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NATURAL LINEAR ADDITIVE FUNCTIONALS OF SUPERPROCESSES[1]

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We investigate natural linear additive (NLA) functionals of a general critical (ξ, K, ψ)-superprocess X. We prove that all of them have only fixed discontinuities. All homogeneous NLA functionals of time-homogeneous superprocesses are continuous (this was known before only in the case of quadratic branching).

We introduce an operator $\mathscr{E}(u)$ defined in terms of (ξ, K, ψ) and we prove that the potential h and the log-potential u of a NLA functional A are connected by the equation $u + \mathscr{E}(u) = h$. The potential is always an exit rule for ξ and the condition $h + \mathscr{E}(h) < \infty$ a.e. is sufficient for an exit rule h to be a potential.

In an accompanying paper, these results are applied to boundary value problems for partial differential equations involving nonlinear operator $Lu - u^\alpha$ where L is a second order elliptic differential operator and $1 < \alpha \le 2$.

1. Introduction.

1.1. *Superprocesses.* We follow definitions and notation of [4].

Let E be a metrizable Luzin space. A *superprocess* X in E is a collection of random measures on $S = \mathbb{R}_+ \times E$ characterized by the following elements.

1. A right Markov process $\xi = (\xi_t, \Pi_{r,x})$ in E.
2. A (positive) continuous additive functional K of ξ.
3. A transformation ψ in the space of positive Borel functions on S.

[This is a mathematical model of a random cloud. The spatial motion of its infinitesimal parts is described by ξ, and the branching mechanism is given by $K(ds)$ (the intensity of branching) and ψ (the branching law).]

We denote by $\mathscr{M}(S)$ the space of all finite measures on a measurable space S. A set Q is called finely open if, for every $(r, x) \in Q$, there exists, $\Pi_{r,x}$-a.s., $t > r$ such that $(s, \xi_s) \in Q$ for all $s \in (r, t)$. To every finely open subset Q of S and to every $\mu \in \mathscr{M}(S)$ there corresponds a random measure (X_Q, P_μ) called the *exit measure from* Q. The quantity X_Q describes the time–space mass distribution of the cloud instantaneously frozen on Q^c and P_μ is a probability measure corresponding to initial time-space mass distribution μ. All P_μ have

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the same domain $\mathscr{F}$. For every positive measurable function f,

$$\begin{aligned} &P_\mu \exp\langle -f, X_Q\rangle = \exp\langle -u, \mu\rangle, \\ &u^r(x) + \Pi_{r,x}\int_r^\tau K(ds)\psi^s(u^s)(\xi_s) = \Pi_{r,x} f^\tau(\xi_\tau) \end{aligned} \tag{1.1}$$

where $\tau = \inf\{t\colon (t,\xi_t)\notin Q\}$ is the first exit time of ξ from Q.

The joint probability distribution of $X_{Q_1},\dots,X_{Q_n}$ is determined by (1.1) and by the Markov property: for every positive $\mathscr{F}_{\supset Q}$-measurable Y,

$$P_\mu\{Y \,|\, \mathscr{F}_{\subset Q}\} = P_{X_Q} Y, \tag{1.2}$$

where $\mathscr{F}_{\subset Q}$ is the σ-algebra generated by $X_{Q'}$ with $Q' \subset Q$ and $\mathscr{F}_{\supset Q}$ is the σ-algebra generated by $X_{Q''}$ with $Q'' \supset Q$.

The existence of a family (X_Q, P_μ) subject to conditions (1.1) and (1.2) is proved in [5], Theorem 5.3.1, under a mild assumption on K for a class of functions ψ which contains the family

$$\psi^s(z)(x) = b^s(x)z(x)^2 + \int_{\mathscr{M}(E)} \operatorname{Exp}\langle z,\nu\rangle n^s(x,d\nu), \tag{1.3}$$

where

$$\operatorname{Exp}(u) = \exp\{-u\} - 1 + u \tag{1.4}$$

and functions $b \geq 0$ and

$$\int_{\mathscr{M}(E)} \langle 1,\nu\rangle \wedge \langle 1,\nu\rangle^2 n^s(x,d\nu) \tag{1.5}$$

are bounded. There is an additional condition which relates $n^s(x,d\nu)$ to the Lévy measure of ξ (see 3.2.A).

Formula (1.3) describes the branching mechanism of the most general critical superprocess. Formulas (1.1) and (1.3) imply

$$P_\mu\langle f, X_Q\rangle = \Pi_\mu f(\tau,\xi_\tau), \tag{1.6}$$

where

$$\Pi_\mu = \int_E \Pi_{r,x}\mu(dr,dx).$$

We put $S_{<t} = [0,t)\times E$ and we denote by $X_{<t}$ the corresponding exit measure. Let X_t stand for the restriction of $X_{<t}$ to $\{t\}\times E$ and let $P_{r,\nu}$ mean the measure P_μ with $\mu(ds,dx) = \delta_r(ds)\nu(dx)$. ($\delta_z$ is Dirac's measure at the point z.) The collection $(X_t, P_{r,x})$ determines a measure-valued Markov process which can be chosen to be a right process (see, e.g., [21]).

In this paper we assume the following conditions.

1.1.A. The process ξ is a Hunt process.

1.1.B. There exists a measure m (called the *reference measure*) such that, if $m(B) = 0$, then $\Pi_{r,x}\{\xi_t \in B\} = 0$ for all $r < t$ and all x.

Under condition 1.1.A, the right version of the Markov process $(X_t, P_{r,x})$ is a Hunt process (see [17]). By [11], under condition 1.1.B, there exists a measurable function $p(r,x;t,y)$ such that, for all $r<t$ and all positive Borel f,

$$T_t^r f(x) = \Pi_{r,x} f(\xi_t) = \int_E p(r,x;t,y) f(y) m(dy) \tag{1.7}$$

and, for all $r<s<t$ and all $x, z \in E$,

$$\int_E p(r,x;s,y) m(dy) p(s,y;t,z) = p(r,x;t,z).$$

We call p the *transition density of* ξ.

1.2. *Additive functionals.* We put $\mathscr{M} = \mathscr{M}(S)$ and we denote by $\mathscr{F}^0(I)$ the σ-algebra in Ω generated by X_t with $t \in I$. Let $A(\omega, \cdot)$ be a measure on $(0,\infty)$ which depends on parameter $\omega \in \Omega$ and let $\mathscr{M}^* \subset \mathscr{M}$. We say that A is an *additive functional of* X *with determining set* $\mathscr{M}^*$ if, for every open interval I and every $\mu \in \mathscr{M}^*$, $A(I)$ is measurable relative to the P_μ-completion of $\mathscr{F}^0(I)$. We assume that set $\mathscr{M}^*$ has the following properties.

1.2.A. If $\mu \in \mathscr{M}^*$ and if $\tilde{\mu} \le \mu$, then $\tilde{\mu} \in \mathscr{M}^*$.

1.2.B. For every $\mu \in \mathscr{M}^*$ and for an arbitrary Q, $P_\mu\{X_Q \in \mathscr{M}^*\} = 1$. Moreover, $P_\mu\{X_t \text{ and } X_{t-} \in \mathscr{M}^* \text{ for all } t\} = 1$.

1.2.C. The set $S^* = \{(r,x) : \delta_{(r,x)} \in \mathscr{M}^*\}$ is the complement of a ξ-polar set. (A set $\tilde{S} \subset S$ is called ξ-polar if $\Pi_{r,x}\{\xi_t \in \tilde{S} \text{ for some } t\} = 0$ for all $(r,x) \notin \tilde{S}$.)

1.2.D. Every measure $\mu \in \mathscr{M}^*$ is concentrated on S^*.

We call sets with properties 1.2.A–D *total*. Note that the intersection of any countable family of total sets is a total set.

A positive measurable function h on S is called an *exit rule* if

$$T_s^r h^s \le h^r \quad \text{and} \quad T_s^r h^s \to h^r \text{ as } s \downarrow r.$$

We say that h is a *pure exit rule* if, in addition, $T_s^r h^s \downarrow 0$ as $s \to \infty$. Denote by H the set of all pure exit rules h such that $h^r(x) < \infty$ m-a.e. for all r. To every $h \in H$ there corresponds a total set $\mathscr{M}(h) = \{\mu \in \mathscr{M} : \langle h, \mu \rangle < \infty\}$.

Denote by $\mathscr{P}_r$ the σ-algebra in $(r,\infty) \times \Omega$ generated by functions $F(t,\omega)$ which are left continuous in t and adapted to $\mathscr{F}^0(r,t)$. An additive functional A is called *natural* if, for every r and every $\mu \in \mathscr{M}^*$, the function $A(r,t]$, $t \ge r$ is P_μ-indistinguishable from an $\mathscr{P}_r$-measurable function.

1.3. *Natural linear additive functionals.* Let $h \in H$. We say that A is a *natural linear additive* (*NLA*) *functional with potential* h if A is a natural additive functional with determining set $\mathscr{M}^* \subset \mathscr{M}(h)$ and if, for all $\mu \in \mathscr{M}^*$,

$$P_\mu A(0, \infty) = \langle h, \mu \rangle \tag{1.8}$$

and

$$P_\mu \{A(0, r] = 0\} = 1 \quad \text{if } \mu(S_{<r}) = 0. \tag{1.9}$$

[A heuristic interpretation of (1.9): nothing is accumulated by A before the cloud is born.]

Put $S^* = \{(r, x)\colon \delta_{r,x} \in \mathscr{M}^*\}$. The *log-potential* of an NLA functional A is defined by

$$u^r(x) = -\log P_{r,x} e^{-A(r,\infty)}, \qquad (r, x) \in S^*. \tag{1.10}$$

By Jensen's inequality,

$$u^r(x) \le h^r(x) \quad \text{on } S^*. \tag{1.11}$$

The *characteristic* of A is defined by

$$h_t^r(x) = P_{r,x} A(r, t] \quad \text{for } r < t, (r, x) \in S^*. \tag{1.12}$$

It can be continued to $\{r < t,\ (r, x) \notin S^*\}$ by

$$h_t^r(x) = \lim_{s \downarrow r} \int_E p(r, x; s, y) h_t^s(y) m(dy). \tag{1.13}$$

(Since m does not charge the complement of S^*, the integrand is defined m-a.e. The limit exists because the integral is monotone decreasing in s.) We set $h_t^r(x) = 0$ for $r \ge t$.

An important class of NLA functionals is given by

$$A(I) = \sum_1^n \delta_{t_k}(I) \langle f_k, X_{t_k -} \rangle \tag{1.14}$$

where $0 < t_1 < \cdots < t_n < \infty$ and $f_1, \ldots, f_n$ are positive functions. We call them *discrete functionals*. The potential of A is equal to

$$h^r(x) = \sum_{t_k > r} T_{t_k}^r f_k. \tag{1.15}$$

1.4. *Operator $\mathscr{E}$.* A superprocess X can be characterized by a nonlinear operator acting on positive Borel functions on S by

$$\mathscr{E}(u)(r, x) = \Pi_{r,x} \int_r^\infty K(ds) \psi^s(u^s)(\xi_s). \tag{1.16}$$

The expression $\mathscr{E}(u, \mu) = \langle \mathscr{E}(u), \mu \rangle$ can be considered as a generalized energy integral. A similar generalization is introduced in nonlinear potential theory [see, e.g., [1], (2.2.6)]. We set

$$S_{\mathscr{E}}(h) = \{(r, x)\colon (h + \mathscr{E}(h))(r, x) < \infty\}, \tag{1.17}$$

$$\mathscr{M}_{\mathscr{E}}(h) = \{\mu\colon \langle h + \mathscr{E}(h), \mu \rangle < \infty\}. \tag{1.18}$$

For every total set $\mathscr{M}^*$, we put $\mathscr{M}^*_{\mathscr{E}}(h) = \mathscr{M}^* \cap \mathscr{M}_{\mathscr{E}}(h)$ and $S^*_{\mathscr{E}}(h) = S^* \cap S_{\mathscr{E}}(h)$. Note that all measures $\mu \in \mathscr{M}^*_{\mathscr{E}}(h)$ are concentrated on $S^*_{\mathscr{E}}(h)$.

1.5. *Continuity properties.* An additive functional A with determining set $\mathscr{M}^*$ is called *continuous* if there exists a set Ω' such that: (1) the measure $A(\omega, \cdot)$ is diffuse (i.e., it does not charge single points) for every $\omega \notin \Omega'$; (2) $P_\mu(\Omega') = 0$ for every $\mu \in \mathscr{M}^*$.

THEOREM 1.1. *Let A be an NLA functional with potential h and determining set $\mathscr{M}^* \subset \mathscr{M}_{\mathscr{E}}(h)$. If the characteristic $h^r_t(x)$ is continuous in t for every $(r, x) \in S^*$, then functional A is continuous.*

We say an additive functional A with determining set $\mathscr{M}^*$ has only *fixed discontinuities* if there exists a set Ω' and a set Φ, at most countable independent of ω such that: (1) $A(\omega, \{t\}) = 0$ for all $\omega \notin \Omega'$ and all $t \notin \Phi$; (2) $P_\mu(\Omega') = 0$ for all $\mu \in \mathscr{M}^*$.

THEOREM 1.2. *An NLA functional with determining set $\mathscr{M}^* \subset \mathscr{M}_{\mathscr{E}}(h)$ has only fixed discontinuities.*

1.6. *Discrete approximation.* Let A be an NLA functional with potential h and let $\Lambda = \{0 = t_0 < t_1 < \cdots < t_n\}$. Denote by A_Λ the discrete NLA functional given by

$$A_\Lambda(ds) = \sum_1^{n-1} \delta_{t_k}(ds)\langle h^{t_k}_{t_{k+1}}, X_{t_k-}\rangle + \delta_{t_n}(ds)\langle h^{t_n}, X_{t_n-}\rangle. \tag{1.19}$$

We say that Λ_n is a *standard sequence of partitions* of $\mathbb{R}_+$ if $\Lambda_1 \subset \Lambda_2 \subset \cdots$ and the union of Λ_n is everywhere dense in $\mathbb{R}_+$.

THEOREM 1.3. *Let A be an NLA functional with potential h and determining set $\mathscr{M}^*$. Let Λ_n be a standard sequence of partitions. Then, for every $0 \le r < t \le \infty$ and for every $\mu \in \mathscr{M}^*_{\mathscr{E}}(h)$,*

$$A_{\Lambda_n}(r, t] \to A(r, t] \quad \textit{in } L^1(P_\mu). \tag{1.20}$$

[Formula (1.20) with weak convergence follows easily from [3], VII. 8 and VII.21. The strong convergence was proved in [16], Lemma 3.1, in the case when A is piece-wise continuous.]

REMARK. Let A_1, A_2 be two NLA functionals with potentials h_1, h_2 and determining sets $\mathscr{M}^*_1, \mathscr{M}^*_2$. Theorem 1.3 implies that, if $h_1 = h_2$, then the functionals A_1 and A_2 are indistinguishable with respect to all the measures P_μ with $\mu \in \mathscr{M}^*_1 \cap \mathscr{M}^*_2$.

1.7. *$\mathscr{E}$-equation.* By [4], Theorem I.1.8, discrete functionals have the properties:

1. for every $\mu \in \mathscr{M}$,

$$P_\mu e^{-A(0,\infty)} = e^{-\langle u,\mu\rangle}; \tag{1.21}$$

2. the log-potential u satisfies equation

$$u + \mathscr{E}(u) = h \quad \text{on } S. \tag{1.22}$$

We prove the following theorem.

THEOREM 1.4. *Let A be a NLA functional with potential h, log-potential u and determining set $\mathscr{M}^*$. Then*

$$P_\mu e^{-A(0,\infty)} = e^{-\langle u,\mu\rangle} \quad \textit{for every } \mu \in \mathscr{M}^*_{\mathscr{E}}(h) \tag{1.23}$$

and

$$u + \mathscr{E}(u) = h \quad \textit{on } S^*_{\mathscr{E}}(h). \tag{1.24}$$

We call (1.24) the $\mathscr{E}$-equation.

1.8. *Lifting and projection.* Let a be a natural additive functional of ξ (see, e.g., [5], Section 2.4.1). We consider only functionals satisfying the following finiteness condition

$$h(r,x) = \Pi_{r,x} a(r,\infty) < \infty, \quad m\text{-a.s.} \tag{1.25}$$

for all r. The function h is called the potential of a. It defines a uniquely up to indistinguishability (see, e.g., [5], Theorem 2.4.1).

Let A be a NLA functional with determining set $\mathscr{M}^*$ and a be a natural additive functional for ξ. We say that a is the *projection* of A and A is the *lifting* of a if their potentials coincide, that is, if

$$P_{r,x} A(r,\infty) = \Pi_{r,x} a(r,\infty). \tag{1.26}$$

for all $(r,x) \in S^*$.

THEOREM 1.5. *Let h be a potential of a natural additive functional a of the process ξ. There exists a NLA functional A with potential h and determining set $\mathscr{M}^* = \mathscr{M}(h)$.*

In other words, the lifting can be constructed for every natural additive functional subject to the finiteness condition (1.25). For instance, Theorem 1.5 is applicable to every bounded $h \in H$.

1.9. *Existence of linear additive functionals.* Not every NLA functional has a projection (an example could be given by Iscoe's local times [19] for a super-Brownian motion in dimensions 2 and 3). To consider general functionals, we need some preparations. Let $w(t)$, $0 < t \le 1$ be a positive continuous function. We call it *admissible* if

$$\int_0^1 tw(t)\,dt = \infty, \qquad \int_0^1 t^2 w(t)\,dt < \infty. \tag{1.27}$$

For every admissible function w, we set

$$\mathscr{E}_w(u) = \int_0^1 w(t)\mathscr{E}(tu)\,dt \tag{1.28}$$

and $\mathscr{E}_w(u,\mu) = \langle \mathscr{E}_w(u), \mu\rangle$. By (1.3), $\psi^s(tz) \ge t^2\psi^s(z)$ for $t < 1$. Therefore $\mathscr{E}_w(h) \ge \text{const.}\,\mathscr{E}(h)$ and the set $\mathscr{M}(h + \mathscr{E}_w(h)) \subset \mathscr{M}_{\mathscr{E}}(h)$.

THEOREM 1.6. *Let $h \in H$. Assume that the function $\mathscr{E}_w(h)$ belongs to H for some admissible w. Then there exists a NLA functional A with potential h and determining set $\mathscr{M}(h + \mathscr{E}_w(h))$.*

A stronger result can be established under the following condition.

1.8.A. There exists an increasing positive function $q(\lambda) \uparrow \infty$ as $\lambda \to \infty$, such that

$$\psi^s(\lambda z) \ge \lambda q(\lambda)\psi^s(z) \quad \text{for all } \lambda > 1,\ z \ge 0,\ 0 < s < \infty.$$

[It follows from (1.3) that $\lambda\psi^s(z) \le \psi^s(\lambda z) \le \lambda^2\psi^s(z)$ on the same set.]

THEOREM 1.7. *Let* 1.8.A *be valid and let $h, \mathscr{E}(h) \in H$. Then there exists a NLA functional A with potential h and determining set $\mathscr{M}_{\mathscr{E}}(h)$.*

2. Continuity properties.

2.1. To prove Theorem 1.1, we use the following elementary observations.

2.1.A. Suppose that $\varphi \ge 0$ and

$$\varphi(t) = o(t) \quad \text{as } t \downarrow 0. \tag{2.1}$$

If $a_{kn} \ge 0$, $k = 1, \dots, m_n$, $\sum_k a_{kn} \le C$ for all n and if $\lim_n \sup_k a_{kn} = 0$, then

$$\lim_n \sum_k \varphi(a_{kn}) = 0. \tag{2.2}$$

This follows from an obvious inequality

$$\sum \varphi(a_k) \le \left(\sum a_k\right) \sup_k \frac{\varphi(a_k)}{a_k}. \tag{2.3}$$

2.1.B. Suppose that φ is a convex function on $\mathbb{R}_+$ and $\varphi(0) = 0$. Then

$$\sum \varphi(a_k) \le \varphi\Big(\sum a_k\Big) \tag{2.4}$$

for every positive $a_1, \dots, a_m$.

Indeed, for every y, $\varphi(x + y) - \varphi(x)$ is an increasing function in x and therefore $\varphi(x + y) - \varphi(x) \ge \varphi(y)$.

2.2. First, we prove several lemmas.

LEMMA 2.1. *Let A be a NLA functional with potential h. Then*

$$P_\mu \operatorname{Exp} A(0, \infty) \le \operatorname{Exp}\langle h, \mu\rangle + \mathscr{E}(h, \mu) \tag{2.5}$$

for every $\mu \in \mathscr{M}^*$.

PROOF. (i) Put $u(\mu) = -\log P_\mu \exp(-A(0, \infty))$. By Jensen's inequality, $P_\mu \exp(-A(0, \infty)) \ge \exp(-P_\mu A(0, \infty))$, and therefore

$$u(\mu) \le \langle h, \mu\rangle. \tag{2.6}$$

(ii) Let Λ_n be a standard sequence of partitions. Consider discrete functionals $A_n = A_{\Lambda_n}$ given by (1.19). Denote by h_n and u_n their potentials and log-potentials. We claim that

$$u(\mu) \ge \limsup_n \langle u_n, \mu\rangle. \tag{2.7}$$

Indeed,

$$\exp(-A_n(0, \infty)) \ge \exp(-A(0, \infty)) - \exp(-A(0, \infty))\big[A_n(0, \infty) - A(0, \infty)\big]$$

and therefore

$$P_\mu \exp(-A_n(0, \infty)) \ge P_\mu \exp(-A(0, \infty)) - P_\mu \exp(-A(0, \infty))\big[A_n(0, \infty) - A(0, \infty)\big].$$

Since (1.20) holds with weak convergence in $L^1(P_\mu)$, we have

$$P_\mu \exp(-A(0, \infty))\big[A_n(0, \infty) - A(0, \infty)\big] \to 0$$

as $n \to \infty$, and therefore

$$\liminf P_\mu \exp(-A_n(0, \infty)) \ge P_\mu \exp(-A(0, \infty)),$$

which implies (2.7).

(iii) By Jensen's inequality, $u_n \le h_n$. It follows from (1.15) and (1.19) that $h_n \le h$. By (1.22),

$$\langle u_n, \mu\rangle + \mathscr{E}(u_n, \mu) = \langle h_n, \mu\rangle,$$

and therefore

$$\langle u_n, \mu\rangle + \mathscr{E}(h, \mu) \ge \langle h_n, \mu\rangle.$$

Since $h_n \uparrow h$, we have

$$\limsup \langle u_n, \mu\rangle + \mathscr{E}(h, \mu) \geq \langle h, \mu\rangle.$$

By (2.7),

$$u(\mu) + \mathscr{E}(h, \mu) \geq \langle h, \mu\rangle. \tag{2.8}$$

By (1.4),

$$P_\mu \operatorname{Exp} A(0, \infty) = \exp(-u(\mu)) - 1 + \langle h, \mu\rangle = \operatorname{Exp} u(\mu) + \langle h, \mu\rangle - u(\mu)$$

and (2.5) follows from (2.6) and (2.8). □

LEMMA 2.2. *Let $h \in H$ and $\mu \in \mathscr{M}(h)$. Put $S(h) = \{h < \infty\}$. If the function*

$$h_t^r(x) = 1_{r<t}[h^r(x) - T_t^r h^t(x)] \tag{2.9}$$

is continuous in t for every $(r, x) \in S(h)$, then

$$J_\delta^\mu(t) = \int [h_{t+\delta}^r(x) - h_t^r(x)]\mu(dr, dx) \tag{2.10}$$

converges to 0 *as $\delta \to 0$ uniformly on every finite interval.*

PROOF. The function $J_\delta^\mu(t)$ is increasing in δ. The dominated convergence theorem implies that $J_\delta^\mu(t) \to 0$ for every t as $\delta \to 0$. The uniform convergence follows from an observation: $J_\delta^\mu(t_k) \to 0$ as $\delta \to 0$ and $t_k \to t$. □

LEMMA 2.3. *Let Y be measurable relative to the σ-algebra $\mathscr{F}_{>t}^0$. Then, for every $\mu \in \mathscr{M}$,*

$$P_\mu Y = P_\mu P_{X_{<t}} Y. \tag{2.11}$$

PROOF. By the multiplicative systems theorem, it is sufficient to prove (2.11) for $Y = e^{-A(t,\infty)}$ where A is a discrete NLA functional. Put $Q = S_{<t}$. For every $s > t$, $X_s = X_{<s} - \mu_{S_{>s}}$ P_μ-a.s. Therefore $\mathscr{F}_{>t}^0$ is contained in the σ-algebra generated by $\mathscr{F}_{\supset Q}$ and sets of P_μ-measure 0 and (2.11) follows from (1.2). □

LEMMA 2.4. *Let A be a NLA functional with determining set $\mathscr{M}^*$ and potential h. The characteristic of A is given on S^* by (2.9). For every $\mu \in \mathscr{M}^*$, $s < t \in \mathbb{R}_+$,*

$$P_\mu A(s, t] = \int [h_t^r(x) - h_s^r(x)]\mu(dr, dx) \tag{2.12}$$

and

$$P_\mu A\{t\} = \int [h_t^r(x) - h_{t-}^r(x)]\mu(dr, dx). \tag{2.13}$$

Denote by A_Δ the restriction

$$A_\Delta(I) = A(I \cap \Delta) \tag{2.14}$$

of A to the interval Δ. If $\Delta = (s, t]$, then A_Δ is a NLA functional with determining set $\mathscr{M}^*$ and potential

$$h^r_\Delta = h^r_t - h^r_s \quad \text{on } S^*. \tag{2.15}$$

If $\Delta = \{t\}$, then A_Δ is a NLA functional with potential

$$h^r_{\{t\}} = h^r_t - h^r_{t-} \quad \text{on } S^*. \tag{2.16}$$

PROOF. If $\mu \in \mathscr{M}^*$, then by (2.11),

$$P_\mu A(t, \infty) = P_\mu P_{X_{<t}} A(t, \infty). \tag{2.17}$$

The measure $X_{<t}$ belongs, P_μ-a.s., to $\mathscr{M}^*$ and it is concentrated on $S_{\geq t}$. By (1.9),

$$P_{X_{<t}} A(0, t] = 0$$

and, by (1.8),

$$P_{X_{<t}} A(t, \infty) = P_{X_{<t}} A(0, \infty) = \langle h, X_{<t} \rangle. \tag{2.18}$$

By (2.17), (2.18) and (1.6),

$$P_\mu A(t, \infty) = \Pi_\mu h^\tau(\xi_\tau),$$

where τ is the first exit time from $S_{<t}$. Note that $\Pi_{r,x}\{\tau = t\} = 1$ for $r < t$ and $\Pi_{r,x}\{\tau = r\} = 1$ for $r \geq t$. Therefore

$$\begin{aligned} P_\mu A(t, \infty) &= \int [1_{r<t} T^r_t h^t(x) + 1_{r \geq t} h^r(x)] \mu(dr, dx) \\ &= \int [h^r(x) - h^r_t(x)] \mu(dr, dx). \end{aligned} \tag{2.19}$$

Expression (2.9) for the characteristic on set S^* and formulas (2.12), (2.13) follow from (2.19). Formulas (2.12) and (2.13) imply that A_Δ satisfies (1.8) and (2.15) or (2.16). Formula (1.9) holds because $A_\Delta(I) \leq A(I)$. □

2.3. *Proof of Theorem* 1.1. (i) Fix $b \in \mathbb{R}_+$ and $\mu \in \mathscr{M}^*$. Put

$$\sigma = \sup_{t \in (0, b]} A\{t\}$$

and

$$\Phi(\Lambda) = P_\mu \sum_{i=0}^{n-1} \operatorname{Exp}(A(\Delta_i))$$

for $\Lambda = \{0 = t_0 < t_1 < \cdots < t_n = b\}$ and $\Delta_i = (t_i, t_{i+1}]$. The function Exp is monotone increasing and therefore $P_\mu \operatorname{Exp}(\sigma) \leq \Phi(\Lambda)$ for all Λ. Since $\operatorname{Exp}(u) > 0$ for $u > 0$, Theorem 1.1 will be proved if we show that, for every $b \in \mathbb{R}_+$ and every standard sequence Λ_n of partitions of $[0, b]$,

$$\Phi(\Lambda_n) \to 0. \tag{2.20}$$

Let A_{Δ_i} be the NLA functional corresponding to Δ_i by Lemma 2.4 and let h_{Δ_i} be its potential. By Lemma 2.1,

$$P_\nu \operatorname{Exp}(A(\Delta_i)) \le \operatorname{Exp}\langle h_{\Delta_i}, \nu\rangle + \mathscr{E}(h_{\Delta_i}, \nu)$$

for every $\nu \in \mathscr{M}^*$. Since $X_{<t_i} \in \mathscr{M}^*$ P_μ-a.s., Lemma 2.3 implies that

$$\Phi(\Lambda) = P_\mu \sum P_{X_{<t_i}} \operatorname{Exp}(A(\Delta_i)) \le \Phi_1(\Lambda) + \Phi_2(\Lambda) + \Phi_3(\Lambda),$$

where

(2.21) $$\Phi_1(\Lambda) = \sum P_\mu \mathscr{E}(h_{\Delta_i}, X_{<t_i}),$$

(2.22) $$\Phi_2(\Lambda) = \sum \operatorname{Exp}(P_\mu A(\Delta_i)),$$

(2.23) $$\Phi_3(\Lambda) = P_\mu \sum \operatorname{Exp}(P_{X_{<t_i}} A(\Delta_i)) - \sum \operatorname{Exp}(P_\mu A(\Delta_i)).$$

(ii) First we prove that

(2.24) $$\Phi_1(\Lambda_n) \to 0.$$

By (2.15), $h^s_{\Delta_i} = h^s_{t_{i+1}}$ for $s \in [t_i, t_{i+1})$ and $h^s_{\Delta_i} = 0$ for $s \ge t_{i+1}$. Let τ_i be be the first exit time from $S_{<t_i}$. By (1.6),

$$\Phi_1(\Lambda) = \int \mu(dr, dx) F_\Lambda(r, x),$$

where

$$F_\Lambda = \sum_i F^i_\Lambda$$

with

$$F^i_\Lambda(r, x) = \Pi_{r,x} \mathscr{E}(h_{\Delta_i})(\tau_i, \xi_{\tau_i}) = 1_{r<t_i} \Pi_{r,x} \mathscr{E}(h_{\Delta_i})(t_i, \xi_{t_i}) + 1_{t_i \le r < t_{i+1}} \mathscr{E}(h_{\Delta_i})(r, x).$$

By (1.16) and the Markov property of ξ,

$$F^i_\Lambda(r, x) = \Pi_{r,x} \int_{t_i \vee r}^{t_{i+1} \vee r} K(ds) \psi^s(h^s_{t_{i+1}})(\xi_s)$$

and therefore

(2.25) $$\Phi_1(\Lambda) = \Pi_{r,x} \int_r^b K(ds) \psi^s(h^s_{\alpha(s)})(\xi_s) \le \int \gamma(ds, dx) \psi^s(h^s_{\alpha(s)})(x)$$

where $\alpha(s) = t_{i+1}$ if $t_i \le s < t_{i+1}$ and

(2.26) $$\gamma(B) = \int \mu(dr, dx) \Pi_{r,x} \int_r^\infty K(ds) 1_B(s, \xi_s).$$

By (1.12) $h^s_t \le h^s$ for all t and therefore

(2.27) $$\psi^s(h^s_{\alpha(s)}) \le \psi^s(h^s).$$

We have

$$\int \gamma(dr, dx)\psi^r(h^r)(x) = \mathscr{E}(h, \mu) < \infty, \tag{2.28}$$

which implies

$$\psi^s(h^s)(x) < \infty, \quad \gamma\text{-a.e.} \tag{2.29}$$

If α_n corresponds to partition Λ_n, then $\alpha_n(s) \downarrow s$ and

$$\psi^s(h^s_{\alpha_n(s)}) \to 0 \quad \text{on } \{(s, x)\colon \psi^s(h^s)(x) < \infty\} \tag{2.30}$$

by (1.3) and the dominated convergence theorem. Formula (2.24) follows from (2.25), (2.27), (2.28), (2.30) and the dominated convergence theorem.

(iii) By Lemma 2.4,

$$P_\mu A\{t\} = \int [h^r_t - h^r_{t-}]\mu(dr, dx) = 0 \quad \text{for all } t. \tag{2.31}$$

By (1.12),

$$\sum P_\mu A(\Delta_i) \le \langle h, \mu\rangle < \infty$$

and, by (2.12) and Lemma 2.2,

$$\sup_{0<t\le b} P_\mu A(t, t+\delta] \downarrow 0 \quad \text{as } \delta \to 0. \tag{2.32}$$

Hence,

$$\Phi_2(\Lambda_n) \to 0 \tag{2.33}$$

by 2.1.A.

(iv) It remains to prove that

$$\Phi_3(\Lambda_n) \to 0. \tag{2.34}$$

By (2.11), $P_\mu P_{X_{<t_i}} A(\Delta_i) = P_\mu A(\Delta_i)$ and therefore

$$\begin{aligned} &P_\mu \operatorname{Exp} P_{X_{<t_i}} A(\Delta_i) - \operatorname{Exp} P_\mu A(\Delta_i)\\ &\quad = P_\mu \exp\{-P_{X_{<t_i}} A(\Delta_i)\} - \exp\{-P_\mu A(\Delta_i)\}\\ &\quad = P_\mu \exp\{-\langle h_{\Delta_i}, X_{<t_i}\rangle\} - \exp\{-\langle h_{\Delta_i}, \mu\rangle\}. \end{aligned} \tag{2.35}$$

For $r < t_i$,

$$\Pi_{r,x} h^{\tau_i}_{\Delta_i}(\xi_{\tau_i}) = \Pi_{r,x} h^{t_i}_{\Delta_i}(\xi_{t_i}) = P_{r,x} P_{t_i, X_{<t_i}} A(\Delta_i) = P_{r,x} A(\Delta_i) = h^r_{\Delta_i}(x)$$

and, by (1.1),

$$\begin{aligned} &P_\mu \exp\{-\langle h_{\Delta_i}, X_{<t_i}\rangle\} = \exp\{-\langle v_{\Delta_i}, \mu\rangle\},\\ &v^r_{\Delta_i}(x) + \Pi_{r,x} \int_r^{t_i \vee r} K(ds)\psi^s(v^s_{\Delta_i})(\xi_s) = h^r_{\Delta_i}(x). \end{aligned} \tag{2.36}$$

Since $0 \le e^{-y} - e^{-x} \le x - y$ for $0 \le y \le x$, (2.35) and (2.36) imply

$$0 \le \Phi_3(\Lambda) \le \sum_i \int G^i_\Lambda(r, x)\mu(dr, dx), \tag{2.37}$$

where

$$G^i_\Lambda(r, x) = \Pi_{r,x} \int_r^{t_i \vee r} K(ds)\psi^s(v^s_{\Delta_i})(\xi_s) \le \Pi_{r,x} \int_r^{t_i \vee r} K(ds)\psi^s(h^s_{\Delta_i})(\xi_s). \tag{2.38}$$

Hence

$$0 \le \Phi_3(\Lambda) \le \int \gamma(ds, dx) Q_\Lambda(s, x), \tag{2.39}$$

where

$$Q_\Lambda(s, x) = \sum_{t_i > s} \psi^s(h^s_{\Delta_i})(x). \tag{2.40}$$

Note that

$$\sum_{t_i > s} h^s_{\Delta_i}(x) = P_{s,x} \sum_{t_i > s} A(\Delta_i) \le P_{s,x} A(s, \infty) = h^s(x) \tag{2.41}$$

and, by 2.1.B applied to convex functions u^2 and $\operatorname{Exp}(u)$,

$$Q_\Lambda \le \psi^s(h^s). \tag{2.42}$$

The function $\psi^s(h^s)$ is γ-integrable by (2.28). To get (2.34) from (2.39), it is sufficient to show that

$$Q_{\Lambda_n}(s, x) \to 0, \quad \gamma\text{-a.e.} \tag{2.43}$$

(v) If $\max(t_{i+1} - t_i) < \delta$, then

$$\sup_i \langle h_{\Delta_i}, \nu \rangle \le \sup_{t \le b} J^\nu_\delta(t), \tag{2.44}$$

where J is given by (2.10). For every $\nu \in \mathscr{M}(h)$, the right-hand side in (2.44) tends to 0 as $\delta \to 0$ by Lemma 2.2 and therefore

$$\sup_i \langle h_{\Delta_i}, \nu \rangle \to 0 \quad \text{for all } \nu \in \mathscr{M}(h). \tag{2.45}$$

In particular,

$$\sup_i h^r_{\Delta_i}(x) \to 0 \quad \text{for all } (r, x) \in S^*. \tag{2.46}$$

It is sufficient to prove (2.43) when

$$\psi^s(z)(x) = b^s(x) z(x)^2 \tag{2.47}$$

or if

$$\psi^s(z)(x) = \int_{\mathscr{M}} \operatorname{Exp}\langle z, \nu \rangle n^s(x, d\nu). \tag{2.48}$$

In the first case, (2.43) follows from 2.1.A, (2.41) and (2.46). In the second case,

$$Q_\Lambda(s,x) = \int q_\Lambda(s,\nu) n^s(x,d\nu), \tag{2.49}$$

where

$$q_\Lambda(s,\nu) = \sum_i \operatorname{Exp}\langle h^s_{\Delta_i}, \nu\rangle.$$

By (2.41),

$$\sum_i \langle h^s_{\Delta_i}, \nu\rangle \le \langle h^s, \nu\rangle. \tag{2.50}$$

Put $B_s = \{\nu\colon \langle h^s,\nu\rangle < \infty\}$. By 2.1.A, (2.50) and (2.45), $q_{\Lambda_n}(s,\nu) \to 0$ for $\nu \in B_s$. Note that $\operatorname{Exp}\langle h^s,\nu\rangle = \infty$ on the complement of B_s. By (2.48), $n^s(x,\cdot)$ is concentrated on B_s if $\psi^s(h^s)(x) < \infty$. By (2.28), the measure γ is concentrated on the set $\{\psi(h) < \infty\}$, and (2.43) follows from (2.49) and the dominated convergence theorem. □

2.4. Theorem 1.2 is an immediate implication of the following.

THEOREM 2.1. *Every NLA functional A with determining set $\mathscr{M}^* \subset \mathscr{M}_{\mathscr{E}}(h)$ has the form*

$$A = \tilde{A} + \sum_{t\in\Phi} A_{\{t\}}, \tag{2.51}$$

where $\tilde{A}$ is a continuous NLA functional, Φ is at most countable set and $A_{\{t\}}$ is a NLA functional which corresponds by Lemma 2.4 to the interval $\{t\}$.

PROOF. There exists a measure $\mu_0(dr,dx) = \rho(r,x)\,dr\,m(dx)$ with strictly positive ρ such that $\langle h,\mu_0\rangle < \infty$. By Lemma 2.4, the potential $h_{\{t\}}$ of $A_{\{t\}}$ is given by (2.15) and therefore

$$\langle h_{\{t\}}, \mu_0\rangle = F(t) - F(t-),$$

where

$$F(t) = \int h^r_t(x)\mu_0(dr,dx)$$

is a bounded right continuous monotone increasing function. The set

$$\Phi = \{t\colon F(t) > F(t-)\}$$

is at most countable. If $t \notin \Phi$, then $\langle h_{\{t\}},\mu_0\rangle = 0$ and, since $h_{\{t\}}$ is an exit rule, it is equal to 0 identically. Formula

$$\tilde{A} = A - \sum_{t\in\Phi} A_{\{t\}}$$

defines a NLA functional with characteristic

$$\tilde{h}^r_t(x) = h^r_t(x) - \sum_{s\in\Phi} h^r_{\{s\}}(x).$$

Clearly $\tilde{h}^r_t(x)$ is continuous in t and $\tilde{A}$ is continuous by Theorem 1.1. □

3. Discrete approximation, $\mathscr{E}$-equation.

3.1. *Proof of Theorem* 1.3. The theorem has been already proved in [16], Lemma 3.1, if the set Φ in (2.51) is finite and h is bounded. The second condition can be dropped without any substantial change in the proof. If Φ is countable, we consider a sequence of finite sets $\Phi_n \uparrow \Phi$ and we denote by A_n functional of the form (2.51) with Φ replaced by Φ_n. For every $\mu \in \mathscr{M}^*$ and every partition Λ of $\mathbb{R}_+$,

$$P_\mu|A_\Lambda(r,t] - A(r,t]| \le P_\mu|A_n(r,t] - A(r,t]| + P_\mu|A_{n\Lambda}(r,t] - A_n(r,t]| + P_\mu|A_{n\Lambda}(r,t] - A_\Lambda(r,t]|, \tag{3.1}$$

where A_Λ corresponds to A and $A_{n\Lambda}$ corresponds to A_n by (1.19). Note that

$$P_\mu|A(r,t] - A_n(r,t]| \to 0 \quad \text{as } n \to \infty. \tag{3.2}$$

A computation based on (1.15) shows that

$$P_\mu|A_\Lambda(r,t] - A_{n\Lambda}(r,t]| \le \sum_{t\in\Phi\setminus\Phi_n} P_\mu A_{\{t\}}. \tag{3.3}$$

Since (1.20) holds for every functional A_n, it holds for A by (3.1), (3.2) and (3.3). □

3.2. In the proof of Theorem 1.4 we use an additional condition on $n^s(x, d\nu)$ mentioned in Section 1.1.

Recall that the Lévy measure of ξ is a random measure $N(ds, dy)$ on S concentrated on the set $\{(s, y)\colon \xi_{s-} \ne y\}$ and such that

$$\sum_{r<s\le t} f^s(\xi_{s-}, \xi_s) - \int_r^t \int_E f^s(\xi_s, y) N(ds, dy)$$

is a martingale relative to $(\mathscr{F}^0[r,t], \Pi_{r,x})$ for every r, x and every bounded positive Borel function f on $\mathbb{R}_+ \times E \times E$ with the property $f^s(x,x) = 0$ for all s, x. We introduce a random measure on S

$$\bar{n}(A, B) = \int_A K(ds) \int_{\mathscr{M}(E)} n^s(\xi_s, d\nu)\nu(B).$$

The existence of a superprocess is proved in [5], Theorem 5.3.1, for ψ of the form (1.3) under the following additional assumption.

3.2.A. The measure $\bar{n}$ is dominated by the Lévy measure N in the following sense: for every positive Borel function $f^s(x, y)$ such that $f^s(x,x) = 0$ for all s, x,

$$\int f^s(\xi_{s-}, y)\bar{n}(ds, dy) \le \int f^s(\xi_{s-}, y) N(ds, dy) \quad \text{a.s.}$$

(By [5], Theorem 6.1.1, condition 3.2.A holds for all critical superprocesses with finite second moments.)

LEMMA 3.1. *Let* $(r, x) \in S$ *and let* $\Gamma \subset S$ *be a set with the property*

$$\Pi_{r,x}\{(t, \xi_t) \in \Gamma \quad \text{for all } t \ge r\} = 1.$$

Then the measure

$$\gamma(r, x; B) = \Pi_{r,x} \int_r^\infty K(ds) 1_B(s, \xi_s) \tag{3.4}$$

is concentrated on Γ *and the measure*

$$\begin{aligned} \bar\gamma(r, x; C) &= \Pi_{r,x} \int \bar n(ds, dz) 1_C(s, \xi_s, z) \\ &= \int \gamma(r, x; ds, dy) n^s(y, d\nu)\nu(dz) 1_C(s, y, z) \end{aligned} \tag{3.5}$$

is concentrated on the set $\{(s, y, z): (s, z) \in \Gamma\}$.

PROOF. The first statement is obvious since the right side of (3.4) vanishes for $B = \Gamma^c$.

It follows from 3.2.A, (3.5) and the definition of the Lévy measure that

$$\int \bar\gamma(r, x; ds, dy, dz) f^s(y, z) \le \Pi_{r,x} \sum_{s>r} f^s(\xi_{s-}, \xi_s) \tag{3.6}$$

if $f^s(x, x) = 0$. Put $C' = \{(s, y, z): y \ne z, (s, z) \notin \Gamma\}$, $C'' = \{(s, y, z): y = z, (s, z) \notin \Gamma\}$. Measure $\bar\gamma$ does not charge C' because the right-hand side in (3.6) vanishes for $f = 1_{C'}$. It is clear from (3.5) that $\bar\gamma$ does not charge C'' [because $(s, \xi_s) \notin \Gamma$ if $(s, \xi_s, z) \in C''$]. □

3.3. *Proof of Theorem* 1.4. (i) Let A_Λ be given by (1.19). By (1.15), its potential

$$h_\Lambda = T^r_{t_i} h^{t_i}, \tag{3.7}$$

where $i = \min\{k: t_k > r\}$. Fix $\mu \in \mathscr{M}^*_{\mathscr{E}}(h)$. By (1.21),

$$P_\mu \exp(-A_\Lambda(0, \infty)) = \exp(-\langle u_\Lambda, \mu\rangle),$$

where u_Λ is the log-potential of A_Λ.

Let Λ_n be a standard sequence of partitions of $\mathbb{R}_+$ and let $u_n = u_{\Lambda_n}$. By Theorem 1.3,

$$P_\mu |A_{\Lambda_n}(0, \infty) - A(0, \infty)| \to 0$$

and therefore

$$\langle u_n, \mu\rangle \to -\log P_\mu e^{-A(0,\infty)}. \tag{3.8}$$

Hence u_n tends to the log-potential u of A on $\Gamma = S^*_{\mathscr{E}}(h)$. By (3.7) and (1.11),

$$u_n \le h. \tag{3.9}$$

Therefore $\langle u_n, \mu\rangle \to \langle u, \mu\rangle$ and (1.23) follows from (3.8).

(ii) By (1.22),

$$u_n + \mathscr{E}(u_n) = h_{\Lambda_n}. \tag{3.10}$$

By (3.7), $h_{\Lambda_n} \to h$. We know that $u_n \to u$ on Γ. We get (1.24) if we prove that

(3.11) $$\mathscr{E}(u_n) \to \mathscr{E}(u) \quad \text{on } \Gamma.$$

First, we show that, if $(r, x) \in \Gamma$, then

(3.12) $$\psi(u_n) \to \psi(u), \quad \gamma(r, x; \cdot)\text{-a.e.}$$

Since ψ is represented by (1.3), we need to check that

(3.13) $$u_n^s(y) \to u^s(y) \quad \text{for } \gamma(r, x; \cdot)\text{-almost all } (s, y)$$

and

(3.14) $$\int \operatorname{Exp}\langle u_n^s, \nu\rangle n^s(y, d\nu) \to \int \operatorname{Exp}\langle u^s, \nu\rangle n^s(y, d\nu) \quad \text{for } \gamma(r, x; \cdot)\text{-almost all } (s, y).$$

Formula (3.13) follows immediately from Lemma 3.1. To prove (3.14), we put $\nu \in B_s$ if ν does not charge $\{y: (s, y) \notin \Gamma\}$ and $\langle h^s, \nu\rangle < \infty$. If $\nu \in B_s$, then $(s, y) \in \Gamma$ for ν-almost all y and $u_n^s(y) \to u^s(y)$ ν-a.e. By (3.9) and the dominated convergence theorem, $\langle u_n^s, \nu\rangle \to \langle u^s, \nu\rangle$.

Fix $(r, x) \in \Gamma$. By Lemma 3.1,

(3.15) $$\int \bar{\gamma}(r, x; ds, dy, dz) 1_{\Gamma^c}(s, z) = 0$$

and, by (3.4) and (2.48),

(3.16) $$\int \gamma(r, x; ds, dy) \operatorname{Exp}\langle h^s, \nu\rangle n^s(y, d\nu) \le \mathscr{E}(h)(r, x) < \infty.$$

Formulas (3.15) and (3.16) imply that, for $\gamma(r, x; \cdot)$-almost all (s, y), measure $n^s(y, d\nu)$ is concentrated on B_s and therefore (3.14) follows from the bounds (3.9) and (3.16) and the dominated convergence theorem.

To get (3.11) from (3.12), it is sufficient to note that

$$\mathscr{E}(u_n)(r, x) = \int \gamma(r, x; ds, d\nu) \psi^s(u_n^s)(y)$$

and

$$\int \gamma(r, x; ds, d\nu) \psi^s(h^s)(y) = \mathscr{E}(h)(r, x) < \infty \quad \text{on } \Gamma$$

and to use the bound (3.9). □

4. Existence of NLA functionals.

4.1. *A general existence theorem.*

THEOREM 4.1. *Let h be a pure exit rule for ξ and let $\mathscr{M}^*$ be a total subset of $\mathscr{M}(h)$. Suppose that, for every $\mu \in \mathscr{M}^*$, the process $Y_t = \langle h, X_{<t}\rangle$ belongs to class (D) relative to $(\mathscr{F}_t^\mu, P_\mu)$ where $\mathscr{F}_t^\mu$ is the σ-algebra generated by $X_{<s}$, $s \le t$. Then there exists a NLA functional A with determining set $\mathscr{M}^*$ and potential h.*

PROOF (cf. proof of Theorem 2.4.2 in [5]). By (1.2), Y_t is a supermartingale relative to $(\mathscr{F}_t^\mu, P_\mu)$. Let $A^t(\mu)$ be the compensator of Y [that is, a predictable increasing process such that $A^0(\mu) = 0$ and $Y - A$ is a martingale].

To every $\Lambda = \{0 = t_0 < t_1 < \cdots < t_n\}$ there corresponds an additive functional A_Λ given by (1.19) with h_t^s defined by (2.9). For every $s < t \in \mathbb{R}_+$,

$$P_\mu\{Y_s - Y_t | \mathscr{F}_{s-}\} = \langle h_t^s, X_{<s}\rangle$$

and therefore

$$A_\Lambda(0, t] = \sum_{k=1}^{n-1} P_\mu\{Y_{t_k} - Y_{t_{k+1}} | \mathscr{F}_{t_k-}\} + P_\mu\{Y_{t_n} | \mathscr{F}_{t_n-}\}.$$

Let Λ_n be a standard sequence of partitions. By [3], VII.8 and VII.22, there exists, P_μ-a.s. a weak limit $a_t(\mu)$ of $A_{\Lambda_n}(0, t]$ in $L^1(P_\mu)$. By Lemma 2.4.2 in [5], there exists a natural additive functional A with the properties described in Theorem 4.1. □

4.2. *Proof of Theorem* 1.5. Theorem 1.5 will follow from Theorem 4.1 if we show that $Y_t = \langle h, X_{<t}\rangle$ belongs to class (D) relative to $(\mathscr{F}_t, P_\mu)$ for every $\mu \in \mathscr{M}$. According to [3], Theorem VI.25, it is enough to prove that $P_\mu Y_{T_n} \to 0$ for every increasing sequence of stopping times with $\lim_n T_n = \infty$.

Put $\rho(t) = \langle h 1_{r>t}, \mu\rangle$. By [10], there exists a sequence of randomized stopping times σ_n for the process ξ such that $\sigma_n \le \sigma_{n+1}$, $\lim_n \sigma_n = \infty$ Π_μ-a.s. and

$$P_\mu Y_{T_n} = P_\mu \rho(T_n) + \Pi_\mu h^{\sigma_n}(\xi_{\sigma_n}) = P_\mu \rho(T_n) + \Pi_\mu a(\sigma_n, \infty) \tag{4.1}$$

and the right-hand side of (4.1) tends to 0 since $\langle h, \mu\rangle < \infty$. □

4.3. *Proof of Theorem* 1.6. Denote by $\mathscr{T}$ the family of all stopping times relative to the filtration $\mathscr{F}_t$ in Theorem 4.1. By Theorem 4.1, we need to check that, for every $\mu \in \mathscr{M}(h + \mathscr{E}_w(h))$, the family Y_T, $T \in \mathscr{T}$ is uniformly P_μ-integrable. To this end, it is sufficient to show that

$$\sup_{T \in \mathscr{T}} P_\mu \varphi(Y_T) < \infty \tag{4.2}$$

for a positive function φ such that $\varphi(u)/u \to \infty$ as $u \to \infty$ (see, e.g., [3], Theorem II.22).

Put

$$\varphi(u) = \int_0^1 w(t)\, \mathrm{Exp}(tu)\, dt. \tag{4.3}$$

There exists a constant $c > 0$ such that, for all $u \in \mathbb{R}_+$,

$$c^{-1} q(u) \le \mathrm{Exp}(u) \le c q(u), \tag{4.4}$$

where $q(u) = u \wedge u^2$. It follows from (1.27) and (4.4) that $\varphi(u) < \infty$ and

$$\varphi(u)/u \ge c^{-1} \int_{1/u}^1 t w(t)\, dt \to \infty$$

as $u \to \infty$. We get (4.2) if we prove

$$P_\mu \varphi(Y_T) \le \varphi(\langle h, \mu\rangle) + \mathscr{E}_w(h, \mu). \tag{4.5}$$

Since $h \wedge N \in H$ if $h \in H$, it is sufficient to prove (4.5) for bounded h. By Theorem 1.5, every bounded pure exit rule h is a potential of a NLA functional A. Note that $\langle h, X_T\rangle = P_\mu(A(T,\infty)|\mathscr{F}_T^0)$ and, by Jensen's inequality,

$$P_\mu\varphi(\langle h, X_T\rangle) \le P_\mu\varphi(A(T,\infty)) \le P_\mu\varphi(A(0,\infty)).$$

On the other hand, by Theorem 1.4,

$$P_\mu \operatorname{Exp}(A(0,\infty)) = \exp(-\langle u,\mu\rangle) - 1 + \langle h,\mu\rangle = \operatorname{Exp}\langle u,\mu\rangle + \mathscr{E}(u,\mu)$$

and therefore

$$\begin{aligned} P_\mu\varphi(A(0,\infty)) &= \int_0^1 dt\, w(t) P_\mu \operatorname{Exp}(tA(0,\infty)) \\ &= \int_0^1 dt\, w(t) \operatorname{Exp}\langle u_t,\mu\rangle + \int_0^1 dt\, w(t)\mathscr{E}(u_t,\mu), \end{aligned}$$

where u_t is the log-potential of NLA functional tA. Since $u_t \le th$, we get

$$\begin{aligned} P_\mu\varphi(A(0,\infty)) &\le \int_0^1 dt\, w(t) \operatorname{Exp}(t\langle h,\mu\rangle) + \int_0^1 dt\, w(t)\mathscr{E}(th,\mu) \\ &= \varphi(\langle h,\mu\rangle) + \mathscr{E}_w(h,\mu), \end{aligned}$$

which implies (4.5). □

4.4. *Proof of Theorem* 1.7. Condition 1.8.A implies the existence of admissible function w such that $\mathscr{E}_w(h) \le \text{const.}\,\mathscr{E}(h)$. Indeed, if $q(u) \uparrow \infty$ as $u \to \infty$, then there is a function $W(u) > 0$ with the properties

$$\int_1^\infty W(u)\,du = \infty, \qquad \int_1^\infty \frac{W(u)}{u \wedge q(u)} du < \infty.$$

Put $w(t) = t^{-3}W(t^{-1})$. By construction, the function w is admissible and

$$\int_0^1 tw(t)q^{-1}(t^{-1})\,dt < \infty. \tag{4.6}$$

By 1.8.A,

$$\mathscr{E}(th) \le tq^{-1}(t^{-1})\,\mathscr{E}(h),$$

for $0 < t < 1$, and Theorem 1.7 follows from (1.28), (4.6) and Theorem 1.6. □

5. Concluding remarks.

5.1. *Class SL.* Let Z be a positive $\mathscr{F}$-measurable function on Ω. We say that Z belongs to class SL if there exists a total set $\mathscr{M}^*$ and positive functions h, u such that

$$P_\mu Z = \langle h,\mu\rangle, \qquad P_\mu e^{-Z} = e^{-\langle u,\mu\rangle} \quad \text{for all } \mu \in \mathscr{M}^*. \tag{5.1}$$

It follows from [4], Section I.1.6, that SL contains all finite sums

(5.2) $$\langle \varphi_1, X_{Q_1}\rangle + \cdots + \langle \varphi_n, X_{Q_n}\rangle,$$

where Q_i are finely open sets and φ_i are positive Borel functions. By Theorem 1.4, SL contains $A(0, \infty)$ for every NLA functional with potential h and determining set $\mathscr{M}^* \subset \mathscr{M}_{\mathscr{E}}(h)$. Using Theorem 1.3, it is easy to show that SL contains the convex cone generated by these functionals and sums (5.2). We do not know if SL is a convex cone itself.

5.2. *More general processes* ξ. Theorems 1.1–1.7 can be restated to cover the case when ξ does not satisfy condition 1.1.B. Let $h \in H$ and let h_t^r be defined by (2.9). There exists a unique exit rule $h_{\{t\}}$ such that

(5.3) $$h_{\{t\}}^r(x) = h_t^r(x) - h_{t-}^r(x) \quad \text{for } (r, x) \in S(h).$$

Put

$$\Phi(h) = \{t\colon h_{\{t\}}^r(x) \neq 0 \text{ for some } (r, x)\}.$$

Let A be a NLA functional with potential h and determining set $\mathscr{M}^* \subset \mathscr{M}_{\mathscr{E}}(h)$. By revising the proofs of Theorems 1.1–1.7, we get the following.

5.2.A. The functional A is continuous if $\Phi(h)$ is empty.

5.2.B. Under condition 1.1.B, $\Phi(h)$ is at most countable.

5.2.C. Theorems 1.3 and 1.4 hold if $\Phi(h)$ is at most countable.

5.2.D. Theorems 1.5–1.7 remain valid.

Propositions 5.2.A–5.2.D can be applied, in particular, to the so-called historical process which does not obey 1.1.B except in trivial cases. (For the definition and properties of historical processes and historical superprocesses we refer to [7] (cf. [2]). Suppose that Q is a finely open set in S, τ is the first exit time from Q and φ is a bounded positive measurable function on S. Put

$$h(w) = \Pi_{r,x}\varphi(\tau, \xi_\tau)$$

if w is a path which is contained in Q and (r, x) is its end. Put $h(w) = 0$ if w is not contained in Q. Consider the historical process $\hat{\xi}$ and the historical superprocess $\hat{Z}$ corresponding to the part of ξ in Q. Function h is an exit rule for $\hat{\xi}$ and

$$\Phi(h) = \big\{t\colon \Pi_{r,x}\{\tau = t\big\} \neq 0 \text{ for some } r, x\}.$$

If condition 1.1.B holds for ξ, then

$$\Phi(h) \subset \big\{t\colon \Pi_{r,m}\{\tau = t\big\} \neq 0 \text{ for some rational } r\}$$

and therefore $\Phi(h)$ is at most countable. Therefore Theorems 1.3–1.7 can be applied to the h. The NLA functional A of $\hat{X}$ can be expressed by the formula

$$A(0, t] = \langle \varphi, X'_t \rangle,$$

where X'_t is the so-called absorption process. (Heuristicly, X'_t describes the mass distribution of the part of random cloud which exited from Q during time interval $[0, t]$ and was frozen at the exit. A precise definition is given in [15], Section 2.4.)

5.3. *Superdiffusions.* A special class of superprocesses (we call them superdiffusions) is investigated in [15]. The corresponding process ξ is a diffusion and the branching mechanism is given by $K(ds) = ds$ and $\psi(u) = u^\alpha$ with $1 < \alpha \le 2$. In [15], the general theory developed in the present paper is used, in combination with methods of the theory of p.d.e. and the theory of capacity to get stronger results on relationship between additive functionals and the $\mathscr{E}$-equation. In particular, we prove that, for every NLA functional, (1.24) in Theorem 1.4 holds on S^* [even if $S^*_{\mathscr{E}}(h)$ is trivial]. We also prove that $h \in H$ is the potential of a NLA functional if $h = u + \mathscr{E}(u)$ m-a.e. for some u.

5.4. *Continuous functionals with discontinuous projections.* It follows from the results of this paper that, if a is the projection of A, then A is continuous if and only if a has no fixed discontinuities. Indeed, by Theorem 2.1, A is continuous if and only if $P_{r,x}A\{t\} = 0$ a.e. for every t. On the other hand, (1.26) implies $P_{r,x}A\{t\} = \Pi_{r,x}a\{t\}$. The class of continuous additive functionals of a Hunt process ξ is, in general, smaller than the class of additive functionals with no fixed discontinuities (see, e.g., Theorem 2.4.2 in [5] which is a modification of a well-known result of M. Shur). This implies the existence of continuous NLA functionals with discontinuous projections.

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