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SYMMETRIC LANGEVIN SPIN GLASS DYNAMICS

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We study the asymptotic behavior of symmetric spin glass dynamics in the Sherrington–Kirkpatrick model as proposed by Sompolinsky–Zippelius. We prove that the averaged law of the empirical measure on the path space of these dynamics satisfies a large deviation upper bound in the high temperature regime. We study the rate function which governs this large deviation upper bound and prove that it achieves its minimum value at a unique probability measure Q which is not Markovian. We deduce an averaged and a quenched law of large numbers. We then study the evolution of the Gibbs measure of a spin glass under Sompolinsky–Zippelius dynamics. We also prove a large deviation upper bound for the law of the empirical measure and describe the asymptotic behavior of a spin on path space under this dynamic in the high temperature regime.

1. Introduction. The Sherrington–Kirkpatrick (S–K) model is a mean field simplification of the spin glass model of Edwards–Anderson. The behavior of its static characteristics, such as its partition function, has been intensively studied by physicists (see [12] for a broad survey). There are few mathematical results available (except for [1], [6], [9] and [17]).

In [12], it is argued that studying dynamics might be simpler since it avoids using the "replica trick" and the Parisi ansatz for symmetry breaking, which are yet to be put on firm ground. It seems that, in the physics literature, the first attempt to study the dynamics of S–K is due to Sompolinsky and Zippelius (see [15]), who chose a Langevin dynamics scheme.

In [3], we followed this strategy for asymmetric dynamics (which are not directly relevant to the study of statics for the S–K model). We obtained there a full large deviation principle for path space empirical measure averaged on the Gaussian couplings (for short times or large temperatures). This large deviation principle enabled us to derive the so-called self-consistent limiting dynamics, which proved to be non-Markovian.

Here we want to attack the real problem, that is, symmetric dynamics. We prove only a strong large deviation upper bound with a good rate function. Minimizing this rate function gives a theorem on convergence to self-consistent limiting dynamics, which we identify, though in a rather cryptic form.

We can do this only in a short time or high temperature regime, and so this prevents us from drawing any conclusion for the behavior in large time, at fixed temperature, which would be a line of attack to study the equilibrium

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measure. Weaker results concerning these dynamics are proved in [11] for any time and temperature.

To be more specific, let us recall that the S–K Hamiltonian is given, for $x = (x^1, \ldots, x^N) \in \{-1, 1\}^N$, by

$$H_J^N(x) = \frac{-1}{\sqrt{N}} \sum_{i,\, j=1}^{N} J_{ij} x^i x^j,$$

where the randomness in the spin glass is here modeled by the $(J_{ij})_{i \le j}$ which are independent centered Gaussian random variables, and where $J_{ij} = J_{ji}$. The Gibbs probability measure one would like to study (for N large) is given by

$$\frac{\exp(-\beta H_J^N(x))}{Z_N(J)} \alpha^{\otimes N}(dx),$$

where $\alpha = \frac{1}{2}(\delta_{-1} + \delta_1)$ and β is the inverse of temperature.

Here $Z_N(J)$ is the partition function

$$Z_N(J) = \frac{1}{2^N} \sum_{x \in \{-1,\, 1\}^N} \exp(-\beta H_J^N(x)).$$

If one replaces the hard spins $\{-1, +1\}$ by continuous spins, that is, by spins taking values in $\mathbb{R}$, or as we shall see in a bounded interval of $\mathbb{R}$, and if one replaces the measure $\alpha = \frac{1}{2}(\delta_{-1} + \delta_1)$ by $\alpha(dx) = (e^{-2U(x)} / \int e^{-2U(x)}\, dx)\, dx$, where U is, for instance, a double well potential on $\mathbb{R}$, then the Langevin dynamics for this problem are given by

$$dx_t^j = dB_t^j - \nabla U(x_t^j)\, dt + \frac{\beta}{\sqrt{N}} \sum_{1 \le i \le N} J_{ji} x_t^i \, dt, \tag{1}$$

where B is an N-dimensional Brownian motion.

We want to understand the limiting behavior (for large N) of the law, say $P_\beta^N(J)$, of these randomly interacting diffusions given the initial law, say $\mu_0^{\otimes N}$.

As in [3], we will only study bounded spins; that is, we will assume that $U(x)$ is defined on a bounded interval $[-A, A]$ and tends to infinity when $|x| \to A$ sufficiently fast to insure our spins x^j stay in the interval $[-A, A]$.

However, we will not assume as in [3] that the the whole matrix $(J_{ij})_{i,j}$ is made of i.i.d $N(0, 1)$ random variables but rather assume the symmetry of couplings; that is, we will here suppose that the random matrix $(J_{ij})_{i,\, j}$ is symmetric, that is, $J_{ij} = J_{ji}$. More precisely, we will suppose that under the diagonal, the J_{ij}'s are i.i.d $N(0, 1)$ and $N(0, 2)$ on the diagonal. Such a choice of covariance is nice from the technical point of view since it makes the law of the J_{ij}'s invariant by rotation. On the other hand, it does not interfere with the limit behavior of the spin glass.

So, under this symmetry hypothesis, our dynamics (1) are reversible and their invariant measure is given by the Gibbs measure:

$$\mu_J^N(dx) = \exp\left\{-\beta H_J^N(x) - 2 \sum_{i=1}^{N} U(x^i)\right\} \prod_{i=1}^{N} dx^i.$$

Thus the symmetry hypothesis is crucial to understanding S–K dynamics. On the other hand, this model is much more difficult to understand than the asymmetric one.

Our first goal is to study the empirical measure $\hat{\mu}^N = (1/N)\sum_{i=1}^N \delta_{x^i}$ on path space. There is no reason for this to be a simple problem, since, for fixed interaction J, the variables $(x^1, \dots, x^N)$ are not exchangeable. So we first study the law of the empirical measure $\hat{\mu}^N$ averaged on the interaction, leaving for a later work the study of J almost sure properties of this law.

The main result of this paper is large deviation upper bounds for this averaged law in a large temperature (or short time) regime, which entails a propagation of chaos result, that is, a theorem on convergence to a probability measure on path space that we describe explicitly as the law of a non-Markovian, highly nonlinear, solution of a stochastic differential equation (see Corollary 3.2). The existence and uniqueness problems for this limit law are not obvious and are the analogue here of the existence and uniqueness problem for asymmetric spin glass dynamics as obtained in [3].

As in [3], we then deduce that the quenched law of the empirical measure converges exponentially fast to δ_Q, which entails quenched laws of large numbers.

We finally underline how our method can be used to study the evolution of the Gibbs measure μ_J^N under Sompolinski–Zippelius dynamics and prove that, in the high temperature or short time regime, the quenched law of the empirical measure converges to the weak solution of a new nonlinear stochastic differential equation.

The organization of the paper is as follows.

In Section 2, we state and prove the strong large deviation upper bound. For more detail, see the following.

1. In Section 2.1, we introduce the rate function and state the strong large deviation upper bound (see Proposition 2.2 and Theorem 2.3).
2. In Section 2.2, we prove that the law of the path space dynamics averaged on the couplings is absolutely continuous with respect to the law of these dynamics with no couplings and show that its Radon–Nikodym derivative is a function of the empirical measure.
3. In Section 2.3, we study the continuity properties of this density.
4. In Section 2.4, these continuity properties enable us to prove that the rate function is a good rate function in the short time or high temperature regime.
5. In Section 2.5, we prove the strong large deviation upper bound in the short time or high temperature regime by first proving an exponential tightness result and then a weak large deviation upper bound.

In Section 3, we study the minima of the good rate function and prove that it achieves its minimum value at a unique probability measure, say Q. We describe Q as the unique solution of a fixed point problem in Theorem 3.14. This gives a propagation of chaos result stated in Corollary 3.3. In order to give a hint about what kind of result this approach leads to, let us state here Corollary 3.3.(ii): *For any bounded continuous functions* $(f_1,\dots,f_m)$ *on* $C([0,T],[-A,+A])$,

$$\lim_{N\to\infty}\mathscr{E}\Big[\int f_1(x^1)\cdots f_m(x^m)\,dP_\beta^N(J)(x)\Big]=\prod_{i=1}^m\int f_i(x)\,dQ(x),$$

where $\mathscr{E}$ *is the expectation over the Gaussian couplings.*

In Section 3.1, we characterize the minima of the good rate function.

In Section 3.2, we reduce the problem of finding these minima to a fixed point problem and then we show that this fixed point problem has at most one solution.

In Section 4, we apply our strategy to the stationary law of spin glass dynamics starting from the Gibbs measure. To this end, we need to suppose that β is small enough so that we are below the phase transition and that the free energy concentrates as proved by Talagrand (see [17]). Then, the study of the law of the empirical measure is reduced to that of the law of the empirical measure starting from the nonnormalized Gibbs measure $Z_J^N\times\mu_J^N$, which can be studied following the above procedure. We then describe the asymptotic behavior of the empirical measure.

2. Averaged and quenched large deviation upper bounds.

2.1. *Statement of the large deviation upper bound.* We first make precise the setting of our model: let A be a strictly positive real and U be a C^2 function on the interval $]-A,A[$ such that U tends to infinity, when $|x|\to A$, sufficiently fast to insure that

$$\lim_{|x|\to A}\int_0^x\exp 2U(y)\bigg(\int_0^y\exp -2U(z)\,dz\bigg)\,dy=+\infty.$$

For any number N of particles, any temperature $(=1/\beta)$ and $J=(J_{ij})_{1\le i,\,j\le N}\in\mathbb{R}^{N\times N}$, we consider the following system $\mathscr{S}_\beta^N(J)$ of interacting diffusions. For $j\in\{1,\dots,N\}$,

$$\mathscr{S}_\beta^N(J)=\begin{cases} dx_t^j=-\nabla U(x_t^j)\,dt+dB_t^j+\dfrac{\beta}{\sqrt{N}}\displaystyle\sum_{i=1}^N J_{ji}\;x_t^i\,dt,\\[2ex] \text{Law of } x_0=\mu_0^{\otimes N},\end{cases}$$

where $(B^j)_{1\le j\le N}$ is an N-dimensional Brownian motion and μ_0 is a probability measure on $[-A,A]$ which does not put mass on the boundary $\{-A,+A\}$. Under these assumptions, we recall Proposition 2.1 of [3].

PROPOSITION 2.1. *For each* $J\in\mathbb{R}^{N\otimes N}$, $\mathscr{S}_\beta^N(J)$ *has a unique weak solution and, almost surely,* $\sup_{s\le T}\sup_{1\le j\le N}|x_s^j|$ *does not reach* A.

In the following pages, we will focus on the evolution of this dynamical system until a time T and denote by $P_\beta^N(J)$ the weak solution of $\mathscr{S}_\beta^N(J)$

restricted to the sigma algebra $\mathscr{F}_T = \sigma(x_s^i,\ 1 \le i \le N,\ s \le T)$, and by $P^{\otimes N}$ the weak solution of $\mathscr{S}_0^N(J)$ restricted to $\mathscr{F}_T$.

Let W_T^A be the space of continuous functions from $[0, T]$ into $[-A, A]$. Then Proposition 2.1 insures that $P_\beta^N(J)$ is a probability measure on $\big(W_T^A\big)^N$.

We now suppose that the J_{ij}'s are random and that their distribution is given by the following.

1. For any integer numbers (i, j), $J_{ij} = J_{ji}$.
2. If $i < j$, the J_{ij}'s are independent centered Gaussian variables with covariance 1.
3. The J_{ii}'s are independent centered Gaussian variables with covariance 2. They are also independent of the $(J_{ij})_{i<j}$.

We shall denote by γ the law of the J_{ij}'s and by $\mathscr{E}$ expectation under γ. We have already noticed in [3] that $P_\beta^N(J)$ is a measurable function of the J_{ij}'s.

Further, we will be interested in the averaged law Q_β^N:

$$Q_\beta^N = \int P_\beta^N(J(\omega))\,d\gamma(\omega).$$

The aim of this section is to prove that the law of the empirical measure under Q_β^N satisfies a large deviation upper bound, which entails a quenched large deviation upper bound. To this end, we first define the rate function H which governs this upper bound (see Proposition 2.2). In order to define H, we need some notation and definitions that will also be useful later.

1. Let

$$\mathscr{M} = \left\{\mu \in \mathscr{M}_1^+(W_T^A) \Big/ \int \left(\int_0^T |\nabla U(x_s)|\,ds\right)^2 d\mu(x) < +\infty\right\}.$$

2. Let μ be a probability measure in $\mathscr{M}$. We denote by $L_\mu^2(W_T^A)$ the space of the square integrable functions under μ. Hence $L_\mu^2(W_T^A)$ is a Hilbert space with scalar product $\langle f, g\rangle_\mu = \int gf\,d\mu$.
3. Let I be the identity on $L_\mu^2(W_T^A)$.
4. Let $\mathscr{B}_T$ be an integral operator on $L_\mu^2(W_T^A)$ with kernel

$$b_T(x, y) = \int_0^T x_t y_t\,dt.$$

   Then $\mathscr{B}_T$ is a symmetric nonnegative Hilbert–Schmidt operator in $L_\mu^2(W_T^A)$ [for any $\mu \in \mathscr{M}_1^+(W_T^A)$].
5. Let λ_i be the eigenvalues of $\mathscr{B}_T$ in $L_\mu^2(W_T^A)$, and $(E_i)_{i\in\mathbb{N}}$ be an orthonormal basis of eigenvectors of $\mathscr{B}_T$ such that $\mathscr{B}_T E_i = \lambda_i E_i$. Since $\mathscr{B}_T$ is nonnegative, the λ_i's are nonnegative so that we can define a symmetric positive Hilbert–Schmidt operator $\log(I + \beta^2\mathscr{B}_T)$ in $L_\mu^2(W_T^A)$ by

$$\forall\ i \in \mathbb{N}, \qquad \log(I + \beta^2\mathscr{B}_T)E_i = \log(1 + \beta^2\lambda_i)E_i.$$

6. We define another integral operator $\mathscr{A}_T$ with kernel

$$a_T(x, y) = \tfrac{1}{2}\left(x_T y_T - x_0 y_0 + \int_0^T x_s \nabla U(y_s)\, ds + \int_0^T y_s \nabla U(x_s)\, ds\right).$$

Then $\mathscr{A}_T$ is a symmetric Hilbert–Schmidt operator in $L^2_\mu(W_T^A)$, since $\int (\int_0^T |\nabla U(x_s)|\, ds)^2\, d\mu$ is finite.
7. We denote by tr_μ the trace in $L^2_\mu(W_T^A)$.
8. Let $I(\ |P)$ be the relative entropy with respect to P:

$$I(\mu|P) = \begin{cases} \displaystyle\int \log \frac{d\mu}{dP}\, d\mu, & \text{if } \mu \ll P, \\ +\infty, & \text{otherwise.} \end{cases}$$

PROPOSITION 2.2 (Definition). *We can define a map Γ from $\mathscr{M}$ into $\mathbb{R}$ by*

$$\Gamma(\mu) = -\frac{1}{2}\, \mathrm{tr}_\mu \log\big(I + \beta^2 \mathscr{B}_T\big) + \int_0^\infty \mathrm{tr}_\mu\big(\mathscr{A}_T \exp\{-\lambda \mathscr{B}_T\}\big)^2 \exp\left\{\frac{-\lambda}{\beta^2}\right\} d\lambda \tag{2}$$

and a map H from $\mathscr{M}_1^+(W_T^A)$ into $\mathbb{R}$ by

$$H(\mu) = \begin{cases} I(\mu|P) - \Gamma(\mu), & \textit{if } I(\mu|P) < \infty, \\ +\infty, & \textit{otherwise.} \end{cases}$$

PROOF. We first show that Γ is well defined and finite for any μ in $\mathscr{M}$ [see (11) too].

Indeed, as $\mathscr{B}_T$ is a nonnegative Hilbert–Schmidt operator, $\mathrm{tr}_\mu \log(I + \beta^2 \mathscr{B}_T)$ is well defined and is finite according to (11) for any $\mu \in \mathscr{M}_1^+(W_T^A)$.

Moreover, since $\exp\{-\lambda \mathscr{B}_T\}$ is a bounded operator and $\mathscr{A}_T$ is Hilbert–Schmidt for $\mu \in \mathscr{M}$, $\mathscr{A}_T \exp\{-\lambda \mathscr{B}_T\}$ is Hilbert–Schmidt and its square is trace class. Further, since $\mathscr{B}_T$ is nonnegative, $\mathrm{tr}_\mu(\mathscr{A}_T \exp\{-\lambda \mathscr{B}_T\})^2 \le \mathrm{tr}_\mu(\mathscr{A}_T)^2$. So, for any μ in $\mathscr{M}$, the second term in the right-hand side of (2) exists and is bounded.

Moreover, we will see later (see Lemma A.8) that, when $I(\mu|P)$ is finite, $\int (\int_0^T |\nabla U(x_s)|\, ds)^2\, d\mu$ is finite so that $\{\mu \in \mathscr{M}_1^+(W_T^A)/I(\mu|P) < +\infty\} \subset \mathscr{M}$. Thus, H is well defined and finite on $\{\mu \in \mathscr{M}_1^+(W_T^A)/I(\mu|P) < +\infty\}$. $\square$

We shall prove the following theorem.

THEOREM 2.3. *If $2\beta^2 A^2 T < 1$, then we have the following:*

(i) *H is a good rate function; that is, H takes its values in $[0, +\infty]$ and, for all $M \in \mathbb{R}$, $\{H \le M\}$ is a compact subset of $\mathscr{M}_1^+(W_T^A)$.*

(ii) *For any closed subset F of $\mathscr{M}_1^+(W_T^A)$,*

$$\limsup_{N\to\infty} \frac{1}{N} \log Q_\beta^N\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^i} \in F\right) \le -\inf_F H.$$

From Theorem 2.3, we can deduce the following quenched large deviation upper bound as in [3].

THEOREM 2.4. *If $2\beta^2 A^2 T < 1$, for any closed subset F of $\mathscr{M}_1^+(W_T^A)$ and for almost all J,*

$$\limsup_{N\to\infty} \frac{1}{N} \log P_\beta^N(J)\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^i} \in F\right) \le -\inf_F H.$$

We omit the proof that Theorem 2.3 implies Theorem 2.4 since it parallels the proof given in [3], Appendix C. The strategy of the proof of Theorem 2.3 is the following.

1. First, we prove (see Section 2.2) that Q_β^N is absolutely continuous with respect to $P^{\otimes N}$ and that the Radon–Nikodym derivative of Q_β^N with respect to $P^{\otimes N}$ is equal, in the large deviation scaling, to $\exp\{N\Gamma(\hat\mu^N)\}$. Hence, according to Laplace-type methods, Theorem 2.3(ii) is not surprising (see [2] and [7]).
2. Once we are motivated by this last result, we study H and prove that it is a good rate function.
3. Finally, following a method very similar to the one we developed in [3], Section 3, we prove the upper bound result.

2.2. *Study of Q_β^N.* We first show that Q_β^N is absolutely continuous with respect to $P^{\otimes N}$ and give the Radon–Nykodim derivative of Q_β^N with respect to $P^{\otimes N}$.

The Girsanov theorem implies that, for almost all couplings J, $P_\beta^N(J)$ is absolutely continuous with respect to $P^{\otimes N}$ and describes its Radon–Nikodym derivative. Thus, it is not hard to see that, if we denote by B^i the process defined by $B_t^i = B_t(x^i) = x_t^i - x_0^i + \int_0^t \nabla U(x_s^i)\,ds$, then

$$\begin{aligned} M_{\beta,T}^N = \mathscr{E}\left[\frac{dP_\beta^N(J)}{dP^{\otimes N}}\right] = \mathscr{E}\Bigg[\exp \sum_{j=1}^N \Bigg\{\beta \int_0^T \left(\frac{1}{\sqrt{N}}\sum_{i=1}^N J_{ji} x_t^i\right) dB_t^j \\ - \frac{\beta^2}{2}\int_0^T \left(\frac{1}{\sqrt{N}}\sum_{i=1}^N J_{ji} x_t^i\right)^2 dt\Bigg\}\Bigg], \end{aligned} \tag{3}$$

and we have the following proposition.

PROPOSITION 2.5. *We have $Q_\beta^N \ll P^{\otimes N}$ and*

$$\frac{dQ_\beta^N}{dP^{\otimes N}} = M_{\beta,T}^N.$$

In order to study the law of the empirical measure under Q_β^N, we want to prove that $M_{\beta,T}^N$ is a function of the empirical measure. More precisely, let $\mathbb{I}$ be the identity in the tensor product space $L_\mu^2(W_T^A) \otimes L_\mu^2(W_T^A)$ and $\mathrm{tr}_{\mu\otimes\mu}$ the trace in $L^2_\mu(W^A_T)\otimes L^2_\mu(W^A_T)$. We then define

$$\overline{\Gamma}(\mu)=-\frac{1}{4}\operatorname{tr}_{\mu\otimes\mu}\log\left(\frac{(\mathbb{I}+\beta^2\mathscr{B}_T\otimes I+\beta^2 I\otimes\mathscr{B}_T)}{(I+\beta^2\mathscr{B}_T)\otimes(I+\beta^2\mathscr{B}_T)}\right)$$
$$-\frac{1}{4}\operatorname{tr}_\mu\log(I+2\beta^2\mathscr{B}_T)-\beta^2T\operatorname{tr}_\mu((I+2\beta^2\mathscr{B}_T)^{-1}\mathscr{A}_T)+\frac{\beta^2T}{4}.$$

Denote in short $\hat{\mu}^N$ for the empirical measure $\frac{1}{N}\sum_{i=1}^N\delta_{x^i}$.

We are going to prove the following statement.

THEOREM 2.6.

(i) *We have, $P^{\otimes N}$ almost surely,*

$$M^N_{\beta,T}=\exp\{N\Gamma(\hat{\mu}^N)+\overline{\Gamma}(\hat{\mu}^N)\}.$$

(ii) *There exists a finite constant $C=C(\beta,T,A)$ such that, for any discrete probability measure on W^A_T, $\mu\in\mathscr{M}_1^+(W^A_T)$, if $\dim(\mu)$ denotes the dimension of the image of $\mathscr{B}_T$ in $L^2(\mu)$,*

$$\left|\overline{\Gamma}(\mu)\right|\le\tfrac{1}{2}C(1+\dim(\mu)^{1/2})(\Gamma(\mu)+1),$$

so, if $D=\exp C$, $P^{\otimes N}$ almost surely,

$$D^{-1-\sqrt{N}}\exp\left\{N\left(1-\frac{C}{\sqrt{N}}\right)\Gamma(\hat{\mu}^N)\right\}\le\frac{dQ^N_\beta}{dP^{\otimes N}}\le D^{1+\sqrt{N}}\exp\left\{N\left(1+\frac{C}{\sqrt{N}}\right)\Gamma(\hat{\mu}^N)\right\}.$$

REMARK. It is obvious that $\int(\int_0^T|\nabla U(x_s)|\,ds)^2\,dP(x)$ is finite. Hence, $\int(\int_0^T|\nabla U(x_s)|\,ds)^2\,d\hat{\mu}^N(x) = (1/N)\sum_{i=1}^N(\int_0^T|\nabla U(x^i_s)|\,ds)^2$ is $P^{\otimes N}$ almost surely finite, that is, $\hat{\mu}^N\in\mathscr{M}$, $P^{\otimes N}$ almost surely. Thus, $\Gamma(\hat{\mu}^N)$ is well defined, $P^{\otimes N}$ almost surely.

To prove Theorem 2.6, we shall use spectral theory.

2.2.1. *Spectral calculus.* In the following pages, an integer N will be given. We may regard $J=(J_{ij})_{1\le i,\,j\le N}$ as an element of the space $\mathscr{S}^N$ of the $N\times N$ real symmetric matrices. For any $(x^1,\dots,x^N)$ such that $\int_0^T|\nabla U(x^i_s)|\,ds$ is finite for any $i\in\{1,\dots,N\}$, we define two other symmetric matrices A and B in $\mathbb{R}^{N\times N}$ by

$$A_{ij}=\frac{1}{2\sqrt{N}}\left(\int_0^T x^i_t\,dB^j_t+\int_0^T x^j_t\,dB^i_t\right)$$
$$=\frac{1}{2\sqrt{N}}\left(x^i_Tx^j_T-x^i_0x^j_0+\int_0^T x^i_s\nabla U(x^j_s)\,ds+\int_0^T x^j_s\nabla U(x^i_s)\,ds-\delta_{ij}T\right),$$
$$B_{ij}=\frac{1}{N}\int_0^T x^i_tx^j_t\,dt.$$

Let λ_i be the eigenvalues of B and e_i be the eigenvectors of B in $\mathbb{R}^N$ such that $Be_i=\lambda_ie_i$. We prove the following.

PROPOSITION 2.7. *We have, $P^{\otimes N}$ almost surely,*

$$M^N_{\beta,T} = \exp\left\{\beta^2 \sum_{i,j=1}^N \frac{(e_i^* A e_j)^2}{1+\beta^2\lambda_i+\beta^2\lambda_j} - \frac{1}{4}\sum_{i,j=1}^N \log(1+\beta^2\lambda_i+\beta^2\lambda_j) - \frac{1}{4}\sum_{i=1}^N \log(1+2\beta^2\lambda_i)\right\}.$$

PROOF. If tr denotes the trace in $\mathscr{S}^N$, since $J = J^*$, we get

$$\frac{1}{\sqrt{N}}\sum_{i,j=1}^N J_{ji}\int_0^T x_t^i\,dB_t^j = \sum_{i,j=1}^N A_{ij}J_{ji} = \mathrm{tr}(AJ),$$

$$\frac{1}{N}\sum_{i,j,k=1}^N J_{ji}J_{jk}\int_0^T x_t^i x_t^k\,dt = \mathrm{tr}(JBJ^*) = \mathrm{tr}(JBJ).$$

So, since $\mathscr{E}$ denotes the expectation with respect to the Gaussian variable J, we get that, for any $x = (x^1,\dots,x^N)$ such that $\int_0^T |\nabla U(x_s^i)|\,ds$ is finite for $1 \le i \le N$ and so $P^{\otimes N}$ almost surely,

$$M^N_{\beta,T} = \mathscr{E}\big[\exp\{\beta\,\mathrm{tr}(JA) - \tfrac{1}{2}\beta^2\,\mathrm{tr}(JBJ)\}\big].$$

Using the usual rules of computation for Gaussian variables (see [13], Proposition 8.4), we get

$$\begin{aligned} M^N_{\beta,T} &= \mathscr{E}\left[\exp\left\{-\frac{1}{2}\beta^2\,\mathrm{tr}(JBJ)\right\}\right] \\ &\quad\times \exp\left\{\frac{1}{2}\beta^2\mathscr{E}\left[(\mathrm{tr}(JA))^2\frac{\exp\{-(1/2)\beta^2\,\mathrm{tr}(JBJ)\}}{\mathscr{E}[\exp\{-(1/2)\beta^2\,\mathrm{tr}(JBJ)\}]}\right]\right\}. \end{aligned} \tag{4}$$

LEMMA 2.8.

$$\begin{aligned} &\mathscr{E}\big[\exp\{-\tfrac{1}{2}\beta^2\,\mathrm{tr}(JBJ)\}\big] \\ &\quad = \exp\left\{-\tfrac{1}{4}\sum_{i,j=1}^N \log(1+\beta^2(\lambda_i+\lambda_j)) - \tfrac{1}{4}\sum_{i=1}^N \log(1+2\beta^2\lambda_i)\right\}. \end{aligned}$$

PROOF. We have chosen the $(J_{ij})_{1\le i,j\le N}$'s so that their law is invariant by rotation on $\mathbb{R}^N$; that is, for any orthogonal matrix O, the law of $(J_{ij})_{1\le i,j\le N}$ is invariant by the action $J \to OJO^*$. Thus, if O is an orthogonal matrix such that OBO^* is a diagonal matrix $D = \mathrm{diag}(\lambda_1,\dots,\lambda_N)$, then

$$\begin{aligned} &\mathscr{E}\big[\exp\{-\tfrac{1}{2}\beta^2\,\mathrm{tr}(JBJ)\}\big] \\ &\quad = \mathscr{E}\big[\exp\{-\tfrac{1}{2}\beta^2\,\mathrm{tr}(JDJ)\}\big] \\ &\quad = \exp\left\{-\tfrac{1}{4}\sum_{i,j=1}^N \log(1+\beta^2(\lambda_i+\lambda_j)) - \tfrac{1}{4}\sum_{i=1}^N \log(1+2\beta^2\lambda_i)\right\}. \qquad \square \end{aligned}$$

LEMMA 2.9.

$$\mathscr{E}\left[(\operatorname{tr}(JA))^2 \frac{\exp\{-(1/2)\beta^2 \operatorname{tr}(JBJ)\}}{\mathscr{E}[\exp\{-(1/2)\beta^2 \operatorname{tr}(JBJ)\}]}\right] = 2 \sum_{i,\, j=1}^{N} \frac{(e_i^* A e_j)^2}{1+\beta^2(\lambda_i+\lambda_j)}.$$

PROOF. Let $\widetilde{A} = OAO^*$. Since the law of J is invariant by rotation,

$$\begin{aligned}
&\mathscr{E}\left[(\operatorname{tr}(JA))^2 \frac{\exp -(1/2)\beta^2 \operatorname{tr}(JBJ)}{\mathscr{E}[\exp -(1/2)\beta^2 \operatorname{tr}(JBJ)]}\right] \\
&\qquad = \mathscr{E}\left[(\operatorname{tr}(J\widetilde{A}))^2 \frac{\exp -(1/2)\beta^2 \operatorname{tr}(JDJ)}{\mathscr{E}[\exp -(1/2)\beta^2 \operatorname{tr}(JDJ)]}\right] \\
&\qquad = \sum_{ijkl} \widetilde{A}_{ij}\widetilde{A}_{kl}\mathscr{E}\left[J_{ji}J_{lk} \frac{\exp -(1/2)\beta^2 \operatorname{tr}(JDJ)}{\mathscr{E}[\exp -(1/2)\beta^2 \operatorname{tr}(JDJ)]}\right].
\end{aligned}$$

However,

$$\begin{aligned}
&\mathscr{E}\left[J_{ij}J_{kl} \frac{\exp -(1/2)\beta^2 \operatorname{tr}(JDJ)}{\mathscr{E}[\exp -(1/2)\beta^2 \operatorname{tr}(JDJ)]}\right] \\
&\qquad = \begin{cases}
0, & \text{if } (j,i) \neq (k,l) \text{ and } (l,k), \\
\dfrac{1}{1+\beta^2(\lambda_i+\lambda_j)}, & \text{if } (j,i) = (k,l) \text{ or } (l,k),\ i \neq j, \\
\dfrac{2}{1+\beta^2(\lambda_i+\lambda_j)}, & \text{if } i=j=k=l.
\end{cases}
\end{aligned}$$

Since $\widetilde{A} = \widetilde{A}^*$, we conclude

$$\mathscr{E}\left[(\operatorname{tr}(JA))^2 \frac{\exp -(1/2)\beta^2 \operatorname{tr}(JBJ)}{\mathscr{E}[\exp -(1/2)\beta^2 \operatorname{tr}(JBJ)]}\right] = 2 \sum_{i,\, j=1}^{N} \frac{\widetilde{A}_{ij}^2}{1+\beta^2(\lambda_i+\lambda_j)}.$$

Finally, according to the definition of O, if e_i is the eigenvector of B associated with the eigenvalue λ_i, then $\widetilde{A}_{ij} = e_i^* A e_j$, so we have proved Lemma 2.9. □

According to (4), Lemmas 2.8 and 2.9 give Proposition 2.7.

2.2.2. *Proof of Theorem* 2.6. We shall now use Proposition 2.7 to express $M_{\beta,T}^N$ as a function of the empirical measure (and of N). To this end, we shall use that $L^2_{\hat{\mu}^N}(W_T^A)$ and $\mathbb{R}^N$ are isomorphic whenever the x^i's are distinct, and so $P^{\otimes N}$-a.s. More precisely, we shall prove that the operator B in $\mathbb{R}^N$ and the integral operator $\mathscr{B}_T$ on $L^2_{\hat{\mu}^N}(W_T^A)$ with kernel $\int_0^T x_t y_t \, dt$ are identical after the natural identification of $\mathbb{R}^N$ and $L^2_{\hat{\mu}^N}$ (when the x^i's are distinct). For convenience, we state without proof the following trivial identification.

PROPOSITION 2.10. *Let* $(x^1, \dots, x^N) \in (W_T^A)^N$ *and* $\hat{\mu}^N = (1/N)\sum_{i=1}^N \delta_{x^i}$.

(i) *Let*

$$\psi\colon L^2_{\hat{\mu}^N}(W_T^A) \mapsto \mathbb{R}^N,$$
$$Z \to \frac{1}{\sqrt{N}}(Z(x^1), \dots, Z(x^N)).$$

Then ψ *is an isomorphism from* $(L^2_{\hat{\mu}^N}(W_T^A), \langle\,,\,\rangle_{\hat{\mu}^N})$ *into* $\mathbb{R}^N$ *endowed with the Euclidean scalar product. Moreover,*

$$\psi\mathscr{B}_T = B\psi.$$

As a consequence, if (E_1, E_2) *are eigenvectors of* $\mathscr{B}_T$ *with eigenvalues* (λ_1, λ_2), *then* $(\psi(E_1), \psi(E_2))$ *are eigenvectors of* B *with eigenvalues* (λ_1, λ_2) *and, for* $(i, j) \in \{1, 2\}$, $\psi(E_i)^*\psi(E_j) = \langle E_i, E_j\rangle_{\hat{\mu}^N}$.

(ii) *If the* x^i *are distinct, there exists an orthonormal basis* $(E_i)_{1\le i\le N}$ *of eigenvectors of* $\mathscr{B}_T$ *in* $L^2_{\hat{\mu}^N}(W_T^A)$ *with eigenvalues* $(\lambda_i)_{1\le i\le N}$; $\mathscr{B}_T E_i = \lambda_i E_i$. *Then* $(\psi(E_i))_{1\le i\le N}$ *is an orthonormal basis of eigenvectors of* B *and* $B\psi(E_i) = \lambda_i\psi(E_i)$.

COROLLARY 2.11. *Almost surely* $P^{\otimes N}$ *the operators* $\mathscr{B}_T$ *on* $L^2_{\hat{\mu}^N}(W_T^A)$ *and* B *on* $\mathbb{R}^N$ *have the same eigenvalues and there exists a one-to-one map between their* N *eigenvectors.*

Corollary 2.11 is a direct consequence of Proposition 2.10(ii) since, as P is the law of a diffusion, P does not put mass on points of W_T^A so that the x^i are $P^{\otimes N}$ almost surely distinct.

As a consequence of Proposition 2.7, Proposition 2.10 and Corollary 2.11, we find the following.

PROPOSITION 2.12. *We have,* $P^{\otimes N}$ *almost surely,*

$$\begin{aligned}\log M^N_{\beta,T} = &-\frac{1}{4}\operatorname{tr}_{\hat{\mu}^N\otimes\hat{\mu}^N}\log\bigl(\mathbb{I} + \beta^2\mathscr{B}_T\otimes I + \beta^2 I\otimes\mathscr{B}_T\bigr)\\ &+ \beta^2 N\operatorname{tr}_{\hat{\mu}^N\otimes\hat{\mu}^N}\bigl(\bigl(\mathbb{I} + \beta^2 I\otimes\mathscr{B}_T + \beta^2\mathscr{B}_T\otimes I\bigr)^{-1}\mathscr{A}_T\otimes\mathscr{A}_T\circ\mathscr{S}\bigr)\\ &- \beta^2 T\operatorname{tr}_{\hat{\mu}^N}\bigl(\bigl(I + 2\beta^2\mathscr{B}_T\bigr)^{-1}\mathscr{A}_T\bigr) - \frac{1}{4}\operatorname{tr}_{\hat{\mu}^N}\log\bigl(I + 2\beta^2\mathscr{B}_T\bigr) + \frac{\beta^2T^2}{4},\end{aligned}$$

where $\mathbb{I}$ *denotes the identity in the tensor product space* $L^2_{\hat{\mu}^N}(W_T^A)\otimes L^2_{\hat{\mu}^N}(W_T^A)$ *and* $\mathscr{S}$ *the symmetry operator in* $L^2_{\hat{\mu}^N}(W_T^A)\otimes L^2_{\hat{\mu}^N}(W_T^A)$ *such that, for any* $(f, g) \in L^2_{\hat{\mu}^N}(W_T^A)$,

$$\mathscr{S}(f\otimes g) = g\otimes f.$$

PROOF. We stated in Proposition 2.7 that

$$
\begin{aligned}
\log M^N_{\beta,T} = {} & -\frac{1}{4}\sum_{i,\,j=1}^{N}\log(1+\beta^2\lambda_i+\beta^2\lambda_j) - \frac{1}{4}\sum_{i=1}^{N}\log(1+2\beta^2\lambda_i) \\
& + \beta^2\sum_{i,\,j=1}^{N}\frac{(e_i^*Ae_j)^2}{1+\beta^2(\lambda_i+\lambda_j)}.
\end{aligned} \tag{5}
$$

According to Corollary 2.11, $P^{\otimes N}$-a.s, the operators B in $\mathbb{R}^N$ and $\mathscr{B}_T$ in $L^2_{\hat\mu^N}(W^A_T)$ have the same eigenvalues $(\lambda_i)_{1\le i\le N}$, so that

$$
\sum_{i=1}^{N}\log(1+2\beta^2\lambda_i) = \mathrm{tr}_{\hat\mu^N}\log(I+2\beta^2\mathscr{B}_T) \tag{6}
$$

and

$$
\sum_{i,\,j=1}^{N}\log(1+\beta^2\lambda_i+\beta^2\lambda_j) = \mathrm{tr}_{\hat\mu^N\otimes\hat\mu^N}\log(\mathbb{I}+\beta^2\mathscr{B}_T\otimes I+\beta^2 I\otimes\mathscr{B}_T). \tag{7}
$$

We now focus on $\sum_{i,\,j=1}^{N}(e_i^*Ae_j)^2/1+\beta^2(\lambda_i+\lambda_j)$. It is an easy matter to see that this term does not depend on the choice of the basis of eigenvectors of B. Let $(E_i)_{1\le i\le N}$ be an orthonormal basis of eigenvectors of $\mathscr{B}_T$. We choose $(e_i=\psi(E_i))_{1\le i\le N}$ as in Proposition 2.10(ii).

Then

$$
\begin{aligned}
e_i^*Ae_j &= N^{-1}\sum_{k,\,l=1}^{N}E_i(x^k)A_{k,\,l}E_j(x^l) \\
&= N^{-3/2}\sum_{k,\,l=1}^{N}E_i(x^k)\frac{1}{2}\biggl(\int_0^T x_t^k\,dB_t(x^l)+\int_0^T x_t^l\,dB_t(x^k)\biggr)E_j(x^l) \\
&= N^{1/2}\langle E_i,\mathscr{A}_T E_j\rangle_{\hat\mu^N} - \frac{T}{2}N^{-1/2}\delta_{ij},
\end{aligned}
$$

so that

$$
\begin{aligned}
\sum_{i,\,j=1}^{N}\frac{(e_i^*Ae_j)^2}{1+\beta^2(\lambda_i+\lambda_j)} = {} & N\sum_{i,\,j=1}^{N}\frac{\langle E_i,\mathscr{A}_T E_j\rangle^2_{\hat\mu^N}}{1+\beta^2(\lambda_i+\lambda_j)} \\
& - T\sum_{i=1}^{N}\frac{\langle E_i,\mathscr{A}_T E_i\rangle_{\hat\mu^N}}{1+2\beta^2\lambda_i} + \frac{T^2}{4}.
\end{aligned} \tag{8}
$$

However,

$$
\langle E_i,\mathscr{A}_T E_j\rangle^2_{\hat\mu^N} = \langle E_i\otimes E_j,(\mathscr{A}_T\otimes\mathscr{A}_T\circ\mathscr{S})E_i\otimes E_j\rangle_{\hat\mu^N\otimes\hat\mu^N},
$$

$$
\frac{1}{1+\beta^2(\lambda_i+\lambda_j)} = \langle E_i\otimes E_j,(\mathbb{I}+\beta^2 I\otimes\mathscr{B}_T+\beta^2\mathscr{B}_T\otimes I)^{-1}E_i\otimes E_j\rangle_{\hat\mu^N\otimes\hat\mu^N}
$$

and since $(E_i \otimes E_j)_{1\le i,\, j\le N}$ is an orthonormal basis of the tensor product space $L^2_{\hat\mu^N}(W_T^A) \otimes L^2_{\hat\mu^N}(W_T^A)$, we deduce that

$$
\begin{aligned}
(9)\qquad & \sum_{i,\, j=1}^{N} \frac{\langle E_i, \mathscr{A}_T E_j\rangle^2_{\hat\mu^N}}{1+\beta^2(\lambda_i+\lambda_j)} \\
&\quad = \mathrm{tr}_{\hat\mu^N\otimes\hat\mu^N}\big(\big(\mathbb{I}+\beta^2 I\otimes\mathscr{B}_T+\beta^2\mathscr{B}_T\otimes I\big)^{-1}\mathscr{A}_T\otimes\mathscr{A}_T\circ\mathscr{S}\big).
\end{aligned}
$$

Equations (5)–(9) achieve the proof of Proposition 2.12. □

PROOF OF THEOREM 2.6(i). We show here that Theorem 2.6(i) is equivalent to Proposition 2.12. In fact, we can see that

$$
\begin{aligned}
&\beta^2\, \mathrm{tr}_{\hat\mu^N\otimes\hat\mu^N}\big(\big(\mathbb{I}+\beta^2 I\otimes\mathscr{B}_T+\beta^2\mathscr{B}_T\otimes I\big)^{-1}\mathscr{A}_T\otimes\mathscr{A}_T\circ\mathscr{S}\big) \\
&\quad = \int_0^\infty \mathrm{tr}_{\hat\mu^N}(\mathscr{A}_T \exp\{-\lambda\mathscr{B}_T\})^2 \exp\left\{-\frac{\lambda}{\beta^2}\right\} d\lambda
\end{aligned}
$$

in view of the following resolvent formula.

LEMMA 2.13.

$$
\begin{aligned}
&\beta^2\left(\mathbb{I}+\beta^2 I\otimes\mathscr{B}_T+\beta^2\mathscr{B}_T\otimes I\right)^{-1} \\
&\quad = \int_0^\infty \exp-\{\lambda\mathscr{B}_T\}\otimes\exp-\{\lambda\mathscr{B}_T\}\exp\left\{-\frac{\lambda}{\beta^2}\right\} d\lambda.
\end{aligned}
$$

The proof of this lemma is trivial as soon as we notice that this equality is true on the orthonormal basis $(E_i\otimes E_j)_{1\le i,\, j\le N}$ of $L^2_{\hat\mu^N}(W_T^A)\otimes L^2_{\hat\mu^N}(W_T^A)$.

Thus, by definition of $\overline{\Gamma}$, Proposition 2.12 implies that

$$
\begin{aligned}
\log M^N_{\beta,T} &= -\frac{1}{2}N\,\mathrm{tr}_{\hat\mu^N}\log\big(I+\beta^2\mathscr{B}_T\big)+\overline{\Gamma}(\hat\mu^N) \\
&\qquad + N\int_0^\infty tr_{\hat\mu^N}(\mathscr{A}_T\exp\{-\lambda\mathscr{B}_T\})^2\exp\left\{-\frac{\lambda}{\beta^2}\right\}d\lambda \\
&= N\Gamma(\hat\mu^N)+\overline{\Gamma}(\hat\mu^N).
\end{aligned}
$$

PROOF OF THEOREM 2.6(ii). We finally bound $\overline{\Gamma}$.

LEMMA 2.14. *There exists a finite constant $C=C(\beta,A,T)$ such that, for any probability measure μ on W_T^A,*

$$
\big|\bar\Gamma(\mu)\big| \le C(\dim(\mu)^{1/2}+1)(1+\Gamma(\mu)),
$$

where $\dim(\mu)$ *denotes the dimension of the image of $\mathscr{B}_T$ in $L^2(\mu)$.*

PROOF. Let λ_i be the eigenvalues of $\mathscr{B}_T$ in $L^2_\mu(W_T^A)$. Then

$$\overline{\Gamma}(\mu) = -\frac{1}{4}\sum_{i,j=0}^{\infty}\log\left(\frac{(1+\beta^2\lambda_i+\beta^2\lambda_j)}{(1+\beta^2\lambda_i)(1+\beta^2\lambda_j)}\right) - \frac{1}{4}\sum_{i=0}^{\infty}\log(1+2\beta^2\lambda_i) + \frac{\beta^2T^2}{4} - T\beta^2\sum_{i=1}^{\infty}\frac{\langle E_i, \mathscr{A}_T E_i\rangle_\mu}{1+2\beta^2\lambda_i}. \tag{10}$$

Since the λ_i's are nonnegative, for any $i \in \mathbb{N}$, $0 \le \log(1+2\beta^2\lambda_i) \le 2\beta^2\lambda_i$. So

$$0 \le \sum_{i=0}^{\infty}\log(1+2\beta^2\lambda_i) \le 2\beta^2\sum_{i=0}^{\infty}\lambda_i = 2\beta^2 \operatorname{tr}_\mu \mathscr{B}_T \le 2\beta^2A^2T. \tag{11}$$

Moreover, for any positive real numbers (a, b), we have the elementary inequality

$$\exp\{-ab\} \le \frac{1+a+b}{(1+a)(1+b)} \le 1.$$

So

$$-(\beta^2A^2T)^2 \le -\beta^4(\operatorname{tr}_\mu(\mathscr{B}_T))^2 = -\beta^4\sum_{i,j=0}^{\infty}\lambda_i\lambda_j \le \sum_{i,j=0}^{\infty}\log\left(\frac{1+\beta^2(\lambda_i+\lambda_j)}{(1+\beta^2\lambda_i)(1+\beta^2\lambda_j)}\right) \le 0. \tag{12}$$

Finally, we observe that for any real numbers (a, b) and any positive α, we have

$$2|ab| \le \alpha a^2 + \alpha^{-1}b^2$$

so that

$$\sum_i\left|\frac{\langle E_i, \mathscr{A}_T E_i\rangle_\mu}{1+2\beta^2\lambda_i}\right| \le \dim(\mu)^{1/2}\sum_i\frac{\langle E_i, \mathscr{A}_T E_i\rangle_\mu^2}{1+2\beta^2\lambda_i} + \dim(\mu)^{1/2} \le \dim(\mu)^{1/2}(\Gamma(\mu) + \beta^2A^2T) + \dim(\mu)^{1/2}, \tag{13}$$

where we have used the bound (11) in the last line.

Lemma 2.14 is a direct consequence of (10)–(13). □

2.3. *Continuity properties of* Γ. In order to study the rate function H and to prove the large deviations upper bound theorem, we first have to study the map Γ. Since this study is rather heavy and technical, we will only state the results here, leaving the proofs and details in the Appendix. To this end, let us first define linear functions Λ_ν which are given, for any probability measure ν on W_T^A, by

$$\Lambda_\nu(\mu) = \tfrac{1}{2}\beta^2\int_0^T ds\int_0^T dt\langle (I+\beta^2\mathscr{B}_T)^{-1}X_s, X_t\rangle_\nu\langle \nabla U(x_s), \nabla U(x_t)\rangle_\mu.$$

Then Γ can be approximated by the sum of a continuous function and a linear function in the following sense.

PROPOSITION 2.15. *There exists a finite constant C_0 such that, for any probability measure μ in $\mathscr{M}$, for any probability measure ν in $\mathscr{M}_1^+(W_T^A)$, for any positive real number M, there exists a bounded continuous function Γ^M such that*

$$\left|\Gamma(\mu) - \Gamma^M(\mu) - \Lambda_\nu(\mu)\right| \le C_0\left(\frac{1}{M} + d_T(\mu, \nu)\right)C(\mu),$$

where d_T is the Wasserstein distance which is defined by

$$d_T(\mu, \nu) = \inf\left\{\int \sup_{s\le T} |x_s^1 - x_s^2|^2 \, d\xi(x^1, x^2)\right\}^{1/2} \tag{14}$$

where the infimum is taken on the probability measures ξ on $W_T^A \times W_T^A$ with marginals μ and ν and $C(\mu) = (\int(\int_0^T \nabla \cup (x_t) dt)^2 d\mu(x))^{3/2} + s$.

2.4. *H is a good rate function.* Let us now show that H is a good rate function, that is, Theorem 2.3(i). We first prove that H is nonnegative. This fact is not trivial since we cannot prove a large deviation lower bound. In order to see that, we first derive an alternative expression for Γ, which will also be useful for identifying the minima of H.

2.4.1. *An alternative expression for Γ.* We denote by X_s the evaluation at time s, that is, the map from W_T^A into $\mathbb{R}$ such that for any $x \in W_T^A$, $X_s(x) = x_s$.

We denote by a_t the function in $L_\mu^2(W_T^A) \otimes L_\mu^2(W_T^A)$ defined by

$$a_t = \tfrac{1}{2}\left(X_t \otimes X_t - X_0 \otimes X_0 + \int_0^t X_s \otimes \nabla U(X_s)\, ds + \int_0^t \nabla U(X_s) \otimes X_s \, ds\right).$$

According to Itô's formula, a_t is also given, under any probability measure $\mu \ll P$, by

$$a_t = \tfrac{1}{2}\left(\int_0^t dB_s \otimes X_s + \int_o^t X_s \otimes dB_s\right),$$

where $B_t(x) = x_t - x_0 + \int_0^t \nabla U(x_s)\, ds$.

We then define, for any probability measure μ in $\mathscr{M}$, a function F^μ in $L_\mu^2(W_T^A)$ by

$$F_t^\mu(x) = 2\int y_t\big(\mathbb{I} + \beta^2 \mathscr{B}_t \otimes I + \beta^2 I \otimes \mathscr{B}_t\big)^{-1} a_t(x, y)\, d\mu(y).$$

Let μ satisfy $I(\ |P) < \infty$. Then B is a semimartingale under μ according to Girsanov's theorem. Moreover, F^μ is previsible and belongs to $L_\mu^2(W_T^A)$, so that $\int_0^T F_s^\mu \, dB_s$ is well defined under μ and belongs to $L_\mu^1(W_T^A)$.

LEMMA 2.16. *Let $\mu \in \{I(\ |P) < +\infty\}$; then*

$$\Gamma(\mu) = \int\left(\beta^2 \int_0^T F_t^\mu(x)\, dB_t(x) - \frac{\beta^4}{2}\int_0^T (F_t^\mu(x))^2\, dt\right) d\mu(x).$$

PROOF. Let $\mathscr{C}_T = (\mathbb{I} + \beta^2 \mathscr{B}_T \otimes I + \beta^2 I \otimes \mathscr{B}_T)^{-1}$.

By the definition of Γ_2, we have

$$\frac{1}{\beta^2}\Gamma_2(\mu) = \mathrm{tr}_{\mu\otimes\mu}(\mathscr{C}_T \mathscr{A}_T \otimes \mathscr{A}_T \circ \mathscr{S}).$$

Since a_T is the kernel of $\mathscr{A}_T$, we get

$$\begin{aligned}\frac{1}{\beta^2}\Gamma_2(\mu) &= \langle a_T, \mathscr{C}_T a_T\rangle_\mu \\ &= \frac{1}{4}\bigg\langle \int_0^T X_t \otimes dB_t + \int_0^T dB_t \otimes X_t, \mathscr{C}_T \\ &\qquad \times \left(\int_0^T X_t \otimes dB_t + \int_0^T dB_t \otimes X_t\right)\bigg\rangle_\mu,\end{aligned}$$

where we write $\langle\,,\rangle_\mu$ instead of $\langle\,,\rangle_{\mu\otimes\mu}$ for the scalar product in $L^2_\mu(W_T^A) \otimes L^2_\mu(W_T^A)$ for simplification.

At this point, we have not used the existence of stochastic integrals against B since a_T is pointwise defined. We shall now take into account that we suppose that $I(\mu|P)$ is finite, so that $\mu \ll P$ and $\int_0^T X_t \otimes dB_t$ is well defined in $L^2_\mu(W_T^A) \otimes L^2_\mu(W_T^A)$.

Since $\mathscr{C}_T$ is symmetric, we get

$$\frac{1}{\beta^2}\Gamma_2(\mu) = \frac{1}{2}\bigg\langle \int_0^T X_t \otimes dB_t, \mathscr{C}_T\left(\int_0^T X_t \otimes dB_t + \int_0^T dB_t \otimes X_t\right)\bigg\rangle_\mu. \tag{15}$$

We want to apply Itô's formula in (15). To this end, we study the martingale properties of the processes contained in the bracket of the right-hand side of (15). We first observe that

$$\begin{aligned}\mathscr{C}_T \int_0^T X_t \otimes dB_t &= (\mathbb{I} + \beta^2 \mathscr{B}_T \otimes I)^{-1} \int_0^T X_t \otimes dB_t \\ &\quad - \beta^2 (\mathbb{I} + \beta^2 \mathscr{B}_T \otimes I)^{-1} \mathscr{C}_T I \otimes \mathscr{B}_T \int_0^T X_t \otimes dB_t.\end{aligned} \tag{16}$$

However,

$$I \otimes \mathscr{B}_T \int_0^T X_t \otimes dB_t(x, y) = \int d\mu(z) \int_0^T y_t z_t\, dt \int_0^T x_t\, dB_t(z),$$

so that, using the semimartingale representation of B, we see that $I \otimes \mathscr{B}_T \int_0^T X_t \otimes dB_t$ has finite variations. As a consequence, $(\mathbb{I}+\beta^2\mathscr{B}_T \otimes I)^{-1}\mathscr{C}_T I \otimes \mathscr{B}_T \int_0^T X_t \otimes dB_t$ has finite variations. Moreover, $(\mathbb{I}+\beta^2\mathscr{B}_T\otimes I)^{-1}\int_0^T X_t \otimes dB_t = \int_0^T (I+\beta^2\mathscr{B}_T)^{-1} X_t \otimes dB_t$ and, for any $y \in W_T^A$, $(\int_0^s (I+\beta^2\mathscr{B}_T)^{-1} X_t(y) \otimes dB_t)_{s\le T}$ is a martingale under P with martingale bracket with $\int_0^s y_t\, dB_t$ equal to $\int_0^s \{y_t (I + \beta^2 \mathscr{B}_T)^{-1} X_t(y)\}\, dt$.

As a conclusion, (16) implies that, for any $y \in W_T^A$, $(\mathscr{C}_T \int_0^s X_t \otimes dB_t(y,.))_{s\le T}$ is a semimartingale whose martingale bracket with $\int_0^s y_t\, dB_t$ is equal to $\int_0^s \{y_t (I+\beta^2 \mathscr{B}_T)^{-1} X_t(y)\}\, dt$. Hence, Itô's formula implies that

$$
\begin{aligned}
&\left\langle \int_0^T X_t \otimes dB_t, \mathscr{C}_T \int_0^T X_t \otimes dB_t \right\rangle_\mu \\
&\qquad = \int \left(\int \int_0^T y_t\, dB_t(x) \mathscr{C}_T \int_0^T X_t \otimes dB_t(y,x)\, d\mu(x) \right) d\mu(y) \\
&\qquad = 2 \left\langle \int_0^T X_t \otimes dB_t, \mathscr{C}_T \int_0^t X_s \otimes dB_s \right\rangle_\mu + \int_0^T \langle X_t, (I+\beta^2 \mathscr{B}_T)^{-1} X_t \rangle_\mu \, dt.
\end{aligned}
$$

Similarly, we find

$$
\left\langle \int_0^T X_t \otimes dB_t,\ \mathscr{C}_T \int_0^T dB_t \otimes X_t \right\rangle_\mu = 2 \left\langle \int_0^T X_t \otimes dB_t,\ \mathscr{C}_T \int_0^t dB_s \otimes X_s \right\rangle_\mu,
$$

so that we have proved

$$
\frac{1}{\beta^2} \Gamma_2(\mu) = 2 \left\langle \int_0^T dB_t \otimes X_t, \mathscr{C}_T a_t \right\rangle_\mu + \frac{1}{2} \int_0^T \langle X_t, (I+\beta^2 \mathscr{B}_T)^{-1} X_t \rangle_\mu \, dt. \tag{17}
$$

We now focus on the dependence of $\mathscr{C}_T$ on the time variable T. Let $\mathscr{D}_s$ be an integral operator in $L_\mu^2(W_T^A)$ with kernel $d_s(x,y) = x_s y_s$. Then we state the following.

LEMMA 2.17. *For any probability measure μ in W_T^A, for any (f,g) in $L_\mu^2(W_T^A) \otimes L_\mu^2(W_T^A)$,*

$$
\langle f, \mathscr{C}_T g \rangle_\mu = \langle f, g \rangle_\mu - \beta^2 \int_0^T \langle f, \mathscr{C}_t (I \otimes \mathscr{D}_t + \mathscr{D}_t \otimes I) \mathscr{C}_t g \rangle_\mu \, dt.
$$

PROOF. Let $\Delta_n = \{0 = t_0 < t_1 < \cdots < t_{n+1} = T\}$ be a subdivision of $[0,T]$. Let $|\Delta_n| = \max_{0\le k\le n} |t_{k+1} - t_k|$ and let

$$
\mathscr{B}_m = \sum_{k=1}^{m} \mathscr{D}_{t_{k-1}} (t_k - t_{k-1}), \qquad \mathscr{B}_0 = 0
$$

and

$$
\mathscr{C}_m = \left(\mathbb{I} + \beta^2 \mathscr{B}_m \otimes I + \beta^2 I \otimes \mathscr{B}_m \right)^{-1}.
$$

Then

$$
\beta^2 \sum_{k=0}^{n} \mathscr{C}_k (I \otimes \mathscr{D}_{t_k} + \mathscr{D}_{t_k} \otimes I) \mathscr{C}_{k+1} (t_{k+1} - t_k) = \mathbb{I} - \mathscr{C}_{n+1}. \tag{18}
$$

To prove Lemma 2.17, we can assume without loss of generality that $f=g=f_1\otimes f_2$ where (f_1,f_2) are in $L^2_\mu(W^A_T)$ and satisfy $\int f_1^2\,d\mu=\int f_2^2\,d\mu=1$. Then

$$\begin{aligned}
&\left|\langle f_1\otimes f_2, (\mathscr{B}_T\otimes I+I\otimes\mathscr{B}_T-\mathscr{B}_{n+1}\otimes I-I\otimes\mathscr{B}_{n+1})f_1\otimes f_2\rangle_\mu\right|\\
&\qquad=\sum_{k=0}^n\int_{t_k}^{t_{k+1}}\big(\langle f_1,(X_t-X_{t_k})\rangle_\mu^2+\langle f_2,(X_t-X_{t_k})\rangle_\mu^2\big)\,dt\\
&\qquad\le 2\int\sum_{k=0}^n\int_{t_k}^{t_{k+1}}(x_t-x_{t_k})^2\,dt\,d\mu(x).
\end{aligned}$$

But the canonical process is bounded and continuous under μ, so that

$$\lim_{|\Delta_n|\to 0}\int\sum_{k=0}^n\int_{t_k}^{t_{k+1}}(x_s-x_{t_k})^2\,ds\,d\mu(x)=0.$$

Hence

$$\lim_{|\Delta_n|\to 0}\left|\langle f_1\otimes f_2,(\mathscr{B}_T\otimes I+I\otimes\mathscr{B}_T-\mathscr{B}_{n+1}\otimes I-I\otimes\mathscr{B}_{n+1})f_1\otimes f_2\rangle_\mu\right|=0.$$

Since $\mathscr{C}_T$ and $\mathscr{C}_{n+1}$ are positive operators, we deduce

$$\lim_{|\Delta_n|\to 0}\left|\langle f_1\otimes f_2,(\mathscr{C}_{n+1}-\mathscr{C}_T)f_1\otimes f_2\rangle_\mu\right|=0$$

and, similarly,

$$\begin{aligned}
&\lim_{|\Delta_n|\to 0}\langle f_1\otimes f_2,\sum_{k=0}^n\mathscr{C}_k(I\otimes\mathscr{D}_{t_k}+\mathscr{D}_{t_k}\otimes I)\mathscr{C}_{k+1}(t_{k+1}-t_k)f_1\otimes f_2\rangle_\mu\\
&\qquad=\int_0^T\langle f_1\otimes f_2,\mathscr{C}_t(I\otimes\mathscr{D}_t+\mathscr{D}_t\otimes I)\mathscr{C}_t f_1\otimes f_2\rangle_\mu\,dt.
\end{aligned}$$

So (18) gives Lemma 2.17 when $|\Delta_n|$ tends to zero. □

Since $\int_0^t X_s\otimes dB_s$ and $\int_0^t dB_s\otimes X_s$ (and so a_t) belong to $L^2_\mu(W^A_T)\otimes L^2_\mu(W^A_T)$ for any $t\le T$, we can apply Lemma 2.17 in (17). We find

$$\begin{aligned}
\frac{1}{\beta^2}\Gamma_2(\mu)&=2\left\langle\int_0^T dB_t\otimes X_t,\mathscr{C}_t a_t\right\rangle_\mu+\frac12\int_0^T\langle X_t,(I+\beta^2\mathscr{B}_T)^{-1}X_t\rangle_\mu\,dt\\
&\quad-\beta^2\int_0^T\left\langle\mathscr{C}_t(I\otimes\mathscr{D}_t+\mathscr{D}_t\otimes I)\mathscr{C}_t\right.\\
&\qquad\left.\times\left(\int_0^t dB_s\otimes X_s+\int_0^t X_s\otimes dB_s\right),\int_0^s dB_u\otimes X_u\right\rangle_\mu dt.
\end{aligned}\tag{19}$$

We can use Itô's formula for the last term of the right-hand side of (19) as we did previously to prove (17). We find

$$\begin{aligned}
&\left\langle \mathscr{C}_t(I \otimes \mathscr{D}_t + \mathscr{D}_t \otimes I)\mathscr{C}_t\left(\int_0^t dB_s \otimes X_s + \int_0^t X_s \otimes dB_s\right), \int_0^s dB_u \otimes X_u\right\rangle_\mu \\
&\qquad = \left\langle (I \otimes \mathscr{D}_t + \mathscr{D}_t \otimes I)\mathscr{C}_t a_t, \mathscr{C}_t \int_0^t dB_u \otimes X_u\right\rangle_\mu \qquad (20)\\
&\qquad\quad - \tfrac{1}{2}\int_0^t \langle \mathscr{D}_t(I + \beta^2 \mathscr{B}_t)^{-1} X_s, (I + \beta^2 \mathscr{B}_t)^{-1} X_s\rangle_\mu \, ds.
\end{aligned}$$

Moreover, since the kernel of $\mathscr{D}_t$ is $d_t(x, y) = x_t y_t$, we find

$$\langle \mathscr{D}_t(I + \beta^2 \mathscr{B}_t)^{-1} X_s, (I + \beta^2 \mathscr{B}_t)^{-1} X_s\rangle_\mu = \langle X_t, (I + \beta^2 \mathscr{B}_t)^{-1} X_s\rangle_\mu^2$$

and

$$\begin{aligned}
&\left\langle (I \otimes \mathscr{D}_t + \mathscr{D}_t \otimes I)\mathscr{C}_t a_t, \mathscr{C}_t \int_0^t dB_u \otimes X_u\right\rangle_\mu \\
&\qquad = \int d\mu(x) \int d\mu(y) y_t \mathscr{C}_t \int_0^t dB_u \otimes X_u(x, y) \int d\mu(z) z_t \mathscr{C}_t a_t(x, z) \\
&\qquad\quad + \int d\mu(x) \int d\mu(y) y_t \mathscr{C}_t \int_0^t dB_u \otimes X_u(y, x) \int d\mu(z) z_t \mathscr{C}_t a_t(z, x) \qquad (21)\\
&\qquad = 2 \int d\mu(x) \left(\int d\mu(z) z_t \mathscr{C}_t a_t(z, x)\right)^2,
\end{aligned}$$

where (21) comes from the symmetry of the function $(x, z) \to \mathscr{C}_t a_t(x, z)$. Let

$$F_t^\mu(x) = 2 \int d\mu(y) y_t \mathscr{C}_t \, a_t(x, y).$$

Then (21) reads

$$\left\langle (I \otimes \mathscr{D}_t + \mathscr{D}_t \otimes I)\mathscr{C}_t a_t; \mathscr{C}_t \int_0^t dB_u \otimes X_u\right\rangle_\mu = \tfrac{1}{2}\int (F_t^\mu(x))^2 \, d\mu(x).$$

Thus, (20) becomes

$$\begin{aligned}
&\left\langle \mathscr{C}_t(I \otimes \mathscr{D}_t + \mathscr{D}_t \otimes I)\mathscr{C}_t\left(\int_0^t dB_s \otimes X_s + \int_0^t X_s \otimes dB_s\right), \int_0^s dB_u \otimes X_u\right\rangle_\mu \\
&\qquad = \tfrac{1}{2}\int (F_t^\mu(x))^2 \, d\mu(x) - \tfrac{1}{2}\langle X_t, (I + \beta^2 \mathscr{B}_t)^{-1} X_s\rangle_\mu^2,
\end{aligned}$$

and so (19) shows

$$\begin{aligned}
\Gamma_2(\mu) &= \int d\mu(x)\left(\beta^2 \int_0^T (F_t^\mu(x)) \, dB_t(x) - \frac{\beta^4}{2}\int_0^T (F_t^\mu(x))^2 \, dt\right) \\
&\quad + \frac{\beta^2}{2}\int_0^T \langle X_t, (I + \beta^2 \mathscr{B}_T)^{-1} X_t\rangle_\mu \, dt \qquad (22)\\
&\quad + \frac{\beta^4}{2}\int_0^T dt \int_0^t \langle X_t, (I + \beta^2 \mathscr{B}_t)^{-1} X_s\rangle_\mu^2 \, ds.
\end{aligned}$$

We can compute

$$\beta^2 \int_0^T \langle X_t, (I+\beta^2\mathscr{B}_T)^{-1} X_t\rangle_\mu \, dt + \beta^4 \int_0^T \, dt \int_0^t \langle X_t, (I+\beta^2\mathscr{B}_t)^{-1} X_s\rangle_\mu^2 \, ds$$
$$= \beta^2 \int_0^T \langle X_t, (I+\beta^2\mathscr{B}_t)^{-1} X_t\rangle_\mu \, dt = \operatorname{tr}_\mu \log(I+\beta^2\mathscr{B}_T). \tag{23}$$

Equations (22) and (23) complete the proof of Lemma 2.16. □

2.4.2. *H is nonnegative.*

LEMMA 2.18. *The rate function H maps $\mathscr{M}_1^+(W_T^A)$ into $\mathbb{R}^+$; that is, for any μ satisfying $I(\ |P) < +\infty$,*

$$\Gamma(\mu) \le I(\mu|P). \tag{24}$$

PROOF. Let $\mu \in \{I(\ |P) < +\infty\}$. We can apply Lemma 2.16. Since $F_t^\mu(x)$ is a previsible function along the canonical filtration $\mathscr{F}_t = \sigma(x_s,\ s \le t)$, under P, $M_t^\mu(x) = \beta^2 \int_0^t F_s^\mu(x)\, dB_s(x)$ is a local continuous martingale along the filtration $(\mathscr{F}_t)_{t\le T}$, with quadratic variation $\langle M^\mu\rangle_t = \beta^4 \int_0^t (F_s^\mu(x))^2 \, ds$.

Let $\tau_K = \inf\{t/|M_t^\mu - \frac{1}{2}\langle M^\mu\rangle_t| > K\}$. Since M^μ is continuous, τ_K is a stopping time for the canonical filtration. As a consequence, $m^\mu_{T\wedge\tau_K} = M^\mu_{T\wedge\tau_K} - \frac{1}{2} < M^\mu >_{T\wedge\tau_K}$ is measurable. According to the definition of τ_K, $m^\mu_{T\wedge\tau_K}$ is bounded by K.

We now apply the relative entropy property,

$$\int m^\mu_{T\wedge\tau_K}(x)\, d\mu(x) \le I(\mu|P) + \log \int \exp m^\mu_{T\wedge\tau_K}(x)\, dP(x). \tag{25}$$

But $(\exp m^\mu_{t\wedge\tau_K})_{t\le T}$ is a bounded martingale with respect to the filtration $(\mathscr{F}_{t\wedge\tau_K})_{t\le T}$. Hence, for any positive real number K,

$$\int \exp m^\mu_{T\wedge\tau_K}(x)\, dP(x) = \int \exp m^\mu_0(x)\, dP(x) = 1,$$

so that (25) becomes

$$\int m^\mu_{T\wedge\tau_K}(x)\, d\mu(x) \le I(\mu|P). \tag{26}$$

Thus, to deduce Lemma 2.18 from (26), we need to show that:

$$\lim_{K\to\infty} \int m^\mu_{T\wedge\tau_K}(x)\, d\mu(x) = \int m^\mu_T(x)\, d\mu(x). \tag{27}$$

Since $|m^\mu_{T\wedge\tau_K}| \le |m^\mu_T|$ and $m^\mu_{T\wedge\tau_K}(x)$ converges to $m^\mu_T(x)$ when K tends to infinity for any x such that $m^\mu_T(x)$ is finite, the dominated convergence theorem shows that (27) is satisfied as soon as m^μ_T belongs to $L^1(\mu)$. To establish this last point, we only need to prove that $\langle M^\mu\rangle_T = \int_0^T (F_t^\mu(x))^2\,dt$ belongs to $L^1(\mu)$. However, since $\mathscr{B}_T$ is positive,

$$\begin{aligned}
&\int\int_0^T (F_t^\mu(x))^2\,dt\,d\mu(x)\\
&\quad\le 4A^2\int_0^T\int(\mathscr{C}_t a_t(x,y))^2\,d\mu^{\otimes 2}(x,y)\,dt\\
&\quad\le A^2\int_0^T\int\left(\int_0^t y_s\,dB_s(x)+\int_0^t x_s\,dB_s(y)\right)^2 d\mu^{\otimes 2}(x,y)\,dt.
\end{aligned}$$

Using the relative entropy property and the monotone convergence theorem, we conclude that [see (29)], for any positive real number α small enough, there exists a finite constant ξ such that

$$\int\int_0^T (F_t^\mu(x))^2\,dt\,d\mu(x) \le \frac{2}{\alpha}A^2 T I(\mu|P)+\xi.$$

Thus, for any μ in $\{I(\ |P)<+\infty\}$, m_T^μ belongs to $L^1(\mu)$ so that (26) and (27) imply

$$\int m_T^\mu(x)\,d\mu(x)\le I(\mu|P),$$

that is, Corollary 2.18. □

2.4.3. *H is a good rate function.* We first show that the entropy relative to P is bounded in terms of H.

LEMMA 2.19. *If $2\beta^2A^2T<1$, there exists a strictly positive real number α and a finite constant C, $C>0$, such that*

$$H(\mu)\ge \alpha I(\mu|P)-C.$$

PROOF. Let $\mu\in\mathscr{M}_1^+(W_T^A)$.

If $I(\mu|P)=+\infty$, then $H(\mu)=+\infty$ so that Lemma 2.19 is true. Otherwise, $I(\mu|P)$ is finite so that $H(\mu)=I(\mu|P)-\Gamma(\mu)$. Moreover,

$$\Gamma(\mu)=\Gamma_1(\mu)+\Gamma_2(\mu)\le\Gamma_2(\mu),$$

but

$$\begin{aligned}
\Gamma_2(\mu)&=\beta^2\,\mathrm{tr}_{\mu\otimes\mu}\big((\mathbb{I}+\beta^2\mathscr{B}_t\otimes I+\beta^2 I\otimes\mathscr{B}_t)^{-1}\mathscr{A}_T\otimes\mathscr{A}_T\circ\mathscr{S}\big)\\
&\le\beta^2\,\mathrm{tr}_\mu(\mathscr{A}_T^2)\\
&=\frac{\beta^2}{4}\int\left(\int_0^T x_t\,dB_t(y)+\int_0^T y_t\,dB_t(x)\right)^2 d\mu^{\otimes 2}(x,y).
\end{aligned}$$

Thanks to the relative entropy properties, for any $x \in W_T^A$ and any positive real number κ:

$$\kappa\frac{\beta^2}{4}\int\left(\int_0^T x_t\,dB_t(y)+\int_0^T y_t\,dB_t(x)\right)^2 d\mu(y)$$
$$\le I(\mu|P)+\log\int\exp\left\{\kappa\frac{\beta^2}{4}\left(\int_0^T x_t\,dB_t(y)+\int_0^T y_t\,dB_t(x)\right)^2\right\}dP(y),$$

and then, for any positive real number $\varepsilon \ge 0$,

$$\begin{aligned}\kappa\varepsilon\frac{\beta^2}{4}&\int\left(\int_0^T x_t\,dB_t(y)+\int_0^T y_t\,dB_t(x)\right)^2 d\mu^{\otimes 2}(x,y)\\ &\le (1+\varepsilon)I(\mu|P)\\ &\quad+\log\int\exp\left\{\kappa\varepsilon\frac{\beta^2}{4}\left(\int_0^T x_t\,dB_t(y)+\int_0^T y_t\,dB_t(x)\right)^2\right\}dP^{\otimes 2}(x,y).\end{aligned}\tag{28}$$

Let J be a centered Gaussian variable with covariance 1.

$$\begin{aligned}\int&\exp\left\{\kappa\varepsilon\frac{\beta^2}{4}\left(\int_0^T x_t\,dB_t(y)+\int_0^T y_t\,dB_t(x)\right)^2\right\}dP^{\otimes 2}(x,y)\\ &=\mathscr{E}\left[\int\exp\left\{\sqrt{\frac{\kappa\varepsilon}{2}}\beta J\left(\int_0^T x_t\,dB_t(y)+\int_0^T y_t\,dB_t(x)\right)\right\}dP^{\otimes 2}(x,y)\right]\\ &\le\mathscr{E}\left[\left(\int\exp\left\{\kappa\varepsilon\beta^2J^2\left(\int_0^T x_t^2\,dt+\int_0^T y_t^2\,dt\right)\right\}dP^{\otimes 2}(x,y)\right)^{1/2}\right]\\ &=\mathscr{E}\left[\int\exp\left\{\kappa\varepsilon\beta^2J^2\int_0^T x_t^2\,dt\right\}dP(x)\right]\\ &\le\mathscr{E}[\exp\{\kappa\varepsilon\beta^2A^2TJ^2\}]=\frac{1}{\sqrt{1-2\kappa\varepsilon\beta^2A^2T}},\end{aligned}\tag{29}$$

where the last equality holds as soon as $2\kappa\varepsilon\beta^2A^2T<1$. However, we supposed that $2\beta^2A^2T<1$ in order to choose $\kappa\varepsilon>1$ small enough so that $2\kappa\varepsilon\beta^2A^2T<1$. We then choose $\varepsilon>0$ so that $1+\varepsilon<\kappa\varepsilon$. Hence, inequalities (28) and (29) show that we can find a strictly positive real number $\alpha=(\kappa\varepsilon-1-\varepsilon)/\kappa\varepsilon$ and a finite constant $C=1/(\kappa\varepsilon\sqrt{1-2\kappa\varepsilon\beta^2A^2T})$ such that

$$\Gamma(\mu)\le(1-\alpha)I(\mu|P)+C,$$

so that

$$H(\mu)\ge\alpha I(\mu|P)-C. \qquad\square$$

We now prove that H is lower semicontinuous. We assume in the following that $2\beta^2A^2T<1$. Take a sequence μ_k of probability measures converging to a probability measure μ and choose a subsequence (n_k) such that $\liminf_k H(\mu_k)=\lim_k H(\mu_{n_k})$.

We distinguish the case where $I(\mu_{n_k} \mid P)$ stay bounded for large k from the case where we can find a subsequence $\mu_{n_k(K)}$ such that $I(\mu_{n_k(K)} \mid P) \geq K$.

In the first case, we suppose that $I(\mu_{n_k} \mid P)$ stay bounded for k larger than some k_0. Then $C(\mu)$ is uniformly bounded by a finite constant L for $k \geq k_0$, according to Lemma A.8. Moreover, Lemma 2.15 says that, for any positive real number M, for any $k \geq k_0$,

$$\left|\Gamma(\mu_{n_k}) - \Gamma^M(\mu_{n_k}) - \Lambda_\mu(\mu_{n_k})\right| \leq C_0 L\left(\frac{1}{M} + d_T(\mu, \mu_{n_k})\right). \tag{30}$$

Hence, for any $k \geq k_0$,

$$H(\mu_{n_k}) \geq I(\mu_{n_k}|P) - \Gamma^M(\mu_{n_k}) - \Lambda_\mu(\mu_{n_k}) - C_0 L\left(\frac{1}{M} + d_T(\mu, \mu_{n_k})\right). \tag{31}$$

Let Q_μ be a probability measure on W_T^A, absolutely continuous with respect to P, such that

$$\frac{dQ_\mu}{dP}(x) = \frac{1}{Z_\mu} \exp\left\{\frac{1}{2}\beta^2 \int_0^T \int_0^T \langle (I + \beta^2 \mathscr{B}_T)^{-1} x_s, x_t\rangle_\mu \nabla U(x_s) \nabla U(x_t)\, dt\, ds\right\},$$

where

$$Z_\mu = \int \exp\left\{\tfrac{1}{2}\beta^2 \int_0^T \int_0^T \langle (I + \beta^2 \mathscr{B}_T)^{-1} x_s, x_t\rangle_\mu \nabla U(x_s) \nabla U(x_t)\, dt\, ds\right\} dP(x).$$

In the regime $2\beta^2 A^2 T < 1$, Z_μ is finite. Note that

$$\frac{dQ_\mu}{dP}(x) = \frac{1}{Z_\mu} \exp \Lambda_\mu(\delta_x).$$

Then, we can prove as in Appendix B of [3] that

$$I(\cdot|P) - \Lambda_\mu = I(\cdot|Q_\mu) - \log Z_\mu, \tag{32}$$

so that (31) becomes

$$H(\mu_{n_k}) \geq I(\mu_{n_k}|Q_\mu) - \log Z_\mu - \Gamma^M(\mu_{n_k}) - C_0 L\left(\frac{1}{M} + d_T(\mu, \mu_{n_k})\right). \tag{33}$$

Since $I(\cdot|Q_\mu)$ is l.s.c and Γ^M is continuous, (33) gives

$$\begin{aligned}
\liminf_{n\to\infty} H(\mu_n) &= \lim_{k\to\infty} H(\mu_{n_k}) \\
&\geq I(\mu|Q_\mu) - \log Z_\mu - \Gamma^M(\mu) - C_0 L \frac{1}{M} \\
&= I(\mu|P) - \Lambda_\mu(\mu) - \Gamma^M(\mu) - C_0 L \frac{1}{M} \quad \text{by (32)} \\
&\geq H(\mu) - 2C_0 L \frac{1}{M} \quad \text{by (30).}
\end{aligned}$$

Since the last inequality holds for any real number M, we conclude that $\liminf_{n\to\infty} H(\mu_n) \geq H(\mu)$.

In the other case, we can find a subsequence $n_p(K)$ such that $\lim_K I(\mu_{n_p(K)} \mid P) = +\infty$ and then Lemma 2.19 implies that

$$\liminf_{k\to\infty} H(\mu_k) = \lim_{k\to\infty} H(\mu_{n_k}) = \lim_{K\to\infty} H(\mu_{n_p(K)}) = +\infty,$$

so that we also get $\liminf_{k\to\infty} H(\mu_k) \geq H(\mu)$.

Moreover, H is a good rate function. Indeed, for any positive real number R, $\{H \leq R\}$ is a compact set as it is a closed set (H is l.s.c) which is included in a compact set, since, by Lemma 2.19, the relative entropy $I(\ \mid P)$ is bounded on $\{H \leq R\}$.

2.5. *Proof of the large deviation upper bound.* As in the asymmetric version of dynamics, we first prove an exponential tightness result, and we then prove a weak large deviation upper bound, that is, Theorem 2.3(ii) when F is compact. We finally deduce from these two results Theorem 2.3(ii) for any closed subset F.

LEMMA 2.20. *If* $2\beta^2 A^2 T < 1$, *there exists* $\alpha > 1$ *and a finite constant* C *such that*

$$\int \left(\frac{dQ_\beta^N}{dP^{\otimes N}}\right)^\alpha dP^{\otimes N} \leq C^N.$$

PROOF. With the notation of Section 2.2.1,

$$\begin{aligned}\int \left(\frac{dQ_\beta^N}{dP^{\otimes N}}\right)^\alpha dP^{\otimes N} &= \int \mathscr{E}\left[\exp\left\{\beta Tr(JA) - \frac{1}{2}\beta^2 Tr(JBJ)\right\}\right]^\alpha dP^{\otimes N} \\ &\leq \int \mathscr{E}\left[\exp\left\{\alpha\beta Tr(JA) - \frac{1}{2}\beta^2 \alpha Tr(JBJ)\right\}\right] dP^{\otimes N}.\end{aligned}$$

Let (p, q) be conjugate exponents. The Hölder inequality gives

$$\begin{aligned}\int \left(\frac{dQ_\beta^N}{dP^{\otimes N}}\right)^\alpha dP^{\otimes N} &\leq \mathscr{E}\left[\int dP^{\otimes N} \exp\left\{\alpha p\beta Tr(JA) - \frac{1}{2}\beta^2\alpha^2 p^2 Tr(JBJ)\right\}\right]^{1/p} \\ &\quad \times \mathscr{E}\left[\int dP^{\otimes N} \exp\left\{\frac{1}{2} q\beta^2\alpha(p\alpha - 1) Tr(JBJ)\right\}\right]^{1/q}.\end{aligned}$$

Recall that

$$\begin{aligned}&\exp\left\{\alpha p\beta Tr(JA) - \frac{1}{2}\beta^2\alpha^2 p^2 Tr(JBJ)\right\} \\ &\quad = \exp\left\{\alpha p\beta \sum_{i=1}^N \int_0^T \left(\frac{1}{\sqrt{N}} \sum_{j=1}^N J_{ij} x_t^j\right) dB_t^j \right. \\ &\qquad \left. - \frac{1}{2}\beta^2\alpha^2 p^2 \sum_{i=1}^N \int_0^T \left(\frac{1}{\sqrt{N}} \sum_{j=1}^N J_{ij} x_t^j\right)^2 dt\right\}\end{aligned}$$

is a supermartingale, so that we find, for conjugate exponents (p, q),

$$\int \left(\frac{dQ_\beta^N}{dP^{\otimes N}}\right)^\alpha dP^{\otimes N} \le \mathscr{E}\left[\int dP^{\otimes N} \exp \frac{1}{2} q\beta^2 \alpha(p\alpha - 1) Tr(JBJ)\right]^{1/q}. \tag{34}$$

But, if $(\lambda_i)_{1\le i\le N}$ are the eigenvalues of B, we can prove as in Lemma 2.8 that

$$\begin{aligned} &\mathscr{E}\left[\exp \tfrac{1}{2} q\beta^2\alpha(p\alpha-1)Tr(JBJ)\right] \\ &\quad = \exp\Biggl(-\tfrac{1}{4}\sum_{i,j=1}^N \log(1 - q\beta^2\alpha(p\alpha-1)(\lambda_i+\lambda_j)) \\ &\qquad\qquad - \tfrac{1}{4}\sum_{i=1}^N \log(1 - 2q\beta^2\alpha(p\alpha-1)\lambda_i)\Biggr), \end{aligned} \tag{35}$$

whenever α is close enough to one. Indeed, since the λ_i's are positive,

$$\lambda_i \le \sum_{i=1}^N \lambda_i = \frac{1}{N}\sum_{i=1}^N \int_0^T (x_t^i)^2\, dt \le A^2 T, \quad P^{\otimes N}\text{-a.s.}$$

But we supposed that $2\beta^2 A^2 T < 1$, so we can find $\alpha > 1$ small enough and two conjugate exponents p and q such that $\max_{i,j}\left(q\beta^2\alpha(p\alpha-1)(\lambda_i+\lambda_j)\right) \le 2q\beta^2\alpha(p\alpha-1)A^2T < 1$. Then, the right-hand side of (35) is finite.

More precisely, we can find a finite constant c such that, for any x smaller than $2q\beta^2\alpha(p\alpha-1)A^2T < 1$ (see Appendix B of [3]),

$$-\log(1-x) \le cx,$$

so that equality (35) implies

$$\begin{aligned} &\mathscr{E}\left[\exp \tfrac{1}{2} q\beta^2\alpha(p\alpha-1)Tr(JBJ)\right] \\ &\quad \le \exp\{cq\beta^2\alpha(p\alpha-1)A^2T\}\left(\exp\{\tfrac{1}{2}cq\beta^2\alpha(p\alpha-1)A^2T\}\right)^N, \end{aligned}$$

which proves Lemma 2.20. □

We turn to the proof of the weak upper bound.

LEMMA 2.21. *If $2\beta^2A^2T < 1$, for any compact subset K of $\mathscr{M}_1^+(W_T^A)$,*

$$\limsup_{N\to\infty} \frac{1}{N} \log Q_\beta^N(\hat{\mu}^N \in K) \le -\inf_K H.$$

PROOF. Let K be a compact subset of $\mathscr{M}_1^+(W_T^A)$. For any positive real number δ, we can cover K by a finite number p of open balls $B(\mu_i, \delta)$ for Wasserstein's distance d_T:

$$B(\mu_i,\delta) = \{\nu \in \mathscr{M}_1^+(W_T^A)/d_T(\mu_i,\nu) < \delta\}, \quad K \subset \bigcup_{1\le i\le p} B(\mu_i,\delta).$$

Let L be a positive real number and let $\mathscr{L}_L$ be defined by

$$\mathscr{L}_L = \left\{ \mu \in \mathscr{M}_1^+(W_T^A) \Big/ \int \left(\int_0^T |\nabla U(x_t)|\, dt \right)^2 d\mu(x) \le L \right\}.$$

We then bound $Q_\beta^N(\hat{\mu}^N \in K)$:

$$Q_\beta^N(\hat{\mu}^N \in K) \le Q_\beta^N(\hat{\mu}^N \in \mathscr{L}_L^c) + \sum_{i=1}^p Q_\beta^N\left(\hat{\mu}^N \in K \cap \mathscr{L}_L \cap B(\mu_i, \delta)\right). \tag{36}$$

(a) Estimate of $Q_\beta^N(\hat{\mu}^N \in \mathscr{L}_L^c)$. We use the Hölder inequality with the real number $\alpha > 1$ introduced in Lemma 2.20 and its conjugate exponent σ:

$$\begin{aligned} Q_\beta^N(\hat{\mu}^N \in \mathscr{L}_L^c) &= \int_{\hat{\mu}^N \in \mathscr{L}_L^c} \frac{dQ_\beta^N}{dP^{\otimes N}}\, dP^{\otimes N} \\ &\le \left(\int_{\hat{\mu}^N \in \mathscr{L}_L^c} \left(\frac{dQ_\beta^N}{dP^{\otimes N}} \right)^\alpha dP^{\otimes N} \right)^{1/\alpha} P^{\otimes N}(\hat{\mu}^N \in \mathscr{L}_L^c)^{1/\sigma} \\ &\le C^N P^{\otimes N}(\hat{\mu}^N \in \mathscr{L}_L^c)^{1/\sigma}. \end{aligned} \tag{37}$$

Using Chebyshev's inequality, we get, for any positive real number r,

$$\begin{aligned} P^{\otimes N}(\hat{\mu}^N \in \mathscr{L}_L^c) &\le \exp\{-rNL\} \int \exp\left[r \sum_{i=1}^N \left(\int_0^T |\nabla U(x_t^i)|\, dt \right)^2 \right] dP^{\otimes N} \\ &\le \exp\{-rNL\} \left(\int \exp\left[r \left(\int_0^T |\nabla U(x_t)|\, dt \right)^2 \right] dP \right)^N. \end{aligned} \tag{38}$$

But, if r is small enough, $\int \exp[r(\int_0^T |\nabla U(x_t)|\, dt)^2]\, dP$ is finite (see the proof of Lemma A.8), so that, in conclusion of (37) and (38), we find, in the high temperature regime $2\beta^2 A^2 T < 1$, a strictly positive real number r and a finite constant D such that we can state the following.

LEMMA 2.22. *For any positive real number L,*

$$Q_\beta^N(\hat{\mu}^N \in \mathscr{L}_L^c) \le \exp[-r(L-D)N].$$

(b) Estimate of $Q_\beta^N(\hat{\mu}^N \in \mathscr{L}_L \cap K \cap B(\mu_i, \delta))$. According to Theorem 2.6,

$$\begin{aligned} &Q_\beta^N(\hat{\mu}^N \in \mathscr{L}_L \cap K \cap B(\mu_i, \delta)) \\ &\qquad \le D^{1+\sqrt{N}} \int_{\mathscr{L}_L \cap K \cap B(\mu_i, \delta)} \exp N\left(1 + \frac{C}{\sqrt{N}}\right) \Gamma(\hat{\mu}^N)\, dP^{\otimes N}, \end{aligned}$$

so that the Hölder inequality and Lemma 2.20 imply

$$\begin{aligned} &\limsup_{N\to\infty} \frac{1}{N} \log Q_\beta^N(\hat{\mu}^N \in \mathscr{L}_L \cap K \cap B(\mu_i, \delta)) \\ &\qquad \le \limsup_{N\to\infty} \int_{\mathscr{L}_L \cap K \cap B(\mu_i, \delta)} \exp N\Gamma(\hat{\mu}^N)\, dP^{\otimes N}. \end{aligned}$$

However, we saw in Proposition 2.15 that, for any probability measure μ in $\mathscr{L}_L \subset \mathscr{M}$,

$$|\Gamma(\mu) - \Gamma^M(\mu) - \Lambda_{\mu_i}(\mu)| \le C_0\left(\frac{1}{M} + d_T(\mu, \mu_i)\right)C(\mu).$$

On the subset $\mathscr{L}_L \cap K \cap B(\mu_i, \delta)$, $C(\mu)$ is uniformly bounded by $m_L = L^{3/2}+1$ and $d_T(\mu_i, \mu)$ by δ, so that

$$\begin{aligned}
&\limsup_{N\to\infty} \int_{\mathscr{L}_L\cap K\cap B(\mu_i,\delta)} \exp[N\Gamma(\hat{\mu}^N)]\, dP^{\otimes N} \\
&\quad \le C_0\left(\frac{1}{M} + \delta\right)m_L \\
&\qquad + \limsup_{N\to\infty} \int_{K\cap B(\mu_i,\delta)} \exp[N\{\Gamma^M(\hat{\mu}^N) + \Lambda_{\mu_i}(\hat{\mu}^N)\}]\, dP^{\otimes N}.
\end{aligned} \tag{39}$$

Let Q_{μ_i} be a probability measure on W_T^A, absolutely continuous with respect to P such that

$$\frac{dQ_{\mu_i}}{dP} = \frac{1}{Z_{\mu_i}} \exp \Lambda_{\mu_i}(\delta_x).$$

The measure Q_{μ_i} is well defined in the regime $2\beta^2 A^2 T < 1$ and Z_{μ_i} is then finite. Then (38) reads

$$\begin{aligned}
&Q_\beta^N(\hat{\mu}^N \in \mathscr{L}_L \cap K \cap B(\mu_i, \delta)) \\
&\quad \le C Z_{\mu_i}^N \exp\left\{C_0\left(\frac{1}{M} + \delta\right)m_L N\right\} \\
&\qquad \times \int_{K\cap \overline{B(\mu_i,\delta)}} \exp N\{\Gamma^M(\hat{\mu}^N)\} d(Q_{\mu_i})^{\otimes N}.
\end{aligned} \tag{40}$$

Using Sanov's theorem, we deduce

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \log Q_\beta^N(\hat{\mu}^N \in \mathscr{L}_L \cap K \cap B(\mu_i, \delta)) \\
&\quad \le \log Z_{\mu_i} + C_0\left(\frac{1}{M} + \delta\right)m_L - \inf_{K\cap\overline{B(\mu_i,\delta)}} (I(\ |Q_{\mu_i}) - \Gamma^M).
\end{aligned} \tag{41}$$

As in Appendix B of [3], we find that

$$I(\mu|Q_{\mu_i}) = \begin{cases} I(\mu|P) - \Lambda_{\mu_i}(\mu) + \log Z_{\mu_i}, & \text{if } I(\mu|P) < +\infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

However, we recall (see Lemma A.5) that

$$|\Lambda_{\mu_i}(\mu) - \Lambda(\mu)| \le c' d_T(\mu, \mu_i) \int \left(\int_0^T |\nabla U(x_t)|\, dt\right)^2 d\mu,$$

and since we saw in the proof of Lemma A.8 that there exist two finite constants c_1 and c_2 such that

$$\int\biggl(\int_0^T |\nabla U(x_t)|\,dt\biggr)^2 d\mu \le c_1 I(\mu|P) + c_2,$$

we find two finite constants C_1 and C_2 such that (41) becomes

$$\begin{aligned}&\limsup_{N\to\infty}\frac{1}{N}\log Q_\beta^N(\hat\mu^N\in\mathscr{L}_L\cap K\cap B(\mu_i,\delta))\\ &\qquad\le C_2\biggl(\frac{1}{M}+\delta\biggr)m_L-\inf_{\mu\in K\cap\overline{B(\mu_i,\delta)}}((1-C_1\delta)I(\mu|P)-\Lambda(\mu)-\Gamma^M(\mu)).\end{aligned}\tag{42}$$

If we recall (36), Lemma 2.22 and (42), we proved that

$$\begin{aligned}&\limsup_{N\to\infty}\frac{1}{N}\log Q_\beta^N(\hat\mu^N\in K)\\ &\qquad\le\max\biggl\{-r(L-D),\\ &\qquad\qquad C_2\biggl(\frac{1}{M}+\delta\biggr)m_L-\inf_{\mu\in K}((1-C_1\delta)I(\mu|P)-\Lambda(\mu)-\Gamma^M(\mu))\biggr\}.\end{aligned}\tag{43}$$

We now need to show the following.

COROLLARY 2.23.

$$\lim_{\delta\downarrow 0}\lim_{M\uparrow\infty}-\inf_{\mu\in K}\bigl((1-C_1\delta)I(\mu|P)-\Lambda_\mu(\mu)-\Gamma^M(\mu)\bigr)\le-\inf_K H.$$

To this end, we give a technical lemma.

LEMMA 2.24. *If $2\beta^2A^2T<1$, there exists $\alpha<1$ and a finite constant ξ such that, for any positive real number M, for any probability measure μ,*

$$\Gamma^M(\mu)+\Lambda(\mu)\le\alpha I(\mu|P)+\xi.$$

The proof of Lemma 2.24 follows the lines of the proof of Lemma 2.19; we omit it.

PROOF OF COROLLARY 2.23. We choose δ small enough such that $\kappa=1-C_1\delta-\alpha>0$, so that Lemma 2.24 implies

$$(1-C_1\delta)I(\mu|P)-\Lambda(\mu)-\Gamma^M(\mu)\ge\kappa I(\mu|P)-\xi,\tag{44}$$

so that, if we distinguish the case where $\inf_K H=\inf_K(I(\mu|P)-\Gamma(\mu))$ is finite from the case where it is not, we find as follows:

(i) if $\inf_K H=+\infty$, then $\inf_K I(\mu|P)=+\infty$ so that (44) implies that, for any positive real number M, $\inf_{\mu\in K}((1-C_1\delta)I(\mu|P)-\Lambda_\mu(\mu)-\Gamma^M(\mu))=+\infty$;

(ii) if $\inf_K H < +\infty$, since H is a good rate function, we can find a finite real number R such that

$$\inf_K H = \inf_{K\cap\{I\le R\}} H.$$

However, for any real number R', (44) implies

$$\begin{aligned} &-\inf_{\mu\in K}((1-C_1\delta)I(\mu|P)-\Lambda_\mu(\mu)-\Gamma^M(\mu)) \\ &\qquad \le \max\Big\{-\inf_{\mu\in K\cap\{I\le R'\}}((1-C_1\delta)I(\mu|P)-\Lambda(\mu)-\Gamma^M(\mu)); -\kappa R'+\xi\Big\}. \end{aligned} \tag{45}$$

By Lemma 2.15, we know that, for any probability measure μ in $\mathscr{M}$,

$$\big|\Gamma(\mu)-\Gamma^M(\mu)-\Lambda(\mu)\big| \le \frac{C_0}{M}C(\mu),$$

so that Lemma A.8 shows that there exists a finite constant k such that, for any real number R',

$$\sup_{\{I(\ |P)\le R'\}}\big|\Gamma-\Gamma^M-\Lambda\big| \le \frac{k}{M}\big((R')^{3/2}+1\big).$$

Thus

$$\begin{aligned} &\inf_{K\cap\{I\le R'\}}((1-C_1\delta)I(\ |P)-\Lambda-\Gamma^M) \\ &\qquad \ge \inf_{K\cap\{I\le R'\}}(I(\ |P)-\Gamma)-C_1\delta R'-\frac{k}{M}((R')^{3/2}+1) \\ &\qquad = \inf_{K\cap\{I\le R'\}} H - C_1\delta R' - \frac{k}{M}((R')^{3/2}+1). \end{aligned}$$

Therefore, (45) implies that, if $c=\max\{k, C_1\}$,

$$\begin{aligned} &-\inf_{\mu\in K}\big((1-C_1\delta)I(\mu|P)-\Lambda(\mu)-\Gamma^M(\mu)\big) \\ &\qquad \le \max\Big\{-\inf_{K\cap\{I\le R'\}} H + c\Big(\delta+\frac{1}{M}\Big)\big((R')^{3/2}+1\big); -\kappa R'+\xi\Big\}, \end{aligned}$$

so that, for any real number $R'\ge R$,

$$\begin{aligned} &\lim_{\delta\downarrow 0}\lim_{M\uparrow\infty} -\inf_K((1-C_1\delta)I(\ |P)-\Lambda-\Gamma^M) \\ &\qquad \le \max\Big\{-\inf_{K\cap\{I\le R'\}}(H); -\kappa R'+\xi\Big\} = \max\Big\{-\inf_K H; -\kappa R'+\xi\Big\}. \end{aligned}$$

We finally let $R'\uparrow+\infty$ so that we get Corollary 2.23. □

We can end the proof of Lemma 2.21. If we let δ tend to zero and M tend to infinity in (43), Corollary 2.23 implies that, for any positive real number L:

$$\limsup_{N\to\infty}\frac{1}{N}\log Q_\beta^N(\hat\mu^N\in K) \le \max\Big\{-r(L-D);\ -\inf_K H\Big\}.$$

So that, letting L tend to infinity proves Lemma 2.21. □

We finally deduce Theorem 2.3(ii) from Lemmas 2.20 and 2.21; that is, we show the following.

LEMMA 2.25. *If $2\beta^2 A^2 T < 1$, for any closed set F of $\mathscr{M}_1^+(W_T^A)$,*

$$\limsup_{N\to\infty} \frac{1}{N} \log Q_\beta^N(\hat{\mu}^N \in F) \le -\inf_F H.$$

PROOF. It is well known (see, for instance, [8], Lemma 3.2.7) that the law of the empirical measure under $P^{\otimes N}$ is exponentially tight; that is, for any real number L, there exists a compact subset K_L of $\mathscr{M}_1^+(W_T^A)$ so that

$$P^{\otimes N}(\hat{\mu}^N \in K_L^c) \le \exp\{-LN\}.$$

Then Lemma 2.20 implies that the law of the empirical measure under Q_β^N is exponentially tight since, if $\alpha > 1$ is chosen as in Lemma 2.20 and σ is the conjugate exponent of α, we have

$$\begin{aligned} Q_\beta^N(\hat{\mu}^N \in K_L^c) &\le \left(\int \left(\frac{dQ_\beta^N}{dP^{\otimes N}}\right)^\alpha dP^{\otimes N}\right)^{1/\alpha} P^{\otimes N}(\hat{\mu}^N \in K_L^c)^{1/\sigma} \\ &\le (C^{1/\alpha})^N \exp\left\{-\frac{L}{\sigma}N\right\}. \end{aligned}$$

Thus, the weak large deviation upper bound of Lemma 2.21 implies Lemma 2.25 (see [8], Lemma 2.15, page 40). □

3. Existence and uniqueness of the minima of the rate function. We shall use Theorem 2.3 to study the convergence of the law $\Pi_{\beta,T}^N$ of the empirical measure under Q_β^N.

We recall that, for any probability measure μ in $\mathscr{M}$, we defined in Lemma 2.16 a function F_t^μ on W_T^A by

$$F_t^\mu(x) = 2\int d\mu(y) y_t \left(\mathbb{I} + \beta^2 \mathscr{B}_t \otimes I + \beta^2 I \otimes \mathscr{B}_t\right)^{-1} a_t(x, y).$$

Then we have the theorem.

THEOREM 3.1. *The rate function H achieves its minimum value $(= 0)$ at a unique probability measure Q on W_T^A which is implicitly defined by*

$$Q \ll P \frac{dQ}{dP}(x) = \exp\left\{\beta^2 \int_0^T F_t^Q(x)\, dB_t(x) - \frac{\beta^4}{2}\int_0^T (F_t^Q(x))^2\, dt\right\}.$$

We can also give a pathwise description of the minima of H.

COROLLARY 3.2. *The good rate function H achieves its minimum value at a unique probability measure Q which is the solution of the nonlinear stochastic differential equation*

$$\begin{gathered} dx_t = -\nabla U(x_t)\, dt + dB_t + \beta^2 F_t^Q(x)\, dt, \\ \text{Law of } x = Q, \qquad \text{Law of } x_0 = Q|_{\mathscr{F}_0} = \mu_0. \end{gathered}$$

The proof is a direct consequence of Girsanov's Theorem which implies that Theorem 3.1 and Corollary 3.2 are equivalent (see [3], Theorem 6.13, for more details).

Moreover, when H is l.s.c., we know that H achieves its minimum value. As a consequence, if $2\beta^2 A^2 T < 1$, there exists a unique solution of the nonlinear stochastic differential equation described above.

Furthermore, Q_β^N being an exchangeable law, a result due to Sznitman (see Lemma 3.1 in [15]) allows deducing from Theorem 3.1 the propagation of chaos result.

COROLLARY 3.3. *Let* $2\beta^2 A^2 T < 1$.

(i) $\Pi_{\beta,T}^N$ *converges weakly to* δ_Q. *In particular, if* F *is a bounded continuous function on* $\mathscr{M}_1^+(\mathrm{W}_T^A)$, *then*

$$\lim_{N\to\infty} \int F(\hat{\mu}^N)\, dQ_\beta^N = F(Q).$$

(ii) *For any bounded continuous functions* $(f_1, \dots, f_m)$ *on* W_T^A,

$$\lim_{N\to\infty} \int\int f_1(x^1)\cdots f_m(x^m)\, dP_\beta^N(J)(x)\, d\gamma = \prod_{i=1}^m \int f_i(x)\, dQ.$$

We can also deduce (as in [3], Appendix C) from Theorems 2.4 and 3.1 that the quenched law of the empirical measure converges exponentially fast to δ_Q, so we have the following.

COROLLARY 3.4. *Let* $2\beta^2 A^2 T < 1$.

(i) *If* F *is a bounded continuous function on* $\mathscr{M}_1^+(\mathrm{W}_T^A)$, *then, for almost all* J,

$$\lim_{N\to\infty} \int F(\hat{\mu}^N)\, dP_\beta^N(J) = F(Q).$$

(ii) *For almost all* J *and for any bounded continuous function* f *on* W_T^A,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N f(x^i) = \int f\, dQ \quad \textit{a.s.}$$

The proof of Theorem 3.1 will need two steps.

First, we shall prove that H achieves its minimum value on the set M_1 of probability measures on W_T^A defined by

$$M_1 = \left\{ Q/Q \ll P \frac{dQ}{dP}(x) = \exp\left\{ \beta^2 \int_0^T F_t^Q(x)\, dB_t(x) - \frac{\beta^4}{2} \int_0^T \left(F_t^Q(x)\right)^2 dt \right\}\right\}.$$

In a second step, we shall prove that M_1 is reduced to a unique probability measure.

3.1. *Study of the minima of H.* We first prove that any minimum of H is equivalent to P.

LEMMA 3.5. *If Q minimizes H, then Q is equivalent to P.*

Lemma 3.5 is a straightforward consequence of Lemma 3.6.

LEMMA 3.6. *Let Q be a probability measure on W_T^A which minimizes H. Then we have the following conditions:*

(i) $Q \ll P$;
(ii) *denote* $B = \{x \in W_T^A/(dQ/dP)(x) = 0\}$ *and* $\delta = P(B)$,
 (a) $I((Q + s\mathbb{1}_B P/1 + s\delta) \mid P) = I(Q \mid P) + s\delta \log s + O(s)$,
 (b) *if* $2\beta^2 A^2 T < 1$, $\Gamma((Q + s\mathbb{1}_B P)/1 + s\delta) = \Gamma(Q) + O(s)$,

so that

$$H\left(\frac{Q + s\mathbb{1}_B P}{1 + s\delta}\right) - H(Q) = \delta s \log s + O(s).$$

REMARK 3.7. We do not think that the condition $2\beta^2 A^2 T < 1$ is really crucial in Lemma 3.6(ii)(b) but we leave it since we are not able to prove any large deviation upper bound result without it.

PROOF. (i) Since $I(Q \mid P)$ is finite, $Q \ll P$.
(ii)(a) One can compute

$$I\left(\frac{Q + s\mathbb{1}_B P}{1 + s\delta} \mid P\right) = \frac{1}{1 + s\delta} I(Q \mid P) - \frac{\log(1 + s\delta)}{1 + s\delta} + \frac{s\delta}{1 + s\delta} \log \frac{s}{1 + s\delta}. \tag{46}$$

which gives (ii)(a).
(ii)(b) We state a result even stronger than Lemma 3.6(ii)(b).

LEMMA 3.8. *If $2\beta^2 A^2 T < 1$, for any probability measure μ in $\mathscr{M}$, and for any signed measure ν such that $\nu(W_T^A) = 0$ and $\int (\int_0^T |\nabla U(x_s)|\, ds)^2 d|\nu|$ is finite and for which $\mu + \delta\nu$ is a probability measure when δ is small enough, Γ is Gateaux-differentiable at μ in the direction ν.*

This lemma can be proved by expanding Γ_1 and Γ_2 in powers of β (which can be done under the assumption that $2\beta^2 A^2 T < 1$) and then by showing that each term of these expansions are Gateaux-differentiable in a neighborhood $\{\mu + \kappa\nu,\ \kappa \le \delta\}$ of μ and that the series of these derivatives is absolutely and uniformly bounded on this neighborhood. We leave the proof to the reader.

We now prove that, if Q minimizes H, then Q belongs to M_1.

LEMMA 3.9. *If Q minimizes H, then Q is the solution of the nonlinear equation*

$$Q \ll P \frac{dQ}{dP} = \exp\left\{\beta^2 \int_0^T F_t^Q \, dB_t - \frac{\beta^4}{2} \int_0^T (F_t^Q)^2 \, dt\right\}.$$

To prove Lemma 3.9, we study the Taylor expansion of H at Q in the direction of $\nu = \psi \cdot Q$, for bounded measurable functions ψ such that $\int \psi\, dQ = 0$.

LEMMA 3.10. *Let* $2\beta^2 A^2 T < 1$.

(i) $I((1+s\psi)\cdot Q|P) = I(Q|P) + s\int(\log(dQ/dP))\psi\, dQ + o(s)$.

(ii) $\Gamma((1+s\psi)\cdot Q) = \Gamma(Q) + s\int\{\beta^2\int_0^T F_t^Q\, dB_t - (\beta^4/2)\int_0^T (F_t^Q)^2\, dt + Y_T\}\psi\, dQ + o(s)$,

where $(Y_s)_{s\le T}$ *is the previsible process with finite variations defined by*

$$Y_s(y) = \int\left\{\beta^2\int_0^s h_t^Q(x,y)\, dB_t(x) - \beta^4\int_0^s F_t^Q(x) h_t^Q(x,y)\, dt\right\} dQ(x),$$

if $h_t^Q(x,y) = DF_t^Q[\delta_y](x)$.

The reader can prove Lemma 3.10 using Lemma 2.16.

PROOF OF LEMMA 3.9. Since Q minimizes H,

$$\lim_{s\to 0}\frac{1}{s}(H((1+s\psi)\cdot Q) - H(Q)) = 0.$$

Hence, according to Lemma 3.10,

$$\int\left\{\log\frac{dQ}{dP} - \beta^2\int_0^T F_t^Q\, dB_t + \frac{\beta^4}{2}\int_0^T (F_t^Q)^2\, dt - Y_T\right\}\psi\, dQ = 0.$$

Since this equality is true for any bounded measurable function ψ such that $\int \psi\, dQ = 0$, we deduce that there exists a finite constant c_Q such that, Q almost surely, and so P almost surely by Lemma 3.5,

$$\log\frac{dQ}{dP} = \beta^2\int_0^T F_t^Q\, dB_t - \frac{\beta^4}{2}\int_0^T (F_t^Q)^2\, dt + Y_T + c_Q.$$

However, $(dQ/dP|_{\mathscr{F}_t})_{t\le T}$ must be a local martingale (see [14], Chapter VIII) so that, by uniqueness of the semimartingale decomposition,

$$\log\frac{dQ}{dP} = \beta^2\int_0^T F_t^Q dB_t - \frac{\beta^4}{2}\int_0^T (F_t^Q)^2\, dt. \qquad \square$$

3.2. *Existence and uniqueness problem for the minima of H.* The aim of this section is to prove that M_1 is reduced to a unique probability measure Q, that is, that the rate function H achieves its minimum value at a unique probability measure Q. We will first show that H achieves its minimum value at a unique probability measure. Independently, we can construct this minimum in the regime $3\beta^2 A^2 T < 1$.

THEOREM 3.11. (i) *For any time and temperature, there exists at most one probability measure Q such that $I(Q|P) < +\infty$ which is a solution of*

$$Q \ll P\frac{dQ}{dP} = \exp\left\{\beta^2\int_0^T F_t^Q(x)\, dB_t(x) - \frac{\beta^4}{2}\int_0^T (F_t^Q(x))^2\, dt\right\}.$$

(ii) *If* $3\beta^2 A^2 T < 1$, *there exists a unique probability measure* Q *such that* $I(Q|P) < +\infty$ *which is a solution of*

$$Q \ll P \frac{dQ}{dP} = \exp\left\{\beta^2 \int_0^T F_t^Q(x)\, dB_t(x) - \frac{\beta^4}{2} \int_0^T (F_t^Q(x))^2\, dt\right\}.$$

We shall use a fixed point argument to prove Theorem 3.11. To this end, we first study the functions F^μ, and, more precisely, show the following.

LEMMA 3.12. *For any probability measure* μ *in* $\{I(\ |P) < +\infty\}$ *and for any* $s \le T$,

$$E_P\left[\exp\left\{\beta^2 \int_0^s F_t^\mu\, dB_t - \frac{\beta^4}{2} \int_0^s (F_t^\mu)^2\, dt\right\}\right] \equiv 1,$$

so that $(\exp\{\beta^2 \int_0^s F_t^\mu(x)\, dB_t(x) - (\beta^4/2)\int_0^s (F_t^\mu(x))^2\, dt\})_{s\le T}$ *is a* $(P, \mathscr{F}_t)$-*martingale.*

To prove Lemma 3.12, we show the following.

LEMMA 3.13. *For any probability measure* μ *such that* $I(\mu|P) < \infty$, *there exists a bounded previsible process* f^μ *such that*

$$F_t^\mu(x) = \int_0^t \langle (I + \beta^2 \mathscr{B}_t)^{-1} X_t, X_s\rangle_\mu\, dB_s + f_t^\mu(x)$$

and there exists a finite constant c_1 *such that, for any* μ *such that* $I(\mu|P) < \infty$,

$$\sup_{x\in W_T^A} \sup_{t\le T} |f_t^\mu(x)|^2 \le c_1 (1 + I(\mu|P)).$$

PROOF. Denote $V_t = \nabla U \circ X_t$ and recall that

$$F_t^\mu(x) = \int d\mu(y) y_t \mathscr{C}_t \left(\int_0^t X_s \otimes dB_s + \int_0^t dB_s \otimes X_s\right)(y, x),$$

where, according to Lemma 2.13,

$$\mathscr{C}_t = \frac{1}{\beta^2} \int_0^\infty d\lambda \exp\left\{-\frac{\lambda}{\beta^2}\right\} \exp\{-\lambda \mathscr{B}_t\} \otimes \exp\{-\lambda \mathscr{B}_t\}. \tag{47}$$

Let $\mu \in \{I(\ |P) < +\infty\}$. Then B is a semimartingale under μ so that we can write

$$\begin{aligned} F_t^\mu(x) &= \int d\mu(y) y_t \mathscr{C}_t \int_0^t X_u \otimes dB_u(x, y) \\ &\quad + \int d\mu(y) y_t \mathscr{C}_t \int_0^t X_u \otimes dB_u(y, x). \end{aligned} \tag{48}$$

Moreover,

$$
\begin{aligned}
&\int d\mu(y) y_t \mathscr{C}_t \int_0^t X_u \otimes dB_{(}y, x) \\
&\qquad = \int d\mu(y)(I+\beta^2\mathscr{B}_t)^{-1} X_t(y) \int_0^t y_u \, dB_u(x) \\
&\qquad\quad - \beta^2 \int_0^t ds\, x_s \int d\mu(y)(I+\beta^2\mathscr{B}_t)^{-1} X_t(y) \\
&\qquad\qquad \times \int d\mu(z) z_t \mathscr{C}_t \int_0^t X_u \otimes dB_u(y, z).
\end{aligned} \tag{49}
$$

Let

$$
m^\mu_{s,t}(x) = \int d\mu(y) y_s \mathscr{C}_t \int_0^t X_u \otimes dB_u(x, y).
$$

Then we deduce from (48) and (49) that

$$
\begin{aligned}
F^\mu_t(x) &= m^\mu_{t,t}(x) + \int_0^t \langle (I+\beta^2\mathscr{B}_t)^{-1} X_t, X_s\rangle_\mu \, dB_s \\
&\quad - \beta^2 \int_0^t ds\, x_s \langle m^\mu_{s,t}(X); (I+\beta^2\mathscr{B}_t)^{-1} X_t\rangle_\mu.
\end{aligned}
$$

Denote

$$
f^\mu_t(x) = m^\mu_{t,t}(x) - \beta^2 \int_0^t ds\, x_s \langle m^\mu_{s,t}(X); (I+\beta^2\mathscr{B}_t)^{-1} X_t\rangle_\mu.
$$

It is obvious that f^μ is a continuous previsible process. To bound f^μ, we first bound $m^\mu_{s,t}(x)$. According to the definition (47) of $\mathscr{C}_t$, if $\|\ \|_\mu$ denotes the norm in L^2_μ, then

$$
\begin{aligned}
|m^\mu_{s,t}(x)| &= \left| \frac{1}{\beta^2} \int_0^\infty d\lambda \exp\left\{-\frac{\lambda}{\beta^2}\right\} \left\langle \exp\{-\lambda\mathscr{B}_t\} X_s; \int_0^t \exp\{-\lambda\mathscr{B}_t\} X_u(x)\, dB_u \right\rangle_\mu \right| \\
&\le \frac{A}{\beta^2} \int_0^\infty d\lambda \exp\left\{-\frac{\lambda}{\beta^2}\right\} \left\| \int_0^t \exp\{-\lambda\mathscr{B}_t\} X_u(x)\, dB_u \right\|_\mu .
\end{aligned}
$$

However, we prove in (67) that $\exp\{-\lambda\mathscr{B}_t\} X_u(x)$ is bounded by $A(1+\lambda A^2 t)$. So that, if one follows the strategy of the proof of Lemma 2.19, one finds that, for any $p > 1$, there exists a finite constant $C_p = -\frac{1}{2}\log(1-(1/p))$ such that

$$
\left\| \int_0^t \exp\{-\lambda\mathscr{B}_t\} X_u(x)\, dB_u \right\|^2_\mu \le 2pA^2(1+\lambda A^2 t)^2 t \left\{ I(\mu|P) + C_p \right\}.
$$

Thus

$$
|m^\mu_{s,t}(x)| \le A^2(1+\beta^2 A^2 t)\sqrt{2pt}\sqrt{I(\mu|P)+C_p},
$$

so that

$$
|f^\mu_t(x)| \le A^2(1+\beta^2 A^2 t)^2 \sqrt{2pt}\sqrt{I(\mu|P)+C_p}. \tag{50}
$$

□

We now turn to the proof of Lemma 3.12

Since $(M_s = \exp\{\beta^2 \int_0^s F_t^\mu \, dB_t - (\beta^4/2)\int_0^s (F_t^\mu)^2 \, dt\})_{s\le T}$ is a supermartingale, it is enough to prove that, for any $t \le T$, $\int M_t \, dP = 1$. We denote

$$\widetilde{K}_\mu^t(s,u) = \langle (I+\beta^2 \mathscr{B}_t)^{-1} X_s, X_u\rangle_\mu.$$

According to Lemma 3.13, we have

$$\begin{aligned} M_T = \exp&\left\{\beta^2 \int_0^T \int_0^t \widetilde{K}_\mu^t(t,s)\, dB_s\, dB_t - \frac{\beta^4}{2}\int_0^T \left(\int_0^t \widetilde{K}_\mu^t(t,s)\, dB_s\right)^2 dt\right\} \\ &\times \exp\left\{\beta^2 \int_0^T f_t^\mu \, dB_t - \frac{\beta^4}{2}\int_0^T (f_t^\mu)^2\, dt\right\} \\ &\times \exp\left\{-\beta^4 \int_0^T f_t^\mu \int_0^t \widetilde{K}_\mu^t(t,s)\, dB_s\, dt\right\}. \end{aligned}$$

We have already studied in [3], Section 6, the local martingale

$$M_u^1 = \exp\left\{\beta^2 \int_0^u \int_0^t \widetilde{K}_\mu^t(t,s)\, dB_s\, dB_t - \frac{\beta^4}{2}\int_0^u \left(\int_0^t \widetilde{K}_\mu^t(t,s)\, dB_s\right)^2 dt\right\},$$

and we proved there that M^1 is a uniformly integrable $(P, \mathscr{F}_t)$-martingale. Thus, we can define a probability measure P', absolutely continuous with respect to P, such that $P' = M_T^1 \cdot P$. So that, for any $u \le T$,

$$\int M_u\, dP = \int \exp\left\{\beta^2 \int_0^u f_t^\mu \left(dB_t - \beta^2 \int_0^t \widetilde{K}_\mu^t(t,s)\, dB_s\, dt\right) - \frac{\beta^4}{2}\int_0^u (f_t^\mu)^2\, dt\right\} dP'.$$

Girsanov's theorem implies that $B'_u = B_u - \beta^2 \int_0^u \int_0^t \widetilde{K}_\mu^t(t,s)\, dB_s\, dt$ is a Brownian motion under P'. But f^μ is bounded, so that $(\exp\{\beta^2 \int_0^u f_t^\mu \, dB'_t - (\beta^4/2)\int_0^u (f_t^\mu)^2\, dt\})_{u\le T}$ is a $(P', \mathscr{F}_t)$-martingale. Thus, for any $u \le T$,

$$\int M_u\, dP = 1.$$

As a consequence of Lemma 3.12, we can define, for any $\mu \in \{I(\ |P) < +\infty\}$, a probability measure $L(\mu)$ on W_T^A, absolutely continuous with respect to P, such that

$$\frac{dL(\mu)}{dP} = \exp\left\{\beta^2 \int_0^T F_t^\mu\, dB_t - \frac{\beta^4}{2}\int_0^T (F_t^\mu)^2\, dt\right\}.$$

It is clear that M_1 can be characterized as the set of the fixed points of the map L. Hence, Theorem 3.11 is equivalent to Theorem 3.14.

THEOREM 3.14. (i) *For any time and temperature, there exists at most one probability measure Q in $\{I(\ |P) < +\infty\}$ such that $L(Q) = Q$.*

(ii) *If $2\beta^2 A^2 T < 1$, there exists a unique probability measure Q in $\{I(\ |P) < +\infty\}$ such that $L(Q) = Q$.*

We shall prove Theorem 3.14 through a contraction argument.

Let $(\mathscr{F}_t)_{t\leq T}$ be the natural filtration on $\mathscr{M}_1^+(W_T^A)$ defined by $\mathscr{F}_t = \sigma(x_s, s \leq t)$. For any real number $a \geq 1$, let $\mathscr{L}^{(a)}$ be the subset of $\mathscr{M}_1^+(W_T^A)$ defined by

$$\mathscr{L}^{(a)} = \left\{\mu \in \mathscr{M}_1^+(W_T^A)/\mu \ll P \int \left(\frac{d\mu}{dP}\right)^a dP < +\infty\right\}.$$

For positive real number a, $a \geq 1$, we define an L^a variational distance $D_T^{(a)}$ on $\mathscr{L}_T^{(a)}$ by

$$D_T^{(a)}(\mu,\nu) = \left(\int \left|\frac{d\mu}{dP} - \frac{d\nu}{dP}\right|^a dP\right)^{1/a}.$$

On $\mathscr{L}_T^{(a)}$, the variational topology induced by $D_T^{(a)}$ is stronger than the weak topology. More precisely, for any positive real number a, $a \geq 1$, for any μ and ν in $\mathscr{L}_T^{(a)} \subset \mathscr{L}_T^{(1)}$,

$$d_T(\mu,\nu) \leq D_T^{(1)}(\mu,\nu) \leq D_T^{(a)}(\mu,\nu). \tag{51}$$

We will denote, for any time $t \leq T$ and for any probability measures μ and ν in $\mathscr{L}_T^{(a)}$, $D_t^{(a)}(\mu,\nu) = D_T^{(a)}(\mu|_{\mathscr{F}_t}, \nu|_{\mathscr{F}_t})$.

We shall prove the following.

PROPOSITION 3.15. *We can find a real number a, $a > 1$, a strictly positive real number q such that for any probability measures μ and ν in $\{I(\ |P) < \infty\}$ and for any $t \leq T$,*

$$D_t^{(a)}(L(\mu), L(\nu))^{2aq} \leq Z(\mu,\nu) \int_0^t \left(D_s^{(a)}(\mu,\nu)\right)^{2aq} ds,$$

where, if we denote by b the conjugate exponent of a, we can find finite constants C and C' so that we can choose

$$Z(\mu,\nu) = C\left(1 + \int\int_0^T |\nabla U(x_s)|\, ds\, d(\mu+\nu) + \left(\int \left(\int_0^T |\nabla U(x_s)|\, ds\right)^b dP\right)^{1/b}\right)^{4aq} \times \exp C'\{I(\mu|P) + I(\nu|P)\}.$$

REMARK 3.16. We notice that Lemma 3.19 implies that, for $a > 1$ small enough, $L^{-1}(\mathscr{L}^{(a)})$ is included in $\{I(\ |P) < \infty\}$ so that $D_s^{(a)}(L(\mu), L(\nu))$ is well defined for $I(\mu|P) < \infty$, $I(\nu|P) < \infty$ when a is small enough.

Proposition 3.15 already implies that L has at most one fixed point in $\{\mu\colon I(\mu|P) < \infty\}$ according to Gronwall's lemma. The proof of the existence is slightly more demanding since we construct a sequence of probability measures converging to the fixed point of L for which we need to make sure that the entropy is uniformly bounded. The control of the entropy necessitates a high temperature assumption. Indeed, if we define a sequence $(\mu_n)_{n\geq 0}$ by

$$\mu_0 = P, \qquad \mu_{n+1} = L(\mu_n),$$

then $I(\mu_n|P)$ is finite for any integer n. In fact, $I(\mu_0|P) = 0$ is finite, and, by induction over n, $I(\mu_{n+1}|P) = (\beta^4/2)\int\int_0^T (F_t^{\mu_n})^2\,dt\,d\mu_{n+1}$ is finite, according to Lemma 3.13. More precisely, we have the lemma.

LEMMA 3.17. *Let, for $\lambda > 0$, $\xi(\lambda) = \frac{2}{5}(\lambda/1-\lambda^2)\{(1+\lambda)^5 - 1\}$. Then, for any $\xi > \xi(\beta^2 A^2 T)$, there exists a finite constant $\alpha(\xi)$ such that*

$$I(\mu_{n+1}|P) \leq \xi I(\mu_n|P) + \alpha(\xi).$$

One can notice ξ is increasing and that, if $\lambda = 1/3$, $\xi(\lambda) < 1$ so that, if $3\beta^2 A^2 T \leq 1$, we can bound the entropy of μ_n with respect to P uniformly in n. It is now trivial to deduce the existence of Q for β small enough.

We will not prove Lemma 3.17 (see [10] for details), but will turn to the proof of Proposition 3.15, which necessitates several technical lemmas.

Let

$$X_t^\mu(x) = \beta^2 \int_0^t F_s^\mu(x)\,dB_s(x) - \frac{\beta^4}{2}\int_0^t (F_s^\mu(x))^2\,ds.$$

Then we obtain the following result.

LEMMA 3.18. *For any conjugate exponents (p, q), for any probability measures μ and ν in $\{I(\ |P) < +\infty\}$:*

$$\begin{aligned} &D_t^{(a)}(L(\mu), L(\nu))^a \\ &\quad \leq \left(\int |X_t^\mu - X_t^\nu|^{aq}\,dP_t\right)^{1/q} \\ &\qquad \times \left(\int_0^1 d\alpha\left(\int \exp\{apX_t^\mu\}\,dP_t\right)^{1-\alpha}\left(\int \exp\{apX_t^\nu\}\,dP_t\right)^{\alpha}\right)^{1/p}. \end{aligned} \tag{52}$$

PROOF. The proof is identical to that of Lemma 7.5 in [3].

We first bound the second term in the right-hand side of (52).

LEMMA 3.19. *If $ap-1$ is small enough (more precisely if $\beta^2 ap(ap-1)A^2 T < 1$), we can find a finite constant C_1 such that, for $I(\mu|P) < \infty$ and for any $t \leq T$,*

$$\int \exp\{apX_t^\mu\}\,dP \leq \exp\{C_1(I(\mu|P) + 1)\}.$$

PROOF. We have

$$\int \exp\{apX_t^\mu\}\,dP = \int \exp\left\{ap\beta^2 \int_0^t F_s^\mu\,dB_s - ap\frac{\beta^4}{2}\int_0^t (F_s^\mu)^2\,ds\right\}dP.$$

With the notation of the proof of Lemma 3.12,

$$\int \exp\{apX_t^\mu\}\,dP = \int (M_t^1)^{ap} \exp\left\{ap\beta^2 \int_0^t f_s^\mu dB'_s - ap\frac{\beta^4}{2}\int_0^t (f_s^\mu)^2\,ds\right\}dP.$$

Let (p', q') be conjugate exponents. The Hölder inequality gives

$$\begin{aligned}
&\int \exp\{apX_t^\mu\}\,dP \\
&\quad\le \left(\int (M_t^1)^{app'}\,dP\right)^{1/p'} \\
&\qquad \times \left(\int \exp\left\{apq'\beta^2\int_0^t f_s^\mu\,dB'_s - apq'\frac{\beta^4}{2}\int_0^t (f_s^\mu)^2\,ds\right\}dP\right)^{1/q'}.
\end{aligned} \tag{53}$$

We have already proved in [3], Lemma 6.10, that, if $\beta^2 app'(app'-1)A^2T < 1$, there exists a finite constant $c(app')$, independent of $t \le T$, and $\mu \in \mathscr{M}_1^+(W_T^A)$, such that

$$\left(\int (M_t^1)^{app'}\,dP\right)^{1/p'} \le c(app').$$

Since we supposed that $ap-1$ is small enough so that $\beta^2 ap(ap-1)A^2T < 1$, we can choose p' close enough to 1 so that $\beta^2 app'(app'-1) < 1$. With such a choice of p', (53) becomes

$$\begin{aligned}
&\int \exp\{apX_t^\mu\}\,dP \\
&\quad\le c(app')\left(\int \exp\left\{apq'\beta^2\int_0^t f_s^\mu\,dB'_s - apq'\frac{\beta^4}{2}\int_0^t (f_s^\mu)^2\,ds\right\}dP\right)^{1/q'}.
\end{aligned} \tag{54}$$

We now bound the second term in the right-hand side of (54). We recall that

$$B'_t = B_t + \beta^2 \int_0^t \int_0^s \widetilde{K}_\mu^s(s,u)\,dB_u\,ds,$$

so that the Cauchy–Schwarz inequality gives

$$\begin{aligned}
&\int \exp\left\{apq'\beta^2\int_0^t f_s^\mu\,dB'_s - apq'\frac{\beta^4}{2}\int_0^t (f_s^\mu)^2\,ds\right\}dP \\
&\quad\le \left(\int \exp apq'\beta^4\left\{(2apq'-1)\int_0^t (f_s^\mu)^2\,ds\right.\right. \\
&\qquad\qquad \left.\left. - 2\int_0^t f_s^\mu \int_0^s \widetilde{K}_\mu^s(s,u)\,dB_u\,ds\right\}dP\right)^{1/2}.
\end{aligned}$$

By Lemma 3.13 and since B is a Brownian motion under P and using the Jensen inequality, we find

$$\begin{aligned}\int \exp\biggl\{2apq'\beta^4 c_1^{1/2}(I(\mu|P)+1)^{1/2}\int_0^t\biggl|\int_0^s \widetilde{K}_\mu^s(s,u)\,dB_u\biggr|\,ds\biggr\}\,dP\\ \le \frac{2}{t}\int_0^t\int \exp\biggl\{2apq'\beta^4 c_1^{1/2}(I(\mu|P)+1)^{1/2}t\int_0^s \widetilde{K}_\mu^s(s,u)\,dB_u\biggr\}\,dP\,ds\\ \le 2\exp\{2(apq'\beta^4)^2 c_1(I(\mu|P)+1)t^3A^4\},\end{aligned}$$

so that (54) shows that, for any $t \le T$,

$$\int \exp apX_t^\mu\,dP \le 2c(app')\exp\bigl\{2ap\beta^4 c_1(apq'(1+\beta^4T^3A^2)-1)(I(\mu|P)+1)\bigr\},$$

which gives Lemma 3.19. □

In the following pages, we will choose ap close enough to 1 so that Lemma 3.19 holds and, for later convenience, so that $aq \ge 2$.

We now bound the first term in the right-hand side of (52).

LEMMA 3.20. *Let $b = a/(a-1)$. We can find a finite constant C_2 such that, for any probability measures (μ,ν) with $I(\mu|P)<\infty$, $I(\nu|P)<\infty$*

$$\begin{aligned}\int |X_t^\mu - X_t^\nu|^{aq}\,dP\\ \le C_2\biggl(1+\int\int_0^T|\nabla U(x_s)|\,ds d(\mu+\nu)+\biggl(\int\biggl(\int_0^T|\nabla U(x_s)|\,ds\biggr)^b dP\biggr)^{1/b}\biggr)^{2aq}\\ \times\biggl(\int_0^t D_s^{(a)}(\mu,\nu)^{2aq}\,ds\biggr)^{1/2}.\end{aligned}$$

PROOF. One can see, using the Burkholder–Davis–Gundy inequality (see, for more details, the proof of Lemma 6.10 in [3]), that there exists a finite constant c_q such that

$$\begin{aligned}\int |X_t^\mu - X_t^\nu|^{aq}\,dP \le c_q\biggl(1+\biggl(\int_0^t\int|F_s^\mu+F_s^\nu|^{2aq}\,dP\,ds\biggr)^{1/2}\biggr)\\ \times\biggl(\int_0^t\int|F_s^\mu-F_s^\nu|^{2aq}\,dP\,ds\biggr)^{1/2}.\end{aligned}\tag{55}$$

We focus on the second term in the right-hand side of (55). We want to prove that F^μ satisfies a Lipschitz type property.

LEMMA 3.21. *For any time T, there exists a finite constant A_T such that, for any paths x and y, for any probability measures μ and ν and for any time*

$t \le T$,

$$\bigl|F_t^\mu(x) - F_t^\nu(y)\bigr| \le A_T \int_0^t (\langle|\nabla U(X_u)|\rangle_\mu + \langle|\nabla U(X_u)|\rangle_\nu)\, du \\ \times \Bigl(\sup_{u\le t} |x_u - y_u| + d_t(\mu,\nu)\Bigr) \\ + A_T\biggl(K_t(\mu,\nu) + \int_0^t |\nabla U(x_s) - \nabla U(y_s)|\, ds\biggr),$$

where d_t is the Wasserstein's distance between μ and ν and

$$K_T(\mu,\nu) = \sup_{u,\,v\le T}\biggl\{\biggl|\int \nabla U(X_u)X_v\, d\nu - \int \nabla U(X_u)X_v\, d\mu\biggr|\biggr\}.$$

As a consequence, for any positive a, b such that $a^{-1} + b^{-1} = 1$, there exists a finite constant C_3 such that, for any probability measures μ and ν with $I(\mu|P) < \infty$, $I(\nu|P) < \infty$, for any x in W_T^A and for any $s \le T$,

$$\bigl|F_s^\mu(x) - F_s^\nu(x)\bigr| \le C_3\biggl(1 + \int\int_0^T |\nabla U(y_s)|\, ds\, d(\mu+\nu)(y) \\ + \biggl(\int\biggl(\int_0^T |\nabla U_s|\, ds\biggr)^b dP\biggr)^{1/b}\biggr) D_s^{(a)}(\mu,\nu).$$

PROOF.

$$F_s^\mu(x) = \frac{1}{\beta^2}\int d\mu(y) y_s \int_0^\infty d\lambda \exp\biggl\{-\frac{\lambda}{\beta^2}\biggr\} \\ \times\biggl[\exp\{-\lambda\mathscr{B}_s\}X_s(x)\exp\{-\lambda\mathscr{B}_s\}X_s(y) \\ - \exp\{-\lambda\mathscr{B}_s\}X_0(x)\exp\{-\lambda\mathscr{B}_s\}X_0(y) \\ + \exp\{-\lambda\mathscr{B}_s\}\biggl(\int_0^s \exp\{-\lambda\mathscr{B}_s\}X_u(x)V_u\, du\biggr)(y) \\ + \exp\{-\lambda\mathscr{B}_s\}\biggl(\int_0^s \exp\{-\lambda\mathscr{B}_s\}X_u(y)V_u\, du\biggr)(x)\biggr].$$

For any probability measure μ in $\mathscr{M}$, the right-hand side of the last equality belongs to $L_\mu^1(W_T^A)\otimes L^1_{\exp\{-(\lambda/\beta^2)d\lambda\}}(\mathbb{R}^+)$, so that we can use Fubini's theorem, which gives

$$F_s^\mu(x) = \frac{1}{\beta^2}\int_0^\infty d\lambda \exp\biggl\{-\frac{\lambda}{\beta^2}\biggr\} \\ \times\biggl[\langle\exp\{-\lambda\mathscr{B}_s\}X_s, X_s\rangle_\mu \exp\{-\lambda\mathscr{B}_s\}X_s(x) \\ - \langle\exp\{-\lambda\mathscr{B}_s\}X_s, X_0\rangle_\mu \exp\{-\lambda\mathscr{B}_s\}X_0(x) \\ + \int_0^s \exp\{-\lambda\mathscr{B}_s\}X_u(x)\langle\exp\{-\lambda\mathscr{B}_s\}X_s, V_u\rangle_\mu\, du \\ + \exp\{-\lambda\mathscr{B}_s\}\biggl(\int_0^s \langle\exp\{-\lambda\mathscr{B}_s\}X_u, X_s\rangle_\mu V_u\, du\biggr)(x)\biggr]. \tag{56}$$

Since we shall compare the action of $\mathscr{B}_s$ in L^2_μ with its action in L^2_ν, we shall be more precise and denote $\mathscr{B}^\mu_s$ the operator in L^2_μ with kernel $b_s(x, y) = \int_0^s x_u y_u \, du$. Moreover, since the second inequality of the lemma can be deduced from the first one by the Hölder inequality, we will concentrate on the first only. According to (56), it is not hard to deduce Lemma 3.21 from the following.

LEMMA 3.22. (i) *There exists a finite constant k_1 such that, for any positive real number λ and for any probability measures μ and ν on W^A_T,*

$$\sup_{u\le s, v\le s} \left|\langle X_u, \exp\{-\lambda \mathscr{B}^\mu_s\} X_v\rangle_\mu - \langle X_u, \exp\{-\lambda \mathscr{B}^\nu_s\} X_v\rangle_\nu\right| \le k_1(1+\lambda)^2 d_s(\mu,\nu).$$

(ii) *There exists a finite constant k_2 such that, for any $\mu, \nu \in \mathscr{M}^+_1(W^A_T)$, for any measurable function h in $L^2_\mu(W^A_s) \cap L^2_\nu(W^A_s)$ and for any $(x, \bar{x})$ in W^A_T,*

$$\begin{aligned}
&\left|\exp\{-\lambda \mathscr{B}^\mu_s\} h(x) - \exp\{-\lambda \mathscr{B}^\nu_s\} h(\bar{x})\right| \\
&\quad \le k_2(1+\lambda)^2 \left\{ |h(x) - h(\bar{x})| + \int_0^s \left| \int h(y) y_u d(\nu - \mu)(y)\right| du \right. \\
&\qquad\qquad \left. + \left(\int |h| d(\nu + \mu)\right)\left(d_s(\mu,\nu) + \int_0^s |x_v - \bar{x}_v| \, dv\right)\right\}.
\end{aligned}$$

(iii) *There exists a finite constant k_3 such that, for any probability measure μ in $\mathscr{M}$,*

$$\left|\exp\{-\lambda \mathscr{B}^\mu_s\} h(x) - h(x)\right| \le k_3(1+\lambda)^2 \int |h| \, d\mu.$$

PROOF. (i) has already been proved in [3], Lemma A.4. The proof of (ii) is quite similar to that of (67). We write

$$\begin{aligned}
\exp\{-\lambda \mathscr{B}^\mu_s\} h(x) &= h(x) - \lambda \int_0^s du \, x_u \int y_u h(y) \, d\mu(y) \\
&\quad + \int_0^\lambda d\alpha \int_0^\alpha d\alpha' \int_0^s \int_0^s du \, dv \, x_u \langle \exp\{-\alpha' \mathscr{B}^\mu_s\} X_u, X_v\rangle_\mu \\
&\qquad \times \int h(y) y_v \, d\mu(y).
\end{aligned} \tag{57}$$

Thus

$$\left|\exp\{-\lambda \mathscr{B}^\mu_s\} h(x) - \exp\{-\lambda \mathscr{B}^\nu_s\} h(x)\right| \le \sum_{i=1}^4 L^i_\lambda(\mu,\nu), \tag{58}$$

where

$$L^1_\lambda(\mu,\nu) = \lambda \left| \int_0^s du \, x_u \int y_u h(y) \, d\mu(y) - \int_0^s du \, x_u \int y_u h(y) \, d\nu(y)\right|,$$

$$L^2_\lambda(\mu,\nu) = A\int_0^\lambda d\alpha \int_0^\alpha d\alpha' \int_0^s \times \int_0^s du\,dv \Big|\langle \exp\{-\alpha'\mathscr{B}^\mu_s\}X_u, X_v\rangle_\mu - \langle \exp\{-\alpha'\mathscr{B}^\mu_s\}X_u, X_v\rangle_\nu\Big| \left|\int h(y)y_v\,d\mu(y)\right|,$$

$$L^3_\lambda(\mu,\nu) = A^3T\lambda^2\int_0^s dv\left|\int h(y)y_v\,d\mu(y) - \int h(y)y_v\,d\nu(y)\right|,$$

$$L^4_\lambda(\mu,\nu) = A^3\lambda^2T\int |h|d(\mu+\nu)\int_0^s |x_u-\bar{x}_u|\,du.$$

It is not difficult to bound L^1_λ:

$$L^1_\lambda(\mu,\nu) \le \lambda A^2\int |h|d(\mu+\nu)\int_0^s |x_u-\bar{x}_u|\,du + \lambda A\int_0^s |y_u h(y)d(\mu-\nu)(y)|\,du.$$

Moreover, we use Lemma 3.22(i) to bound L^2_λ:

$$\begin{aligned} L^2_\lambda(\mu,\nu) &\le A^2\int_0^\lambda d\alpha\int_0^\alpha d\alpha'\int_0^s\int_0^s du\,dv \\ &\quad\times \big|\langle \exp\{-\alpha'\mathscr{B}^\mu_s\}X_u, X_v\rangle_\mu - \langle \exp\{-\alpha'\mathscr{B}^\mu_s\}X_u, X_v\rangle_\nu\big|\int |h(y)|\,d\mu(y) \\ &\le \tfrac{1}{2}A^2k_1T^2\lambda^2(1+\lambda)^2\int |h|\,d\mu\,d_s(\mu,\nu). \end{aligned}$$

Putting these bounds together gives Lemma 3.22(ii). The proof of (iii) is very similar; we omit it. □

As a consequence of Lemma 3.21, we find

$$\begin{aligned} &\int_0^t\int |F^\mu_s - F^\nu_s|^{2aq}\,ds\,dP \\ &\quad\le C_3^{2aq}\left(1+\int\int_0^T |V_s|\,ds\,d(\mu+\nu) + \left(\int\left(\int_0^T |V_s|\,ds\right)^b dP\right)^{1/b}\right)^{2aq} \\ &\qquad\times\int_0^t D^{(a)}_s(\mu,\nu)^{2aq}\,ds. \end{aligned} \tag{59}$$

Similarly, we can bound the first term in the right-hand side of (55) and prove that there exists a finite constant C_4 such that for any time $t \le T$,

$$\begin{aligned} &\int_0^t\int dP|F^\mu_s + F^\nu_s|^{2aq}\,ds \\ &\quad\le C_4\left(1+\int\int_0^T |V_s|\,ds\,d(\mu+\nu) + \left(\int\left(\int_0^T |V_s|\,ds\right)^b dP\right)^{1/b}\right)^{2aq}. \end{aligned} \tag{60}$$

Thus, (55), (59), (60) give Lemma 3.20. □

Finally, Lemmas 3.18, 3.19 and 3.20 imply that we can find finite constants C and C_1 such that, for any probability measures μ and ν in $\{I(\ |P) < \infty\}$,

$$\begin{aligned}
&D_t^{(a)}(L(\mu), L(\nu))^a \\
&\quad \le C\biggl(1 + \int\int_0^T |V_s|\,ds\,d(\mu+\nu) + \biggl(\int\biggl(\int_0^T |V_s|\,ds\biggr)^b dP\biggr)^{1/b}\biggr)^{4aq} \\
&\qquad \times \exp\frac{C_1}{p}\{I(\mu|P) + I(\nu|P)\}\biggl(\int_0^t \bigl(D_s^{(a)}(\mu,\nu)\bigr)^{2aq}\,ds\biggr)^{1/2q},
\end{aligned}$$

which is Lemma 3.15.

REMARK 3.23. According to the proof of Lemma A.8, we can see the following:

(i) $\int_0^T |\nabla U(x_s)| ds$ belongs to $L_P^b(W_T^A)$, for any positive real b;
(ii) there exist real numbers ξ and η such that, for any probability measure μ in $\mathscr{M}_1^+(W_T^A)$, $\int\int_0^T |\nabla U(x_s)|\,ds\,d\mu \le \sqrt{\xi I(\mu|P) + \eta}$.

Thus, Lemma 3.15 implies that there exist finite constants c and c' such that

$$\begin{aligned}
&D_t^{(a)}(L(\mu), L(\nu))^{2aq} \le c\Bigl(1 + \sqrt{I(\mu|P)} + \sqrt{I(\nu|P)}\Bigr)^{4aq} \\
&\qquad\qquad \times \exp c'\{I(\mu|P) + I(\nu|P)\}\biggl(\int_0^T \bigl(D_s^{(a)}(\mu,\nu)\bigr)^{2aq}\,ds\biggr).
\end{aligned} \tag{61}$$

4. Averaged evolution of the Gibbs measure.

In this section, we study Sompolinski–Zippelius dynamics, starting from the Gibbs measure μ_J^N:

$$\mu_J^N(dx) = \frac{1}{Z_J^N}\exp\biggl\{-\beta H_J^N(x) - 2\sum_{i=1}^N U(x^i)\biggr\}\prod_{i=1}^N dx^i,$$

where we recall that

$$H_J^N(x) = \frac{-1}{\sqrt{N}}\sum_{i,\,j=1}^N J_{ij}x^i x^j$$

and

$$Z_J^N = \int \exp\biggl\{-\beta H_J^N(x) - 2\sum_{i=1}^N U(x^i)\biggr\}\prod_{i=1}^N dx^i.$$

Let $\widetilde{P}_\beta^N(J)$ be the weak solution on W_T^A of the stochastic differential system

$$dx_t^i = -\nabla U(x_t^i)\,dt + dB_t^i + \frac{\beta}{\sqrt{N}}\sum_{i=1}^N J_{ij}x_t^j\,dt,$$
$$\text{Law of } x_0 = \mu_J^N;$$

$\widetilde{P}_\beta^N(J)$ exists and is unique for any finite couplings $(J_{ij})_{1\le i\le j\le N}$.

We shall prove that the law of the empirical measure under $\widetilde{P}^N_\beta(J)$ converges in the high temperature regime and when the potential U is even, which entails an averaged propagation of chaos result.

More precisely, with the notations of Section 2, we define, for any probability measure μ in $\mathscr{M}$, a function $\widetilde{F}^\mu$ by

$$\widetilde{F}^\mu_t(x) = 2\int y_t \mathscr{C}_t(a_t + X_0 \otimes X_0)(x, y)\, d\mu(y).$$

Then the asymptotic and averaged behavior of a spin is described by the nonlinear stochastic differential system

$$\begin{gathered} dx_t = -\nabla U(x_t)\, dt + dB_t + \beta^2 \widetilde{F}^\mu_t(x)\, dt, \\ \text{Law of } x = \mu, \qquad \text{Law of } x_0 = q, \\ dq(x_0) = \exp\{2\beta^2 \langle X_0, X_0\rangle_q x_0^2\} \exp\{-2U(x_0)\}\, dx_0. \end{gathered} \tag{62}$$

One can prove as in Section 3 that this nonlinear system admits a unique solution, say $\widetilde{Q}$, in the high temperature regime $\{2\beta^2 A^2 T < 1 \text{ and } 4\beta^2 A^4 < 1\}$.

The main theorem of this section is the following.

THEOREM 4.1. *If U is even, if β is small enough and if $2\beta^2 A^2 T < 1$, for almost all J, the law of the empirical measure under $\widetilde{P}^N_\beta(J)$ converges to $\delta_{\widetilde{Q}}$ exponentially fast.*

So we deduce the averaged propagation of chaos result.

COROLLARY 4.2. *If U is even, if β is small enough and if $2\beta^2 A^2 T < 1$, $\widetilde{Q}^N_\beta := \mathscr{E}[\widetilde{P}^N_\beta(J)]$ is $\widetilde{Q}$ chaotic; that is, for any bounded continuous functions $(f_1, \dots, f_m)$ on W^A_T,*

$$\lim_{N\to\infty} \mathscr{E}\left[\int f_1(x^1)\cdots f_m(x^m)\, d\widetilde{P}^N_\beta(J)(x)\right] = \prod_{i=1}^m \int f_i(x)\, d\widetilde{Q}.$$

REMARK 4.3. The restriction on high temperature is due to the statement that we are below the phase transition, that is, that the free energy $(1/N)\log Z^N_J$ converges and that the initial law $\mathscr{E}[\mu^N_J]$ is chaotic [more precisely, that (3) admits a unique solution at time $T = 0$]. The condition of an even potential is needed to prove that the free energy $(1/N)\log Z^N_J$ converges.

REMARK 4.4. Since the Sompolinski–Zippelius dynamics are reversible and μ^N_J is an invariant measure for these dynamics, $\widetilde{Q}^N_\beta$ is stationary. As a consequence, Theorem 4.1 implies that $\widetilde{Q}$ is stationary in the high temperature regime. This property, which was not trivial a priori, gives us a new strategy to study $\widetilde{Q}$.

Theorem 4.1 comes from Theorem 4.5 where we state a quenched large deviation upper bound for the law of the empirical measure under $\widetilde{P}_{\beta}^{N}(J)$.

The rate function which governs this large deviation upper bound is slightly different from that which governs the deviations of the empirical measure under the quenched law $P_{\beta}^{N}(J)$ starting from an independent law as stated in Theorem 2.4. Indeed, a new term appears from the interaction between the couplings $(J_{ij})_{i,j}$ of the initial distribution and those of the dynamic. More precisely, for any μ in $\mathscr{M}$, let

$$f(\mu) = \beta^2 \langle X_0 \otimes X_0, \mathscr{C}_T X_0 \otimes X_0 \rangle_{\mu\otimes\mu} + 2\beta^2 \langle X_0 \otimes X_0, \mathscr{C}_T a_T \rangle_{\mu\otimes\mu}.$$

With the definition of Γ given in Section 2, we define a new map $\widetilde{\Gamma}$ from $\mathscr{M}$ into $\mathbb{R}$ by

$$\widetilde{\Gamma}(\mu) = \Gamma(\mu) + f(\mu).$$

Let P be the weak solution of the stochastic differential equation

$$dx_t = -\nabla U(x_t)\,dt + dB_t,$$

$$\text{Law of } x_0 = \alpha = (1/Z)\exp\{-2U(x)\}\,dx \quad \text{where } Z = \int \exp\{-2U(x)\}\,dx.$$

Let $\widetilde{H}$ be the map from $\mathscr{M}_1^+(W_T^A)$ into $\mathbb{R} \cup \{+\infty\}$ defined by

$$\widetilde{H}(\mu) = \begin{cases} I(\mu|P) - \widetilde{\Gamma}(\mu) + \inf(I(\mu|P) - \beta^2\langle X_0, X_0\rangle_\mu^2), & \text{if } I(\mu|P) < +\infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

In other words, if H denotes the rate function which governs the large deviation upper bound of Theorem 2.4 under the quenched law $P_{\beta}^{N}(J)$, starting from the independent law $\mu_0^{\otimes N}$ with the specific choice of $\mu_0(dx) = \alpha(dx) = (1/Z)\exp\{-2U(x)\}\,dx$, then

$$\widetilde{H}(\mu) = \begin{cases} H(\mu) - f(\mu) + \inf(I(\mu|P) - \beta^2\langle X_0, X_0\rangle_\mu^2), & \text{if } H(\mu) < +\infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then we have Theorem 4.5.

THEOREM 4.5. (i) *If* $2\beta^2 A^2 T < 1$, $\widetilde{H}$ *is a good rate function.*

(ii) *If* $2\beta^2 A^2 T < 1$, *if* β *is small enough and if the potential* U *is even, for any closed subset* F *of* $\mathscr{M}_1^+(W_T^A)$ *and for almost all* J,

$$\limsup_{N\to\infty} \frac{1}{N} \log \widetilde{P}_{\beta}^{N}(J)(\hat{\mu}^N \in F) \le -\inf_F \widetilde{H}.$$

Theorem 4.1 can be deduced from Theorem 4.5, thanks to the study of the minima of $\widetilde{H}$, which shows that the following.

PROPERTY 4.6. If $4\beta^2A^2 < 1$ and $2\beta^2A^2T < 1$, $\widetilde{H}$ achieves its minimal value at a unique probability measure $\widetilde{Q}$ which is described by the nonlinear system (62).

The strategy of the proof of Theorem 4.5 is the following.

1. We first notice that, when the potential U is even and the temperature is high enough, the following concentration of measure result holds: there exist finite constants C_β and K_β, $C_\beta > 0$ such that, for N large enough and for $0 < v \le 8\beta A^2\sqrt{N}$,

$$\gamma\biggl(J\bigg/\left|\frac{1}{N}\log\frac{Z_J^N}{\mathscr{E}[Z_J^N]}\right| > v + \frac{K_\beta}{\sqrt{N}}\biggr) \le \exp\{-C_\beta v^2 N\}. \tag{63}$$

   The concentration of the free energy to its mean value has been proved by Bovier, Gayrard and Picco ([5], Section 3) and by Talagrand ([17], Chapter 12), in the Ising spin model. Its extension to the continuous setting was shown in [11].

   As a consequence, the Borel–Cantelli lemma shows that a large deviation upper bound for the law of the empirical measure under $\overline{Q_\beta^N} := \mathscr{E}\left[\frac{Z_J^N}{\mathscr{E}[Z_J^N]}\widetilde{P}_\beta^N(J)\right]$ with rate function $\widetilde{H}$ will give Theorem 4.5.

2. Then, we prove that the law of the empirical measure under $\overline{Q_\beta^N}$ satisfies a large deviation upper bound.

PROPERTY 4.7. If $2\beta^2A^2T < 1$, for any closed subset F of $\mathscr{M}_1^+(W_T^A)$,

$$\limsup_{N\to\infty}\frac{1}{N}\log\overline{Q_\beta^N}(\hat{\mu}^N \in F) \le -\inf_F \widetilde{H}.$$

The proof of Property 4.7 is quite similar to that of Theorem 2.3; its main steps are to show that the probability measure $\overline{Q_\beta^N}$ is absolutely continuous with respect to Q_β^N and express its density as a function of the empirical measure, to study the continuity properties of this density, to deduce from this study that $\widetilde{H}$ is a good rate function and to get Property 4.7 from the large deviation upper bound stated for Q_β^N in Theorem 2.3 in the high temperature regime $2\beta^2A^2T < 1$. The details are given in [10].

APPENDIX

In order to study the rate function H and to prove the large deviations upper bound theorem, we need to study the map Γ. We recall that, for any μ in $\mathscr{M}$, we defined $\Gamma(\mu)$ by

$$\Gamma(\mu) = -\frac{1}{2}\operatorname{tr}_\mu\log(I+\beta^2\mathscr{B}_T) + \int_0^\infty \operatorname{tr}_\mu(\mathscr{A}_T\exp\{-\lambda\mathscr{B}_t\})^2\exp-\left\{\frac{\lambda}{\beta^2}\right\}d\lambda.$$

Let

$$\Gamma_1(\mu) = -\tfrac{1}{2}\,\mathrm{tr}_\mu \log(I + \beta^2 \mathscr{B}_T),$$
$$\Gamma_2(\mu) = \int_0^\infty \mathrm{tr}_\mu(\mathscr{A}_T \exp\{-\lambda \mathscr{B}_t\})^2 \exp - \left\{\frac{\lambda}{\beta^2}\right\} d\lambda,$$
$$= \beta^2\,\mathrm{tr}_{\mu\otimes\mu}\big((\mathbb{I} + \beta^2 I \otimes \mathscr{B}_T + \beta^2 \mathscr{B}_T \otimes I)^{-1} \mathscr{A}_T \otimes \mathscr{A}_T \circ \mathscr{S}\big),$$

so that

$$\Gamma = \Gamma_1 + \Gamma_2.$$

We will prove first that Γ_1 is a bounded continuous function. In a second step we will approximate Γ_2, on "large" subsets of $\mathscr{M}_1^+(W_T^A)$ (see Lemma A.8), by the sum of a linear function and a bounded continuous function (see Lemma A.6).

A.1. *Continuity of* Γ_1.

LEMMA A.1. *The mapping Γ_1 is a bounded continuous function:*

$$-\tfrac{1}{2}\beta^2 A^2 T \le \Gamma_1(\mu) \le 0.$$

PROOF. It is quite easy to see (see [13], Proposition 8.4) that

$$\Gamma_1(\mu) = \log \mathscr{E}_\mu\left[\exp\left\{-\tfrac{1}{2}\beta^2 \int_0^T G_t^2\,dt\right\}\right],$$

where G is a centered Gaussian process with covariance

$$\mathscr{E}_\mu[G_t G_s] = \int d\mu(x) x_s x_t.$$

However, for any probability measures (μ, ν) on W_T^A, if ξ denotes a probability measure on $W_T^A \times W_T^A$ with marginals μ and ν and if we denote by $\mathscr{E}_\xi$ the expectation over the centered bidimensionnal Gaussian process with covariance $\mathscr{E}_\xi[G_t^i G_s^j] = \int x_t^i x_s^j\,d\xi(x^1, x^2)$, where $(i, j) \in \{1, 2\}$, then

$$\begin{aligned}
|\exp\Gamma_1(\mu) - \exp\Gamma_1(\nu)| &\le \mathscr{E}_\xi\left[\left|\exp\left\{-\tfrac{1}{2}\beta^2\int_0^T (G_t^1)^2\,dt\right\}\right.\right.\\
&\qquad \left.\left. - \exp\left\{-\tfrac{1}{2}\beta^2\int_0^T (G_t^2)^2\,dt\right\}\right|\right]\\
&\le \tfrac{1}{2}\beta^2 \mathscr{E}_\xi\left[\left|\int_0^T (G_t^1)^2\,dt - \int_0^T (G_t^2)^2\,dt\right|\right]\\
&\le \tfrac{1}{2}\beta^2 \mathscr{E}_\xi\left[\int_0^T (G_t^1 - G_t^2)^2\,dt\right]^{1/2} \mathscr{E}_\xi\left[\int_0^T (G_t^1 + G_t^2)^2\,dt\right]^{1/2}\\
&\le \beta^2 A T\left(\int \sup_{s\le T} |x_s^1 - x_s^2|^2\,d\xi(x^1, x^2)\right)^{1/2}.
\end{aligned}$$

Since this last inequality holds for any probability measure ξ with marginals μ and ν, we proved that $\exp(\Gamma_1)$ is Lipschitz with respect to the Wasserstein distance.

As a consequence, $\exp(\Gamma_1)$ is continuous with respect to the weak topology. Moreover, the Jensen inequality implies that $\exp(\Gamma_1)$ is lower bounded. In fact, for any $\mu \in \mathscr{M}_1^+(W_T^A)$, we find

$$\exp\Gamma_1(\mu) = \mathscr{E}_\mu\left[\exp\left\{-\tfrac{1}{2}\beta^2\int_0^T G_t^2\,dt\right\}\right] \ge \exp\left\{-\tfrac{1}{2}\beta^2\mathscr{E}_\mu\left[\int_0^T G_t^2\,dt\right]\right\}$$
$$= \exp\left\{-\tfrac{1}{2}\beta^2\int d\mu\int_0^T x_t^2\,dt\right\} \ge \exp\{-\tfrac{1}{2}\beta^2 A^2 T\}.$$

Thus, the continuity of $\exp(\Gamma_1)$ implies that of Γ_1. $\square$

A.2. *Approximation of* Γ_2. Let M be a positive real number and ρ_M be a smooth map from $\mathbb{R}^+$ into $[0,1]$ such that $\rho_M(x) = 1$ if $x \le M$, $\rho_M(x) = 0$ if $x \ge M+1$. For $t \le T$, let V_t^M be the map from W_T^A into $\mathbb{R}$ such that $V_t^M = \rho_M(\int_0^T |V_u|\,du)V_t$. For any probability measure μ on W_T^A, we define an integral operator $\mathscr{A}_T^M$ on $L_\mu^2(W_T^A)$ by its kernel,

$$a_T^M(x,y) = \tfrac{1}{2}\left(x_T y_T - x_0 y_0 + \int_0^T x_s V_s^M(y)\,ds + \int_0^T y_s V_s^M(x)\,ds\right).$$

We then define a map Γ_2^M from $\mathscr{M}_1^+(W_T^A)$ into $\mathbb{R}$ by

$$\Gamma_2^M(\mu) = \int_0^\infty \operatorname{tr}_\mu\big(\mathscr{A}_T^M \exp\{-\lambda\mathscr{B}_T\}\big)^2 \exp\left\{-\frac{\lambda}{\beta^2}\right\}d\lambda.$$

LEMMA A.2. Γ_2^M *is bounded and continuous.*

PROOF. Since $\mathscr{B}_T$ is a positive operator, for any positive real number λ and for any probability measure μ,

$$\operatorname{tr}_\mu\big(\mathscr{A}_T^M \exp\{-\lambda\mathscr{B}_T\}\big)^2 \le \operatorname{tr}_\mu\big(\mathscr{A}_T^M\big)^2 \le 4A^2(A+M+1)^2, \tag{64}$$

so that

$$0 \le \Gamma_2^M(\mu) \le 4\beta^2 A^2(A+M+1)^2.$$

Moreover, thanks to the dominated convergence theorem, (64) implies that it is enough to prove that $\mu \to \operatorname{tr}_\mu(\mathscr{A}_T^M\exp\{-\lambda\mathscr{B}_T\})^2$ is continuous for any positive real number λ to show that Γ_2^M is continuous. Since $\mathscr{B}_T$ is a bounded operator,

$$\operatorname{tr}_\mu\big(\mathscr{A}_T^M\exp\{-\lambda\mathscr{B}_T\}\big)^2 = \sum_{k,m=0}^\infty \frac{(-\lambda)^{k+m}}{k!m!}\operatorname{tr}_\mu\big(\mathscr{A}_T^M(\mathscr{B}_T)^k\mathscr{A}_T^M(\mathscr{B}_T)^m\big).$$

Once again, by the dominated convergence theorem, it is enough to prove that, for any integers (k,m), $\mu \to \operatorname{tr}_\mu(\mathscr{A}_T^M(\mathscr{B}_T)^k\mathscr{A}_T^M(\mathscr{B}_T)^m)$ is continuous. This is obvious since $\mathscr{B}_T$ and $\mathscr{A}_T^M$ have bounded continuous kernels. Hence, Γ_2^M is continuous. □

One would like to prove that $\lim_{M\to\infty}|\Gamma_2-\Gamma_2^M|=0$ when $\mu\in\mathscr{M}$. This is difficult because Γ_2^M contains a singular term, say Λ^M, that we will substract off.

More precisely, let

$$\Lambda^M(\mu)=\tfrac{1}{2}\beta^2\int_0^T ds\int_0^T dt\langle(I+\beta^2\mathscr{B}_T)^{-1}X_s,X_t\rangle_\mu\langle V_s^M,V_t^M\rangle_\mu.$$

Similarly, let

$$\Lambda(\mu)=\tfrac{1}{2}\beta^2\int_0^T ds\int_0^T dt\langle(I+\beta^2\mathscr{B}_T)^{-1}X_s,X_t\rangle_\mu\langle V_s,V_t\rangle_\mu,$$

and define a map C from $\mathscr{M}$ into $\mathbb{R}$ by

$$C(\mu)=\left(\int d\mu(x)\left(\int_0^T|\nabla U(x_t)|\,dt\right)^2\right)^{3/2}+1.$$

LEMMA A.3. *If we define a map F^M from $\mathscr{M}$ into $\mathbb{R}$ by*

$$\Gamma_2-\Gamma_2^M=F^M+\Lambda-\Lambda^M,$$

then there exists a finite constant c such that, for any μ in $\mathscr{M}$,

$$|F^M(\mu)|\le\frac{c}{M}C(\mu).$$

PROOF. Let $\mathscr{A}_c$ be the integral operator with kernel $a_c(x,y)=\frac{1}{2}(x_Ty_T-x_0y_0)$ and $\mathscr{A}_d^M$ be the integral operator with kernel

$$a_d^M(x,y)=\tfrac{1}{2}\left(\int_0^T x_sV_s^M(y)\,ds+\int_0^T y_sV_s^M(x)\,ds\right).$$

Then, $\mathscr{A}_T^M=\mathscr{A}_c+\mathscr{A}_d^M$. We can expand Γ_2^M:

$$\begin{aligned}\Gamma_2^M(\mu)&=\int_0^\infty\left(\mathrm{tr}_\mu\big((\mathscr{A}_d^M+\mathscr{A}_c)\exp\{-\lambda\mathscr{B}_T\}\big)^2\right)\exp-\{(\lambda/\beta^2)\}\,d\lambda\\
&=\int_0^\infty\Bigg\{\frac{1}{2}\int_0^T\int_0^T ds\,du\langle\exp\{-\lambda\mathscr{B}_T\}X_s,V_u^M\rangle_\mu\langle\exp\{-\lambda\mathscr{B}_T\}X_u,V_s^M\rangle_\mu\\
&\qquad+\frac{1}{2}\int_0^T\int_0^T ds\,du\langle\exp\{-\lambda\mathscr{B}_T\}X_s,X_u\rangle_\mu\langle\exp\{-\lambda\mathscr{B}_T\}V_u^M,V_s^M\rangle_\mu\\
&\qquad+\mathrm{tr}_\mu\big(\mathscr{A}_c\exp\{-\lambda\mathscr{B}_T\}\big)^2+2\,\mathrm{tr}_\mu\big(\mathscr{A}_d^M\exp\{-\lambda\mathscr{B}_T\}\mathscr{A}_c\exp\{-\lambda\mathscr{B}_T\}\big)\Bigg\}\\
&\qquad\times\exp\left\{-\frac{\lambda}{\beta^2}\right\}d\lambda.\end{aligned}$$

We can similarly expand Γ_2 so that we find

$$|F^M(\mu)| = \left|\Gamma_2(\mu) - \Gamma_2^M(\mu) - \Lambda(\mu) + \Lambda^M(\mu)\right| \tag{65}$$
$$\leq \sum_{i=1}^{4} \int_0^\infty L^i_{\lambda,M}(\mu) \exp\left\{-\frac{\lambda}{\beta^2}\right\} d\lambda,$$

where

$$L^1_{\lambda,M}(\mu) = \tfrac{1}{2}\left|\int_0^T \int_0^T ds\, du \langle \exp\{-\lambda \mathscr{B}_T\} X_s, V_u - V_u^M\rangle_\mu \langle \exp\{-\lambda \mathscr{B}_T\} X_u, V_s\rangle_\mu\right|,$$

$$L^2_{\lambda,M}(\mu) = \tfrac{1}{2}\left|\int_0^T \int_0^T ds\, du \langle \exp\{-\lambda \mathscr{B}_T\} X_s, V_u^M\rangle_\mu \langle \exp\{-\lambda \mathscr{B}_T\} X_u, V_s - V_s^M\rangle_\mu\right|,$$

$$L^3_{\lambda,M}(\mu) = \tfrac{1}{2}\left|\int_0^T \int_0^T ds\, du \langle \exp\{-\lambda \mathscr{B}_T\} X_s, X_u\rangle_\mu \right.$$
$$\left. \times \big(\langle (\exp\{-\lambda \mathscr{B}_T\} - I) V_u, V_s\rangle_\mu - \langle (\exp\{-\lambda \mathscr{B}_T\} - I) V_u^M, V_s^M\rangle_\mu\big)\right|,$$

$$L^4_{\lambda,M}(\mu) = 2\left|\mathrm{tr}_\mu\big((\mathscr{A}_d^\infty - \mathscr{A}_d^M) \exp\{-\lambda \mathscr{B}_T\} \mathscr{A}_c \exp\{-\lambda \mathscr{B}_T\}\big)\right|.$$

To bound $(L^i_{\lambda,M}(\mu))_{1\leq i\leq 4}$, we first show that $\exp\{-\lambda \mathscr{B}_T\} X_s$ is a uniformly bounded operator on $L^2_\mu(W_T^A)$.

One can see that

$$\exp\{-\lambda \mathscr{B}_T\} = I - \int_0^\lambda \mathscr{B}_T \exp\{-\alpha \mathscr{B}_T\}\, d\alpha, \tag{66}$$

so that, for any x in W_T^A:

$$\exp\{-\lambda \mathscr{B}_T\} X_s(x) = x_s - \int_0^\lambda \mathscr{B}_T \exp\{-\alpha \mathscr{B}_T\} X_s(x)\, d\alpha$$
$$= x_s - \int_0^\lambda \int_0^T x_t \langle X_t, \exp\{-\alpha \mathscr{B}_T\} X_s\rangle_\mu\, dt\, d\alpha$$

However, since $\mathscr{B}_T$ is positive, $|\langle X_t, \exp\{-\alpha \mathscr{B}_T\} X_s\rangle_\mu| \leq ||X_t||_\mu ||X_s||_\mu \leq A^2$ so that

$$\sup_{s\leq T} \sup_{x\in W_T^A} |\exp\{-\lambda \mathscr{B}_T\} X_s(x)| \leq A + A^3 T\lambda. \tag{67}$$

Thus we can bound $L^1_{\lambda,M}(\mu)$:

$$L^1_{\lambda,M}(\mu) \leq \tfrac{1}{2}(A + A^3 T\lambda)^2 E_\mu\left[\int_0^T |V_s - V_s^M|\, ds\right] E_\mu\left[\int_0^T |V_s|\, ds\right]. \tag{68}$$

Similarly,

$$L^2_{\lambda,M}(\mu) \leq \tfrac{1}{2}(A + A^3 T\lambda)^2 E_\mu\left[\int_0^T |V_s - V_s^M|\, ds\right] E_\mu\left[\int_0^T |V_s|\, ds\right].$$

Moreover, according to (66),

$$
\begin{aligned}
&\langle(\exp\{-\lambda\mathscr{B}_T\}-I)V_u, V_s\rangle_\mu \\
&\qquad = -\int_0^\lambda \langle \mathscr{B}_T \exp\{-\alpha\mathscr{B}_T\}V_s, V_u\rangle_\mu \, d\alpha \\
&\qquad = -\int_0^\lambda \int_0^T \langle \exp\{-\alpha\mathscr{B}_T\}V_s, X_t\rangle_\mu \langle V_u, X_t\rangle_\mu \, d\alpha\, dt.
\end{aligned}
$$

So that we can bound $L^3_{\lambda, M}(\mu)$ using (67), we get

$$L^3_{\lambda, M}(\mu) \le A^4 T(1 + A^3 T\lambda) E_\mu\biggl[\int_0^T |V_s - V_s^M|\, ds\biggr] E_\mu\biggl[\int_0^T |V_s|\, ds\biggr].$$

Finally, $L^4_{\lambda, M}(\mu)$ may easily be bounded. More precisely, we find

$$L^4_{\lambda, M}(\mu) \le 4A^2(A + A^3 T\lambda) E_\mu\biggl[\int_0^T |V_s - V_s^M|\, ds\biggr].$$

Therefore, we can find a finite constant c such that, for any $\mu \in \mathscr{M}$, for any positive real number M,

$$
\begin{aligned}
\bigl|F^M(\mu)\bigr| &= \bigl|\Gamma_2(\mu) - \Gamma_2^M(\mu) - \Lambda(\mu) + \lambda^M(\mu)\bigr| \\
&\le \sum_{i=1}^4 \int_0^\infty L^i_{\lambda, M}(\mu) \exp\biggl\{-\frac{\lambda}{\beta^2}\biggr\}\, d\lambda \\
&\le \frac{c}{2} E_\mu\biggl[\int_0^T |V_s - V_s^M|\, ds\biggr]\biggl(E_\mu\biggl[\int_0^T \bigl|V_s\bigr|\, ds\biggr] + 1\biggr).
\end{aligned}
$$

However,

$$E_\mu\biggl[\int_0^T \bigl|V_s - V_s^M\bigr|\, ds\biggr] \le \frac{1}{M} E_\mu\biggl[\biggl(\int_0^T \bigl|V_s\bigr|\, ds\biggr)^2\biggr],$$

so that we have proved

$$\bigl|F^M(\mu)\bigr| \le \frac{c}{M}\biggl(E_\mu\biggl[\biggl(\int_0^T \bigl|V_s\bigr|\, ds\biggr)^2\biggr]^{3/2} + 1\biggr) = \frac{c}{M} C(\mu). \qquad \square$$

Now, we have to control both terms Λ and Λ^M. First, Λ^M is a "good" map since the following holds.

LEMMA A.4. *Λ^M is bounded and continuous.*

PROOF. We write

$$
\begin{aligned}
L^M(\mu) &= \tfrac{1}{2}\beta^2 \int_0^T ds \int_0^T dt \langle (I + \beta^2\mathscr{B}_T)^{-1} X_s, X_t\rangle_\mu \langle V_s^M, V_t^M\rangle_\mu \\
&= \tfrac{1}{2}\beta^2 \int \biggl(\int_0^T ds \int_0^T dt \langle (I + \beta^2\mathscr{B}_T)^{-1} X_s, X_t\rangle_\mu V_s^M(x) V_t^M(x)\biggr)\, d\mu(x).
\end{aligned}
$$

We first remark that $\mu \to \langle (I+\beta^2\mathscr{B}_T)^{-1}X_s, X_t\rangle_\mu$ is continuous for the weak topology. Indeed, if we denote by $\overline{K}_\mu$ the integral operator in $L^2_{dt}([0,T])$ with kernel $\int x_s x_t\, d\mu(x)$, then, for any x in W_T^A, $\overline{K}_\mu x(t) = \mathscr{B}_T X_t(x)$, so that

$$\big\langle (I+\beta^2\mathscr{B}_T)^{-1}X_s, X_t\big\rangle_\mu = \int x(s)\big(I_{L^2_{dt}([0,T])} + \beta^2\overline{K}_\mu\big)^{-1} x(t)\, d\mu(x) = \widetilde{K}_\mu^T(s,t).$$

We have already proved in [3], Appendix A, Lemma A.4, that $\mu \to \widetilde{K}_\mu^T(s,t)$ is Lipschitz for the Wasserstein distance d_T [whose definition is given in (14)]. More precisely, there exists a finite constant k such that, for any probability measures μ and ν on W_T^A,

$$\sup_{s,t\le T}\big|\widetilde{K}_\mu^T(s,t) - \widetilde{K}_\nu^T(s,t)\big| \le k d_T(\mu,\nu). \tag{69}$$

Since d_T is compatible with the weak topology, (69) implies that $\mu \to \langle (I+\beta^2\mathscr{B}_T)^{-1}X_s, X_t\rangle_\mu$ is continuous for the weak topology.

Thus, $(x,\mu) \to \int_0^T ds \int_0^T dt \langle (I+\beta^2\mathscr{B}_T)^{-1}X_s, X_t\rangle_\mu V_s^M(x) V_t^M(x)$ is a bounded continuous function on $W_T^A \times \mathscr{M}_1^+(W_T^A)$ so that Λ^M is bounded and continuous. $\square$

To control Λ, we introduce linear functions Λ_ν which are given, for any probability measure ν on W_T^A, by

$$\Lambda_\nu(\mu) = \tfrac{1}{2}\beta^2 \int_0^T ds \int_0^T dt \langle (I+\beta^2\mathscr{B}_T)^{-1}X_s, X_t\rangle_\nu \langle V_s, V_t\rangle_\mu.$$

Note that $\Lambda(\mu) = \Lambda_\mu(\mu)$.

Then we have the lemma.

LEMMA A.5. *There exists a finite constant c' such that, for any probability measure ν on W_T^A, for any probability measure μ on W_T^A,*

$$|\Lambda_\nu(\mu) - \Lambda(\mu)| \le c' C(\mu) d_T(\mu,\nu).$$

PROOF. According to (69), for any probability measure ν on W_T^A,

$$\begin{aligned}
\big|\Lambda_\nu(\mu) - \Lambda(\mu)\big| &= \tfrac{1}{2}\beta^2 \left| \int_0^T \int_0^T \big(\widetilde{K}_\mu^T(s,t) - \widetilde{K}_\nu^T(s,t)\big)\langle V_s, V_t\rangle_\mu\, ds\, dt \right| \\
&\le \tfrac{1}{2}\beta^2 k d_T(\mu,\nu) \int \left(\int_0^T |V_s|\, ds \right)^2 d\mu \\
&\le \tfrac{1}{2}\beta^2 k d_T(\mu,\nu) C(\mu).
\end{aligned}$$

From Lemmas A.2–A.5, we deduce the following.

LEMMA A.6. (i) $\Gamma_2^M - \Lambda^M$ *is a bounded continuous function.*

(ii) *We can find a finite constant C_0 such that, for any positive real number M, for any probability measure ν on W_T^A, for any $\mu \in \mathscr{M}$,*

$$\left|\Gamma_2(\mu) - \Gamma_2^M(\mu) + \Lambda^M(\mu) - \Lambda_\nu(\mu)\right| \le C_0\left(\frac{1}{M} + d_T(\mu,\nu)\right)C(\mu). \tag{70}$$

Finally, if we recall that we proved in Lemma A.1 that Γ_1 is bounded and continuous and if we write

$$\Gamma^M = \Gamma_1 + \Gamma_2^M - \Lambda^M,$$

then we have proved the crucial result of this section, Proposition A.7.

PROPOSITION A.7. (i) Γ^M *is bounded and continuous.*

(ii) *There exists a finite constant C_0 such that, for any probability measure μ in $\mathscr{M}$, for any probability measure ν in $\mathscr{M}_1^+(W_T^A)$,*

$$\left|\Gamma(\mu) - \Gamma^M(\mu) - \Lambda_\nu(\mu)\right| \le C_0\left(\frac{1}{M} + d_T(\mu,\nu)\right)C(\mu).$$

Finally, we prove that $C(\mu)$ is bounded when the entropy relative to P is bounded.

LEMMA A.8. *For any positive real number M, there exists a finite constant $m(M)$ so that, for any probability measure μ such that $I(\mu|P) \le m(M)$, $C(\mu) \le M$.*

PROOF. Let ρ be a smooth approximation of the sign of ∇U; that is, let ρ be a continuously differentiable function such that there exists a finite constant C so that

$$\rho(x)\nabla U(x) = |\nabla U(x)| \quad \text{if } |\nabla U(x)| > 1, \qquad ||\rho||_\infty \le 1,\ ||\rho'||_\infty \le C.$$

Then

$$\left|\int_0^T |\nabla U(x_s)|\,ds - \int_0^T \rho(x_s)\nabla U(x_s)\,ds\right| \le 2T,$$

and Itô's formula implies that, under P, if r is a twice continuously differentiable function such that $r' = \rho$,

$$r(x_T) = r(x_0) + \int_0^T \rho(x_s)(-\nabla U(x_s)\,ds + dB_s) + \tfrac{1}{2}\int_0^T \rho'(x_s)\,ds,$$

so that

$$\left|\int_0^T |\nabla U(x_s)|\,ds - \int_0^T \rho(x_s)\,dB_s\right| \le 2A + \left(\frac{C}{2}+2\right)T. \tag{71}$$

Moreover, for any positive real number ε such that $\varepsilon < (1/8T)$, we can use relative entropy and supermartingale properties as in Lemma 2.19 to get that

$$\int dP \exp\left\{\varepsilon\left(\int_0^T \rho(x_s)\,dB_s\right)^2\right\} \le \frac{1}{(1-8T\varepsilon)^{1/4}}. \tag{72}$$

Thus, (71) and (72) (with $\varepsilon = 1/16T$) imply that

$$\begin{aligned}
&\int\left(\int_0^T |\nabla U(x_s)|\,ds\right)^2 d\mu(x)\\
&\quad\le 2\left(2A+\left(\frac{C}{2}+2\right)T\right)^2 + 2\int\left(\int_0^T \rho(x_s)\,dB_s(x)\right)^2 d\mu(x)\\
&\quad\le 2\left(2A+\left(\frac{C}{2}+2\right)T\right)^2 + 32TI(\mu|P) + 8T\log 2.
\end{aligned} \tag{73}$$

Since $C(\mu) = (\int(\int_0^T |\nabla U(x_s)|\,ds)^2\,d\mu(x))^{3/2} + 1$, it is obvious that (73) gives Lemma A.8. □

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