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CHASING BALLS THROUGH MARTINGALE FIELDS

BY MICHAEL SCHEUTZOW AND DAVID STEINSALTZ

Technische Universität Berlin and University of California, Berkeley

We consider the way sets are dispersed by the action of stochastic flows derived from martingale fields. Under fairly general continuity and ellipticity conditions, the following dichotomy result is shown: any nontrivial connected set $\mathcal{X}$ either contracts to a point under the action of the flow, or its diameter grows linearly in time, with speed at least a positive deterministic constant Λ. The linear growth may further be identified (again, almost surely), with a much stronger behavior, which we call "ball-chasing": if ψ is any path with Lipschitz constant smaller than Λ, the ball of radius ε around $\psi(t)$ contains points of the image of $\mathcal{X}$ for an asymptotically positive fraction of times t. If the ball grows as the logarithm of time, there are individual points in $\mathcal{X}$ whose images eventually remain in the ball.

1. Introduction.

1.1. *Problems and results.* We have been investigating for some time the often surprising behavior of exceptional individual points in a stochastic flow that acts on Euclidean space of dimension d larger than 1. The flows we consider are driven at every point by martingale increments with bounded variance, so that the image of any individual point is a diffusion, acquiring a displacement on the order of $\sqrt{T}$ from its origin in time T. In our earlier paper [4] we (together with M. Cranston) showed that in the special case of isotropic Brownian flows, a positive Lyapunov exponent will guarantee the existence of points which travel at an exceptional linear rate from their origin. In [5] we proved a conjecture of Carmona [3] that, under very general conditions, no point advances superlinearly in time.

In the present work we extend the results on isotropic Brownian flows to a more general context. We are still talking about stochastic flows of homeomorphisms $\phi_t : \mathbb{R}^d \to \mathbb{R}^d$ of the Kunita type, defined by a field of semimartingales $M(t, x)$ (where t is the time coordinate, running over $[0, \infty)$, and x is the space coordinate, in $\mathbb{R}^d$), and the equation

$$\phi_t(x) = x + \int_0^t M\big(ds, \phi_s(x)\big).$$

We assume here that the semimartingales M are in fact martingales and that they satisfy some basic continuity and uniform ellipticity conditions, which will be stated in Section 1.2.

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Under some conditions on M and the starting set $\mathcal{X} \subset \mathbb{R}^d$, the image $\phi_t(\mathcal{X})$ will have a positive probability of shrinking to a point asymptotically under the action of the flow. This has been shown, for example, by Baxendale and Harris in Section 9 of [2] in dimension 2, when M generates an isotropic Brownian flow whose local longitudinal and transverse parameters β_L and β_N satisfy $3 \geq \beta_L/\beta_N > 5/3$, and $\mathcal{X}$ is a sufficiently small ball. (We show, though, in Proposition 3.1, that this probability is never equal to 1 if the flow satisfies our basic ellipticity conditions.) Under such circumstances, all the paths will ultimately converge to the path of a single point. Since the path of any single point is almost surely diffusive—that is, has separation from its origin on the order of the square root of time—this excludes the possibility that there is a "ballistic" point, defined to be a point x such that

$$\liminf_{t\to\infty} t^{-1}\|\phi_t(x)\| > 0. \tag{1}$$

The main result of this paper, given in several forms, tells us that this is the only way there can fail to be a ballistic point. We show in Proposition 3.1 that the image of a nontrivial set $\mathcal{X}$ (a set will be called *nontrivial* if it is connected and contains more than one point) expands linearly in time precisely when it does not shrink down to a point. That is,

$$\mathrm{P}\Big\{\mathcal{X} \text{ grows linearly or } \lim_{t\to\infty} \operatorname{diam} \phi_t(\mathcal{X}) = 0\Big\} = 1,$$

while the probability that both events occur is 0.

The earlier result on isotropic Brownian flows with a positive Lyapunov exponent is a special case of this one, since under such a flow any nontrivial set almost surely does not shrink to a point. But this result also tells us what may happen in flows with negative Lyapunov exponents, where the probability of a set shrinking to a point is positive, but not 1.

We are also concerned in this article to elucidate the structure of the set of ballistic points. In our earlier work, we showed that there are ballistic points travelling in every direction with a certain minimum speed. In principle, this could have been achieved by a few thin streams of ballistic points. Here we show that there are indeed very many ballistic points, travelling in many different directions. In fact, we show a much more refined property than ballistic growth. We call this property "ball-chasing," by which we mean that if a ball with fixed radius moves in space with speed bounded by a certain constant, even when the path of the ball is itself random, the image of our initial set $\mathcal{X}$ will intersect the ball for a nonzero proportion of the time. For a positive real ε and a point $x \in \mathbb{R}^d$, we use the notation $B_\varepsilon(x)$ to represent the closed ball of radius ε with center x.

Unless otherwise indicated, all stochastic processes are assumed adapted to a filtration of σ-algebras which will be uniformly denoted $(\mathcal{F}_t)_{0\leq t\leq\infty}$. All references to martingale properties are meant to be referred to this filtration.

DEFINITION 1. We are given a positive function $\varepsilon:[0,\infty)\to(0,\infty)$ and a Lipschitz continuous path $\psi:[0,\infty)\to\mathbb{R}^d$ which is adapted with respect to the filtration $\mathcal{F}_t$. A set $\mathcal{X}$ acted on by the flow ϕ_t *chases balls with path* ψ *and radius* ε if the set

$$\mathcal{I}_{\varepsilon,\psi}(\mathcal{X}) := \{t : \phi_t(\mathcal{X}) \cap B_{\varepsilon(t)}(\psi(t)) \neq \varnothing\}$$

has positive density, asymptotically, as $t\to\infty$; that is,

$$\liminf_{T\to\infty} T^{-1} m(\mathcal{I}_{\varepsilon,\psi}\cap[0,T]) > 0,$$

where m is Lebesgue measure. $\mathcal{X}$ *chases balls weakly with path* ψ *and radius* ε if $\mathcal{I}_{\varepsilon,\psi}(\mathcal{X})$ is unbounded. Finally, $\mathcal{X}$ *chases balls strongly with path* ψ *and radius* ε if $\mathcal{I}_{\varepsilon,\psi}(\mathcal{X})$ includes all $t\geq t_0$ for some random t_0.

DEFINITION 2. We say that the set $\mathcal{X}$ *contracts to a point* under the action of the flow ϕ_t if $\lim_{t\to\infty}\operatorname{diam}\phi_t(\mathcal{X})=0$.

We show in Theorem 3.2 that under fairly general conditions on a martingale flow there is a positive deterministic Λ, depending only on the local characteristic bounds, such that whenever $\mathcal{X}$ is a nontrivial subset of $\mathbb{R}^d$ and ψ any path with Lipschitz constant less than Λ, with probability 1 either $\mathcal{X}$ chases balls with path ψ and radius $\varepsilon(t)\equiv\varepsilon$ any constant, or the image of $\mathcal{X}$ shrinks down to a point.

The points which come into these moving balls again and again, may in principle be always new points, so that no individual point actually moves ballistically. In Section 4 we strengthen these results, to show that there are indeed individual random points which enter a moving ball infinitely often, or eventually.

DEFINITION 3. We are given a positive function $\varepsilon:[0,\infty)\to(0,\infty)$ and a Lipschitz continuous path $\psi:[0,\infty)\to\mathbb{R}^d$ which is adapted with respect to the filtration $\mathcal{F}_t$. A set $\mathcal{X}$ acted on by a flow ϕ_t *chases balls pointwise with path* ψ *and radius* ε if there is a point $x\in\mathcal{X}$ such that the set

$$\mathcal{I}_{\varepsilon,\psi}(x) := \{t : \phi_t(x)\in B_{\varepsilon(t)}(\psi(t))\}$$

has positive density, asymptotically, as $t\to\infty$. Weak and strong pointwise ball-chasing are defined analogously.

Theorem 4.1 states that for any nontrivial subset $\mathcal{X}$ of $\mathbb{R}^d$, $\varepsilon(t)\equiv\varepsilon$ any positive constant, and ψ any path with Lipschitz constant less than or equal to Λ, $\mathcal{X}$ either contracts to a point under the action of ϕ_t or chases balls weakly pointwise with path ψ and radius ε. Theorem 4.2 states that we can obtain strong pointwise ball-chasing under the above conditions by allowing $\varepsilon(t)$ to grow as an appropriate constant times $\log t$. Note that this includes paths ψ which do not move, which means that there are random points in a martingale flow which are almost stable, never moving more than a constant times $\log t$ from their starting point in time t.

Strictly speaking, weak or normal ball-chasing implies linear growth of $\phi_t(\mathcal{X})$ only in the sense that $\limsup t^{-1} \operatorname{diam} \phi_t(\mathcal{X}) > 0$. Linear growth in the strict lim inf sense is implied by strong ball-chasing when the function $\varepsilon(t)$ is strictly sublinear. In particular, the existence of ballistic points whenever $\phi_t(\mathcal{X})$ does not contract to a point is a consequence of Theorem 4.2.

1.2. *The standard conditions.* The martingale field M will be assumed to satisfy a set of basic conditions, which we will refer to as the "standard conditions." With no drift, the only local characteristic of the flow is a $d \times d$ matrix $a(s, x, y, \omega)$, defined by

$$\langle M(\cdot, x, \omega), M(\cdot, y, \omega)\rangle_t = \int_0^t a(s, x, y, \omega)\, ds,$$

which we assume to be continuous in (x, y) for almost every (s, ω), and predictable in (s, ω) for each (x, y). We assume that there are some constants a and A such that the "continuity conditions,"

$$\|\mathcal{A}(s, x, y, \omega)\| \le a^2 \|x - y\|^2, \tag{2}$$

$$\|a(s, x, x, \omega)\| \le A^2 \qquad \text{for all } x, \tag{3}$$

are satisfied, where

$$\mathcal{A}(s, x, y, \omega) = a(s, x, x, \omega) - a(s, y, x, \omega) - a(s, x, y, \omega) + a(s, y, y, \omega).$$

In addition, we impose the "local two-point ellipticity condition"

$$\begin{aligned} &v^\top a(s, x, x, \omega)v + 2v^\top a(s, x, y, \omega)w + w^\top a(s, y, y, \omega)w \\ &\qquad \ge \mathcal{E}(\|x - y\|)\big(\|v\|^2 + \|w\|^2\big) \end{aligned} \tag{4}$$

for all $x, y, v, w \in \mathbb{R}^d$, $s \in \mathbb{R}^+$ and $\omega \in \Omega$. Here $\mathcal{E} : [0, \infty) \to [0, \infty)$ is a continuous function with $\mathcal{E}(\rho) > 0$ for $0 < \rho < \rho^*$, where ρ^* is a positive constant. Remember that a is a matrix which gives the derivative relative quadratic variation at two points x and y; that is, the (i, j) component of $a(t, x, y, \omega)$ is

$$\frac{d}{dt}\langle M(\cdot, x, \omega)_i, M(\cdot, y, \omega)_j\rangle_t.$$

This makes $\mathcal{A}(s, x, y, \omega)$ the derivative of the quadratic variation for the difference $M(s, x, \omega) - M(s, y, \omega)$. The left-hand side of inequality (4) is thus the rate of growth in quadratic variation for the linear combination $vM(\cdot, x, \omega) + wM(\cdot, y, \omega)$. What this inequality says, in effect, is that the relative motion of two points separated by a distance ρ has a diffusion component of magnitude at least $\mathcal{E}(\rho)$. This excludes, among others, a flow which is simply a rigid motion of the space by a single Brownian motion. We observe here that this condition implies as well the "global one-point ellipticity condition," that there is a positive constant ϵ [which here may be taken as $\mathcal{E}(\rho^*/2)$] such that

$$v^\top a(s, x, x, \omega)v \ge \epsilon \|v\|^2 \qquad \text{for all } x, v \in \mathbb{R}^d,\ s \in \mathbb{R}^+ \text{ and } \omega \in \Omega. \tag{5}$$

The local two-point ellipticity condition is satisfied, in particular, by all isotropic Brownian flows which are nondegenerate, in the sense that the covariance tensor is not constant. The locality of the condition also allows it to hold for periodic fields, that is, for flows on a torus.

The triple $(a, A, \mathcal{E})$ will be referred to as "local-characteristic bounds." A constant which is a function of these bounds will be called a "flow-bound constant."

REMARK 1.1. Let ϕ be a flow satisfying the standard conditions. If T is any almost surely finite stopping time, then the flow

$$\phi_{T,T+t} := \phi_{T+t} \circ \phi_T^{-1}, \qquad t \geq 0$$

also satisfies the standard conditions. The essential change is in the filtration of the new flow, which is $\mathcal{F}_{T+t}$ of the old flow, at time t.

An extensive account of this kind of stochastic flow, including a proof that these "standard conditions" guarantee their existence, may be found in the book by Kunita [7].

2. The retraction argument.

2.1. *General description.* The primary instrument of our analysis is a technique that we call the "retraction argument." The full action of the stochastic flow on a domain is too complex to comprehend with current technology. Instead, we focus our attention on a tiny subset, consisting of two points plus a connection. That is, we follow the motions of just two points at a time, and allow the continuity of the flow to track the intermediate points. If we follow the trail of a single point, and wait for it to cross a fixed hyperplane, we will be waiting a very long time: the expected arrival time is infinite. If we start with two points and wait for just one of them to cross the hyperplane, the expected arrival time is still infinite, but for a more interesting reason. The leading point, as such, does tend toward the hyperplane with a certain positive speed; what happens, though, is that this boost dissipates with time, as the two points drift apart. It is here that the continuity of the flow is important. When we begin with a connected set $\mathcal{X}$, the images of any two points are eternally connected by a continuous curve of other points, also in the image of the flow. This allows us to counteract the separation of the two points by pulling up the laggard at regular intervals to a fixed distance behind the leader. In fact, we will apply a slightly more sensitive version of this argument, "targeting" our points not only at a hyperplane, but also at the current position of a path which is being chased.

There are two kinds of results which undergird this method. First, we need to know that in a fixed time span, the "leading" point does indeed have a forward drift. Since the point that leads at the start of the time period moves with zero drift, we need only show that the hind point has a positive probability of ending up a positive distance in front. Targeting a point is slightly more complicated, since a spatial martingale in fact has a drift away from any fixed point, but this drift goes to zero as the initial separation goes to ∞, so may be ignored if this initial separation is chosen sufficiently large.

The main difficulty here is that the independence of the two points disappears as their separation shrinks, a behavior which cannot be ruled out. We will use a variant of the "support theorem" for diffusions, which says that any fixed Lipschitz path has a positive probability of being within ε of the entire path of a uniformly elliptic diffusion with bounded growth. We apply this to the two-point motion of the flow, which is not uniformly elliptic, but which confines its nonellipticity to a neighborhood of the diagonal. The path may then simply skirt around this frozen region: if the diffusion stays close to the path, it never experiences the loss of ellipticity.

Once we have established this drift toward the path ψ_t, for points outside a sufficiently large ball, it follows that the image of $\mathcal{X}$ will keep returning to that ball. To make this rigorous and to develop bounds on the rate of return, we use the law of large numbers for martingales. The results are given in Lemma 2.6 and Lemma 2.7.

2.2. *General bounds on flows.* We will need tail bounds on the motions of individual points in a stochastic flow. We have been unable to find bounds of sufficient generality in the published literature, although they are not difficult to derive from standard facts about stochastic integrals (such as those in [7]). We use therefore the bounds which appeared in our earlier paper [5] as Proposition 5.2:

LEMMA 2.1. *If M is a martingale field satisfying the continuity conditions, the flow it generates satisfies*

$$\mathrm{P}\left\{\sup_{0\le t\le T}\|\phi_t(x)-x\|>z\right\}\le\frac{4d}{\sqrt{\pi}}\exp\left\{-\frac{z^2}{2d^2A^2T}\right\}. \tag{6}$$

We also require tail bounds on the expansion of any set under the action of a standard flow.

LEMMA 2.2. *There is a flow-bound constant K such that for any stopping time $\tau\ge 0$, and any random set $\mathcal{X}$ which is measurable with respect to $\mathcal{F}_\tau$,*

$$\mathrm{P}\left\{\sup_{\tau\le t\le\tau+1}\operatorname{diam}\phi_t(\mathcal{X})\ge z+z\operatorname{diam}\phi_\tau(\mathcal{X})\,\Big|\,\mathcal{F}_\tau\right\}\le Ke^{-z^2/\log^3 z} \tag{7}$$

for all $z>2$, on the event $\{\tau<\infty\}$.

This result is a component of Theorem 2.1, in our earlier paper [5]. The proof is a tedious chaining argument which does not bear repeating here. As a result for finite times, it does not depend on the linear asymptotic growth which is the main result of that theorem. Furthermore, the exponential tail bounds are relevant only in the last step of Theorem 4.2. For the more essential application, that in Proposition 3.1, polynomial bounds would suffice.

LEMMA 2.3. *Let ϕ_t be a stochastic flow satisfying the standard conditions. For each positive integer k there is a flow-bound constant η_k such that for any stopping time $\tau \geq 0$, and any random bounded set $\mathcal{X}$ which is measurable with respect to $\mathcal{F}_\tau$,*

$$\mathrm{P}\left\{\sup_{\tau \leq t \leq \tau+1} \operatorname{diam} \phi_t(\mathcal{X}) \geq z \operatorname{diam} \phi_\tau(\mathcal{X}) \,\middle|\, \mathcal{F}_\tau\right\} \leq \eta_k z^{-k} \tag{8}$$

for all positive z, on the event $\{\tau < \infty\}$.

PROOF. For $x, y \in \mathcal{X}$ let

$$D(x, y) := \sup_{0 \leq t \leq 1} \|\phi_{\tau+t}(x) - \phi_{\tau+t}(y)\|.$$

Let $\psi(z) = z^k$ for z positive real. By Proposition 5.2 of [5], there are positive flow-bound constants b_1, b_2 and b_3 such that the following two-point bound holds:

$$\mathrm{P}\{D(x, y) > z \mid \mathcal{F}_\tau\} \leq b_1 \exp\Big[-b_2\big((\log z - \log \|\phi_\tau(x) - \phi_\tau(y)\| - b_3)^+\big)^2\Big] \tag{9}$$

for all positive z. Integrating this with respect to the function kz^{k-1} gives us a flow-bound constant C_k' such that

$$\mathrm{E}\big[D(x, y)^k\big] \leq C_k'^k \|\phi_\tau(x) - \phi_\tau(y)\|^k.$$

Thus $\|D(x, y)\|_\psi \leq C_k' \|\phi_\tau(x) - \phi_\tau(y)\|$, where $\|\cdot\|_\psi$ is the Orlicz norm corresponding to $\phi_{\tau,\tau+t}$, conditioned on $\mathcal{F}_\tau$.

Let $N(\varepsilon)$ be the number of balls of radius ε needed to cover $\phi_\tau(\mathcal{X})$. Then

$$N(\varepsilon) \leq \left(\frac{2\sqrt{d} \operatorname{diam} \phi_\tau(\mathcal{X})}{\varepsilon}\right)^d \qquad \text{for } \varepsilon \leq \operatorname{diam} \phi_t(\mathcal{X}).$$

Then by Theorem 2.2.4 of [10] there is a constant C_k, depending only on k and C_k', such that

$$\begin{aligned} \mathrm{E}\left[\sup_{x, y \in \mathcal{X}} D(x, y)^k \,\middle|\, \mathcal{F}_\tau\right] &\leq \left(C_k' \int_0^{\operatorname{diam} \phi_\tau(\mathcal{X})} \psi^{-1}(N(\varepsilon))\, d\varepsilon\right)^k \\ &\leq C_k \big(2\sqrt{d} \operatorname{diam} \phi_\tau(\mathcal{X})\big)^d \operatorname{diam} \phi_\tau(\mathcal{X})^{k-d}. \end{aligned}$$

[As stated, the theorem refers only to processes $D(x, y)$ which are differences of a process X between two different points: $D(x, y) = |X_x - X_y|$. But as Ledoux and Talagrand point out in a related context, in Remark 11.5 of [8], the result and its proof require nothing about this particular form of D except that it be symmetric and satisfy the triangle inequality.] The lemma thus holds with $\eta_k = (4d)^{d/2}C_k$. $\square$

2.3. *Support lemma.* We begin with a version of the support theorem for diffusions, adapted for application to stochastic differential equations defined by a martingale field, à la Kunita. This says that the support of a nondegenerate diffusion is the entire space of continuous functions. Our proof is based on one given by Bass ([1], Theorem 1.8.5). One important difference from the standard version here is that we abandon the assumption of uniform ellipticity. Since our process is the joint motion of two points acted on by the flow, the diffusion term goes to zero in a neighborhood of the diagonal. On the other hand, it is unsurprising that the probability of a path avoiding a neighborhood of the origin is unaffected by the situation inside the neighborhood.

LEMMA 2.4. *There exists a positive $p = p(\alpha_1, \alpha_2, \beta, \varepsilon, t_0)$ (a function from $\mathbb{R}_+^5 \to \mathbb{R}_+$, where $\mathbb{R}_+$ is understood to be the strictly positive real numbers) with the following property: let $(Z_t)_{t\geq 0}$ be any $\mathbb{R}^d$-valued continuous semimartingale, with Doob-decomposition $Z_t = N_t + V_t$, N_t being the local martingale part and V_t having locally bounded variation, and $Z_0 = 0$. Let ε positive be given, and define $\tau = \inf\{t : \|Z_t\| > \varepsilon\}$. Let $\alpha_1, \alpha_2, \beta$ be positive, such that V has Lipschitz constant no more than β on $[0, \tau]$ and such that the quadratic variation $a_t = d\langle N\rangle/dt$ satisfies*

$$\alpha_2\|z\|^2 \geq z^\top a_t z \geq \alpha_1\|z\|^2 \tag{10}$$

for $0 \leq t \leq \tau$ and $z \in \mathbb{R}^d$. Then $\mathrm{P}\{\tau > t_0\} \geq p$ for every positive t_0. We may take p to be continuous in all the parameters, decreasing in α_2, β and t_0, and increasing in ε and $\alpha_1 \leq \alpha_2$.

PROOF. Let $y = (\varepsilon/4, 0, 0, \dots, 0)$, $f(z) = \|z - y\|^2$, B_t a standard one-dimensional Brownian motion, and $D_t = \|Z_t - y\|^2 - (\varepsilon/4)^2$. By Itô's formula,

$$dD_t = 2\sum_{i=1}^d (Z_t^i - y_i)\, dZ_t^i + \sum_{i=1}^d d\langle Z^i\rangle_t,$$

while $D_0 = 0$. Let $\tilde\tau = \inf\{t \geq 0 : |D_t| > (\varepsilon/8)^2\}$. Then

$$Y_t := D_{t\wedge\tilde\tau} + \alpha_1(B_t - B_{\tilde\tau})\mathbf{1}_{\{t\geq\tilde\tau\}}$$

is a semimartingale with $Y_0 = 0$, which is uniformly elliptic, hence fulfills the conditions of Lemma I.8.3 of Bass [1] [with $(\varepsilon/8)^2$ in place of ε]. Thus, there is a $p = p(\alpha_1, \alpha_2, \beta, \varepsilon, t_0) > 0$ such that

$$\mathrm{P}\left\{\sup_{0\le t\le t_0} |Y_t| < (\varepsilon/8)^2\right\} \ge p.$$

From the definition of Y it follows that this is a lower bound for $\mathrm{P}\{\tau > t_0\}$ as well.

The parameters enter into the definition of p only as bounds. Since

$$\sup_{\alpha_2'>\alpha_2}\ \sup_{\beta'>\beta}\ \sup_{\varepsilon'<\varepsilon}\ \sup_{t_0'>t_0}\ \sup_{0<\alpha_1'<\alpha_1} p(\alpha_1', \alpha_2', \beta', \varepsilon', t_0')$$

is also positive, and is also a lower bound for $\mathrm{P}\{\tau > t_0\}$, we may define a version which has the claimed monotonicity properties, is left-continuous in α_1 and ε and right-continuous in each of the other parameters. We may then apply a smoothing kernel to create a continuous function $\tilde{p}$ which also has the monotonicity property, and such that

$$p \ge \tilde{p} > 0. \qquad \square$$

As a corollary, we show that if we follow the motion of two points in the flow, the minimum distance to a target point has a downward drift. This is the core of the retraction argument. We imagine that the flow has run up to a stopping time τ, at which time the path being tracked is at $z := \psi(\tau)$. We follow it then up to a later time $\tau + s$. We want to conclude that, on average, the nearest point of $\phi_{\tau+s}(\mathcal{X})$ has then come a little bit closer to z, but the path has moved off to $\psi(\tau + s)$, a distance of no more than λs, where λ is a bound for the Lipschitz constant of ψ. If λs is smaller than the average rate of approach, then we expect the image of $\mathcal{X}$ to track the path ψ.

We will be considering martingale fields which satisfy the two-point local ellipticity condition (4), and then we will define

$$\varepsilon(\mathcal{E}, \rho) := \tfrac{1}{3} \min\{\rho, \rho^* - \rho\} \tag{11}$$

and

$$\zeta(\mathcal{E}, \rho) := \inf\bigl\{\mathcal{E}(\rho') : \rho - 2\varepsilon(\mathcal{E}, \rho) \le \rho' \le \rho + 2\varepsilon(\mathcal{E}, \rho)\bigr\}. \tag{12}$$

Note that $\varepsilon(\mathcal{E}, \rho)$ and $\zeta(\mathcal{E}, \rho)$ are both positive when $0 < \rho < \rho^*$.

LEMMA 2.5. *For all positive ζ, ε, a and A there are functions $G'_{\zeta,\varepsilon,a,A}$, $G''_{\zeta,\varepsilon,a,A} : [0, \infty) \times [0, \infty) \to [0, \infty)$ [the subscripted parameters will generally be suppressed; ε and ζ have the default values $\varepsilon = \varepsilon(\mathcal{E}, \rho)$ and $\zeta = \zeta(\mathcal{E}, \rho)$, when not otherwise specified], which have the following properties:*

(i) *For each s, $G'(s, r)$ is continuous and nonincreasing in r, and converges to 0 as $r \to \infty$.*

(ii) *For all positive s, the function $G''(s,\cdot)$ is continuous and $G''(s,\rho)>0$ when $\rho\in(0,\rho^*)$.*

(iii) *Let M be a martingale field satisfying the standard conditions with a and A as bounds on the local characteristics given in* (2) *and* (3), *and $\mathcal{E}$ as the ellipticity bound of* (4), *with ρ satisfying $\varepsilon(\mathcal{E},\rho)\ge\varepsilon$ and $\zeta(\mathcal{E},\rho)\ge\zeta$. The flow generated by M is denoted ϕ. Let $s>0$ be given, τ any finite stopping time for the flow, x,y,z any $\mathcal{F}_\tau$-measurable $\mathbb{R}^d$-valued random points with $\|x-y\|=\rho$. Define*

$$r_1:=\min\{\|x-z\|,\|y-z\|\},$$

$$x_{\tau+s}:=\phi_{\tau,\tau+s}(x),\qquad y_{\tau+s}:=\phi_{\tau,\tau+s}(y)$$

and

$$r_2:=\min\{\|x_{\tau+s}-z\|,\|y_{\tau+s}-z\|\}.$$

Then

$$\mathrm{E}[r_2\,|\,\mathcal{F}_\tau]\le r_1+G'(s,r_1)-G''(s,\rho). \tag{13}$$

(iv) *If v is an $\mathbb{R}^d$-valued $\mathcal{F}_\tau$-measurable random variable with $\|v\|=1$, then*

$$\mathrm{E}\bigl[\max\{\langle x_{\tau+s},v\rangle,\langle y_{\tau+s},v\rangle\}\,|\,\mathcal{F}_\tau\bigr]\ge\max\{\langle x,v\rangle,\langle y,v\rangle\}+G''(s,\rho). \tag{14}$$

PROOF. Without loss of generality we may assume that $\|x-z\|\le\|y-z\|$, and by changing the coordinates we may assume as well that $z=0$ and $x=(x^1,0,\dots,0)$, with $x^1>0$. All probabilities and expectations will be taken to be conditional on $\mathcal{F}_\tau$, but the notation will be supressed for ease of reading. Define

$$N:=x^1_{\tau+s}-x^1$$

and

$$N':=\left(\sum_{i=2}^d(x^i_{\tau+s})^2\right)^{1/2}.$$

Clearly $\mathrm{E}[N]=0$, and by Lemma 2.1 there is a constant c_1, depending only on s, a and A, such that

$$\mathrm{E}[N^2+N'^2]\le c_1.$$

Now let $\gamma(t):=t/3$ for $0\le t\le 1$, $\gamma(t):=\sqrt[3]{t}-2/3$ for $t>1$, and let $\mathcal{A}$ be the event that $N^2+N'^2\le\gamma^2(x^1)$. Then by Cauchy–Schwarz,

$$\begin{aligned}\mathrm{E}\Bigl[\mathbf{1}_{\mathcal{A}^C}\Bigl(\sqrt{(x^1+N)^2+N'^2}-x^1\Bigr)\Bigr]&\\ \le\mathrm{E}\bigl[\mathbf{1}_{\mathcal{A}^C}(|N|+N')\bigr]&\\ \le\sqrt{\frac{c_1}{\gamma^2(x^1)}}\cdot\sqrt{2c_1}\le\frac{\sqrt{2}c_1}{\gamma(x^1)}&\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{E}\Big[\mathbf{1}_{\mathcal{A}}\Big(\sqrt{(x^1+N)^2+N'^2}-x^1\Big)\Big]\\
&\quad=\mathrm{E}\left[\mathbf{1}_{\mathcal{A}}(x^1+N)\left(\sqrt{1+\left(\frac{N'}{x^1+N}\right)^2}-1\right)\right]+\mathrm{E}[\mathbf{1}_{\mathcal{A}}N]\\
&\quad\le\sqrt{2}(x^1+\sqrt{c_1})\left(\sqrt{1+\left(\frac{\gamma(x^1)}{x^1-\gamma(x^1)}\right)^2}-1\right)+\frac{\sqrt{2}c_1}{\gamma(x^1)},
\end{aligned}$$

since $\mathrm{E}\,\mathbf{1}_{\mathcal{A}}N=\mathrm{E}\,N-\mathrm{E}\,\mathbf{1}_{\mathcal{A}^C}N=-\mathrm{E}\,\mathbf{1}_{\mathcal{A}^C}N$. These bounds allow us to define a function $g(x)$, which is continuous and nonincreasing, and which converges to 0 as $x\to\infty$, depending only on s, a, and A, such that

$$\mathrm{E}\,\|x_{\tau+s}\|-x^1=\mathrm{E}\Big[\sqrt{(x^1+N)^2+N'^2}-x^1\Big]\le g(x^1). \tag{15}$$

Suppose that $\rho=\|x-y\|<\rho^*$ and $\rho<x^1/2$. Let $\psi:[0,s]\to\mathbb{R}^d$ be a path which moves with constant speed along a circle of radius ρ around x, from $\psi(0)=y$ to $\psi(s)=(x^1-\rho,0,0,\dots,0)$. Then the $\mathbb{R}^{2d}$-valued semimartingale

$$Z_t:=\big(\phi_{\tau,\tau+t}(x),\phi_{\tau,\tau+t}(y)\big)-\big(x,\psi(t)\big)$$

satisfies the conditions of Lemma 2.4, with $\varepsilon=\varepsilon(\mathcal{E},\rho)$; $\zeta_1=\zeta(\mathcal{E},\rho)$; $\zeta_2=2A^2$, $\beta=\pi\rho/s$ and $t_0=s$. Thus there is a positive number $p(\rho)$, a continuous function of these parameters, such that

$$\mathrm{P}\big\{\|Z_t\|\le\varepsilon\text{ for all }t\in[0,s]\big\}\ge p(\rho).$$

Consequently, if we let B_1 be the closed ball of radius ε around x and B_2 the closed ball of radius ε around $\psi(s)$, then

$$\mathrm{P}\{x_{\tau+s}\in B_1\text{ and }y_{\tau+s}\in B_2\}\ge p(\rho).$$

This means that

$$\mathrm{E}\big[(\|x_{\tau+s}-z\|-\|y_{\tau+s}-z\|)^+\big]\ge p(\rho)\frac{\rho}{3}, \tag{16}$$

when $\rho<\rho^*\wedge\frac{x^1}{2}$. By linear interpolation we may parlay this into a continuous function $h(r,\rho)$ which is nondecreasing in r, zero for $\rho\ge\rho^*\wedge\frac{r}{2}$, $h(r,\rho)=h(2\rho^*,\rho)$ for $r\ge2\rho^*$, and

$$0<h(r,\rho)\le p(\rho)\frac{\rho}{3}$$

for $\rho<\rho^*\wedge\frac{r}{2}$.

Putting together (15) and (16), we get

$$\begin{aligned}
\mathrm{E}[r_2\,|\,\mathcal{F}_\tau]&=\mathrm{E}\big[\|x_{\tau+s}-z\|\,\big|\,\mathcal{F}_\tau\big]-\mathrm{E}\big[(\|x_{\tau+s}-z\|-\|y_{\tau+s}-z\|)^+\,\big|\,\mathcal{F}_\tau\big]\\
&\le r_1+g(r_1)-h(r_1,\rho).
\end{aligned}$$

Define

$$H(\rho) := \inf_{r\geq 0}\left\{h(r,\rho)+\frac{1}{r+1}\right\}.$$

Observe that for each $\rho \in (0, \rho^*/2)$,

$$\inf_{\rho\leq\rho'\leq\rho^*-\rho} h(x,\rho')$$

is positive for x sufficiently large. This means that

$$\inf_{\rho\leq\rho'\leq\rho^*-\rho} H(\rho')$$

is positive. We can define a continuous function $G''(s,\cdot)$ such that $0 < G''(s,\rho) < H(\rho)$ for $0<\rho<\rho^*$. If we then take $G'(s,r) := g(r)+1/(r+1)$,

$$\begin{aligned}\mathrm{E}[r_2 \mid \mathcal{F}_\tau] &\leq r_1 + g(r_1) + \frac{1}{r_1+1} - \left(h(r_1,\rho)+\frac{1}{r_1+1}\right)\\ &\leq r_1 + G'(s,r_1) - G''(s,\rho),\end{aligned}$$

so conclusions 1–3 of the lemma hold. Conclusion 4 follows if we take z to be equal to $x+Kv$, for K a large real number, and let K go to infinity. □

2.4. *Supermartingales.*

LEMMA 2.6. *Let $(X_n,\mathcal{F}_n)_{n\geq 0}$ be an adapted real-valued process such that the conditional distributions of the increments satisfy almost surely*

$$\mathrm{E}[X_{n+1}-X_n \mid \mathcal{F}_n] \geq c_1\mathbf{1}\{X_n\geq\alpha\} \tag{17}$$

and

$$\mathrm{P}\{|X_{n+1}-X_n|\geq\lambda \mid \mathcal{F}_n\} \leq b_1 e^{-b_2\lambda} \qquad \forall\lambda>0 \tag{18}$$

for positive constants α, c_1, b_1, b_2. If we define

$$\gamma(c_1,b_1,b_2) := \min\left\{\frac{c_1b_2^2}{2b_1+b_2^2}, \frac{b_2}{5}\right\}, \tag{19}$$

then $e^{-\gamma X_{n\wedge\tau}+\gamma^2 n\wedge\tau}$ is a supermartingale (*with respect to the filtration $\mathcal{F}_n$*), *where $\tau := \min\{n\geq 0: X_n\leq\alpha\}$ and $\gamma\leq\gamma(c_1,b_1,b_2)$, as long as*

$$\mathrm{E}[e^{-\gamma X_0}] < \infty. \tag{20}$$

Consequently, *for all $\beta\geq\alpha'\geq\alpha$ and any finite stopping time T such that $\mathrm{E}[e^{-\gamma X_T}]$ is finite*,

$$\mathrm{P}\left\{\lim_{n\to\infty} X_n=\infty \text{ and } X_n\geq\alpha'\ \forall n\geq T \,\middle|\, \mathcal{F}_T\right\} \geq 1-e^{-\gamma(\beta-\alpha')} \tag{21}$$

on $\{X_T \geq \beta\}$. *In particular*, *the probability in* (21) *is positive whenever* $\beta > \alpha'$. *Finally*,

$$\mathrm{P}\left\{\liminf_{n\to\infty} \frac{1}{n} X_n \geq c_1 \,\Big|\, \limsup_{n\to\infty} X_n > \alpha\right\} = 1. \tag{22}$$

PROOF. We know that for all $0 < \gamma < b_2$,

$$\begin{aligned} \mathrm{E}\big[e^{-\gamma(X_{n+1}-X_n)} \,\big|\, \mathcal{F}_n\big] \leq \mathrm{E}\big[e^{\gamma|X_{n+1}-X_n|} \,\big|\, \mathcal{F}_n\big] &\leq \frac{b_1\gamma}{b_2-\gamma} + 1 \leq \frac{b_1}{4} + 1, \\ \mathrm{E}\big[|X_{n+1}-X_n| e^{\gamma|X_{n+1}-X_n|} \,\big|\, \mathcal{F}_n\big] &\leq \frac{b_1 b_2}{(b_2-\gamma)^2}, \\ \mathrm{E}\big[(X_{n+1}-X_n)^2 e^{\gamma|X_{n+1}-X_n|} \,\big|\, \mathcal{F}_n\big] &\leq \frac{2b_1 b_2}{(b_2-\gamma)^3} \end{aligned} \tag{23}$$

almost surely, for all n. These uniform bounds allow us to differentiate inside the integral, so that

$$\begin{aligned} \frac{d}{d\gamma}\bigg|_{\gamma=0} \mathrm{E}\big[e^{-\gamma(X_{n+1}-X_n)} \,\big|\, \mathcal{F}_n\big] &= -\mathrm{E}[X_{n+1}-X_n \mid \mathcal{F}_n] \\ &\leq -c_1 \mathbf{1}\{X_n \geq \alpha\}; \\ \frac{d^2}{d\gamma^2} \mathrm{E}\big[e^{-\gamma(X_{n+1}-X_n)} \,\big|\, \mathcal{F}_n\big] &= \mathrm{E}\big[(X_{n+1}-X_n)^2 e^{-\gamma(X_{n+1}-X_n)} \,\big|\, \mathcal{F}_n\big] \\ &\leq \frac{2b_1 b_2}{(b_2-\gamma)^3}. \end{aligned}$$

The specified γ satisfies

$$1 - c_1\gamma + \frac{b_1 b_2}{(b_2-\gamma)^3}\gamma^2 \leq 1 - \gamma^2 \leq e^{-\gamma^2},$$

so that by Taylor's theorem, separating the cases $\tau \leq n$ and $\tau > n$,

$$\mathrm{E}\big[\exp\{-\gamma(X_{(n+1)\wedge\tau} - X_{n\wedge\tau})\} \,\big|\, \mathcal{F}_n\big] \leq \exp\{-\gamma^2[(n+1)\wedge\tau - n\wedge\tau]\}.$$

Together with (20), this proves that $e^{-\gamma X_{n\wedge\tau} + \gamma^2 n\wedge\tau}$ is a supermartingale.

Define τ_K to be the first time after T when X_n leaves the interval $[\alpha', K)$. The first part of the lemma implies, via Remark 1.1, that $\exp\{-\gamma X_{T+n} + \gamma^2 n\}$ is a supermartingale. As a positive supermartingale, it converges almost surely to a finite number, which cannot happen if τ_K is infinite. (In that case, X_{T+n} would remain bounded, which would imply that $\exp\{-\gamma X_{T+n} + \gamma^2 n\}$ goes to infinity.)

By the optional stopping theorem for positive supermartingales (Theorem II-2-13 of [9]), on $\{X_T \geq \beta\}$,

$$\begin{aligned} e^{-\gamma\beta} &\geq \mathrm{E}\big[e^{-\gamma X_{\tau_K}} \,\big|\, \mathcal{F}_T\big] \\ &\geq \mathrm{E}\big[e^{-\gamma X_{\tau_K}} \mathbf{1}\{X_{\tau_K} < \alpha'\} \,\big|\, \mathcal{F}_T\big] + \mathrm{E}\big[e^{-\gamma X_{\tau_K}} \mathbf{1}\{X_{\tau_K} > K\} \,\big|\, \mathcal{F}_T\big] \\ &\geq e^{-\gamma\alpha'} \mathrm{P}\big\{X_{\tau_K} < \alpha' \,\big|\, \mathcal{F}_T\big\}. \end{aligned}$$

Thus $\mathrm{P}\{X_{\tau_K} < \alpha' \mid \mathcal{F}_T\} \leq e^{-\gamma(\beta-\alpha')}$ on $\{X_T \geq \beta\}$. Letting $K \to \infty$ and applying the monotone convergence theorem, this implies that

$$\begin{aligned} \mathrm{P}\Big\{\limsup_{n\to\infty} X_n = \infty \text{ and } X_n \geq \alpha' \ \forall n \geq T \,\Big|\, \mathcal{F}_T\Big\} &= \mathrm{P}\big\{X_n \geq \alpha' \ \forall n \geq T \,\big|\, \mathcal{F}_T\big\} \\ &\geq 1 - e^{-\gamma(\beta-\alpha')}. \end{aligned}$$

We need to show that, irrespective of X_T,

$$\mathrm{P}\Big\{\liminf_{n\to\infty} X_n < \infty \,\Big|\, \limsup_{n\to\infty} X_n = \infty\Big\} = 0.$$

The set where $\limsup X_n = \infty > \liminf X_n$ is contained in

$$\bigcup_{x=\lceil\alpha\rceil}^{\infty} \bigcap_{K=1}^{\infty} \{\exists\, n, m \text{ with } n > m \text{ such that } X_m > K + x \text{ and } X_n < x\}.$$

By the above argument, the events in the curly brackets above have probabilities smaller than $e^{-\gamma K}$, which means that the intersection has probability 0, for every x, completing the proof of (21).

Choose any positive $\varepsilon < c_1$ and $\beta > \alpha$, and define $\tau_0 = 0$,

$$\sigma_i := \min\{k \geq \tau_{i-1} : X_k > \beta\}$$

and

$$\tau_i := \min\big\{k \geq \sigma_i : X_k \leq \alpha + (c_1 - \varepsilon)(k - \sigma_i)\big\}.$$

Applying the above argument to the process $X_{\sigma_i+k} - (c_1 - \varepsilon)k$ (now with respect to the filtration $\mathcal{F}_{\sigma_i+k}$),

$$\mathrm{P}\{\tau_i < \infty \mid \sigma_i < \infty\} \leq \exp\big\{(\alpha - \beta)\gamma(\varepsilon, b_1 e^{c_1 b_2}, b_2)\big\}.$$

Thus the probability is 0 that all the stopping times are finite. If σ_i is finite and τ_i infinite, then $\liminf n^{-1} X_n \geq c_1 - \varepsilon$. If τ_i is finite and σ_{i+1} infinite, then $\limsup X_n \leq \beta$. Since this is true for all $\beta > \alpha$ and all positive ε, (22) follows. $\square$

LEMMA 2.7. *Let* $(X_n, \mathcal{F}_n)$ *be an adapted real-valued process, and* σ *a stopping time, such that almost surely* (18) *holds. Suppose there are* $\mathcal{F}_n$*-measurable random variables* ξ_n, *and positive constants* α, c_1 *and* c_2, *such that*

$$\mathrm{E}\big[X_{n+1} - X_n \,|\, \mathcal{F}_n\big] \le -c_1 \mathbf{1}\{X_n \ge \alpha\} + c_2 \mathbf{1}\{X_n < \alpha\} + \xi_n \tag{24}$$

on the event $\{\sigma > n\}$. *For fixed positive* $\varepsilon \le c_1$, *let*

$$\tau^{(\varepsilon)} := \sigma \wedge \min\big\{n \ge 0 : X_n < \alpha \text{ or } \xi_n > c_1 - \varepsilon\big\}.$$

Then for $\gamma \le \gamma(\varepsilon, b_1, b_2)$, $\exp\{\gamma X_{n\wedge\tau^{(\varepsilon)}} + \gamma^2(n \wedge \tau^{(\varepsilon)})\}$ *is a supermartingale, as long as* $\mathrm{E}[e^{\gamma X_0}] < \infty$.

Furthermore, if X_n *is almost surely nonnegative for all* n, *then on the event* $\{\sigma = \infty$ *and* $\limsup_{n\to\infty} \xi_n \le c_1 - \varepsilon\}$,

$$\liminf_{t\to\infty} \frac{1}{t} \#\{k \le t : X_k < \alpha\} > 0 \qquad \textit{almost surely,} \tag{25}$$

and there is a positive constant C, *depending on* b_1, b_2 *and* α, *such that*

$$\limsup_{t\to\infty} \frac{1}{t} \#\{k \le t : X_k \ge R\} \le C e^{-\gamma R/2} \tag{26}$$

for all positive R, *almost surely on the event* $\{\sigma = \infty$ *and* $\limsup_{n\to\infty} \xi_n \le c_1 - \varepsilon\}$.

Let $\gamma = \gamma(c_1, b_1, b_2)$. *For* $c > 1/\gamma$ *and* $N > 0$, *let*

$$\sigma^{(c)}(N) := \sigma \wedge \min\big\{n : X_n \ge c \log(n + N)\big\}.$$

Then

$$\begin{aligned} &\mathrm{P}\big\{\sigma^{(c)}(N) < \infty \,\big|\, \mathcal{F}_0\big\} \\ &\quad \le e^{\gamma X_0} N^{-\gamma c} + e^{\gamma\alpha}\left(\frac{b_1}{4} + 1\right)(\gamma c - 1)^{-1} N^{1-\gamma c} \\ &\qquad + \mathrm{P}\big\{\sigma = \sigma^{(c)}(N) < \infty \;\text{ or }\; \exists n < \sigma^{(c)}(N) < \infty \text{ with } \xi_n > 0 \,\big|\, \mathcal{F}_0\big\}. \end{aligned} \tag{27}$$

PROOF. Proof of the supermartingale property is identical to Lemma 2.6. Suppose now that X is nonnegative. Define $\tau_0 = 0$, and then recursively, for $i \ge 1$,

$$\begin{aligned} \tau_i &:= \sigma \wedge \min\big\{n \ge \tau_{i-1} + 1 : X_n < \alpha \text{ or } \xi_n > c_1 - \varepsilon\big\}, \\ s_i &:= \#\{\tau_{i-1} < k \le \tau_i : X_k \ge R\}. \end{aligned}$$

Observe that by the optional stopping theorem,

$$\mathrm{E}\big[\exp\{\gamma^2(\tau_i - \tau_{i-1} - 1)\} \,\big|\, \mathcal{F}_{\tau_{i-1}+1}\big] \le \exp\{\gamma X_{\tau_{i-1}+1}\}$$

and

$$\mathrm{E}\big[\exp\{\gamma X_{\tau_i}\} \,\big|\, \mathcal{F}_{\tau_{i-1}+1}\big] \le \mathrm{E}\big[\exp\{\gamma X_{(\tau_{i-1}+k)\wedge\tau_i}\} \,\big|\, \mathcal{F}_{\tau_{i-1}+1}\big] \le \exp\{\gamma X_{\tau_{i-1}+1}\}.$$

By assumption (18) and the bound (23),

$$\mathrm{E}\big[\exp\{\gamma X_{\tau_{i-1}+1}\} \,\big|\, \mathcal{F}_{\tau_{i-1}}\big] \le \exp\{\gamma X_{\tau_{i-1}}\}\left(\frac{b_1}{4}+1\right)$$

for $i > 1$. (All of these hold, strictly speaking, only on the event $\{\sigma > \tau_{i-1}\}$.) By induction, it follows that $\mathrm{E}\exp\{\gamma X_{\tau_i}\}$ is finite for all i.

This immediately tells us that

$$(\tau_i - \tau_{i-1})\mathbf{1}\{X_{\tau_{i-1}} < \alpha\}$$

has uniformly bounded moments of all orders, when conditioned on $\mathcal{F}_{\tau_{i-1}}$. The strong law of large numbers for martingales (Theorem 2.19 of [6]) then says that

$$\limsup_{n\to\infty} n^{-1}\sum_{i=1}^{n}(\tau_i - \tau_{i-1})\mathbf{1}\{X_{\tau_{i-1}} < \alpha\} < \infty$$

almost surely. Observe that on the event $\{\sigma = \infty \text{ and } \limsup_{n\to\infty} \xi_n \le c_1 - \varepsilon\}$, the indicator $\mathbf{1}\{X_{\tau_{i-1}} < \alpha\}$ is 0 for only finitely many i. This means that

$$\liminf_{t\to\infty} \frac{1}{t}\#\{k \le t : X_k < \alpha\} \ge \liminf_{n\to\infty} \frac{n}{\tau_{n+1}} > 0,$$

proving (25).

For each $i > 1$, by an application of the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\mathrm{E}\big[s_i \,|\, \mathcal{F}_{\tau_{i-1}+1}\big] \\
&\quad \le \mathrm{E}\left[\sum_{k=1}^{\tau_i-\tau_{i-1}} \mathbf{1}\{X_{\tau_{i-1}+k} \ge R\} \,\middle|\, \mathcal{F}_{\tau_{i-1}+1}\right] \\
&\quad = \sum_{k=1}^{\infty} \mathrm{E}\big[\mathbf{1}\{X_{\tau_{i-1}+k} \ge R\}\mathbf{1}\{\tau_i \ge \tau_{i-1}+k\} \,\big|\, \mathcal{F}_{\tau_{i-1}+1}\big] \\
&\quad \le \sum_{k=1}^{\infty} \mathrm{P}\big\{X_{(\tau_{i-1}+k)\wedge\tau_i} \ge R \,\big|\, \mathcal{F}_{\tau_{i-1}+1}\big\}^{1/2} \mathrm{P}\big\{\tau_i - \tau_{i-1} \ge k \,\big|\, \mathcal{F}_{\tau_{i-1}+1}\big\}^{1/2} \\
&\quad \le \sum_{k=1}^{\infty} \exp\left\{\gamma X_{\tau_{i-1}+1} - \frac{\gamma R}{2} - \frac{\gamma^2(k-1)}{2}\right\} \\
&\quad \le \big(1 - e^{-\gamma^2/2}\big)^{-1} \exp\left\{\gamma X_{\tau_{i-1}+1} - \frac{\gamma R}{2}\right\}
\end{aligned}$$

on the event $\{\sigma > \tau_{i-1}\}$. Taking expectations with respect to $\mathcal{F}_{\tau_{i-1}}$, we get almost surely

$$\mathrm{E}\big[s_i \cdot \mathbf{1}\{X_{\tau_{i-1}} < \alpha\} \cdot \mathbf{1}\{\sigma > \tau_{i-1}\} \,|\, \mathcal{F}_{\tau_{i-1}}\big] \le C e^{-\gamma R/2},$$

where $C = e^{\gamma\alpha}(b_1/4+1)(1-e^{-\gamma^2/2})^{-1}$.

Now define

$$M_n := \sum_{i=2}^{n} \bigl(s_i \cdot \mathbf{1}\{X_{\tau_{i-1}} < \alpha\} \cdot \mathbf{1}\{\sigma > \tau_{i-1}\} - Ce^{-\gamma R/2} \bigr).$$

This is a supermartingale with respect to the filtration $(\mathcal{F}_{\tau_n})$. The increments are bounded by a constant plus $(\tau_i - \tau_{i-1})\mathbf{1}\{X_{\tau_{i-1}} < \alpha\}$, which has bounded second moments (almost surely, when conditioned on $\mathcal{F}_{\tau_{i-1}}$). This allows us to apply the strong law of large numbers for martingales to obtain $\limsup_{n\to\infty} n^{-1} M_n \le 0$ almost surely.

As before, on the event $\{\sigma = \infty \text{ and } \limsup_{n\to\infty} \xi_n \le c_1 - \varepsilon\}$, the product $\mathbf{1}\{\sigma > \tau_{i-1}\} \cdot \mathbf{1}\{X_{\tau_{i-1}} < \alpha\}$ is 0 for only finitely many i. As a consequence, since $\tau_n \ge n$, we have almost surely

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t} \#\{k \le t : X_k \ge R\} &\le \limsup_{n\to\infty} \frac{1}{n-1} \#\{k \le \tau_n : X_k \ge R\} \\
&= \limsup_{n\to\infty} \frac{1}{n-1} \sum_{i=2}^{n} s_i \\
&= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=2}^{n} s_i \cdot \mathbf{1}\{X_{\tau_{i-1}} < \alpha\} \\
&\le Ce^{-\gamma R/2},
\end{aligned}$$

completing the proof of (26).

To prove the final part, we define a slightly modified sequence of stopping times $\tau_0 = -1$,

$$\tau_i := \sigma \wedge \min\bigl\{n \ge \tau_{i-1} + 1 : X_n < \alpha \text{ or } \xi_n > 0 \text{ or } X_n \ge c \log(n+N)\bigr\}.$$

We are trying to bound the probability that for some $i \ge 1$, $X_{\tau_i} \ge \alpha$. Since $\exp\{\gamma X_{(\tau_{i-1}+n)\wedge \tau_i}\}$ is a positive supermartingale, by the optional stopping theorem,

$$\begin{aligned}
\mathrm{E}\bigl[\exp\{\gamma X_{\tau_i}\} \bigm| \mathcal{F}_{\tau_{i-1}}\bigr] &\le \mathrm{E}\bigl[\exp\{\gamma X_{\tau_{i-1}+1}\} \bigm| \mathcal{F}_{\tau_{i-1}}\bigr] \\
&\le \left(\frac{b_1}{4} + 1\right) \exp\{\gamma X_{\tau_{i-1}}\},
\end{aligned}$$

which is bounded by $(b_1/4 + 1)e^{\gamma\alpha}$ on the event $\{X_{\tau_{i-1}} < \alpha\}$. Since $\tau_i \ge i - 1$ on $\{\sigma > \tau_{i-1}\}$,

$$\begin{aligned}
&\mathrm{P}\bigl\{X_{\tau_i} \ge c \log(N + \tau_i) \bigm| \mathcal{F}_{\tau_{i-1}}\bigr\} \\
&\qquad \le \left(\frac{b_1}{4} + 1\right) e^{\gamma\alpha} (N + i - 1)^{-c\gamma} + \mathbf{1}\{\sigma \le \tau_{i-1}\} + \mathbf{1}\bigl\{X_{\tau_{i-1}} \ge \alpha\bigr\}.
\end{aligned} \tag{28}$$

Thus,

$$
\begin{aligned}
&\mathrm{P}\bigl\{\exists\, i \geq 2 \text{ such that } X_{\tau_i} \geq c\log(N+\tau_i) > \alpha > X_{\tau_{i-1}} \text{ and } \sigma > \tau_{i-1} \bigm| \mathcal{F}_{\tau_1}\bigr\}\\
&\qquad \leq \left(\frac{b_1}{4}+1\right)e^{\gamma\alpha}\sum_{i=1}^{\infty}(N+i)^{-\gamma c}\\
&\qquad \leq \left(\frac{b_1}{4}+1\right)e^{\gamma\alpha}\bigl(N^{\gamma c-1}(\gamma c-1)\bigr)^{-1}
\end{aligned}
$$

and

$$\mathrm{P}\bigl\{X_{\tau_1} \geq c\log(N+\tau_1) \bigm| \mathcal{F}_0\bigr\} \leq e^{\gamma X_0}N^{-\gamma c}.$$

Since the event $\{\sigma < \infty \text{ or } X_k \geq c\log(k+N) \text{ for some } k\}$ is contained in the union of the events,

$$
\begin{gathered}
\bigl\{X_{\tau_1} \geq c\log(N+\tau_1)\bigr\},\\
\bigl\{\exists\, i \geq 2 \text{ such that } X_{\tau_i} \geq c\log(N+\tau_i) > \alpha > X_{\tau_{i-1}} \text{ and } \sigma > \tau_{i-1}\bigr\}
\end{gathered}
$$

and

$$
\begin{gathered}
\bigl\{\sigma = \sigma^{(c)}(N) < \infty\bigr\},\\
\bigl\{\exists\, i \text{ such that } \xi_{\tau_i} > 0,\ X_k < c\log(N+k)\ \forall\, k \leq \tau_i \text{ and } X_{\tau_{i+1}} \geq c\log(N+\tau_{i+1})\bigr\}.
\end{gathered}
$$

This completes the proof. □

3. Images of nontrivial sets.

3. Images of nontrivial sets. For ease of presentation we separate here the theorems on ball-chasing behavior of whole sets, from the stronger results on pointwise ball-chasing, which appear in Section 4. We warm up with a result that guarantees the appropriateness of our definition of "contracting to a point."

PROPOSITION 3.1. *Under the standard conditions, the diameter of $\phi_t(\mathcal{X})$ has a limit almost surely, for any bounded, connected $\mathcal{X} \subset \mathbb{R}^d$. This limit is almost surely either 0 or ∞. Furthermore, there is a positive flow-bound constant c_1 such that*

$$\mathrm{P}\left\{\liminf_{t\to\infty} t^{-1}\operatorname{diam}\phi_t(\mathcal{X}) \geq c_1 \Bigm| \limsup_{t\to\infty}\operatorname{diam}\phi_t(\mathcal{X}) > 0\right\} = 1. \tag{29}$$

In addition, for any positive α there are flow-bound constants $q(\alpha)$ and $\gamma(\alpha)$ such that

$$\mathrm{P}\bigl\{\exists\, t \geq 0 \textit{ such that } \operatorname{diam}\phi_t(\mathcal{X}) \leq \alpha \mid \mathcal{F}_0\bigr\} \leq q(\alpha)e^{-\gamma(\alpha)\operatorname{diam}\mathcal{X}}. \tag{30}$$

Furthermore, there are flow-bound constants γ_1 and γ_2, such that for all positive α',

$$\begin{aligned} &\mathrm{P}\{\operatorname{diam}\phi_n(\mathcal{X}) \geq \alpha' \ \forall n \in \mathbb{N}\} \\ &\qquad \geq 1 - \inf_{\{\alpha \leq \operatorname{diam}\mathcal{X} \wedge 2\rho^* \wedge \alpha'\}} \exp\left\{-\left[\left(\gamma_1 G''\left(1, \frac{\alpha}{2}\right)\right) \wedge \gamma_2\right](\operatorname{diam}\,\mathcal{X} - \alpha')\right\}. \end{aligned} \tag{31}$$

In particular, the probability that a nontrivial connected set $\mathcal{X}$ contracts to a point under the action of the flow is always smaller than 1. *If the set does not contract, it grows linearly in time.*

PROOF. We know from Lemma 2.5 that the diameter has a positive drift, whose magnitude is bounded away from zero when the diameter itself is bounded away from zero. This means that every time the diameter passes a given threshold, whatever this may be, it has a certain fixed probability of running off to infinity without ever coming back. Conversely, if the diameter fails to run off to infinity, it can only be because the drift converged to zero, which must mean that the diameter itself converged to zero.

If $\mathcal{X}$ is unbounded, then $\phi_t(\mathcal{X})$ is unbounded as well, for all t, so the proposition becomes trivial. Since a set has the same diameter as its closure, and since ϕ_t is continuous, we may assume without loss of generality that $\mathcal{X}$ is compact.

We begin by proving the statements like those of the proposition when the times n are integers, in place of the continuous t. Fix any positive $\alpha < \operatorname{diam}\mathcal{X} \wedge 2\rho^*$. Let x^0 and $\tilde{x}^0$ be two points in $\mathcal{X}$ such that $\|x^0 - \tilde{x}^0\| = \operatorname{diam}\mathcal{X}$. Let $v = (x^0 - \tilde{x}^0)/\|x^0 - \tilde{x}^0\|$. Since $\mathcal{X}$ is connected, we may find points y^0 and $\tilde{y}^0$ in $\mathcal{X}$ such that

$$\|x^0 - y^0\| = \|\tilde{x}^0 - \tilde{y}^0\| = \frac{\alpha}{2}.$$

Starting from x^0 and y^0, we now define a *retraction sequence* tending in the direction v. This is a sequence of $\mathcal{X}$-valued random variables x^n and y^n determined by the flow at time n, where at most one of the points may be replaced at each integer time. Whichever one of x^{n-1} and y^{n-1} is farthest forward (under the action of ϕ_n) in the v direction is carried over to time n, while the other is retracted so that $\|\phi_n(x^n) - \phi_n(y^n)\| = \alpha/2$. To consider one case, suppose that $\operatorname{diam}\phi_n(\mathcal{X}) \geq \alpha$ and

$$\langle\phi_n(x^{n-1}), v\rangle \geq \langle\phi_n(y^{n-1}), v\rangle.$$

We define x^n to be equal to x^{n-1}, and let y^n be a point in $\mathcal{X}$ such that $\|\phi_n(x^n) - \phi_n(y^n)\|$ is exactly $\alpha/2$. Because the diameter of $\phi_n(\mathcal{X})$ is at least α, such a point must exist. When $\phi_n(y^{n-1})$ is farther forward in the v direction, we take $x^n = y^{n-1}$, and again let y^n be defined by retraction. If diam $\phi_n(\mathcal{X}) < \alpha$, we stipulate that $x^n = x^{n-1}$ and $y^n = y^{n-1}$. The construction can now be advanced to the next unit of time.

An identical procedure generates a retraction sequence $(\tilde{x}^n, \tilde{y}^n)$, starting from $(\tilde{x}^0, \tilde{y}^0)$ and tending in the direction of $-v$. It may come to pass that the two retraction sequences find themselves in the wrong order; that is, for some n,

$$\langle \phi_n(x^n), v\rangle < \langle \phi_n(\tilde{x}^n), v\rangle.$$

In this case we simply exchange the two sequences: what was x^n becomes $\tilde{x}^n$ and vice versa; what was y^n becomes $\tilde{y}^n$ and vice versa. We will refer to this as the *swapping rule*.

Define $X_n^+ := \langle \phi_n(x^n), v\rangle$ and $X_n^- := \langle \phi_n(\tilde{x}^n), v\rangle$. Suppose the swapping rule is not applied at time $n+1$, and that $\langle \phi_n(x^n), v\rangle \geq \langle \phi_n(y^n), v\rangle$. The difference $X_{n+1}^+ - X_n^+$ is simply the maximum advance beyond $\phi_n(x^n)$ in direction v of the two points $\phi_n(x^n)$ and $\phi_n(y^n)$, during the time interval $(n, n+1]$. When the diameter of $\phi_n(\mathcal{X})$ is at least α, the separation of these two points at time n is $\alpha/2$. From Lemma 2.5 (with $s = 1$) it follows that

$$\mathrm{E}\big[X_{n+1}^+ - X_n^+ \,\big|\, \mathcal{F}_n\big] \geq G''(1, \alpha/2)\mathbf{1}\{\operatorname{diam}\phi_n(\mathcal{X}) \geq \alpha\}. \tag{32}$$

If y^n is the leader in direction v at time n (which it can be, by as much as $\alpha/2$), this will only increase the expectation in (32). Likewise the swapping rule can only increase the difference, switching it from negative to positive.

Let $X_n := X_n^+ - X_n^-$. Observe that X_n is a lower bound for the diameter of $\phi_n(\mathcal{X})$. Applying the same reasoning as above to X_n^-, we see that

$$\begin{aligned} \mathrm{E}\big[X_{n+1} - X_n \mid \mathcal{F}_n\big] &\geq 2G''(1, \alpha/2)\mathbf{1}\{\operatorname{diam}\phi_n(\mathcal{X}) \geq \alpha\} \\ &\geq 2G''(1, \alpha/2)\mathbf{1}\{X_n \geq \alpha\}. \end{aligned} \tag{33}$$

Thus X_n satisfies condition (17) of Lemma 2.6, with $c_1(\alpha) = 2G''(1, \alpha/2)$. On the other hand, for any positive λ, when $X_{n+1} - X_n \geq \lambda$ it must be that one of the four one-point displacements,

$$\begin{aligned} &\|\phi_{n+1}(x^n) - \phi_n(x^n)\|, \quad \|\phi_{n+1}(y^n) - \phi_n(y^n)\|, \\ &\|\phi_{n+1}(\tilde{x}^n) - \phi_n(\tilde{x}^n)\|, \quad \|\phi_{n+1}(\tilde{y}^n) - \phi_n(\tilde{y}^n)\|, \end{aligned}$$

is at least $(\lambda - \alpha)/2$. By Lemma 2.1,

$$\mathrm{P}\big\{X_{n+1} - X_n \geq \lambda \mid \mathcal{F}_n\big\} \leq \frac{16d}{\sqrt{\pi}} e^{-(\lambda-\alpha)^2/(8d^2A^2)}.$$

The same bound holds for $\mathrm{P}\{X_{n+1} - X_n < -\lambda\}$. This means that X_n satisfies condition (18), with $b_1 = 32de^{(4\rho^*+1)/(8d^2A^2)}/\sqrt{\pi}$ and $b_2 = (8d^2A^2)^{-1}$.

We may apply (21) with $c_1 = c_1(\alpha)$ for any $\alpha < \operatorname{diam}\mathcal{X} \wedge 2\rho^* \wedge \alpha'$. Taking $\gamma = \gamma(c_1, b_1, b_2)$, and using the fact that $\operatorname{diam}\phi_n(\mathcal{X}) \geq X_n$, we get for any positive α',

$$\begin{aligned} \mathrm{P}\big\{\operatorname{diam}\phi_n(\mathcal{X}) \geq \alpha' \ \forall n \in \mathbb{N}\big\} &\geq \mathrm{P}\big\{X_n \geq \alpha' \ \forall n \in \mathbb{N}\big\} \\ &\geq 1 - e^{-\gamma(\operatorname{diam}\mathcal{X} - \alpha')}. \end{aligned}$$

The bound (31), with $\gamma_1 = 2b_2^2/(2b_1 + b_2^2)$ and $\gamma_2 = b_2/5$, follows then directly from the definition of γ, given in (19).

Lemma 2.6 now says that either $0 \le \limsup_{n\to\infty} X_n \le \alpha$ or $\liminf_{n\to\infty} n^{-1}X_n \ge c_1(\alpha)$, with probability 1. It follows that

$$\mathrm{P}\Big\{\liminf_{n\to\infty} n^{-1}X_n \ge c_1(\alpha) \text{ or } \lim_{n\to\infty} X_n = 0\Big\} = 1. \tag{34}$$

for every α smaller than $2\rho^* \wedge \operatorname{diam} \mathcal{X}$.

This rate $c_1(\alpha)$ is not a flow-bound constant, since the restriction on α depends on the initial diameter. But in fact, we can replace $c_1(\alpha)$ by the flow-bound constant $c_1 := c_1(2\rho^*)$: we simply wait for the diameter to grow larger than $2\rho^*$, and restart the process from those initial data. That is, let $T = \inf\{t : \operatorname{diam} \phi_t(\mathcal{X}) \ge 2\rho^*\}$, and choose any positive integer m. Then

$$\begin{aligned}
&\mathrm{P}\Big\{\liminf_{n\to\infty} n^{-1}X_n \ge c_1 - \frac{1}{m} \text{ or } 0 \le \limsup_{n\to\infty} X_n \le \alpha\Big\}\\
&\quad = \mathrm{P}\Big\{\liminf_{n\to\infty} n^{-1}X_n \ge c_1 - \frac{1}{m} \text{ or } 0 \le \limsup_{n\to\infty} X_n \le \alpha \,\Big|\, T = \infty\Big\} \mathrm{P}\{T = \infty\}\\
&\qquad + \mathrm{P}\Big\{\liminf_{n\to\infty} n^{-1}X_n \ge c_1 - \frac{1}{m} \text{ or } 0 \le \limsup_{n\to\infty} X_n \le \alpha \,\Big|\, T < \infty\Big\} \mathrm{P}\{T < \infty\}.
\end{aligned}$$

The first conditional probability is 1 since (34) implies that the diameter converges to 0 whenever it is bounded, with probability 1. The second probability is seen to be 1 by applying the above result to the flow $\phi_{T+\cdot}$, with starting set $\phi_T(\mathcal{X})$. Taking the intersection over all positive integers m, we see that

$$\mathrm{P}\Big\{\liminf_{n\to\infty} n^{-1}X_n \ge c_1 \text{ or } \lim_{n\to\infty} X_n = 0\Big\} = 1. \tag{35}$$

We need to turn these statements about X_n into statements about the diameter of $\phi_n(\mathcal{X})$, showing that the asymptotic fate of $\operatorname{diam} \phi_n(\mathcal{X})$, whether growing linearly with a certain minimum speed or converging to 0, is the same as that of X_n. On one side this is straightforward, since $\operatorname{diam} \phi_n(\mathcal{X}) \ge X_n$. If X_n grows as fast as $c_1 n$, then $\operatorname{diam} \phi_n(\mathcal{X})$ grows at least as fast. But we need to show that $\operatorname{diam} \phi_n(\mathcal{X})$ cannot stay large while X_n converges to 0.

Fix a positive number λ, and let $p_n := \mathrm{P}\{X_{n+1} - X_n > \lambda \mid \mathcal{F}_n\}$. The exponential bound (18) implies that

$$\mathrm{E}\big[X_{n+1} - X_n \mid \mathcal{F}_n\big] \le \lambda(1 - p_n) + \frac{p_n}{b_2}\bigg(1 + \log \frac{b_1}{p_n}\bigg).$$

The right side converges to λ as $p_n \to 0$, which means that there is a positive flow-bound constant $p(\lambda)$ such that

$$p_n \ge p(\lambda)\mathbf{1}\big\{\mathrm{E}\big[X_{n+1} - X_n \mid \mathcal{F}_n\big] \ge \lambda\big\}.$$

We apply this to the bound (33), to reveal that for positive α sufficiently small, taking $\lambda = c_1(\alpha)$,

$$\mathrm{P}\big\{X_{n+1} - X_n > \lambda \mid \mathcal{F}_n\big\} \ge p(\lambda)\mathbf{1}\{\mathrm{diam}\,\phi_n(\mathcal{X}) \ge \alpha\}.$$

We can now apply Neveu's martingale generalization of the Borel–Cantelli lemma ([9], Corollary VII.2.6). On the event $\{\limsup_{n\to\infty} \mathrm{diam}\ \phi_n(\mathcal{X}) \ge \alpha\}$, the increment $X_{i+1} - X_i$ is almost surely bigger than λ infinitely often, so that X_n does not converge. We may conclude that, up to events of measure 0, diam $\phi_n(\mathcal{X}) \to 0$ whenever $X_n \to 0$. The discrete-time version of (29), that is, where t is taken to run over natural numbers, is then a trivial consequence of (35).

We still need to interpolate for noninteger times. We begin with (30). Let α be given, and let $\tau = \inf\{t : \mathrm{diam}\ \phi_t(\mathcal{X}) < \alpha\}$. Then Lemma 2.3 gives us a flow-bound constant K' such that, on the event $\{\tau < \infty\}$, letting $n = \lceil \tau \rceil$,

$$\mathrm{P}\big\{\mathrm{diam}\ \phi_n(\mathcal{X}) \ge K'\alpha \,\big|\, \mathcal{F}_\tau\big\} \le \tfrac{1}{2}.$$

This means that

$$\mathrm{P}\big\{\exists\, n \in \mathbb{N} \text{ such that } \mathrm{diam}\ \phi_n(\mathcal{X}) \le K'\alpha\big\} \ge \tfrac{1}{2}\mathrm{P}\{\tau < \infty\}.$$

By (31) there is a positive flow-bound constant $\gamma(\alpha)$ such that

$$\begin{aligned}\mathrm{P}\big\{\exists\, t \text{ such that } \mathrm{diam}\ \phi_t(\mathcal{X}) < \alpha\big\} &= \mathrm{P}\{\tau < \infty\}\\ &\le 2\exp\big\{-\gamma(\alpha)(\mathrm{diam}\,\mathcal{X} - K'\alpha)\big\},\end{aligned}$$

which proves (30), taking $q(\alpha) = 2e^{K'\alpha\gamma(\alpha)}$.

We have shown that for discrete times there are, almost surely, only two possible behaviors for $\phi_n(\mathcal{X})$: either $\lim_{n\to\infty} \mathrm{diam}\ \phi_n(\mathcal{X}) = 0$ or $\liminf_{n\to\infty} n^{-1} \times \mathrm{diam}\ \phi_n(\mathcal{X}) \ge c_1$. We will be finished once we have shown that

$$\mathrm{P}\Big\{\liminf_{t\to\infty} t^{-1}\,\mathrm{diam}\ \phi_t(\mathcal{X}) \ge c_1 \,\Big|\, \liminf_{n\to\infty} n^{-1}\,\mathrm{diam}\ \phi_n(\mathcal{X}) \ge c_1\Big\} = 1$$

and

$$\mathrm{P}\Big\{\lim_{t\to\infty} \mathrm{diam}\ \phi_t(\mathcal{X}) = 0 \,\Big|\, \lim_{n\to\infty} \mathrm{diam}\ \phi_n(\mathcal{X}) = 0\Big\} = 1.$$

The first of these is particularly easy. The event that $\liminf_{n\to\infty} n^{-1} \times$ diam $\phi_n(\mathcal{X}) \ge c_1$ and $\liminf_{t\to\infty} t^{-1}\,\mathrm{diam}\ \phi_t(\mathcal{X}) < c_1$ is contained in

$$\bigcup_{K=1}^{\infty} \limsup_{n\to\infty} \Big\{\inf_{n\le t<n+1} \mathrm{diam}\ \phi_t(\mathcal{X}) \le \mathrm{diam}\ \phi_n(\mathcal{X}) - \frac{n}{K}\Big\}.$$

Consider two points x and y (these are random variables measurable with respect to $\mathcal{F}_n$) in $\mathcal{X}$ such that $\|\phi_n(x) - \phi_n(y)\| = \mathrm{diam}\ \phi_n(\mathcal{X})$. We have

$$\begin{aligned}&\Big\{\inf_{n\le t<n+1} \mathrm{diam}\,\phi_t(\mathcal{X}) \le \mathrm{diam}\ \phi_n(\mathcal{X}) - \frac{n}{K}\Big\}\\ &\qquad \subset \Big\{\sup_{n\le t<n+1} \|\phi_t(x) - \phi_n(x)\| \ge \frac{n}{2K}\Big\} \cup \Big\{\sup_{n\le t<n+1} \|\phi_t(y) - \phi_n(y)\| \ge \frac{n}{2K}\Big\}.\end{aligned}$$

By Lemma 2.1, each of the events on the right-hand side has probability smaller than $\sqrt{4d}\sqrt{\pi}\exp(-n^2/8d^2A^2K^2)$. The sum of these bounds is finite, which implies, via the Borel–Cantelli lemma, that the probability of infinitely many of these events occurring is probability 0. It follows that $\lim_{n\to\infty} \operatorname{diam} \phi_n(\mathcal{X}) = 0$ or $\liminf_{t\to\infty} t^{-1} \operatorname{diam} \phi_t(\mathcal{X}) \geq c_1$, almost surely.

Let α be any positive number, and define a sequence of stopping times $0 = \tau_0 \leq \sigma_1 < \tau_1 < \sigma_2 \leq \cdots$ by

$$\sigma_{i+1} := \inf\{t \geq \tau_i \text{ such that } \operatorname{diam} \phi_t(\mathcal{X}) \geq \alpha\}$$

and

$$\tau_i := \sigma_i + \min\left\{n \geq 1 \text{ such that } \operatorname{diam} \phi_{\sigma_i+n}(\mathcal{X}) \leq \frac{\alpha}{2}\right\}.$$

Applying our discrete-time results to the flow ϕ_{σ_i+t}, we see that there is a positive p such that

$$\mathrm{P}\left\{\tau_i = \infty \text{ and } \lim_{n\to\infty} \operatorname{diam} \phi_{\sigma_i+n}(\mathcal{X}) = \infty \,\middle|\, \mathcal{F}_{\sigma_i}\right\} \geq p$$

almost surely on the event $\{\sigma_i < \infty\}$, for all i. This means that these stopping times are almost surely not all finite. If the first stopping time to be infinite is one of the τ_i's, then $\liminf_{n\to\infty} \operatorname{diam} \phi_{\sigma_i+n}(\mathcal{X}) \geq \alpha/2$, which implies, by the remarks of the preceding paragraph, that $\liminf_{t\to\infty} t^{-1} \operatorname{diam} \phi_{\sigma_i+t}(\mathcal{X}) \geq c_1$. On the other hand, if the first stopping time to be infinite is one of the σ_i's, then $\limsup_{t\to\infty} \operatorname{diam} \phi_t(\mathcal{X}) \leq \alpha$. Consequently,

$$\mathrm{P}\left\{\infty > \limsup_{t\to\infty} \operatorname{diam} \phi_t(\mathcal{X}) > 0\right\} = 0. \tag{36}$$

Define another stopping time

$$\rho_K := \inf\{t \text{ such that } \operatorname{diam} \phi_t(\mathcal{X}) \geq K\}.$$

Applying (30) to ϕ_{ρ_K+t} tells us that

$$\begin{aligned}
&\mathrm{P}\left\{\limsup_{t\to\infty} \operatorname{diam} \phi_t(\mathcal{X}) = \infty \text{ and } \liminf_{n\to\infty} \operatorname{diam} \phi_n(\mathcal{X}) = 0\right\} \\
&\quad \leq \mathrm{P}\{\rho_K < \infty \text{ and } \exists\, t \geq \rho_K \text{ such that } \operatorname{diam} \phi_t(\mathcal{X}) \leq 1 \mid \mathcal{F}_{\rho_K}\} \\
&\quad \leq \mathrm{E}\Big[\mathbf{1}_{\{\rho_K<\infty\}} \mathrm{P}\{\rho_K < \infty \text{ and } \exists\, t \geq \rho_K \text{ such that } \operatorname{diam} \phi_t(\mathcal{X}) \leq 1 \mid \mathcal{F}_{\rho_K}\} \mid \\
&\qquad\qquad \rho_K < \infty\Big] \\
&\quad \leq q(1)e^{-\gamma(1)K}.
\end{aligned}$$

Since this is true for all K, it must be that this probability is 0. Combining this with (36) yields

$$\mathrm{P}\Big\{\limsup_{t\to\infty} \operatorname{diam} \phi_t(\mathcal{X}) > 0 \text{ and } \liminf_{n\to\infty} \operatorname{diam} \phi_n(\mathcal{X}) = 0\Big\} = 0,$$

completing the proof. $\square$

THEOREM 3.2. *Let M be a martingale field satisfying the standard conditions, and $\phi_t : \mathbb{R}^d \to \mathbb{R}^d$ the flow of homeomorphisms that it determines; let $\mathcal{X}$ be a compact nontrivial subset of $\mathbb{R}^d$ and let ψ be an adapted path with Lipschitz constant almost surely smaller than Λ, which is itself smaller than*

$$\sup_{s>0} \max_{0\le\rho\le\rho^*} \frac{G''(s,\rho)}{s}. \tag{37}$$

Then $\mathcal{X}$ almost surely either chases balls with path ψ and every radius $\varepsilon(t) \equiv \varepsilon$ a positive constant, or shrinks to a point. That is, taking m to represent Lebesgue measure on $\mathbb{R}$,

$$\mathrm{P}\Big\{\liminf_{t\to\infty} \frac{1}{t} m\{s \le t : \phi_s(\mathcal{X}) \cap B_\varepsilon(\psi(s)) \ne \varnothing\} > 0 \,\Big|\, \lim_{t\to\infty} \operatorname{diam} \phi_t(\mathcal{X}) = \infty\Big\} = 1.$$

Furthermore, if $\mathcal{X}$ does not contract to a point, the fraction of time that it spends in a ball around $\psi(t)$ converges to 1 as the radius goes to infinity. That is,

$$\mathrm{P}\Big\{\lim_{r\to\infty} \liminf_{t\to\infty} \frac{1}{t} m\{s \le t \text{ such that } \phi_s(\mathcal{X}) \cap B_r(\psi(s)) \ne \varnothing\} = 1 \,\Big|\, \lim_{t\to\infty} \operatorname{diam} \phi_t(\mathcal{X}) = \infty\Big\} = 1.$$

PROOF. We assume without loss of generality that the starting point of the path $\psi(0)$ is a bounded random variable. If it is not, we may replace ψ by a new path whose starting point is 0, and which approaches the old ψ at a linear rate and finally merges with it after a finite random time. Since the two paths eventually coincide, the asymptotic results for the two are equivalent.

Let s and ρ be chosen with $0 < \rho < \rho^*$, so that $G''(s,\rho) > s\Lambda$, and let r^* be chosen so that

$$G'(s, r^*) \le \tfrac{1}{2}\big(G''(s,\rho) - s\Lambda\big).$$

Let $\psi_n = \psi(sn)$.

As in the proof of Proposition 3.1 we define a retraction sequence of points (x^n, y^n), but in this case the points are not running in a fixed direction but rather running after the path ψ. Choose x^0 and y^0, any two distinct points in $\mathcal{X}$. We define random points x^n and y^n in $\mathcal{X}$, measurable with respect to $\mathcal{F}_{sn}$, as follows. Suppose x^{n-1} and y^{n-1} have been determined. One of x^{n-1} or y^{n-1}

will have its image under ϕ_{sn} closer to ψ_n. We will let x^n be that point, and take for y^n any point in $\mathcal{X}$ such that $\|\phi_{sn}(x^n) - \phi_{sn}(y^n)\| = \rho$, which must exist if $\operatorname{diam}\phi_{sn}(\mathcal{X}) \geq 2\rho$; if the diameter is smaller, and y^n cannot be chosen to make this distance ρ, it is chosen instead to make the distance as large as possible.

Now define $r_n := \|\phi_{sn}(x^n) - \psi_n\|$. Apply Lemma 2.5 to the points $\phi_{sn}(x^n)$ and $\phi_{sn}(y^n)$, with $z = \psi_n$, to see that

$$\begin{aligned}
\mathrm{E}\big[r_{n+1} \mid \mathcal{F}_{sn}\big] &\leq r_n + s\Lambda + G'(s, r_n) - G''\big(s, \|\phi_{sn}(x^n) - \phi_{sn}(y^n)\|\big) \\
&\leq r_n + s\Lambda + G'(s, r^*) - G''(s, \rho) \\
&\quad + \big(G'(s, 0) - G'(s, r^*)\big)\mathbf{1}\{r_n < r^*\} \\
&\quad + G''(s, \rho)\mathbf{1}\{\operatorname{diam}\phi_{sn}(\mathcal{X}) < 2\rho\}.
\end{aligned}$$

Here we may apply Lemma 2.7 to r_n, with

$$\alpha = r^*,$$

$$\begin{aligned}
c_1 &= -s\Lambda - G'(s, r^*) + G''(s, \rho), \\
c_2 &= s\Lambda + G'(s, 0) - G''(s, \rho)
\end{aligned}$$

and

$$\xi_n = G''(s, \rho)\mathbf{1}\{\operatorname{diam}\phi_{sn}(\mathcal{X}) < 2\rho\}.$$

The exponential tail bounds are trivial consequences of Lemma 2.1. This tells us that, on the event $\operatorname{diam}\phi_{sn}(\mathcal{X}) \to \infty$, since ξ_n is nonzero only finitely often,

$$\liminf_{n\to\infty} \frac{1}{n}\#\{i \leq n : r_i < r\} > 0$$

for all $r > r^*$, and that this asymptotic density goes to 1 as $r \to \infty$.

We have not yet shown that this positive density holds for times which are not integer multiples of s. For $r > r^*$, define $\mathcal{A}_n'$ to be the event that $\phi_{sn}(\mathcal{X}) \cap B_r(\psi_n)$ is nonempty, and $\mathcal{A}_n''$ the event that $\phi_t(\mathcal{X}) \cap B_{2r}(\psi(t))$ is nonempty for all t between sn and $s(n+1)$. By tracking one point from the intersection, we see that

$$\mathrm{P}(\mathcal{A}_n'' \mid \mathcal{F}_{sn}) \geq (1 - b_1 e^{-b_2 r})\mathbf{1}_{\mathcal{A}_n'},$$

where b_1 and b_2 are constants depending only on the flow parameters, as given by Lemma 2.1. Applying the law of large numbers for martingales, we see that

$$\begin{aligned}
&\liminf_{t\to\infty} t^{-1} m\big\{s \leq t : \phi_s(\mathcal{X}) \cap B_{2r}(\psi(s)) \neq \varnothing\big\} \\
&\quad \geq \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\mathcal{A}_i''}
\end{aligned}$$

$$\geq (1 - b_1 e^{-b_2 r}) \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\mathcal{A}'_i}$$

$$\geq (1 - b_1 e^{-b_2 r}) \liminf_{n\to\infty} \frac{1}{n} \#\{i \leq n : r_i < r\}.$$

We have shown that the theorem holds then for $\varepsilon \geq 2r^*$. To extend it to all positive ε requires an argument much like the preceding one. Let $\varepsilon^* = \min\{s/2, \varepsilon/(4\Lambda)\}$. Keep $\mathcal{A}'_n$ as above, with $r = 2r^*$, but now define $\mathcal{A}^\varepsilon_n$ to be the event that $\phi_t(\mathcal{X}) \cap B_\varepsilon(\psi(t))$ is nonempty for all $t \in [sn + \varepsilon^*, sn + 2\varepsilon^*]$. On the event $\mathcal{A}'_n$, there is a point $x \in \mathcal{X}$ such that $\|\phi_{sn}(x) - \psi_n\| \leq 2r^*$. Let

$$V_t = \begin{cases} -\phi_{sn}(x) + \dfrac{-\psi_n + \phi_{sn}(x)}{\varepsilon^*} t, & \text{if } 0 \leq t \leq \varepsilon^*, \\ -\psi_n, & \text{if } t > \varepsilon^*. \end{cases}$$

Then $\mathcal{A}^\varepsilon_n$ contains the event that $\|V_t + \phi_{sn+t}(x)\| \leq \varepsilon/2$ for all $0 \leq t \leq 2\varepsilon^*$. By Lemma 2.4 there is a positive p_ε such that

$$\mathrm{P}(\mathcal{A}^\varepsilon_n \mid \mathcal{F}_{sn}) \geq p_\varepsilon \mathbf{1}_{\mathcal{A}'_n}.$$

The law of large numbers for martingales then implies that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\mathcal{A}^\varepsilon_i} \geq p_\varepsilon \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\mathcal{A}'_n} > 0$$

almost surely. This is true simultaneously for all positive ε. □

4. Images of points. In the previous section we showed that for a positive fraction of the time points almost surely do exist in $\phi_t(\mathcal{X})$ whose distance from $\psi(t)$ is no more than any fixed constant, except in the case when $\phi_t(\mathcal{X})$ shrinks to a point. This does not exclude the possibility that it is always a different point, so that for each fixed x, the distance of $\phi_t(x)$ from $\psi(t)$ would eventually exceed any given constant. We do not know whether there are, in general, individual points which spend a positive fraction of the time in balls of constant radius around $\psi(t)$. We can show, however, that "weak ball-chasing" does occur pointwise; that is, given any fixed positive ε, and a nontrivial set $\mathcal{X}$, there are almost surely points in $\mathcal{X}$ which return infinitely often to a ball of radius ε around $\psi(t)$, as long as $\mathcal{X}$ does not shrink to a point. We also show, in Theorem 4.2, that "strong ball-chasing" occurs pointwise for balls of size constant times $\log t$; that is, conditioned on $\mathcal{X}$ not shrinking to a point, there are almost surely points in $\mathcal{X}$ whose image under ϕ_t eventually occupies a ball of radius $c \log t$ about $\psi(t)$.

THEOREM 4.1. *Let M be a martingale field satisfying the standard conditions, and $\phi_t : \mathbb{R}^d \to \mathbb{R}^d$ the flow of homeomorphisms that it determines, and $\mathcal{X}$*

a compact nontrivial set. Let Λ *and* ψ *satisfy the same conditions as in Theorem* 3.2. *Then if* ε *is any positive constant*,

$$\mathrm{P}\Big\{\exists x \in \mathcal{X} \text{ such that } \{t : \|\phi_t(x) - \psi(t)\| \le \varepsilon\} \text{ is unbounded} \,\Big|\, \lim_{t\to\infty} \operatorname{diam} \phi_t(\mathcal{X}) = \infty\Big\} = 1.$$

PROOF. Let $\varepsilon, \delta > 0$ be given. Proposition 3.1 gives us flow-bound constants q, R and $\rho_0 \le \rho_1 \le \rho_2 \le \cdots$ (depending also on δ), such that for any connected set $\mathcal{X}$ and any finite, nonnegative stopping time t,

$$\mathrm{P}\Big\{\exists t' \ge t \text{ such that } \operatorname{diam} \phi_{t'}(\mathcal{X}) < \frac{\varepsilon}{4} \,\Big|\, \mathcal{F}_t\Big\} \le 1 - q\mathbf{1}\{\operatorname{diam} \phi_t(\mathcal{X}) \ge R\} \tag{38}$$

and

$$\mathrm{P}\{\exists t'' \ge t \text{ such that } \operatorname{diam} \phi_{t''}(\mathcal{X}) < R \,|\, \mathcal{F}_t\} \le \delta 2^{-k} + \mathbf{1}\{\operatorname{diam} \phi_t(\mathcal{X}) < \rho_k\}. \tag{39}$$

That is, as long as the image of the set has diameter at least R at time t, there is probability at least q that it will never fall under $\varepsilon/4$ at any later time; and if the diameter is at least ρ_k at time t, the probability of it falling under R is no more than $\delta 2^{-k}$. Furthermore, the support lemma (Lemma 2.4) gives us a positive flow-bound constant q' such that for any $\mathcal{F}_t$-measurable random points x and y,

$$\mathrm{P}\Big\{\|\phi_{t+1}(x) - \phi_{t+1}(y)\| \ge R \text{ and } \inf_{s\in[t,t+1]} \|\phi_s(x) - \phi_s(y)\| \ge \frac{\varepsilon}{4} \,\Big|\, \mathcal{F}_t\Big\}$$
$$\ge q'\mathbf{1}\Big\{\|\phi_t(x) - \phi_t(y)\| \ge \frac{\varepsilon}{2}\Big\}.$$

If diam $\phi_t(\mathcal{X}) \ge \varepsilon/2$, then there are points x and y in $\mathcal{X}$ whose separation at time t is at least $\varepsilon/2$, and their separation at future times provides a lower bound for the diameter of the image. This allows us to improve (38) to

$$\mathrm{P}\Big\{\exists t' \ge t \text{ such that } \operatorname{diam} \phi_{t'}(\mathcal{X}) < \frac{\varepsilon}{4} \,\Big|\, \mathcal{F}_t\Big\} \le 1 - p\mathbf{1}\Big\{\operatorname{diam} \phi_t(\mathcal{X}) \ge \frac{\varepsilon}{2}\Big\}, \tag{40}$$

where $p = qq'$.

As usual, we define an increasing sequence of stopping times, $\tau_0 < \sigma_1 < \tau_1 < \sigma_2 < \cdots$. This time, we also define a corresponding decreasing sequence of connected sets $\mathcal{X} = \mathcal{X}_0 \supset \mathcal{X}_1 \supset \mathcal{X}_2 \supset \cdots$. These are the candidates that we will consider for the point that successfully tracks ψ. The stopping times and sets are defined according to the following rules: first let

$$\tau_0 := \inf\{t : \operatorname{diam}\, \phi_t(\mathcal{X}) \ge \rho_0\}.$$

Once τ_{k-1} and $\mathcal{X}_{k-1}$ have been determined, we let

$$\sigma_k := \inf\biggl\{t \geq \tau_{k-1} : \operatorname{diam} \phi_t(\mathcal{X}_{k-1}) \geq \varepsilon \text{ and } \exists\, x \in \mathcal{X}_{k-1} \text{ such that } \|\phi_t(x) - \psi(t)\| \leq \frac{\varepsilon}{2}\biggr\}.$$

When σ_k is finite, there is at least one compact connected component of

$$\bigl\{x \in \mathcal{X}_{k-1} : \|\phi_{\sigma_k}(x) - \psi(\sigma_k)\| \leq \varepsilon\bigr\},$$

whose image at time σ_k has diameter at least $\varepsilon/2$. We choose such a component and call it $\mathcal{X}_k^*$. Then

$$\tau_k := \inf\biggl\{t \geq \sigma_k + 1 : \operatorname{diam} \phi_t(\mathcal{X}_k^*) \notin \biggl[\frac{\varepsilon}{4}, \rho_k\biggr] \text{ and } \operatorname{diam} \phi_t(\mathcal{X}_{k-1}) > \rho_k\biggr\} \quad \text{for } k \geq 1.$$

Finally, $\mathcal{X}_k$ is defined to be $\mathcal{X}_k^*$ if $\operatorname{diam} \phi_{\tau_k}(\mathcal{X}_k^*) \geq \rho_k$, and $\mathcal{X}_{k-1}$ otherwise. The infimum over the empty set is taken to be ∞.

We first compute the probability of $\mathcal{A}$, the event that the stopping times are all finite. Conditioned on τ_{k-1} being finite, $\phi_{\tau_{k-1}}(\mathcal{X}_{k-1})$ is a set with diameter at least ρ_{k-1}. By Theorem 3.2 the conditional probability of σ_k being infinite, is no more than $\mathrm{P}\{\lim_{t\to\infty} \operatorname{diam} \phi_t(\mathcal{X}_{k-1}) = 0\}$, which is no bigger than $\delta 2^{-k+1}$. By Proposition 3.1 the event $\{\sigma_k < \infty \text{ and } \tau_k = \infty\}$ is also contained in $\{\lim_{t\to\infty} \operatorname{diam} \phi_t(\mathcal{X}_{k-1}) = 0\}$, which then also has probability no bigger than $\delta 2^{-k+1}$. Thus $\mathrm{P}(\mathcal{A} \mid \tau_0 < \infty) \geq 1 - 4\delta$. On the event $\mathcal{A}$, since the sets $\mathcal{X}_k$ are decreasing, nonempty and compact, there is at least one point x in their intersection.

Now consider the events $\mathcal{B}_k = \{\mathcal{X}_k = \mathcal{X}_k^*\}$ and $\mathcal{B}_* = \limsup \mathcal{B}_k$. If σ_k is finite, the image $\phi_{\sigma_k}(\mathcal{X}_k^*)$ has diameter at least $\varepsilon/2$. By condition (40),

$$\mathrm{P}(\mathcal{B}_{k+1} \mid \mathcal{F}_{\sigma_k}) \geq p$$

on the event $\{\sigma_k < \infty\}$. It follows, by the Neveu–Borel–Cantelli lemma that $\mathrm{P}(\mathcal{B}_* \mid \mathcal{A}) = 1$, so $\mathrm{P}(\mathcal{A} \cap \mathcal{B}_* \mid \tau_0 < \infty) \geq 1 - 4\delta$. Thus, since $\{\tau_0 < \infty\}$ contains the event $\{\operatorname{diam} \phi_t(\mathcal{X}) \to \infty\}$, and since

$$\mathrm{P}\bigl\{\operatorname{diam} \phi_t(\mathcal{X}) \to \infty \mid \tau_0 < \infty\bigr\} \geq 1 - \delta,$$

we get

$$\begin{aligned}
&\mathrm{P}\bigl(\mathcal{A} \cap \mathcal{B}_* \mid \operatorname{diam} \phi_t(\mathcal{X}) \to \infty\bigr) \\
&\quad \geq \frac{1}{\mathrm{P}\{\operatorname{diam} \phi_t(\mathcal{X}) \to \infty\}} \\
&\qquad \times \bigl[\mathrm{P}(\mathcal{A} \cap \mathcal{B}_* \cap \{\tau_0 < \infty\}) + \mathrm{P}\{\operatorname{diam} \phi_t(\mathcal{X}) \to \infty\} - \mathrm{P}\{\tau_0 < \infty\}\bigr]
\end{aligned}$$

$$\begin{aligned}&\geq \frac{\mathrm{P}\{\tau_0 < \infty\}}{\mathrm{P}\{\operatorname{diam} \phi_t(\mathcal{X}) \to \infty\}}(1 - 4\delta + 1 - \delta - 1)\\&\geq (1 - 5\delta).\end{aligned}$$

On the event $\mathcal{A} \cap \mathcal{B}_k$ the point x is in $\mathcal{X}_k = \mathcal{X}_k^*$, which means that

$$\|\phi_{\sigma_k}(x) - \psi(\sigma_k)\| \leq \varepsilon.$$

On $\mathcal{A} \cap \mathcal{B}_*$, there are infinitely many k for which this holds; since $\sigma_k \to \infty$, it follows that

$$\begin{aligned}&\mathrm{P}\Big\{\exists\, x \in \mathcal{X} \text{ such that } \{t : \|\phi_t(x) - \psi(t)\| \leq \varepsilon\} \text{ is unbounded} \,\Big|\, \operatorname{diam} \phi_t(\mathcal{X}) \to \infty\Big\}\\&\qquad\geq \mathrm{P}\big(\mathcal{A} \cap \mathcal{B}_* \mid \operatorname{diam} \phi_t(\mathcal{X}) \to \infty\big)\\&\qquad\geq 1 - 5\delta.\end{aligned}$$

Since δ is an arbitrary positive number, weak ball-chasing is proved for the arbitrary positive radius ε. $\square$

THEOREM 4.2. *Let M be a martingale field satisfying the standard conditions, $\phi_t : \mathbb{R}^d \to \mathbb{R}^d$ the flow of homeomorphisms that it determines and $\mathcal{X}$ a compact nontrivial set. Let Λ and ψ satisfy the same conditions as in Theorem* 3.2. *There is a positive constant c^*, depending on the flow bounds and also on Λ, such that*

$$\mathrm{P}\Big\{\exists\, T \text{ and } x \in \mathcal{X} \text{ such that } \forall\, t \geq T\, \|\phi_t(x) - \psi(t)\| \leq c^* \log t \,\Big|\, \lim_{t\to\infty} \operatorname{diam} \phi_t(\mathcal{X}) = \infty\Big\} = 1.$$

PROOF. Define s, ρ and ψ_n exactly as in the proof of Theorem 3.2. For ease of notation we will take s to be 1. Let $\bar{c}$ be any positive number, and fix a positive integer m such that $\bar{c} \log m \geq 2\rho$. We will define a decreasing sequence of connected sets, intended to catch points which have remained within the desired ball of size constant times $\log n$ without interruption since some fixed time in the past: $\mathcal{X} = \mathcal{X}_m^{(m)} \supset \mathcal{X}_{m+1}^{(m)} \supset \cdots$.

We will define a retraction sequence $(x_n^{(m)}, y_n^{(m)})$ tending toward ψ for $n \geq m$, with $x_n^{(m)}, y_n^{(m)} \in \mathcal{X}_n^{(m)}$. Once it is established, we will let $r_n^{(m)} := \min\{\|\phi_n(x_n^{(m)}) - \psi_n\|$.

For $n = m$, we choose $x_m^{(m)} \in \mathcal{X}_m^{(m)}$ with $\phi_m(x_m^{(m)})$ at a minimum distance from ψ_m, and $y_m^{(m)} \in \mathcal{X}_m^{(m)}$ such that $\|\phi_m(x_m^{(m)}) - \phi_m(y_m^{(m)})\| = \rho$ (if such a point exists in $\mathcal{X}_m^{(m)}$; otherwise, this separation is to be the maximum possible). Suppose now that $x_{n-1}^{(m)}$, $y_{n-1}^{(m)}$ and $\mathcal{X}_{n-1}^{(m)}$ have been defined. We extend the sequence as follows: Suppose first that $r_k^{(m)} < \bar c \log k$ for all $k \in \{m, m+1, \dots, n\}$, and that $\operatorname{diam} \phi_n(\mathcal{X}_{n-1}^{(m)}) \ge 2\rho$. If

$$\|\phi_n(x_{n-1}^{(m)}) - \psi_n\| \le \|\phi_n(y_{n-1}^{(m)}) - \psi_n\|,$$

then $x_n^{(m)} = x_{n-1}^{(m)}$; otherwise $x_n^{(m)} = y_{n-1}^{(m)}$. In either case $y_n^{(m)}$ is chosen to be a point in $\mathcal{X}_{n-1}^{(m)}$ such that $\|\phi_n(x_n^{(m)}) - \phi_n(y_n^{(m)})\| = \rho$. Since $\rho \le \bar c \log n$, they may be chosen to lie both in the same connected component of

$$\mathcal{X}_{n-1}^{(m)} \cap \phi_n^{-1}\big(B_{3\bar c \log n}(\psi_n)\big).$$

That connected component then becomes $\mathcal{X}_n^{(m)}$. If the conditions are not satisfied—that is, if $r_k^{(m)} \ge \bar c \log k$ for some $k \le n$ or $\operatorname{diam} \phi_n(\mathcal{X}_{n-1}^{(m)}) < 2\rho$—we keep $x_n^{(m)} = x_{n-1}^{(m)}$, $y_n^{(m)} = y_{n-1}^{(m)}$, and $\mathcal{X}_n^{(m)} = \mathcal{X}_{n-1}^{(m)}$. We define a stopping time $\sigma^{(m)}$ to be the smallest $n > m$ such that $r_n^{(m)} \ge \bar c \log n$, or ∞ if no such n exists.

At the same time, we define another retraction sequence $\tilde x_n^{(m)}$ and $\tilde y_n^{(m)}$, tending away from ψ_n. These points are also chosen to lie in $\mathcal{X}_n^{(m)}$. We start by choosing $\phi_m(\tilde x_m^{(m)})$ to lie at a maximal distance from ψ_m (within $\mathcal{X}_m^{(m)}$), and $\phi_m(\tilde y_m^{(m)})$ at distance ρ (or the maximum possible distance, if this is smaller than ρ) from $\phi_m(\tilde x_m^{(m)})$. We set

$$R_n^{(m)} := \max\big\{\|\phi_n(\tilde x_{n-1}^{(m)}) - \psi_n\|, \|\phi_n(\tilde y_{n-1}^{(m)}) - \psi_n\|\big\},$$

Once $\tilde x_{n-1}^{(m)}$ and $\tilde y_{n-1}^{(m)}$ have been determined, we extend the sequence as follows:

1. If $R_k^{(m)} > 2\bar c \log k$ for all $m \le k \le n$, if $\sigma^{(m)} > n$, and if both $\tilde x_{n-1}^{(m)}$ and $\tilde y_{n-1}^{(m)}$ lie in $\mathcal{X}_n^{(m)}$, then $\tilde x_n^{(m)}$ is whichever one of $\tilde x_{n-1}^{(m)}$ and $\tilde y_{n-1}^{(m)}$ has its image at time n farther from ψ_n. In this case, $\tilde y_n^{(m)}$ is chosen to be a point in $\mathcal{X}_n^{(m)}$ such that $\|\phi_n(\tilde x_n^{(m)}) - \phi_n(\tilde y_n^{(m)})\| = \rho$. [As we assumed that $\sigma^{(m)} > n$, it must be that $r_n^{(m)} \le \bar c \log n$. This means that there is a point of $\mathcal{X}_n^{(m)}$ inside a ball of radius $\bar c \log n$ around ψ_n. The distance from $\tilde x_n^{(m)}$ must therefore be at least $\bar c \log n \ge 2\rho$. Since $\mathcal{X}_n^{(m)}$ is connected, and includes $\tilde x_n^{(m)}$, there are indeed points in $\phi_n(\mathcal{X}_n^{(m)})$ at distance ρ from $\phi_n(\tilde x_n^{(m)})$.]
2. If $R_k^{(m)} > 2\bar c \log k$ for all $m \le k \le n$, if $\sigma^{(m)} > n$, and if $\tilde x_{n-1}^{(m)}$ or $\tilde y_{n-1}^{(m)}$ lie outside $\mathcal{X}_n^{(m)}$, then $\tilde x_n^{(m)}$ is chosen to be any point in $\mathcal{X}_n^{(m)}$ such that $\|\phi_n(\tilde x_n^{(m)}) - \psi_n\| = 3\bar c \log n$, and $\tilde y_n^{(m)}$ to be a point in $\mathcal{X}_n^{(m)}$ such that $\|\phi_n(\tilde y_n^{(m)}) - \psi_n\| \le 3\bar c \log n$ and $\|\phi_n(\tilde y_n^{(m)}) - \phi_n(\tilde x_n^{(m)})\| = \rho$. Note that this is possible, because the construction of $\mathcal{X}_n^{(m)}$ cuts the precursor $\mathcal{X}_{n-1}^{(m)}$ only at the boundary of its intersection with $\phi_n^{-1}(B_{3\bar c \log n}(\psi_n))$. If either $\tilde x_{n-1}^{(m)}$ or $\tilde y_{n-1}^{(m)}$ drops out in this action, it can only be because $\mathcal{X}_n^{(m)}$ meets the boundary.

3. If $R_k^{(m)} \le 2\bar{c}\log k$ for some $m \le k \le n$, or $\sigma^{(m)} \le n$, then $\tilde{x}_n^{(m)} = \tilde{x}_{n-1}^{(m)}$ and $\tilde{y}_n^{(m)} = \tilde{y}_{n-1}^{(m)}$.

We introduce two other variables:

$$\tilde{r}_n^{(m)} := R_n^{(m)} \wedge 3\bar{c}\log n$$

and

$$s_n^{(m)} := 3\bar{c}\log n - \tilde{r}_n^{(m)}.$$

We define $\tilde{\sigma}^{(m)}$ to be the first time $n > m$ such that $s_n^{(m)} > \bar{c}\log n$, or $\sigma^{(m)}$ if that time occurs first.

For any $n < \tilde{\sigma}^{(m)}$, the distance $\|\phi_n(\tilde{x}_n^{(m)}) - \psi_n\|$ is $\tilde{r}_n^{(m)}$, while $R_{n+1}^{(m)}$ is the larger of $\|\phi_{n+1}(\tilde{x}_n^{(m)}) - \psi_{n+1}\|$ and $\|\phi_{n+1}(\tilde{y}_n^{(m)}) - \psi_{n+1}\|$. Since $\|\psi_{n+1} - \psi_n\|$ is bounded by the constant Λ,

$$\begin{aligned}
\mathrm{P}\{&|R_{n+1}^{(m)} - \tilde{r}_n^{(m)}| \ge \lambda \mid \mathcal{F}_n\} \\
&\le \mathrm{P}\{\|\phi_{n+1}(\tilde{x}_n^{(m)}) - \phi_n(\tilde{x}_n^{(m)})\| + \|\phi_{n+1}(\tilde{y}_n^{(m)}) - \phi_n(\tilde{y}_n^{(m)})\| \\
&\qquad\qquad\qquad + \|\psi_{n+1} - \psi_n\| \ge \lambda - \rho \mid \mathcal{F}_n\} \\
&\le \mathrm{P}\left\{\|\phi_{n+1}(\tilde{x}_n^{(m)}) - \phi_n(\tilde{x}_n^{(m)})\| \ge \frac{\lambda - \Lambda - \rho}{2} \,\Big|\, \mathcal{F}_n\right\} \\
&\quad + \mathrm{P}\left\{\|\phi_{n+1}(\tilde{y}_n^{(m)}) - \phi_n(\tilde{y}_n^{(m)})\| \ge \frac{\lambda - \Lambda - 2\rho}{2} \,\Big|\, \mathcal{F}_n\right\}.
\end{aligned}$$

By Lemma 2.1,

$$\mathrm{P}\{|R_{n+1}^{(m)} - \tilde{r}_n^{(m)}| \ge \lambda \mid \mathcal{F}_n\} \le 2 \cdot \frac{4d}{\sqrt{\pi}} \exp\left\{-\frac{(\lambda - \Lambda - 2\rho)^2}{8d^2A^2}\right\}.$$

Thus there are flow-bound constants $\tilde{b}_1$ and $\tilde{b}_2$ such that

$$\mathrm{P}\{|R_{n+1}^{(m)} - \tilde{r}_n^{(m)}| \ge \lambda \mid \mathcal{F}_n\} \le \tilde{b}_1 e^{-\tilde{b}_2\lambda} \qquad \forall \lambda > 0. \tag{41}$$

Since

$$|s_{n+1}^{(m)} - s_n^{(m)}| \le |R_{n+1}^{(m)} - \tilde{r}_n^{(m)}| + 3\bar{c}\log\left(1 + \frac{1}{n}\right),$$

we may find b_1 and b_2 which satisfy the exponential tail bounds (18) and (20) for $s_n^{(m)}$ in place of X_n. The equivalent statement for $r_n^{(m)}$ is more straightforward. We may take b_1 and b_2 to satisfy the tail bounds for both sequences simultaneously.

Using Lemma 2.5, we see that there are positive flow-bound constants c_1, c_2, c_3 and r^*, such that for $n < \sigma^{(m)}$,

$$\mathrm{E}[R_{n+1}^{(m)} \mid \mathcal{F}_n] \ge \tilde{r}_n^{(m)} + c_1 - c_3\mathbf{1}\{\tilde{r}_n^{(m)} - r_n^{(m)} < 2\rho\}$$

and

$$\mathrm{E}[r_{n+1}^{(m)} \mid \mathcal{F}_n] \le r_n^{(m)} - c_1\mathbf{1}\{r_n^{(m)} \ge r^*\} + c_2\mathbf{1}\{r_n^{(m)} < r^*\} + c_3\mathbf{1}\{\tilde{r}_n^{(m)} - r_n^{(m)} < 2\rho\}.$$

Here we have used the fact that $\operatorname{diam}\phi_n(\mathcal{X}_n^{(m)}) \geq \tilde{r}_n^{(m)} - r_n^{(m)}$. [Naively, the terms $\tilde{r}_n^{(m)}$ and $r_n^{(m)}$ appear to refer to points in $\phi_n(\mathcal{X}_{n-1}^{(m)})$. But these extreme points are the ones which are kept in $\mathcal{X}_n^{(m)}$.] Observe, too, that

$$s_{n+1}^{(m)} - s_n^{(m)} = 3\bar{c}\log\left(1+\frac{1}{n}\right) - \left(R_{n+1}^{(m)} - \tilde{r}_n^{(m)}\right) + \left(R_{n+1}^{(m)} - 3\bar{c}\log(n+1)\right)^+.$$

By the exponential tail bounds (41), we get

$$\begin{aligned}
&\mathrm{E}\left[\left(R_{n+1}^{(m)} - 3\bar{c}\log(n+1)\right)^+ \mid \mathcal{F}_n\right] \\
&\qquad \leq \int_0^\infty \mathrm{P}\left\{R_{n+1}^{(m)} - \tilde{r}_n^{(m)} > 3\bar{c}\log(n+1) - \tilde{r}_n^{(m)} + \lambda \mid \mathcal{F}_n\right\} d\lambda \\
&\qquad \leq \int_{3\bar{c}\log(n+1)-\tilde{r}_n^{(m)}}^\infty b_1 e^{-b_2\lambda}\, d\lambda \\
&\qquad = \frac{b_1}{b_2}\exp\left\{-b_2\left(3\bar{c}\log(n+1) - \tilde{r}_n^{(m)}\right)\right\} \\
&\qquad \leq \frac{b_1}{b_2}\exp\{-b_2 s_n^{(m)}\}.
\end{aligned}$$

Thus we may also choose the constants c_1, c_2, c_3, and additional flow-bound constants s^* and m^*, such that for all $m \geq m^*$,

$$\mathrm{E}\left[s_{n+1}^{(m)} \mid \mathcal{F}_n\right] \leq s_n^{(m)} - c_1\mathbf{1}\{s_n^{(m)} \geq s^*\} + c_2\mathbf{1}\{s_n^{(m)} < s^*\} + c_3\mathbf{1}\{\tilde{r}_n^{(m)} - r_n^{(m)} < 2\rho\}.$$

If we assume that $\bar{c} > \gamma^{-1}$, where $\gamma = \gamma(c_1, b_1, b_2)$, we may apply Lemma 2.7 with $X_k = r_{m+k}^{(m)}$ and $\xi_k = c_3\mathbf{1}\{\tilde{r}_{m+k}^{(m)} - r_{m+k}^{(m)} < 2\rho\}$ ($\sigma \equiv \infty$, $N = m$ and $c = \bar{c}$), and again with $X_k = s_{m+k}^{(m)}$, the same ξ_k, $\sigma = \sigma^{(m)}$, and $\sigma^{(c)}(N) = \tilde{\sigma}^{(m)}$. Note that $\xi_k = 0$ for $k < \tilde{\sigma}^{(m)}$, and so

$$\left\{\exists k, m+k < \sigma_m < \infty \text{ and } \xi_k > 0\right\} \subset \left\{\tilde{\sigma}^{(m)} < \sigma^{(m)} < \infty\right\}$$

and

$$\left\{\sigma^{(m)} = \tilde{\sigma}^{(m)} < \infty\right\} \cup \left\{\exists k, m+k < \tilde{\sigma}_m < \infty \text{ and } \xi_k > 0\right\} \subset \left\{\sigma^{(m)} = \tilde{\sigma}^{(m)} < \infty\right\}.$$

Bound (27) of Lemma 2.7 may now be applied to each of these sequences, giving us a constant k such that

$$\begin{aligned}
\mathrm{P}\left\{\sigma^{(m)} < \infty \mid \mathcal{F}_m\right\} &\leq \exp\{\gamma r_m^{(m)}\} m^{-\gamma\bar{c}} + k m^{1-\gamma\bar{c}} \\
&\qquad + \mathrm{P}\left\{\tilde{\sigma}^{(m)} < \sigma^{(m)} < \infty \mid \mathcal{F}_m\right\}
\end{aligned}$$

and

$$\mathrm{P}\{\tilde{\sigma}^{(m)} < \infty \mid \mathcal{F}_m\} \le \exp\{\gamma s_m^{(m)}\} m^{-\gamma\bar{c}} + k m^{1-\gamma\bar{c}} \\ + \mathrm{P}\{\sigma^{(m)} = \tilde{\sigma}^{(m)} < \infty \mid \mathcal{F}_m\}.$$

Adding these two inequalities yields

$$\mathrm{P}\{\sigma^{(m)} < \infty \text{ or } \tilde{\sigma}^{(m)} < \infty \mid \mathcal{F}_m\} \le (2k+2)\exp\{\gamma(r_m^{(m)} \vee s_m^{(m)} \vee r^*)\} m^{1-\gamma\bar{c}}.$$

Let $\mathcal{A}_m$ be the event

$$\Big\{\inf_{x\in\mathcal{X}} \|\phi_m(x) - \psi_m\| \le r^* \text{ and } \sup_{x\in\mathcal{X}} \|\phi_m(x) - \psi_m\| \ge 3\bar{c}\log m\Big\}.$$

Then

$$\mathrm{P}\{\tilde{\sigma}^{(m)} < \infty \mid \mathcal{F}_m\} \le (2k+2)\exp\{\gamma r^*\} m^{1-\gamma\bar{c}} + \mathbf{1}_{\mathcal{A}_m^C}.$$

Thus, conditioning on $\mathcal{A}_* = \limsup \mathcal{A}_m$, the event that $\mathcal{A}_m$ occurs infinitely often,

$$\mathrm{P}\{\exists m \text{ such that } \tilde{\sigma}^{(m)} = \infty \mid \mathcal{A}_*\} = 1$$

if $\bar{c} > \gamma^{-1}$. By Proposition 3.1, on the event $\{\mathrm{diam}\,\phi_n(\mathcal{X}) \to \infty\}$ the diameter of $\phi_n(\mathcal{X})$ grows linearly, so that eventually $\sup_{x\in\mathcal{X}} \|\phi_m(x) - \psi_m\| \ge 3\bar{c}\log m$. Theorem 3.2 (actually, the discrete-time version, which was proved along the way to the full theorem) then tells us that $\mathrm{P}\{\mathcal{A}_* \mid \mathrm{diam}\,\phi_n(\mathcal{X}) \to \infty\} = 1$. This means that on the event $\{\mathrm{diam}\,\phi_n(\mathcal{X}) \to \infty\}$ there is almost surely an m such that $\tilde{\sigma}^{(m)}$ is infinite.

Observe now that $\mathcal{X}_n^{(m)}$ is a nonempty compact set for all $m \le n$. Since

$$\mathcal{X}_m^{(m)} \supset \mathcal{X}_{m+1}^{(m)} \supset \mathcal{X}_{m+2}^{(m)} \supset \cdots,$$

there is a point x_m in $\bigcap_{n=m}^{\infty} \mathcal{X}_n^{(m)}$. If $\tilde{\sigma}^{(m)} = \infty$, this point satisfies

$$\|\phi_n(x_m) - \psi_n\| < 3\bar{c}\log n$$

for all $n \ge m$. Thus we have proved the statement of this theorem for integer times.

Now let $\mathcal{B}_n$ be the event

$$\big\{\exists x \in \mathcal{X} \text{ and } t \in [0,1] \text{ such that } \|\phi_n(x) - \psi_n\| < 3\bar{c}\log n \text{ and} \\ \|\phi_{n+t}(x) - \psi_{n+t}\| > 3(\bar{c}+\varepsilon)\log(n+t)\big\}.$$

Let $z_n = \lceil 3\bar{c}\sqrt{d}\log n\rceil^d$. We may cover the ball of radius $3\bar{c}\log n$ around $\psi(n)$ with z_n balls of radius 1. We number these in some arbitrary order, $U_1, \dots, U_{z_n}$, and let

$$\mathcal{B}_n^{(i)} = \big\{\exists x \in \mathcal{X} \text{ and } t \in [0,1] \text{ such that } \phi_n(x) \in U_i \text{ and} \\ \|\phi_{n+t}(x) - \psi_{n+t}\| > 3(\bar{c}+\varepsilon)\log(n+t)\big\}.$$

If we let u_i be the center of U_i, $\mathcal{B}_n^{(i)}$ is contained in the union of the events

$$\left\{\sup_{0\le t\le 1}\|\psi(n+t)-\psi(n)\|>\varepsilon\log n\right\},$$

$$\left\{\sup_{0\le t\le 1}\|\phi_{n+t}(u_i)-\phi_n(u_i)\|>\varepsilon\log n\right\}$$

and

$$\left\{\sup_{0\le t\le 1}\operatorname{diam}\phi_{\tau,\tau+t}(U_i)>\varepsilon\log n\right\}.$$

Because of the Lipschitz condition on the path ψ, the first of these has probability 0 for $n>e^{\Lambda/\varepsilon}$. By Lemma 2.1, the second event has probability no more than

$$c_4e^{-c_5(\varepsilon\log n)^2},$$

where c_4 and c_5 are flow-bound constants. By Lemma 2.3 we know that there is a flow-bound constant K such that the third event is bounded by

$$K\exp\left\{-\frac{(\varepsilon\log n)^2}{9\log^3(\varepsilon\log n)}\right\}$$

for $n>e^{6/\varepsilon}$.

The probability of $\mathcal{B}_n$ is bounded by z_n times the sum of these three bounds. Thus $\sum_{n=1}^{\infty}\mathrm{P}(\mathcal{B}_n)$ is finite, implying that almost surely only finitely many of the $\mathcal{B}_n$ occur. The ball of radius $3\bar{c}\log n$ around $\psi(n)$ eventually stays in the ball of radius $3(\bar{c}+\varepsilon)\log t$ around $\psi(t)$ for all times $t\in[n,n+1]$. This includes the point x_m, once it settles into the ball of radius $3\bar{c}\log n$ around $\psi(n)$. The theorem then follows for any c^* larger than $3\bar{c}$. $\square$

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TECHNISCHE UNIVERSITÄT BERLIN
MA 7-5
STR. DES 17. JUNI 136
10623 BERLIN
GERMANY
E-MAIL: ms@math.tu-berlin.de

DEPARTMENT OF STATISTICS
UNIVERSITY OF CALIFORNIA
367 EVANS HALL
BERKELEY, CALIFORNIA 94720
E-MAIL: dstein@demog.berkeley.edu