The Annals of Probability
1997, Vol. 25, No. 3, 1541–1543

CORRECTION

ASYMPTOTICS OF A CLASS OF MARKOV PROCESSES WHICH ARE NOT IN GENERAL IRREDUCIBLE

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The Annals of Probability (1988) **16** 1333–1347

The set $P(S)$ of probability measures on the Borel sigma field of every closed subset S of $\mathbb{R}^k$ is shown to be complete under a Kolmogorov type metric. An assertion in the 1988 article by Bhattacharya and Lee that this completeness holds if S is merely topologically complete is false.

In this note we correct an error in the statement of Lemma 2.2 in the 1988 article by Bhattacharya and Lee. To state the original version and its correction: let S be a nonempty Borel subset of $\mathbb{R}^k$. Let A be the class of all sets of the form

$$\{\mathbf{x} \in \mathbb{R}^k: \gamma(\mathbf{x}) \le \mathbf{c}\}\ \big(\gamma(\mathbf{x}) = (\gamma_1(\mathbf{x}), \dots, \gamma_k(\mathbf{x})),\ \mathbf{c} = (c_1, \dots, c_k)\big), \tag{1}$$

where $\gamma: \mathbb{R}^k \to \mathbb{R}^k$ is continuous and nondecreasing, $\mathbf{c} \in \mathbb{R}^k$ and "$\le$" is the usual lexicographic order; that is, $\gamma_i(\mathbf{x}) \le c_i$ for $1 \le i \le k$. Let $P(S)$ denote the set of all probability measures on (the Borel sigma field of) S. On $P(S)$ define the metric

$$d(\mu, \nu) = \sup_{A \in A} |\mu(A \cap S) - \nu(A \cap S)|. \tag{2}$$

Lemma 2.2 asserts that $(P(S), d)$ is a complete metric space if S is topologically complete. The error in the proof of this lemma occurred in the statement that the metric topology of d is stronger than the topology of weak convergence on $P(S)$. Professor B. V. Rao [2] has provided a counterexample to the assertion in the lemma, in which S is the complement of a Cantor set. We will prove that $(P(S), d)$ is complete if S is a closed subset of $\mathbb{R}^k$.

PROPOSITION. *If S is a closed subset of $\mathbb{R}^k$ then $(P(S), d)$ is a complete metric space.*

PROOF. Suppose $\{\mu_n: n = 1, 2, \dots\}$ is a Cauchy sequence in $P(S)$ in the d-metric, that is,

$$d(\mu_n, \mu_{n'}) \to 0 \quad \text{as } n, n' \to \infty. \tag{3}$$

Received January 1997.
AMS 1991 *subject classifications*. Primary 60F05, 60J05.
Key words and phrases. Nondecreasing maps, completeness of a set of probability measures.

Let μ_∞ be the limiting set function on $\mathbf{A}$. That is,

$$\mu_\infty(A) = \lim_{n\to\infty} \mu_n(A \cap S), \qquad A \in \mathbf{A}, \tag{4}$$

so that

$$\sup_{A\in\mathbf{A}} |\mu_n(A\cap S) - \mu_\infty(A)| \to 0 \quad \text{as } n \to \infty. \tag{5}$$

We will now show that there exists a (unique) probability measure $\overline{\mu}$ on (the Borel sigma field of) $\mathbb{R}^k$ such that $\overline{\mu}(A) = \mu_\infty(A)\ \forall\ A \in \mathbf{A}$. For this, extend μ_n to (the Borel sigma field of) $\mathbb{R}^k$ and denote this extension by $\overline{\mu}_n$: $\overline{\mu}_n(A) = \mu_n(A \cap S)$ for every Borel $A \subset \mathbb{R}^k$. Then $\overline{\mu}_n$ converges weakly to a probability measure $\overline{\mu}$ on $\mathbb{R}^k$. To see this, take γ in (1) to be the identity map and let $\mathbf{c}$ vary over $\mathbb{R}^k$, showing that the sequence of distribution functions of $\overline{\mu}_n$ converges uniformly in $\mathbb{R}^k$ to the distribution function of a probability measure $\overline{\mu}$. In particular, $\overline{\mu}_n$ converges weakly to $\overline{\mu}$. This also shows $\mu_\infty(A) = \overline{\mu}(A)$ for every A of the form $A_{\mathbf{c}} = (-\infty, c_1] \times \cdots \times (-\infty, c_k]\ \forall\ \mathbf{c} = (c_1, \dots, c_k)$. Now for every continuous nondecreasing γ, (i) $\overline{\mu}_n \circ \gamma^{-1}$ converges weakly to $\overline{\mu} \circ \gamma^{-1}$ and (ii) $\overline{\mu}_n \circ \gamma^{-1}(A_{\mathbf{c}})$ converges uniformly to $\mu_\infty(\gamma^{-1}(A_{\mathbf{c}}))$ for $\mathbf{c} \in \mathbb{R}^k$. Therefore, $\overline{\mu}(A) = \mu_\infty(A)$ for each $A \in \mathbf{A}$. We now show that $\overline{\mu}(S) = 1$; that is, the restriction $\tilde{\mu}$ of $\overline{\mu}$ to S belongs to $P(S)$, and $d(\mu_n, \tilde{\mu}) \to 0$.

From the weak convergence of $\overline{\mu}_n$ to $\overline{\mu}$, it follows that

$$\overline{\mu}(S) \geq \limsup_{n\to\infty} \overline{\mu}_n(S) = 1. \tag{6}$$

Thus for the restriction $\tilde{\mu}$ of $\overline{\mu}$ to S, we have $\tilde{\mu} \in P(S)$ and

$$\begin{aligned} d(\mu_n, \tilde{\mu}) &\equiv \sup_{A\in\mathbf{A}} |\mu_n(A\cap S) - \tilde{\mu}(A\cap S)| \\ &= \sup_{A\in\mathbf{A}} |\mu_n(A\cap S) - \overline{\mu}(A\cap S)| \\ &= \sup_{A\in\mathbf{A}} |\mu_n(A\cap S) - \mu_\infty(A)| \to 0 \end{aligned} \tag{7}$$

as $n \to \infty$. □

REMARK 1. Suppose S_1, S_2 are two Borel subsets of $\mathbb{R}^k$ such that there exists a strictly increasing homeomorphism of S_1 onto S_2. Then, clearly, if $(P(S_1), d)$ is complete, so is $(P(S_2), d)$. Thus it may be shown that $(P(S), d)$ is complete for all open rectangles S (with edges parallel to the coordinate axes), for such a rectangle is transformed into $\mathbb{R}^k$ by a continuous strictly increasing map. A complete characterization of those S for which $(P(S), d)$ is complete is an interesting open problem. For a related but stronger metric d_1 considered in the 1988 article along with d, some significant results have been obtained recently by Chakraborty and Rao [1].

REMARK 2. If the hypothesis "S is topologically complete" is replaced by "S is closed," all the results stated in the 1988 article remain valid.

Finally, we correct a typo: on page 1343, in the definition of $B(y)$ in (3.19), *max* should be *min*.

Acknowledgments. The authors are indebted to Professor B. V. Rao for pointing out the error in Lemma 2.2 in the original article and for helpful discussions. The proof of completeness of $(P(S), d)$ for all S belonging to a larger class of sets than the class of closed sets has been omitted at the suggestion of a referee, since the proof is somewhat difficult.

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