [(1993), Section 12] or Dawson and Perkins (1991), Perkins (1992, 1995) and Dynkin (1991).

2.2. *Superprocesses with emigration as functionals of the marked historical process.* Let $X = (Y, N) \in D(D(E \times [0,1]))$ denote the path process of the Hunt process ξ generated by L and an independent Poisson process with uniform jumps on $[0,1]$ (i.e., N is the path process of a Poisson point measure on $[0,\infty) \times [0,1]$ with intensity $ds \times dx$). Denote by $\mathbb{P}$ the distribution of the superprocess G over the one-particle motion (Y, N) starting from G_0, that is, the historical process over the Hunt process ξ and an independent Poisson process. Then $\mathbb{P}$ is a measure on $\Omega := C([0,\infty), M(D_{E\times[0,1]}))$ equipped with the canonical filtration $\mathscr{F}_t$ and canonical σ-algebra. The process G is now the canonical process on Ω. Let us denote by $x = (y, n)$ a generic element in $D_{E\times[0,1]}$. Let n also denote the point measure $\sum_{s\le t,\, n_s \ne n_{s-}} \delta_{s,\, n_s - n_{s-}}$ on $[0,\infty) \times [0,1]$.

Let b be a measurable function from $[0,T] \times E$ to $[-1,0]$, the candidate for the immigration term. (Because b is negative, we may view b now as an emigration rather than an immigration function.)

In order to meet the formulation of Evans and Perkins (1995), we define the $[0,1]$-valued function β on $[0,T] \times D_E \times \Omega$ by

$$\beta(s, y, \omega) = -b(s, y(s)). \tag{2.1}$$

Further we define the functions

$$A(t, x, \omega) = n(\{(s, z) \in]0, t[\times [0,1] \mid \beta(s, y, \omega) > z\}), \tag{2.2}$$

$$B(t, x, \omega) = \mathbf{1}_{\{A=0\}}(t, x, \omega). \tag{2.3}$$

Let K be the martingale measure of the historical process G [for the definition of martingale measures for measure-valued processes, cf. Dawson (1993), Section 7, and for historical processes a definition can be found in Perkins (1992)]. Then we can define a new measure $\mathbb{P}^\beta$ on Ω by

$$\left.\frac{d\mathbb{P}^\beta}{d\mathbb{P}}\right|_{\mathscr{F}_t} := R_t^\beta := \exp\left\{\int_0^t \int_{D(E\times[0,1])} \beta(s, y) K(ds, dx) - \frac{1}{2}\int_0^t \int_{D(E\times[0,1])} \beta^2(s, y) G_s(dx)\, ds\right\} \tag{2.4}$$

[cf. Dawson (1993), Section 7; Evans and Perkins (1995)]. Finally let us define the measure-valued processes

$$H_t^\beta(\Xi) = \int_{D(E\times[0,1])} \mathbf{1}_\Xi(y) B(t, x) G_t(dx), \tag{2.5}$$

$$H_t(\Xi) = \int_{D(E\times[0,1])} \mathbf{1}_\Xi(y) G_t(dx). \tag{2.6}$$

From the definition of $\mathbb{P}$ it is obvious that the distribution of H under $\mathbb{P}$ is the historical process over ξ. The following proposition is basic for us.

PROPOSITION 2.1 [Evans and Perkins (1995), Theorem 5.1]. *Under $\mathbb{P}^{\beta}$ the process H^{β} is the historical process over ξ.*

We need a slightly different version of this result which will be obtained by a Girsanov argument. Let $(L^h, D(L^h))$ be the martingale operator of the path process of ξ.

PROPOSITION 2.2. *For every $\phi \in D(L^h)$ the process $H_t^{\beta}(\phi)$ is under $\mathbb{P}$ a semimartingale with increasing process $V(\phi) - \int_0 H_s^{\beta}(\beta(s)\phi)\,ds$, where $V(\phi) = \int_0^{\bullet} H_s^{\beta}(L^h\phi)\,ds$ is the increasing process of $H^{\beta}(\phi)$ under $\mathbb{P}^{\beta}$. The quadratic variation of the martingale part equals $\int_0^{\bullet} H_s^{\beta}(\phi^2)\,ds$. [Hence under $\mathbb{P}$ the process H^{β} is a historical process with (negative) immigration $-\beta$, or in other words, with an emigration function β.]*

PROOF. Applying the Girsanov transformation for martingales we can calculate the semimartingale decomposition of H^{β} under $\mathbb{P}^{\beta}$ from the semimartingale decomposition of H^{β} under $\mathbb{P}$. In order to do this, we have to consider the martingale Z of the densities $Z_t = (d\mathbb{P}/d\mathbb{P}^{\beta})|_{\mathscr{F}_t}$. Let M denote the martingale measure associated with the historical Brownian motion H under $\mathbb{P}$. Then we have

$$
\begin{aligned}
Z_t &= \exp\biggl\{-\int_0^t\int_{D(E)}\beta(s,y)M(ds,dy) \\
&\qquad\qquad + \tfrac{1}{2}\int_0^t\int_{D(E)}\beta^2(s,y)H_s(dy)\,ds\biggr\} \\
&= \exp\biggl\{-\biggl[\int_0^t\int_{D(E)}\beta(s,y)M(ds,dy) \\
&\qquad\qquad\qquad - \int_0^t\int_{D(E)}\beta^2(s,y)H_s(dy)\,ds\biggr] \\
&\qquad\qquad - \tfrac{1}{2}\int_0^t\int_{D(E)}\beta^2(s,y)H_s(dy)\,ds\biggr\} \\
&= \exp\biggl\{-\int_0^t\int_{D(E)}\beta(s,y)N^{\beta}(ds,dy) \\
&\qquad\qquad - \tfrac{1}{2}\int_0^t\int_{D(E)}\beta^2(s,y)H_s(dy)\,ds\biggr\},
\end{aligned}
\tag{2.7}
$$

where N^{β} is the martingale measure associated with H under $\mathbb{P}^{\beta}$. That is, $(\beta\cdot N^{\beta})_t := \int_0^t\int_{D(E)}\beta(s,y)N^{\beta}(ds,dy)$ is the martingale in the semimartingale decomposition of $H_t(\beta(t))$ under $\mathbb{P}^{\beta}$. This yields, in particular, that under $\mathbb{P}^{\beta}$ Z solves the equation

$$Z_t = 1 - \int_0^t Z_s\, d(\beta \cdot N^{\beta})_s. \tag{2.8}$$

According to Proposition 2.1 and Girsanov's theorem [e.g., Revuz and Yor (1991), page 303],

$$\begin{aligned} &H^{\beta}(\phi) - V(\phi) - \int_0^{\cdot} \frac{1}{Z_s} \langle Z_s, H^{\beta}(\phi) - V(\phi) \rangle_s \\ &\quad = H^{\beta}(\phi) - V(\phi) + \langle \beta \cdot N^{\beta}, H^{\beta}(\phi) - V(\phi) \rangle \end{aligned} \tag{2.9}$$

is a martingale under $\mathbb{P}$. The bracket in the last line equals, again according to Revuz and Yor [(1991), page 303],

$$\begin{aligned} &\langle \text{martingale in the decomposition of } H(\beta) \text{ under } \mathbb{P}^{\beta}, \\ &\qquad \text{martingale in the decomposition of } H^{\beta}(\phi) \text{ under } \mathbb{P}^{\beta} \rangle \\ &\quad = \langle \text{martingale in the decomposition of } H(\beta) \text{ under } \mathbb{P}, \\ &\qquad\quad \text{martingale in the decompostion of } H^{\beta}(\phi) \text{ under } \mathbb{P} \rangle \\ &\quad = \langle \beta \cdot K, B \cdot \phi \cdot K \rangle \\ &\quad = \int_0^{\cdot} G_s(B\beta\phi)\, ds \\ &\quad = \int_0^{\cdot} H_s^{\beta}(\beta\phi)\, ds. \end{aligned}$$

Because the quadratic variation of this martingale remains unchanged under a change of measure, the proposition is proved. □

2.3. *Comparison of two historical processes with different noninteractive emigrations*. We consider two functions b^i, $i = 1, 2$, from $[0, T] \times E$ to $[0, 1]$ and define β^i, A^i and H^{β^i} in terms of b^i as in (2.1), (2.2) and (2.5). Then β^i, $i = 1, 2$, do not depend on ω. For a measure H on $\{x^t | x \in D\}$ [where $x^t(s) := x(s)$, $s < t$, $x^t(s) := x(t)$, $s \geq t$] and a function $f \in C_b(E)$, we define $H(f) := \int_D f(x_t^t) H(dx)$.

LEMMA 2.3. *For every $T > 0$ there exists a constant $C_T < \infty$ such that*

$$\mathbb{E}\left[\left(\sup_{\|f\|_{BL} \leq 1} \left| H_t^{\beta^1}(f) - H_t^{\beta^2}(f) \right| \right)^2\right] \leq C_T \int_0^t E\left[\left| b^1(s, \xi_s) - b^2(s, \xi_s) \right|\right] ds$$

for all $t \leq T$, where ξ is the Hunt process generated by L.

PROOF. Let us write $n = \sum_{i=1}^{N} \delta_{t_i, Z_i}$,

$$\mathbb{E}\left[\left(\sup_{\|f\|_{BL} \le 1} \left| H_t^{\beta^1}(f) - H_t^{\beta^2}(f) \right|\right)^2\right]$$

$$= \mathbb{E}\left[\left(\sup_{\|f\|_{BL} \le 1} \int f(x)\left(\mathbf{1}_{A^1(t,x)=0} - \mathbf{1}_{A^2(t,x)=0}\right) G_t(dx)\right)^2\right]$$

$$\le \mathbb{E}\left[\left(\int |\mathbf{1}_{A^1(t,x)=0} - \mathbf{1}_{A^2(t,x)=0}| G_t(dx)\right)\right] (\mathbb{E}[G_0(1)] + t)$$

$$\le \mathbb{E}\left[\int \mathbf{1}_{\cup_{j=1}^{N}\{b^1(t_i, y(t_i)) \le Z_i, i=1,\dots,N, b^2(t_j, y(t_j)) > Z_j\}} G_t(dx)\right] (\mathbb{E}[G_0(1)] + t)$$

$$+ \mathbb{E}\left[\int \mathbf{1}_{\cup_{j=1}^{N}\{b^1(t_i, y(t_i)) \le Z_i, i=1,\dots,N, b^2(t_j, y(t_j)) > Z_j\}} G_t(dx)\right] (\mathbb{E}[G_0(1)] + t).$$

The term $\mathbb{E}[\int \mathbf{1}_{\cup_{j=1}^{N}\{b^1(t_i, y(t_i)) \le Z_i, i=1,\dots,N, b^2(t_j, y(t_j)) > Z_j\}} G_t(dx)]$ equals

$$P\left[\bigcup_{j=1}^{N} \left\{ b^1(t_i, \xi_{t_i}) \le Z_i,\ i = 1, \dots, N,\ b^2(t_j, \xi_{t_j}) > Z_j \right\}\right], \tag{2.10}$$

where Z_j, t_j are uniformly distributed on $[0,1] \times [0,t]$, N has a Poisson distribution and all random variables are mutually independent. Because

$$p\left[b^1(t_j, \xi_{t_j}) \le Z_j < b^2(t_j, \xi_{t_j}) \middle| (t_j, \xi_{t_j}) \right]$$
$$= \left(b^2(t_j, \xi_{t_j}) - b^1(t_j, \xi_{t_j}) \right) \mathbf{1}_{b^2(t_j, \xi_{t_j}) \ge b^1(t_j, \xi_{t_j})},$$

we obtain by conditioning that the probability (2.10) is bounded by

$$E[N] \cdot \int_0^t E\left[\left(b^2(t_j, \xi_{t_j}) - b^1(t_j, \xi_{t_j}) \right) \mathbf{1}_{b^2(t_j, \xi_{t_j}) \ge b^1(t_j, \xi_{t_j})}\right] ds/t.$$

By the same argument for $P[b^2(t_j, \xi_{t_j}) \le Z_j < b^1(t_j, \xi_{t_j}) | (t_j, \xi_{t_j})]$ we can finally prove the assertion. □

2.4. *Proof of Theorem* 1(a). Let us define the map α on $C([0,T], M_1(M(E)))$ by

$$\alpha(u) := \left(P^u \circ X_s^{-1}\right)_{0 \le s \le T}, \tag{2.11}$$

where P^u is the superprocess with immigration function $b^u(s, x) := b(u_s, x)$ on the canonical space $C([0,T], M(E))$ with coordinate process X. Define now for $u^i \in C([0,T], M_1(M(E)))$, $i = 1, 2$, the process H^{β^i} and H as in Proposition 2.2 and Lemma 2.3 with $\beta^i(t, y, \omega) = (1/R) b(u^i_{t/R}, y(t/R))$ and over the one-particle motion generated by L/R. By an obvious scaling property, the superprocesses projected down from the processes $(H^{\beta^i}_{tR})_{t \le 0}$ have distributions P^{u^i}, $i = 1, 2$. Because H^{β^i}, $i = 1, 2$, satisfy the assumptions of Proposition 2.2 and Lemma 2.3, we can conclude that

$$\begin{aligned}\rho_2^2\big(\alpha(u^1)_t, \alpha(u^2)_t\big) &\le E\left[\left(\sup_{\|f\|_{BL}\le 1}\left|H_{tR}^{\beta_1}(f) - H_{tR}^{\beta_2}(f)\right|\right)^2\right]\\ &\le K_{t,b}\int_0^{tR}\rho_2^2\big(u^1_{s/R}, u^2_{s/R}\big)\,ds \qquad (2.12)\\ &\le K'_{T,b}\int_0^t\rho_2^2\big(u^1_s, u^2_s\big)\,ds.\end{aligned}$$

Hence

$$\sup_{r\le t}\rho_2\big(\alpha(u^1)_r, \alpha(u^2)_r\big) \le K''\int_0^t \sup_{r\le s}\rho_2\big(u^1_r, u^2_r\big)\,ds. \qquad (2.13)$$

A Picard–Lindelöf approximation yields that there is a solution u^F of the equation

$$\alpha(u) = u. \qquad (2.14)$$

The approximation starts with $u^1 := (P^0 \circ X_s^{-1})_{0\le s\le T}$, where P^0 is the superprocess with $b^0(s, x) = b(m_0, x)$ with some $m_0 \in M_1(M(E))$, and for $n \in \mathbb{N}$ we define $u^{n+1} = \alpha(u^n)$. Applying successively the inequality (2.13) with u^{n+1} and u^n we obtain that there exists $u^F := \lim_{n\to\infty} u^{n+1}$, which solves (2.14). By the property (2.14) the superprocess P^{u^F} is a solution of the nonlinear martingale problem (1.1), (1.2). The measure P^{u^F} is the unique solution because if we denote by u^i, $i = 1, 2$, the flow $(P^i \circ X_s^{-1})_{0\le s\le T}$ of two solutions P^i of the martingale problem (1.1), (1.2), then both u^1 and u^2 are solutions of the equation (2.14). The properties (2.14) and (2.13) imply by Gronwall's inequality that $u^1 = u^2$ and therefore $P^i = P^{u^F}$, $i = 1, 2$. □

3. Nonlinear one-particle motion. Now we consider the nonlinear martingale problem (1.1), (1.2), where $E = \mathbb{R}^q$ and $L(m)$ equals a linear partial differential operator. We want to couple two different solutions of the nonlinear martingale problem in order to prove uniqueness. We will use the stochastic calculus "along historical trees" developed by Perkins (1992, 1995). In order to describe interacting superprocesses he constructs a unique solution of a strong integral equation, in which the stochastic integral is an "H-historical integral."

3.1. *Stochastic calculus along historical trees*. Let me recall some of the results in Perkins (1992, 1995) specialized to the case of noninteractive parameters. Let $C = C([0, T], \mathbb{R}^q)$, let $(\mathscr{C}_t)$ be the canonical filtration on C, $\Omega = C([0, T], M(C))$, $\hat{\Omega} = \Omega \times C$ with product σ-algebra and let the Campbell-type measure $\hat{\mathbb{P}}$ be defined by $\hat{\mathbb{P}}[A \times B] := \mathbb{P}[\mathbf{1}_A H_T(B)]\mathbb{P}[H_T(1)]^{-1}$, where the coordinate process H on the filtered probability space $(\Omega, \mathscr{H}, \mathscr{H}_t, \mathbb{P})$ is the historical Brownian motion with branching rate 1 and with starting point H_0. For the definition of H we refer again to Perkins [(1995), page 3].

Let $\hat{\mathscr{F}}_t := \mathscr{H}_t \times \mathscr{C}_t$. Let the functions $\sigma: [0,T] \times \mathbb{R}^q \to \mathbb{R}^{q\times q}$, $d^0: [0,T] \times \mathbb{R}^q \to \mathbb{R}^q$ and $c: [0,T] \times \mathbb{R}^q \to (0,\infty)$ be bounded and Lipschitz continuous in $x \in \mathbb{R}^q$. We assume that $(\partial c/\partial s)(s,\cdot)$ and $(\partial^2 c/\partial x_i\, \partial x_j)(s,\cdot)$ exist and are Lipschitz continuous in x with a Lipschitz constant uniform in s. Define the functions $a := \sigma\sigma^*$, $h(s,x) := (\nabla_x c)(s,x)$ and $g(s,x) := (\partial c/\partial s)(s,x) + \frac{1}{2}\sum_{i,j}(\partial^2 c/\partial x_i\, \partial x_j)(s,x)a_{ij}(s,x)$,

$$d := d^0 + ah^* c^{-1} \quad \text{and} \quad b := (g + hd^0)c^{-1}. \tag{3.1}$$

PROPOSITION 3.1 [Perkins (1995), Theorems 4.10 and 5.1 and Example 4.4]. (a) *Let* $K_0(\cdot) := \int \mathbf{1}_{\{Y_0(y)\in\cdot\}} c(0, Y_0(y))H_0(dy)$, *where* Y_0: $\hat{\Omega} \to \mathbb{R}^q$ *is* $\hat{\mathscr{F}}_0$-*measurable. Then there is an* $\hat{\mathscr{F}}_t$-*predictable* $\mathbb{R}^q$-*valued continuous process* Y *and an* $\mathscr{F}_t$-*predictable* $M(C)$-*valued process* K *such that*

$$Y_t(y) = Y_0(y) + \int_0^t \sigma(s, Y_s(y))\, dy(s) + \int_0^t d^0(s, Y_s(y))\, ds, \tag{3.2}$$

$$K_t(\omega)(\phi) = \int \phi\big(Y(\omega,y)^t\big) c(t, Y_t(\omega,y)) H_t(\omega)(dy), \tag{3.3}$$

where the first equation holds a.s. with respect to the first component of $\hat{\mathbb{P}}$, *that is, w.r.t. Wiener measure with initial distribution* $\mathbb{P}[H_0(\cdot)]$. *The second equality holds for all* $\phi \in C_b(C)$ *and* $0 \le t \le T$, $\mathbb{P}$-*a.s.*

(b) *We define the* $M(\mathbb{R}^q)$-*valued projection* $\Pi(K)$ *of the* $M(C)$-*valued process* K *by*

$$\Pi_s(K)(f) := \int_C f(y(s)) K_s(dy), \qquad f \in C_b(\mathbb{R}^q). \tag{3.4}$$

Under $\mathbb{P}$ *we have that for every* $f \in C_b^2(\mathbb{R}^q)$ *the process*

$$M_t(f) := \Pi_t(K)(f) - \Pi_0(K)(f) - \int_0^t \Pi_s(K)(A(s)f)\, ds \tag{3.5}$$

is a martingale with quadratic variation

$$\int_0^t \int_{\mathbb{R}^q} c(s,x) f^2(x) \Pi_s(K)(dx)\, ds, \tag{3.6}$$

where

$$A(s)f(x) = f(x)b(s,x) + \nabla f(x)\cdot d(s,x) + \frac{1}{2}\sum_{i,j=1}^q a_{ij}(s,x)\frac{\partial^2 f}{\partial x_i\, \partial x_j}(x)$$

for $f \in C_b^2(\mathbb{R}^q)$.

PROOF. The proposition is a special case of Perkins [(1995), Theorems 4.10 and 5.1]. In Example 4.4 in Perkins (1995) the case of noninteractive c is considered. The fact that the expression "H-a.s." used in Theorem 4.10 in Perkins (1995) is equivalent to "a.s. with respect to Wiener measure" if all coefficients of the stochastic equation (3.2), (3.3) do not depend on the process K follows by Remarks 3.3(a) and 3.13(d) in Perkins (1992). □

Since linear martingale problems for noninteractive superprocesses have unique solutions [cf. Dawson (1993)], we obtain the following corollary.

COROLLARY 3.2. *The distribution Q of the process $\Pi(K)$ with $\Pi_0(K) = \nu$, $\nu \in M(\mathbb{R}^q)$ is the unique measure in $M_1(C([0,T], M(\mathbb{R}^q)))$ such that*

$$M_t(f) := X_t(f) - \nu(f) - \int_0^t X_s(A(s)f)\,ds \tag{3.7}$$

is a local martingale with increasing process

$$\int_0^t \int_E f^2(x) c(s,x) X_s(dx)\,ds \tag{3.8}$$

for all $f \in C_b^2(\mathbb{R}^q)$, where A is defined in Proposition 3.1 *and where X is the coordinate process on $C([0,T], M(\mathbb{R}^q))$. Of course, each solution to* (3.7), (3.8) *can be obtained as a solution of a strong equation* (3.2), (3.3) *as in Proposition* 3.1.

3.2. *Nonlinear martingale problem.* In the case of a nonlinear martingale problem we want to consider functions a, b, c, d in (3.5) and (3.6) which depend on the external force caused by the distribution of $\Pi_s(K)$ at time s. Hence we consider bounded functions σ, d^0, c on $M_1(M(\mathbb{R}^q)) \times \mathbb{R}^q$ instead of functions on $[0,T] \times \mathbb{R}^q$. We assume that the functions are Lipschitz continuous with respect to both variables, where in the first variable we use the Wasserstein metric ρ_2 on $M_1(M(\mathbb{R}^q))$ [cf. (1.6)]. That is, we assume

$$|r(m_1, x_1) - r(m_2, x_2)| \le K_r(\rho_2(m_1, m_2) + |x_1 - x_2|) \tag{3.9}$$

with some constant K_r for $r = d^0, \sigma, c$. Note that (3.9) is a stronger condition than the condition (1.8) on b; however, we do not assume (3.9) for b because b will be defined by (3.1). Hence the Lipschitz condition is fulfilled for b if it is fulfilled for the other functions. Because of the differentiability assumption for the function c, there is an additional condition on c which will be formulated in the next proposition.

PROPOSITION 3.3. *Let us suppose that there exist bounded and Lipschitz continuous functions $\tilde{c}_0, \tilde{c}_i, \tilde{c}_{jk}$, $1 \le i, j, k \le q$, on $M_1(M(\mathbb{R}^q)) \times \mathbb{R}^q$ such that for the flow $u_s := P \circ X_s^{-1}$ of any solution P of* (1.1), (1.2) *we have*

$$\begin{aligned} \frac{\partial}{\partial s} c(u_s, x) &= \tilde{c}_0(u_s, x), \\ \frac{\partial}{\partial x_i} c(u_s, x) &= \tilde{c}_i(u_s, x), \\ \frac{\partial^2}{\partial x_j\, \partial x_k} c(u_s, x) &= \tilde{c}_{jk}(u_s, x) \end{aligned} \tag{3.10}$$

for $1 \le i, j, k \le q$. *Assume* (3.9). *Then there exists at most one probability measure* P *on* $C([0,T], M(\mathbb{R}^q))$ *which solves* (1.1), (1.2) *with* $d = d^0 + ah^* c^{-1}$, $b = (g + hd^0)c^{-1}$, *where*

$$h(m, x) = \big(\tilde{c}_1(m, x), \dots, \tilde{c}_q(m, x)\big), \tag{3.11}$$

$$g(m, x) = \tilde{c}_0(m, x) + \tfrac{1}{2} \sum_{j,k=1}^{q} \tilde{c}_{jk}(m, x) a_{jk}(m, x) \tag{3.12}$$

and $a = \sigma\sigma^*$.

Proof. First, we need the following lemma, which is implied by the definition of a solution of a nonlinear martingale problem.

Lemma 3.4. *If* $Q \in M_1(C([0,T], M(\mathbb{R}^q)))$ *is a solution of* (1.1), (1.2), *then it is a* (*linear noninteractive*) *superprocess. That is*, Q *is the only measure on* $C([0,T], M(\mathbb{R}^q))$ *such that* $X_0 = \nu$ *and such that*

$$M_t(f) := X_t(f) - \nu(f) - \int_0^t X_s\big(L_Q(s)f + b_Q(s)f\big)\, ds \tag{3.13}$$

is a local martingale with increasing process

$$\int_0^t \int_E f^2(x) c_Q(s, x) X_s(dx)\, ds, \tag{3.14}$$

where

$$L_Q(s)f(x) = \nabla f(x) \cdot d\big(u_s^Q, x\big) + \frac{1}{2} \sum_{i,j=1}^{q} a_{ij}\big(u_s^Q, x\big) \frac{\partial^2 f}{\partial x_i\, \partial x_j}(x)$$

for $f \in C_b^2(\mathbb{R}^q)$ *with* $u_s^Q := Q \circ X_s^{-1}$, *the distribution of the coordinate process* X *at time* s *under the measure* Q.

Let P^i, $i = 1, 2$, be two solutions of (1.1), (1.2). Define the corresponding flows by $u_s^i = P^i \circ X_s^{-1}$. We want to apply Proposition 3.1 and Corollary 3.2 with the functions $\sigma^i(s, x) := \sigma_{P^i}(s, x) := \sigma(u_s^i, x)$, $d^{0,i}(s, x) := d^0_{P^i}(s, x) := d^0(u_s^i, x)$ and $c^i(s, x) := c_{P^i}(s, x) := c(u_s^i, x)$. We define a^i, h^i, g^i and b^i as in (3.1) with the functions σ^i, d^0, and c^i instead of σ, d^0 and c.

By assumption (3.10) it is sufficient to show that for $i = 1, 2$ the mappings $(s, x) \to \tilde{c}_l(u_s^i, x)$, $l = 1, \dots, q$ and $(s, x) \to \tilde{c}_{jk}(u_s^i, x)$, $j, k = 1, \dots, q$, are Lipschitz continuous with a Lipschitz constant independent of $s \in [0, T]$. This follows from the Lipschitz continuity of $\tilde{c}_{jk}$ and $\tilde{c}_l$, $1 \le l, j, k \le q$. For example,

$$\big|\tilde{c}_l\big(u_s^i, x\big) - \tilde{c}_l\big(u_s^i, y\big)\big| \le K_{\tilde{c}_l}\big(\rho_2\big(u_s^i, u_s^i\big) + |x - y|\big) = K_{\tilde{c}_l}|x - y|.$$

Let (Y^1, K^1) and (Y^2, K^2) now be the solutions of (3.2), (3.3) driven by the same historical Brownian motion H but with different functions σ, d^0 and c, namely, with the functions σ^1, $d^{0,1}$ and c^1 for (Y^1, K^1) and with the functions σ^2, $d^{0,2}$ and c^2 for (Y^2, K^2). By Proposition 3.1(b) and properties (3.5) and (3.6) for each $i = 1, 2$, the distribution of the process $\Pi(K^i)$ as defined in (3.4) solves a linear martingale problem for a noninteractive superprocess with one-particle motion given by the diffusion matrix σ^i and drift vector d^i and with branching parameters c^i and b^i. By Corollary 3.2 and by Lemma 3.4 the distribution of the process $\Pi(K^i)$ coincides with P^i for each $i = 1, 2$. Hence we have constructed one strong coupling of the two different solutions; that is, we have constructed them on the same probability space $\hat{\Omega}$ equipped with one single historical Brownian motion H. By the definition of the Wasserstein metric and by Proposition 3.1(b) we have that

$$\begin{aligned}\rho_2^2(u_t^1, u_t^2) &\le \mathbb{P}\left[\left(\sup_{\|f\|_{BL}\le 1} \left|\Pi_t(K^1)(f) - \Pi_t(K^2)(f)\right|\right)^2\right]\\ &\le \mathbb{P}\left[\left(\sup_{\|f\|_{BL}\le 1} \int \left| f(Y^1(t,y))c(u_t^1, Y^1(t,y))\right.\right.\right.\\ &\qquad\qquad \left.\left.\left. - f(Y^2(t,y))c(u_t^2, Y^2(t,y))\right| H_t(dy)\right)^2\right].\end{aligned}$$

The last expression can be bounded by

$$\begin{aligned}&\mathbb{P}\left[\left(\int \left\{\|c\|_\infty \left|Y^1(t,y) - Y^2(t,y)\right|\right.\right.\right.\\ &\qquad \left.\left.\left. + \left|c(u_t^1, Y^1(t,y)) - c(u_t^2, Y^2(t,y))\right|\right\} H_t(dy)\right)^2\right].\end{aligned} \tag{3.15}$$

Because $Y_j^i(t) - Y_j^i(0) - \int_0^t d^0(u_s^i, Y^i(s,y))\,ds$ are continuous martingales for $1 \le j \le q$ with covariation $\sum_k a_{jk}(u_s^i, Y^i(s,y))\,ds$, $i = 1, 2$, and because $c^i \in C_b^{1,2}$, we have for $i = 1, 2$, by the Itô formula, that

$$\begin{aligned}c^i(t, Y^i(t,y)) = c^i(0, Y^i(0,y)) &+ \int_0^t h^i(s, Y^i(s,y))\, dY^i(s,y)\\ &+ \int_0^t g^i(s, Y^i(s,y))\, ds.\end{aligned}$$

Hence (3.15) equals

$$\begin{aligned}\mathbb{P}\Bigg[\Bigg(\int \Bigg\{ &\left|\int_0^t \left[\sigma(u_s^1, Y^1(s,y)) - \sigma(u_s^2, Y^2(s,y))\right] dy(s)\right.\\ &\quad \left. + \int_0^t \left[d^0(u_s^1, Y^1(s,y)) - d^0(u_s^2, Y^2(s,y))\right] ds \right| \|c\|_\infty\\ &+ \left|\int_0^t \left[h\sigma(u_s^1, Y^1(s,y)) - h\sigma(u_s^2, Y^2(s,y))\right] dy(s)\right.\\ &\quad + \int_0^t \left[(hd^0 + g)(u_s^1, Y^1(s,y))\right.\\ &\qquad\qquad \left.\left. - (hd^0 + g)(u_s^2, Y^2(s,y))\right] ds \right| \Bigg\} H_t(dy)\Bigg)^2\Bigg].\end{aligned}$$

By the Cauchy–Schwarz inequality and the formula for the second moment of a superprocess, this is bounded by

$$\begin{aligned}
&\mathbb{P}\Bigg[\int\Bigg\{\Bigg|\int_0^t\big[\sigma\big(u_s^1, Y^1(s,y)\big) - \sigma\big(u_s^2, Y^2(s,y)\big)\big]\,dy(s)\\
&\qquad + \int_0^t\big[d^0\big(u_s^1, Y^1(s,y)\big) - d^0\big(u_s^2, Y^2(s,y)\big)\big]\,ds\Bigg|\|c\|_\infty\\
&\qquad + \Bigg|\int_0^t\big[h\sigma\big(u_s^1, Y^1(s,y)\big) - h\sigma\big(u_s^2, Y^2(s,y)\big)\big]\,dy(s)\\
&\qquad\quad + \int_0^t\big[\big(hd^0 + g\big)\big(u_s^1, Y^1(s,y)\big)\\
&\qquad\qquad\qquad - \big(hd^0 + g\big)\big(u_s^2, Y^2(s,y)\big)\big]\,ds\Bigg|\Bigg\}^2 H_t(dy)\Bigg]\\
&\qquad \times\big(\mathbb{P}[H_0(1)] + t\big)\\
&\quad = E\Bigg[\int\Bigg\{\Bigg|\int_0^t\big[\sigma\big(u_s^1, Y^1(s,W)\big) - \sigma\big(u_s^2, Y^2(s,W)\big)\big]\,dW(s)\\
&\qquad\qquad + \int_0^t\big[d^0\big(u_s^1, Y^1(s,W)\big) - d^0\big(u_s^2, Y^2(s,W)\big)\big]\,ds\Bigg|\|c\|_\infty\\
&\qquad\qquad + \Bigg|\int_0^t\big[h\sigma\big(u_s^1, Y^1(s,W)\big) - h\sigma\big(u_s^2, Y^2(s,W)\big)\big]\,dW(s)\\
&\qquad\qquad\quad + \int_0^t\big[\big(hd^0 + g\big)\big(u_s^1, Y^1(s,W)\big)\\
&\qquad\qquad\qquad\qquad - \big(hd^0 + g\big)\big(g_s^2, Y^2(s,W)\big)\big]\,ds\Bigg|\Bigg\}^2\Bigg]\\
&\qquad \times\big(\mathbb{P}[H_0(1)] + t\big),
\end{aligned}$$

where W is a Brownian motion with initial distribution $\mathbb{P}[H_0(\cdot)]$ and $Y^i(s,W)$ is a solution of (3.2) with W instead of y and $\sigma^i(s,W_s) = \sigma(u_s^i, W_s)$. Because $hd^0 + g$ and $h\sigma$ also satisfy (3.9), we can bound the last expression by

$$\begin{aligned}
&4\big(\mathbb{P}[H_0(1)] + t\big)\max\big\{K_\sigma^2, tK_{d^0}^2, tK_{(hd^0+g)}^2, K_{\sigma h}^2\big\}\big(\|c\|_\infty^2 \vee 1\big)\\
&\qquad \times\int_0^t\Big(E\Big[\big|Y^1(s,W) - Y^2(s,W)\big|^2\Big]\,ds + \rho_2\big(u_s^1, u_s^2\big)\Big)\,ds\\
&\qquad \le K_T'\int_0^t\Big(E\Big[\big|Y^1(s,W) - Y^2(s,W)\big|^2\Big]\,ds + \rho_2\big(u_s^1, u_s^2\big)\Big)\,ds.
\end{aligned}$$

It remains to prove that

$$E\Big[\sup_{s\le t}\big|Y^1(s,W) - Y^2(s,W)\big|^2\Big] \le K_T\int_0^t \rho_2^2\big(u_s^1, u_s^2\big)\,ds \tag{3.16}$$

with a finite constant K_T. We define similarly as in Funaki (1984),

$$A_t := \int_0^t \left[d^0\left(u_s^1, Y^1(s,W)\right) - d^0\left(u_s^2, Y^2(s,W)\right)\right] ds,$$

$$M_t := \int_0^t \left[\sigma\left(u_s^1, Y^1(s,W)\right) - \sigma\left(u_s^2, Y^2(s,W)\right)\right] dW(s).$$

By the Burkholder–Davis–Gundy inequality we obtain

$$\begin{aligned}
&E\left[\sup_{s\le t}|M_s|^2\right] \\
&\quad\le K(2)\sum_{i=1}^{q} E\left[\int_0^t \sum_{j=1}^{q} \left|\sigma_{ij}\left(u_s^1, Y^1(s,W)\right) - \sigma_{ij}\left(u_s^2, Y^2(s,W)\right)\right|^2 ds\right] \\
&\quad\le K(2)K_\sigma^2 t E\left[\sup_{s\le t}\left|Y^1(s,W) - Y^2(s,W)\right|^2\right] \\
&\qquad + K(2)K_\sigma^2 \int_0^t \rho_2^2\left(u_s^1, u_s^2\right) ds
\end{aligned}$$

with some constant $K(2)$. For A we obtain

$$\begin{aligned}
E\left[\sup_{s\le t}|A_s|^2\right] &\le K_{d^0}^2 E\left[\left(\int_0^t \rho_2\left(u_s^1, u_s^2\right) ds + \int_0^t \left|Y^1(s,W) - Y^2(s,W)\right| ds\right)^2\right] \\
&\le 2K_{d^0}^2 t \int_0^t \rho_2^2\left(u_s^1, u_s^2\right) ds \\
&\quad + 2K_{d^0}^2 t^2 E\left[\sup_{s\le t}\left|Y^1(s,W) - Y^2(s,W)\right|^2\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&E\left[\sup_{s\le t}\left|Y^1(s,W) - Y^2(s,W)\right|^2\right] \\
&\quad\le 2E\left[\sup_{s\le t}|M_s|^2\right] + 2E\left[\sup_{s\le t}|A_s|^2\right] \\
&\quad\le \left(2K(2)K_\sigma^2 + 4K_{d^0}^2 t\right) t E\left[\sup_{s\le t}\left|Y^1(s,W) - Y^2(s,W)\right|^2\right] \\
&\quad\le \left(2K(2)K_\sigma^2 + 4K_{d^0}^2 t\right)\int_0^t \rho_2^2\left(u_s^1, u_s^2\right) ds.
\end{aligned}$$

Hence for $t < [1/(2K(2)K_\sigma^2 + 4K_{d^0}^2)] \wedge 1$ we have (3.16). This implies by the previous calculations that

$$\rho_2^2\left(u_t^1, u_t^2\right) \le K_T' \int_0^t \rho_2^2\left(u_s^1, u_s^2\right) ds$$

for small t. Gronwall's lemma yields that $u_t^1 = u_t^2$ for small t. Exploring now the Markov property of the two solutions, we obtain uniqueness for all $t \le T$ [cf. Funaki (1984)] and the assertion is proved. □

In order to prove an existence result we have to be more specific about the function c; for example, it suffices that c is a *finitely based* function with *finitely based base functions*.

COROLLARY 3.5. *Let* $c(m, x) = \Phi(m(F_1), \dots, m(F_k), x)$ *with* $\Phi \in C_b^3(\mathbb{R}^{k+q})$ *and* $F_i(\mu) = \phi_i(\mu(f_{1i}), \dots, \mu(f_{k_i i}))$ *such that* $f_{ji} \in C_b^2(\mathbb{R}^q)$, $j = 1, \dots, k_i$, $k_i \in \mathbb{N}$, $\phi_i \in C_b^2(\mathbb{R}^{k_i})$, $i = 1, \dots, k$. *We keep on assuming* (3.9) *and the boundedness of the functions* a_{ij}, d_l^0, $1 \le i, j, l \le q$. *Then there exists a unique solution of the nonlinear martingale problem* (1.1), (1.2).

PROOF. Uniqueness follows by Proposition 3.3, if we take for $\tilde{c}$ the appropriate derivatives of c. Let P^u denote the superprocess with parameters depending on the flow $u \in C([0, T], M_1(M(\mathbb{R}^q)))$; for example, $a_{ij}^u(s, x) = a_{ij}(u_s, x)$ and $c^u(s, x) = \Phi(u_s(F_1), \dots, u_s(F_k), x)$. The starting point of the Picard–Lindelöf approximation as in Section 2 is now u^0, the flow of the superprocess $p^{u^{m_0}}$ with parameter depending on some constant flow $u_s^{m_0} = m_0$ for all s. Define $u^{n+1} = \alpha(u^n) = (P^{u^n} \circ X_s^{-1})_{0 \le s \le T}$. We have by the boundedness assumptions that

$$\begin{aligned}\left| \frac{\partial}{\partial s} c^{u^n}(s, x) - \frac{\partial}{\partial s} c^{u^n}(s, y) \right| &\le \sup_{1 \le i \le k} \sup_{t \le T} E^{u^n}\big[\mathscr{A}\big(u_t^{n-1}\big) F_i(X_t)\big] K_\Phi |x - y| \\ &\le \sup_{t \le T} K \mathbb{P}\big[H_t(1)\big] |x - y|\end{aligned}$$

with a finite constant $K = K_{\Phi, \phi_i, f_i, a_{ij}, d_k, b, c}$ [where $\mathscr{A}(m)$ is defined in (1.3)]. Hence the Lipschitz condition for $\partial c^{u^n}/\partial s$ formulated at the beginning of Section 3.1 is satisfied. It is straightforward to see that the other conditions for Proposition 3.1 are all satisfied. Hence we can construct P^{u^n} and $P^{u^{n+1}}$ as strong solutions of stochastic equations driven by a historical process. Proceeding now as in the proof of Proposition 3.2 with $P^1 = P^{u^n}$ and $P^2 = P^{u^{n+1}}$ we are led to

$$\rho_2^2\big(u_t^{n+1}, u_t^n\big) \le K_T' \int_0^t \rho_2^2\big(u_s^n, u_s^{n-1}\big)\, ds,$$

which finally yields a solution u^F of $\alpha(u^F) = u^F$. The superprocess P^{u^F} solves (1.1), (1.2). □

Of course, all assumptions on c are satisfied for constant c and we can restate Theorem 1(b) as the following corollary.

COROLLARY 3.6. *Assume c is constant and σ and d are bounded and Lipschitz continuous in (m, x) with respect to ρ_2 in the first component. Then there exists a unique probability measure P on* $C([0, T], M(\mathbb{R}^q))$ *which solves the nonlinear martingale problem* (1.1), (1.2) *with* $b = 0$.

EXAMPLES. Assume that c satisfies the assumption of Proposition 3.3. Let us now give examples for which we can satisfy condition (3.1) in Proposition 3.1. First of all, a necessary condition is that $hd + g = hah^* c^{-1} + cb$ with h and g as in (3.12) and (3.11). That is,

$$\begin{aligned} c(m,x)\left(\tfrac{1}{2}\sum_{j,k=1}^{q}\tilde{c}_{jk}(m,x)a_{jk}(m,x) + \sum_{i=1}^{q}\tilde{c}_i(m,x)d_i(m,x)\right) \\ = c^2(m,x)b(m,x) + \sum_{j,k=1}^{q}\tilde{c}_j(m,x)a_{jk}(m,x)\tilde{c}_k(m,x). \end{aligned} \tag{3.17}$$

(i) Hence if c, a, d are given with c strictly positive, a possible choice is

$$\begin{aligned} b(m,x) = c(m,x)^{-1}\Bigg(\tfrac{1}{2}\sum_{j,k=1}^{q}\tilde{c}_{jk}(m,x)a_{jk}(m,x) \\ + \sum_{i=1}^{q}\tilde{c}_i(m,x)d_i(m,x)\Bigg) \\ - c(m,x)^{-2}\sum_{j,k=1}^{q}\tilde{c}_j(m,x)a_{jk}(m,x)\tilde{c}_k. \end{aligned} \tag{3.18}$$

It is clear that b is Lipschitz continuous and bounded if $\tilde{c}, a_{jk}, d_i$ are as well. Under the same conditions $d_i^0 := d - c^{-1}\sum_{j=1}^{d} a_{ij}\tilde{c}_j$ is also Lipschitz continuous and bounded. Then the functions a, b, c, d^0 satisfy all assumptions of Proposition 3.1 and 3.3.

(ii) If a and b are given, a possible choice for the functions d and d^0 is

$$\begin{aligned} d_j(m,x) = \left(\sum_{j=1}^{q}\tilde{c}_j(m,x)\right)^{-1} \\ \times\Bigg(\sum_{j,k=1}^{q}\tilde{c}_j(m,x)a_{jk}(m,x)\tilde{c}_k(m,x) \\ - c(m,x)b(m,x) - \tfrac{1}{2}\sum_{j,k=1}^{q}\tilde{c}_{jk}(m,x)a_{jk}(m,x)\Bigg) \end{aligned}$$

and $d_j^0 = d_j - c^{-1}\sum_{i=1}^{d} a_{jk}\tilde{c}_i$ for $1 \le j \le q$.

(iii) If $b = 0$ and a, d are given, then c has to satisfy

$$\begin{aligned} c(m,x)\left(\tfrac{1}{2}\sum_{j,k=1}^{q}\tilde{c}_{jk}(m,x)a_{jk}(m,x) + \sum_{i=1}^{q}\tilde{c}_i(m,x)d_i(m,x)\right) \\ = \sum_{j,k=1}^{q}\tilde{c}_j(m,x)a_{jk}(m,x)\tilde{c}_k(m,x), \end{aligned} \tag{3.19}$$

which seems to be very restrictive.

(iv) If $c(m, x) = c_0(x)$ with $c_0 \in C_b^3(\mathbb{R}^q)$ then the functions in (3.10) are computed as follows: $c_0(m, x) = 0$, $c_i(m, x) = (\partial c_0/\partial x_i)(x)$ and $\tilde{c}_{jk}(m, x) = (\partial^2 c_0/\partial x_j\, \partial x_k)(x)$, $1 \leq i, j, k \leq q$. Inserting these relations into (3.18), a concrete example for a nonlinear superprocess with nonconstant c can be given.

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