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ON SPDE'S AND SUPERDIFFUSIONS[1]

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Several stochastic partial differential equations are derived for multi-dimensional superdiffusions.

0. Introduction. The purpose of this article is to present an approach to studying superdiffusions, particularly the super-Brownian process. This approach is based on the theory of stochastic partial differential equations. For quite a while, superdiffusions and their various properties have been investigated by using the abstract theory of Markov processes, nonstandard analysis, semigroup theory and some other tools. The literature and the results concerning these processes are very impressive. The reader can become acquainted with the area by starting with [3, 4, 5].

Superdiffusions present a special kind of Markov processes, namely measure-valued Markov processes. The theory of Markov processes, specifically the theory of diffusion processes, profited in many ways from connections to stochastic analysis and the theory of partial differential equations. Stochastic analysis provides an "explicit" and instructive way of constructing a wide class of diffusions starting with the simplest diffusion process, which is Brownian motion. The theory of partial differential equations gives a tool to investigate transition densities and occupation times. The relations known at the moment of superdiffusions to analysis appear to be much poorer (see however [3, 4, 5]) and this makes the notion of superdiffusion somewhat unusual.

For instance, take the problem of the existence of superdiffusions. There are two common ways known in the literature to prove existence. One of them is to take the limiting law of branching particles, which is analogous to proving the existence of diffusion processes not through solving stochastic equations but through the passage to the weak limit in a sequence of Markov chains. Another way is to construct superdiffusions as Markov processes by defining their transition functions in the space of measures and then using the general theory of Markov processes to get a process corresponding to this transition function. This way is similar to the one used in the theory of diffusion processes with bad, but not too bad, coefficients when there are "good" results concerning the fundamental solution for corresponding parabolic equations. The natural question arises: are there any stochastic Itô type equations for superdiffusions, at least for "regular" ones? We will see that the answer to this question is positive and the stochastic equations are stochastic *partial*

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differential equations. Actually, this result has been known for quite a while in one-dimensional cases (see, [7, 11]), and our contribution relates to the multidimensional case.

Answering our question, we also answer the question concerning the possibility of including superdiffusions in the framework of more or less classical stochastic analysis, without resorting to the abstract theory of Markov processes or relying on nonstandard analysis. We hope that this will attract more investigators, even those not familiar with either of these two theories in the exciting study of superdiffusions. On the other hand, our results show what kind of rather peculiar stochastic partial differential equations appear in connection with superdiffusions.

To give a better idea about the contents of the article we introduce the following notation. By $\mathscr{M}$ we denote the space of all finite measures on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. One endows $\mathscr{M}$ with the usual measurable structure requiring functions (ψ, μ) to be measurable for any $\psi \in C_0^\infty(\mathbb{R}^d)$, where

$$(\psi, \mu) = \int_{\mathbb{R}^d} \psi(x)\, \mu(dx).$$

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with an increasing filtration of complete σ-algebras $\mathscr{F}_t \subset \mathscr{F}$, $t \geq 0$. Recall that an $\mathscr{M}$-valued $\mathscr{F}_t$-adapted process μ_t is called a super-Brownian process if for any $\psi \in C_0^\infty(\mathbb{R}^d)$ the process (ψ, μ_t) is continuous and the process

$$m_t(\psi) := (\psi, \mu_t) - (\psi, \mu_0) - \int_0^t (\Delta\psi, \mu_s)\, ds \tag{0.1}$$

is a continuous local martingale starting from zero with $d\langle m(\psi)\rangle_t = (\psi^2, \mu_t)\, dt$. To derive a stochastic equation for μ_t we need to find a "canonical" representation for the local martingales $m_t(\psi)$.

A similar situation occurs in the finite-dimensional case when one needs to represent as a stochastic integral a local martingale $m_t(\psi)$ such that

$$d\langle m(\psi)\rangle_t = \sum_{ij} a_t^{ij}(x_t)\psi_{x^i}(x_t)\psi_{x^j}(x_t)\, dt,$$

where $a_t = (a_t^{ij})(x_t)$ is a process with values in the set of symmetric nonnegative matrices and x_t is a random process. One knows that if one defines $\sigma_t(x_t) = \sqrt{a_t}$, then there is a (multidimensional) Wiener process w_t such that $dm_t(\psi) = \sum_{ij} \sigma_t^{ij} \psi_{x^i}(x_t)\, dw_t^j$. Actually, one only needs to have σ such that

$$\sum_{ij} a_t^{ij} \psi_{x^i}(x_t)\psi_{x^j}(x_t) = \sum_j \left| \sum_i \sigma_t^{ij}(x_t)\psi_{x^i}(x_t)\right|^2. \tag{0.2}$$

In our situation we also have to represent the quadratic form (ψ^2, μ) as a sum of squares of expressions linear in ψ. This is done in Section 1 by using the notion of frame function. This notion allows one to have a representation analogous to (0.2) without providing any continuity properties of $\sigma_t(x)$ with respect to x.

A different representation is given in Section 2. This representation is based on a particular frame function which is continuous (see Remark 2.5). This can be useful in constructing solutions of the corresponding SPDEs or their generalizations. The author intends to come back to this idea in the future.

Notice that for the sake of simplicity of presentation we are only dealing with super-Brownian processes. The only exception is Remark 1.12 which provides a SPDE for the multidimensional Fleming–Viot process. In Sections 1 and 2, we consider super-Brownian processes which are related to the simplest branching; we conclude with Section 3 where we consider other super-Brownian processes related to more complicated branching.

One can easily generalize our constructions for more general superdiffusions or superprocesses. This will be seen from the fact that we only need to understand the structure of continuous martingales like $m_t(\psi)$ with $d\langle m(\psi)\rangle_t = (\psi^2, \mu_t)\,dt$ or of some discontinuous martingales of the same nature.

There are very many open questions which arise from what is presented here and which we do not know how to answer. One of them concerns pathwise uniqueness of solutions. Also, it is far from being clear if one can use our equations to prove fine properties of superprocesses as in [3] and [4] (see, however, Corollaries 1.9 and 1.10 and Remark 3.5). By the way, this is one reason why in Section 3 we present a result which is somewhat weaker than those from Sections 1 and 2. In Section 3 we show only that any solution of an appropriate equation is a superdiffusion, whereas in Sections 1 and 2 we show that any super-Brownian process satisfies an SPDE. Probably this discrepancy can be removed on the basis of representation theorems for discontinuous martingales.

1. General SPDE for the super-Brownian process.

Take a countable set of functions ψ_k, $k = 1, 2, \dots$, of class $C_0^\infty(\mathbb{R}^d)$ such that, for any $\psi \in C_0^\infty(\mathbb{R}^d)$, there is a sequence $\psi(n) \in \{\psi_k, k = 1, 2\dots\}$ with

$$\max_{|x|\le n} |\psi(n, x) - \psi(x)| \to 0, \qquad \sup_{n,\, x} |\psi(n, x)| < \infty.$$

If $\mu \in \mathscr{M}$, define $\psi_k(\mu) = \psi_k(\mu, x)$ as the result of Gram–Schmidt ortogonalization method applied to ψ_k, $k = 1, 2, \dots$ in the space $L_2(\mathbb{R}^d, \mu)$. More precisely, for $k \ge 1$, let $\pi_k = \pi_k(\mu)$ be the operator of orthogonal projection on the space orthogonal to $\mathrm{Span}\{\psi_1, \dots, \psi_k\}$, define π_0 as the unit operator and define, for $k \ge 1$,

$$\psi_k(\mu) = \left(\int_{\mathbb{R}^d} |\pi_{k-1}\psi_k|^2\,\mu(dx)\right)^{-1/2} \pi_{k-1}\psi_k$$

if $\int_{\mathbb{R}^d} |\pi_{k-1}\psi_k|^2\,\mu(dx) \ne 0$; otherwise $\psi_k(\mu) := 0$.

The existence of such functions $\psi_k(\mu)$ shows that the following definition makes sense.

DEFINITION 1.1. Let a system $\{\varphi_i(\mu),\ i \geq 1\}$ of Borel functions $\varphi_i(\mu) = \varphi_i(\mu, x)$ on $\mathbb{R}^d$ be given for any $\mu \in \mathscr{M}$. We call it a *frame function* if the following holds:

(i) for any i we have $\varphi_i(\mu) \in L_2(\mathbb{R}^d, \mu)$ and for any $\psi \in C_0^\infty(\mathbb{R}^d)$ the function $(\psi\varphi_i(\mu), \mu)$ is measurable with respect to μ;
(ii) for any $\psi \in L_2(\mathbb{R}^d, \mu)$ we have

$$||\psi||^2_{L_2(\mathbb{R}^d,\mu)} = \sum_{i=1}^{\infty}(\psi, \varphi_i(\mu))^2_{L_2(\mathbb{R}^d,\mu)}.$$

REMARK 1.2. Property (ii) in Definition 1.1 does not imply that $\{\varphi_i(\mu), i \geq 1\}$ is an orthonormal basis in $L_2(\mathbb{R}^d, \mu)$. One can understand why this happens if one considers a more or less arbitrary two-dimensional plane, say A, in a three-dimensional space and takes the orthogonal projection to A of an orthonormal basis in the space ending up with three vectors $a_1, a_2, a_3 \in A$ such that $|b|^2 = (a_1, b)^2 + (a_2, b)^2 + (a_3, b)^2$ for any $b \in A$.

Also notice that, by polarization, property (ii) implies that for any $\psi, \theta \in L_2(\mathbb{R}^d, \mu)$ we have

$$(\psi, \theta)_{L_2(\mathbb{R}^d,\mu)} = \sum_{i=1}^{\infty}(\psi, \varphi_i(\mu))_{L_2(\mathbb{R}^d,\mu)}(\theta, \varphi_i(\mu))_{L_2(\mathbb{R}^d,\mu)}.$$

REMARK 1.3. For a measure μ, writing $\mu \in L_2(\mathbb{R}^d)$ [$L_2(\mathbb{R}^d)$ stands for the usual L_2 space with respect to Lebesgue measure with norm $||\cdot||_{L_2}$] means that the generalized function μ acts as an element of $L_2(\mathbb{R}^d)$, or in other words, μ is absolutely continuous with respect to Lebesgue measure and its density belongs to $L_2(\mathbb{R}^d)$. If $\{\varphi_i\}$ is any orthonormal basis in $L_2(\mathbb{R}^d)$ and $\mu \in L_2(\mathbb{R}^d)$, then for $\rho := \mu(dx)/dx$ and any $\psi \in L_2(\mathbb{R}^d, \mu)$ we have

$$||\psi||^2_{L_2(\mathbb{R}^d,\mu)} = \int_{\mathbb{R}} |\psi\sqrt{\rho}|^2\,dx = \sum_{i=1}^{\infty}\left|\int_{\mathbb{R}} \psi\sqrt{\rho}\varphi_i\,dx\right|^2 = \sum_{i=1}^{\infty}\left|\int_{\mathbb{R}} \psi\varphi_i(\mu)\,\mu(dx)\right|^2,$$

where $\varphi_i(\mu) = \varphi_i\rho^{-1/2}I_{\rho\neq 0}$. This suggests that if we define

$$\varphi_i(\mu) = \begin{cases} \varphi_i\rho^{-1/2}I_{\rho\neq 0}, & \text{for } \mu \in L_2(\mathbb{R}^d),\ \rho = \dfrac{\mu(dx)}{dx}, \\ \psi_i(\mu), & \text{for } \mu \notin L_2(\mathbb{R}^d), \end{cases}$$

then $\{\varphi_i(\mu)\}$ is a frame function.

This is true indeed. Property (ii) has been checked above. As far as measurability is concerned, observe that $L_2(\mathbb{R}^d) \cap \mathscr{M}$ is a measurable subset of $\mathscr{M}$ since

$$L_2(\mathbb{R}^d) \cap \mathscr{M} = \bigcup_{n=1}^{\infty}\bigcap_{i=1}^{\infty}\{\mu \in \mathscr{M} : |(\psi_i, \mu)| \leq n||\psi_i||_{L_2}\}.$$

Also take a nonnegative function $\zeta_1 \in C_0^\infty(\mathbb{R}^d)$ with unit integral and define $\zeta_\varepsilon(x) = \varepsilon^{-d}\zeta_1(x/\varepsilon)$ and for $\mu \in \mathscr{M}$ let $\mu_\varepsilon = \mu * \zeta_\varepsilon$. One knows that if $\mu \in L_2(\mathbb{R}^d)$, then $\|\sqrt{\rho}-\sqrt{\mu_\varepsilon}\|_{L_2}\to 0$ as $\varepsilon\to 0$. Therefore, for $\mu\in L_2(\mathbb{R}^d)$ and $\psi\in C_0^\infty(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d}\psi\varphi_i(\mu)\,\mu(dx)=\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d}\psi\varphi_i\sqrt{\mu_\varepsilon}\,dx.$$

Finally, the function $\mu_\varepsilon(x)$ is measurable with respect to μ (by definition) and infinitely differentiable with respect to x. This proves that property (i) in Definition 1.1 holds as well.

It is interesting that for this frame as well as for $\{\psi_i(\mu)\}$ generally, one cannot get $(\psi\varphi_i(\mu),\mu)$ for $\mu\notin L_2(\mathbb{R}^d)$ as a limit of $(\psi\varphi_i(\mu_n),\mu_n)$ where $\mu_n\in L_2(\mathbb{R}^d)$ and $\mu_n\to\mu$ weakly. However, it turns out (see Remark 2.5) that there are frame functions for which the l_2-valued functions $(\psi\varphi_i(\mu),\mu)$ are continuous in μ for any $\psi\in C_0^\infty(\mathbb{R}^d)$. This may be used in constructing superdiffusions by solving SPDE's. By the way, in the following theorem we do not assume any continuity of $(\psi\varphi_i(\mu),\mu)$.

THEOREM 1.4. *Let μ_t be a super-Brownian process on $(\Omega,\mathscr{F},P)$ and let $(\tilde\Omega,\tilde{\mathscr{F}},\tilde P)$ be a probability space carrying independent one-dimensional Wiener processes $\tilde w_t^k$, $k\geq 1$. Also assume that we are given a frame function $\{\varphi_i(\mu),\ i\geq 1\}$. Then on $(\Omega,\mathscr{F},P)\times(\tilde\Omega,\tilde{\mathscr{F}},\tilde P)$ there exist independent Wiener processes $w_t^1,w_t^2,\dots$ such that*

$$d\mu_t=\Delta\mu_t\,dt+\sum_{i=1}^\infty\varphi_i(\mu_t,\cdot)\mu_t\,dw_t^i. \tag{1.3}$$

REMARK 1.5. As always, we understand (1.3) in the generalized sense; in particular, μ_t is a generalized function. More precisely, by (1.3) we mean that, for any $\psi\in C_0^\infty(\mathbb{R}^d)$ and $t\geq 0$,

$$(\psi,\mu_t)=(\psi,\mu_0)+\int_0^t(\Delta\psi,\mu_s)\,ds+\sum_{i=1}^\infty\int_0^t(\psi\varphi_i(\mu_s),\mu_s)\,dw_s^i\quad\text{a.s.},$$

and additionally we assert that the last series converges in probability uniformly on every finite interval of time.

Because of the properties of $m_t(\psi)$ introduced in (0.1), Theorem 1.4 is a consequence of the following result, which also implies similarly to Theorem 1.4 results for many other superdiffusions.

THEOREM 1.6. *Assume that we are given an $\mathscr{M}$-valued process μ_t such that, for any $\psi\in C_0^\infty(\mathbb{R}^d)$, the process (ψ,μ_t) is predictable and locally integrable in t. Also assume that for any $\psi\in C_0^\infty(\mathbb{R}^d)$ we are given a continuous martingale $m_t(\psi)$ with $m_0(\psi)=0$ such that for any $\psi_1,\psi_2\in C_0^\infty(\mathbb{R}^d)$ and $t\geq 0$,*

$$\langle m(\psi_1),m(\psi_2)\rangle_t=\int_0^t(\psi_1\psi_2,\mu_s)\,ds. \tag{1.4}$$

Let $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ be a probability space carrying independent one-dimensional Wiener processes $\tilde{w}_t^k$, $k \geq 1$. Finally, assume that we are given a frame function $\{\varphi_i(\mu),\ i \geq 1\}$. Then on $(\Omega, \mathscr{F}, P) \times (\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ there exist independent Wiener processes $w_t^1, w_t^2, \dots$ such that for any $\psi \in C_0^\infty(\mathbb{R}^d)$,

$$m_t(\psi) = \sum_{i=1}^{\infty} \int_0^t (\psi\varphi_i(\mu_s), \mu_s)\, dw_s^i \quad \text{a.s.,} \tag{1.5}$$

where the last series converges in probability uniformly on every finite interval of time.

PROOF. The stated character of convergence in (1.5) follows at once from the well-known result on equivalence of convergence of local martingales and their quadratic variations and the fact that

$$\sum_{i=1}^{\infty} \int_0^t (\psi\varphi_i(\mu_s), \mu_s)^2\, ds = \int_0^t \sum_{i=1}^{\infty} (\psi\varphi_i(\mu_s), \mu_s)^2\, ds = \int_0^t (\psi^2, \mu_s)\, ds < \infty \quad \text{a.s.}$$

To prove (1.5) without loss of generality, we may assume that on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ we are given two independent *infinite* sets $\{\tilde{w}_t^i\}$ and $\{\hat{w}_t^i\}$, each consisting of independent Wiener processes. Indeed, in any case we can just split the set of processes $\{\tilde{w}_t^i\}$ into two infinite parts. First we prove (1.5) for our particular frame $\{\psi_i(\mu)\}$. For $k \geq 1$, define

$$m_t^k = m_t(\psi_k), \qquad n_{kt}^2 = \int_{\mathbb{R}^d} |\pi_{k-1}(\mu_t)\psi_k|^2\, \mu_t(dx), \qquad I_t^k = I_{n_{kt} \neq 0}$$

and define w_t^k recursively by ($\sum_{i<1} := 0$),

$$w_t^k = \int_0^t n_{ks}^{-1} I_s^k \Big[dm_s^k - \sum_{i<k} (\psi_k\psi_i(\mu_s), \mu_s)\, dw_s^i \Big] + \int_0^t (1 - I_s^k)\, d\tilde{w}_s^k. \tag{1.6}$$

We need to show that the definitions make sense. Assume that for an integer $k \geq 1$ we have already defined w^i, $i < k$, and for any $i, j < k \leq l$ and $\psi \in C_0^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned} d\langle m(\psi), w^i\rangle_t &= (\psi\psi_i(\mu_t), \mu_t)\, dt, \\ d\langle w^i, w^j\rangle_t &= \delta^{ij}\, dt, \qquad \langle w^i, \tilde{w}^l\rangle_t = 0. \end{aligned} \tag{1.7}$$

As an example of such k, one can take $k = 1$. Under this assumption the quadratic variation of the local martingale

$$m_t^k - \sum_{i<k} \int_0^t (\psi_k\psi_i(\mu_s), \mu_s)\, dw_s^i \tag{1.8}$$

equals

$$\begin{aligned}
&\langle m^k, m^k\rangle_t - 2\sum_{i<k}\int_0^t (\psi_k\psi_i(\mu_s), \mu_s)\, d\langle m^k, w^i\rangle_s + \sum_{i<k}\int_0^t (\psi_k\psi_i(\mu_s), \mu_s)^2\, ds \\
&\quad = \int_0^t (\psi_k^2, \mu_s)\, ds - \sum_{i<k}\int_0^t (\psi_k\psi_i(\mu_s), \mu_s)^2\, ds \\
&\quad = \int_0^t (|\pi_{k-1}\psi_k|^2, \mu_s)\, ds = \int_0^t n_{ks}^2\, ds.
\end{aligned}$$

This implies that the stochastic integral in (1.6) with respect to the process (1.8) makes sense for the given k. Taking into account the last equality in (1.7) and the fact that $m(\psi)$ and $\tilde{w}^k$ are independent by Levy's theorem, one concludes that w^k is a (one-dimensional) Wiener process. Also, for $i < k < l$,

$$d\langle w^i, w^k\rangle_t = n_{kt}^{-1} I_t^k [d\langle w^i, m^k\rangle_t - (\psi_k\psi_i(\mu_t), \mu_t)\, dt] = 0, \qquad \langle w^k, \tilde{w}^l\rangle_t = 0,$$

and for any $\psi \in C_0^\infty(\mathbb{R}^d)$ [remember that the martingales $m(\psi)$ and $\tilde{w}$ are independent],

$$\begin{aligned}
d\langle m(\psi), w^k\rangle_t &= n_{kt}^{-1} I_t^k \Big[d\langle m(\psi), m^k\rangle_t - \sum_{i<k} (\psi_k\psi_i(\mu_t), \mu_t)(\psi\psi_i(\mu_t), \mu_t)\, dt \Big] \\
&= n_{kt}^{-1} I_t^k [(\psi\psi_k, \mu_t) - (\psi(1-\pi_{k-1}(\mu_t))\psi_k, \mu_t)]\, dt \\
&= I_t^k (\psi\psi_k(\mu_t), \mu_t)\, dt = (\psi\psi_k(\mu_t), \mu_t)\, dt.
\end{aligned}$$

This allows us to get (1.7) by induction on k and at the same time proves that w^k are independent Wiener processes.

Next take an integer $r \ge 1$ and $\psi \in \mathrm{Span}\{\psi_1, \dots, \psi_r\}$. Observe that for any ω, t we have $\mathrm{Span}\{\psi_1, \dots, \psi_r\} = \mathrm{Span}\{\psi_1(\mu_t), \dots, \psi_r(\mu_t)\}$ in the sense of $L_2(\mathbb{R}^d, \mu_t)$ so that

$$(\psi^2, \mu_t) = \sum_{i\le r} (\psi\psi_i(\mu_t), \mu_t)^2.$$

Hence and from (1.7) we get

$$\langle m(\psi)\rangle_t = \int_0^t (\psi^2, \mu_s)\, ds = \sum_{i\le r}\int_0^t (\psi\psi_i(\mu_s), \mu_s)^2\, ds,$$

$$\langle m(\psi), w^i\rangle_t = \int_0^t (\psi\psi_i(\mu_s), \mu_s)\, ds.$$

In turn this implies (for instance, by definition in the framework of [9]) that

$$m_t(\psi) = \sum_{i\le r}\int_0^t (\psi\psi_i(\mu_s), \mu_s)\, dw_s^i.$$

Since obviously $(\psi\psi_i(\mu), \mu) = 0$ for $i > r$, we have obtained (1.5) for $\psi \in \mathrm{Span}\{\psi_1, \dots, \psi_r\}$ and our particular frame $\{\psi_i(\mu),\ i \ge 1\}$. In particular (1.5) holds for $\psi = \psi_r$ and any r. In the case of general ψ one gets the result for the same frame $\{\psi_i(\mu),\ i \geq 1\}$ after approximating ψ by ψ_r's and observing that

$$\langle m(\psi)-m(\psi_r)\rangle_t = \langle m(\psi)\rangle_t - 2\langle m(\psi), m(\psi_r)\rangle_t + \langle m(\psi_r)\rangle_t$$
$$= \int_0^t (|\psi-\psi_r|^2, \mu_s)\, ds \quad \text{a.s.}$$

We now consider the case of general frame function $\{\varphi_i(\mu), i \geq 1\}$. Define the following predictable functions

$$\alpha_t^{ij} = \int_{\mathbb{R}^d} \psi_i(\mu_t, x)\varphi_j(\mu_t, x)\, \mu_t(dx),$$

and observe that by definition, the vectors $\alpha_t^i := \{\alpha_t^{ij},\ j \geq 1\}$ as elements of l_2 are orthogonal and have unit length. However, it may happen that they do not form an orthonormal basis. Then by using again the Gram–Schmidt method, one can find vectors $\beta_t^i := \{\beta_t^{ij},\ j \geq 1\}$, $i < N_t$ with some $N_t \leq \infty$, such that the system $\{\alpha_t^i,\ i \geq 1,\ \beta_t^j,\ j < N_t\}$ forms an orthonormal basis in l_2 for any ω, t and all vectors are appropriately measurable with respect to ω, t. It is well known that

$$\sum_{i \geq 1} \alpha_t^{ij}\alpha_t^{ik} + \sum_{i<N_t} \beta_t^{ij}\beta_t^{ik} = \delta^{jk} \quad \forall\ j, k. \tag{1.9}$$

Next define

$$\bar{w}_t^j = \sum_{i=1}^{\infty} \int_0^t \alpha_s^{ij}\, dw_s^i + \sum_{i=1}^{\infty} \int_0^t I_{i<N_s} \beta_s^{ij}\, d\hat{w}_s^i.$$

The processes w and $\hat{w}$ are independent. Therefore from (1.9) it is easy to see that $\bar{w}_t^j$ are independent Wiener processes. Furthermore, well-known theorems on the passage to the limit in stochastic integrals and theorems on stochastic integration with respect to stochastic integrals (see, for instance [9]) show that if γ_t^j are predictable and $S_t := \sum_{j \geq 1} |\gamma_t^j|^2$ is locally integrable, then (a.s.)

$$\sum_{j=1}^{\infty} \int_0^t \gamma_s^j\, d\bar{w}_s^j = \sum_{i=1}^{\infty} \int_0^t \sum_{j=1}^{\infty} \gamma_s^j \alpha_s^{ij}\, dw_s^i + \sum_{i=1}^{\infty} \int_0^t I_{i<N_s} \sum_{j=1}^{\infty} \gamma_s^j \beta_s^{ij}\, d\tilde{w}_s^i. \tag{1.10}$$

For $\psi \in C_0^\infty(\mathbb{R}^d)$ and $\gamma_t^j := (\psi\varphi_j(\mu_t), \mu_t)$ we have $S_t = (\psi^2, \mu_t)$ which is locally integrable by our assumptions. Also for these γ_t^j, by our definitions and properties of Hilbert spaces we have

$$\sum_{j=1}^{\infty} \gamma_s^j \alpha_s^{ij} = \sum_{j=1}^{\infty} (\psi\varphi_j(\mu_s), \mu_s)(\psi_i(\mu_s)\varphi_j(\mu_s), \mu_s) = (\psi\psi_i(\mu_s), \mu_s),$$

$$\gamma_s^j = \sum_{i=1}^{\infty} (\psi\psi_i(\mu_s), \mu_s)\alpha_s^{ij}, \qquad \sum_{j=1}^{\infty} \alpha_s^{ij}\beta_s^{kj} = 0,$$

$$\sum_{j=1}^{\infty} \gamma_s^j \beta_s^{kj} = \sum_{i=1}^{\infty} (\psi\psi_i(\mu_s), \mu_s) \sum_{j=1}^{\infty} \alpha_s^{ij}\beta_s^{kj} = 0.$$

Hence, (1.10) means that

$$\sum_{i=1}^{\infty}\int_0^t(\psi\psi_i(\mu_s),\mu_s)\,dw_s^i=\sum_{i=1}^{\infty}\int_0^t(\psi\varphi_i(\mu_s),\mu_s)\,d\bar{w}_s^i,$$

and this gives us (1.5) with $\bar{w}^i$ in place of w^i. The theorem is proved.

REMARK 1.7. It follows from (1.5) that $m_t(\psi)$ is linear in ψ. Actually this linearity can be easily obtained from (1.4) alone.

REMARK 1.8. One can define a martingale measure $m((0,t]\times\Gamma)$ so that

$$\langle m((0,\cdot]\times\Gamma)\rangle_t=\int_0^t\mu_s(\Gamma)\,ds,\qquad m_t(\psi)=\int_{\mathbb{R}^d}\psi(x)\,m((0,t]\times dx)$$

for all $\psi\in C_0^\infty(\mathbb{R}^d)$. For instance, one can use Theorem 1.6 and let

$$m((0,t]\times\Gamma)=\sum_{i=1}^{\infty}\int_0^t\int_\Gamma\varphi_i(\mu_s)\mu_s(dx)\,dw_s^i. \tag{1.11}$$

In [6] we showed how to reduce the stochastic integrals with respect to martingale measures to the usual stochastic integrals. This was further used in [6] to treat stochastic equations containing integrals against martingale measures in the same way as equations containing just the usual stochastic integrals.

To be more precise, we used the formula

$$\int_0^t\int_{\mathbb{R}^d}\phi(s,x)\,m(ds\times dx)=\sum_{i=1}^{\infty}\int_0^t\phi_i(s)\,dm_s^i, \tag{1.12}$$

where

$$m_t^i=\int_0^t\int_{\mathbb{R}^d}\eta_i(s,x)\,m(ds\times dx),\qquad \phi_i(s)=\int_{\mathbb{R}^d}\eta_i(s,x)\phi(s,x)\,\mu_s(dx),$$

and for any ω,s the system of functions $\{\eta_i(s,\cdot)\}$ form an orthonormal basis in $L_2(\mathbb{R}^d,\mu_s)$. For ϕ being the indicator of $(0,t]\times\Gamma$, from (1.12) we recover (1.11) but only for the case when the frame $\{\varphi_i\}$ coincides with $\{\eta_i\}$ and in particular forms an orthonormal basis. The possibility of treating any frame is important as we will see in the proof of Theorem 2.3.

To discuss our results further, take $\delta>0$ and define

$$R_\delta(x)=c|x|^{\delta-d}\int_0^\infty t^{(\delta-d)/2}\exp(-t|x|^2-1/(4t))\,\frac{dt}{t},$$

so that for an appropriate choice of the constant $c=c(\delta)$, which we fix in this way, the function R_δ is the kernel of the operator $R_\delta:=(1-\Delta)^{-\delta/2}$. Then for any finite measure μ,

$$I(\mu) := \left\| \left\{ \sum_{i=1}^{\infty} \left| \int_{\mathbb{R}^d} R_\delta(\cdot - y)\psi_i(\mu, y)\,\mu(dy) \right|^2 \right\}^{1/2} \right\|_{L_2}$$
$$= \left\| \left\{ \int_{\mathbb{R}^d} R_\delta^2(\cdot - y)\,\mu(dy) \right\}^{1/2} \right\|_{L_2} = \mu^{1/2}(\mathbb{R}^d)||R_\delta||_{L_2}.$$

The norm $||R_\delta||_{L_2}$ is finite if $\delta > d/2$. Also for the super-Brownian process, $\mu_t(\mathbb{R}^d)$ is known to be a continuous process, so that

$$\int_0^T I^2(\mu_t)\,dt < \infty \quad \forall\, T < \infty.$$

Combining this with Theorem 3.2 from [10] and Theorem 1.4, we get the following corollary.

COROLLARY 1.9. *If μ_0 belongs to the space of Bessel potentials $H_2^{1-\delta}(\mathbb{R}^d)$ and $\delta > d/2$, then for almost all (t, ω) we have $\mu_t \in H_2^{1-\delta}(\mathbb{R}^d)$.*

The number $1 - \delta$ can be taken greater than or equal to 0 only if $d = 1$. Also in this case $H_2^{1-\delta}(\mathbb{R}) \subset L_2(\mathbb{R})$.

COROLLARY 1.10 (cf. [7, 11]). *If $d = 1$ and $\mu_0 \in L_2(\mathbb{R})$, then for almost all (t, ω) the measure μ_t has a square integrable density ρ_t which satisfies an equation of the form*

$$d\rho_t = \Delta\rho_t\,dt + \sum_{i=1}^{\infty} \sqrt{\rho_t}\varphi_i\,dw_t^i, \tag{1.13}$$

where $\{\varphi_i,\ i \geq 1\}$ is any orthonormal basis in $L_2(\mathbb{R})$.

The first part of this corollary has been proved above. To prove (1.13), take $\varphi_i(\mu)$ from Remark 1.3, apply Theorem 1.4 and observe that for any $\psi \in C_0^\infty(\mathbb{R})$ we have

$$\int_{\mathbb{R}} \psi\varphi_i(\mu_t)\,\mu_t(dx) = \int_{\mathbb{R}} \psi\sqrt{\rho_t}\varphi_i\,dx.$$

REMARK 1.11. Further continuity properties of ρ_t can be found in [7] for equations with nonrandom coefficients and operators more general than Δ. For equations with random coefficients, we refer the reader to [10].

REMARK 1.12. If instead of the super-Brownian process one takes the Fleming–Viot process in Theorem 1.4, then instead of (1.3) one can prove that

$$d\mu_t = \Delta\mu_t\,dt + \sum_{i=1}^{\infty} \pi(\mu_t)\varphi_i(\mu_t, \cdot)\mu_t\,dw_t^i,$$

where $\pi(\mu)$ is the orthogonal projection in $L_2(\mathbb{R}^d, \mu)$ on the subspace of functions orthogonal to constants. In other words $\pi(\mu)h = h - (h, \mu)\mu^{-1}(\mathbb{R}^d)$. By the way, for the Fleming–Viot process μ_t, one has $\mu_t(\mathbb{R}^d) \equiv 1$.

2. More SPDE's for super-Brownian process. As we have mentioned in the Introduction, the whole idea of the construction relies on representing $(\psi\phi, \mu)$ as a sum of products of linear expressions, one containing only ψ, the other ϕ.

One can look at $(\psi\phi, \mu)$ as a bilinear form generated by a linear operator and take the square root of this operator. Then if in some sense $(\psi\phi, \mu) = \langle B\psi, \phi\rangle$, one has $(\psi\phi, \mu) = \langle\sqrt{B}\psi, \sqrt{B}\phi\rangle$ and assuming that $\langle\cdot\rangle$ is defined by some sort of summation or integration, we get the desired representation of $(\psi\phi, \mu)$ as a sum of products of linear expressions, one containing only ψ, the other ϕ. If μ has a density ρ, then of course

$$(\psi\phi, \mu) = \int_{\mathbb{R}^d} [\psi\sqrt{\rho}][\phi\sqrt{\rho}]\, dx$$

and the operator $\sqrt{B}$ is just the multiplication by $\sqrt{\rho}$. In the general situation we proceed in a more complicated way.

In the general case, it is convenient to change spaces and instead of $(\psi\phi, \mu)$ consider

$$([(1-\Delta)^{-\delta/2}\psi](1-\Delta)^{-\delta/2}\phi, \mu) = \int_{\mathbb{R}^d} \psi(x)K(\mu)\phi(x)\, dx, \tag{2.1}$$

where $K(\mu)$ is the operator defined by

$$K(\mu)\phi(x) = \int_{\mathbb{R}^d} K(\mu, x, y)\phi(y)\, dy,$$

$$K(\mu, x, y) = \int_{\mathbb{R}^d} R_\delta(x-z)R_\delta(z-y)\, \mu(dz).$$

The operator $K(\mu)$ is symmetric positive and one can find its square root, say a symmetric positive operator $Q(\mu)$, such that $Q^2(\mu) = K(\mu)$. Then

$$\int_{\mathbb{R}^d} \psi(x)K(\mu)\phi(x)\, dx = \int_{\mathbb{R}^d} [Q(\mu)\psi]Q(\mu)\phi\, dx$$

and from (2.1) one can expect that

$$\begin{aligned}(\psi\phi, \mu) &= \int_{\mathbb{R}^d} [Q(\mu)(1-\Delta)^{\delta/2}\psi]Q(\mu)(1-\Delta)^{\delta/2}\phi\, dx \\ &= \sum_{i=1}^{\infty} \int_{\mathbb{R}^d} \psi(1-\Delta)^{\delta/2}Q(\mu)\phi_i\, dx \int_{\mathbb{R}^d} \phi(1-\Delta)^{\delta/2}Q(\mu)\phi_i\, dx,\end{aligned}$$

where $\{\phi_i\}$ is any orthonormal basis in $L_2(\mathbb{R}^d)$.

There are several ways to find $Q(\mu)$. One of them is based on the formula

$$Q(\mu) = c\int_0^\infty \frac{e^{-tK(\mu)} - 1}{\sqrt{t}}\, \frac{dt}{t}, \qquad c^{-1} = \int_0^\infty \frac{e^{-t} - 1}{\sqrt{t}}\, \frac{dt}{t}, \tag{2.2}$$

where by $e^{-tK(\mu)}$ we mean the operator which takes any function $\psi \in C_0^\infty(\mathbb{R}^d)$ into the value at time t of the solution of the problem

$$\frac{d}{dt}\psi(t) = -K(\mu)\psi(t), \qquad \psi(0) = \psi. \tag{2.3}$$

LEMMA 2.1. *Let $\delta > d/2$ and μ be a finite measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ with $\mu(\mathbb{R}^d) > 0$. Then:*

(i) *The operator $K(\mu)$ is a bounded and even Hilbert–Schmidt operator from $L_2(\mathbb{R}^d)$ into itself.*

(ii) *For any $\psi \in L_2(\mathbb{R}^d)$, problem (2.3) as an ordinary differential equation for $L_2(\mathbb{R}^d)$-valued functions has a unique solution differentiable in the strong sense. Moreover, $||\psi(t)||_{L_2} \le ||\psi||_{L_2}$, so that the semigroup e^{-tK} is a semigroup of contractions in $L_2(\mathbb{R}^d)$.*

(iii) *The operator $Q(\mu)$ is well defined by (2.2) as a bounded symmetric operator from $L_2(\mathbb{R}^d)$ into itself. Moreover, $Q^2(\mu) = K(\mu)$.*

(iv) *The operator $J(\mu) := Q(\mu)(1-\Delta)^{\delta/2}$ acting from $C_0^\infty(\mathbb{R}^d)$ into $L_2(\mathbb{R}^d)$ can be extended as a unitary operator acting from $L_2(\mathbb{R}^d, \mu)$ into $L_2(\mathbb{R}^d)$. Keep the notation $J(\mu)$ for the extended operator. Then the operator $J^*(\mu)J(\mu)$ is a unit operator on $L_2(\mathbb{R}^d, \mu)$.*

PROOF. (i) We have

$$K^2(\mu, x, y) \le \int_{\mathbb{R}^d} R_\delta^2(x-z)\,\mu(dz) \int_{\mathbb{R}^d} R_\delta^2(z-y)\,\mu(dz).$$

Hence the Hilbert–Schmidt norm of $K(\mu)$ equals

$$\left\{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K^2(\mu, x, y)\,dx\,dy\right\}^{1/2} \le \mu(\mathbb{R}^d)\int_{\mathbb{R}^d} R_\delta^2(x)\,dx, \tag{2.4}$$

which is finite as stated.

(ii) Since the operator $K(\mu)$ is bounded, the first statement in (ii) is well known. To prove the second one, it suffices to notice that

$$d||\psi(t)||_{L_2}^2 = 2(\psi(t), d\psi(t)) = -2(\psi(t), K(\mu)\psi(t))\,dt \le 0.$$

(iii) By virtue of statement (ii), statement (iii) is well known from the theory of fractional powers of operators (see, for instance, [8]).

(iv) To prove the first assertion in (iv) it suffices to observe that, by (iii) and (2.1), we have for $\psi \in C_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned}||Q(\mu)(1-\Delta)^{\delta/2}\psi||_{L_2}^2 &= (Q(\mu)(1-\Delta)^{\delta/2}\psi, Q(\mu)(1-\Delta)^{\delta/2}\psi)_{L_2}\\ &= ((1-\Delta)^{\delta/2}\psi, K(\mu)[(1-\Delta)^{\delta/2}\psi])_{L_2} = (\psi^2, \mu).\end{aligned}$$

The second assertion in (iv) is, actually, true in the general framework of Hilbert spaces, since any time we have a unitary operator I, the operator I^*I is a unit operator. The lemma is proved. □

THEOREM 2.2. *Let the assumptions of Theorem 1.4 be satisfied and let $\{\varphi_i,\ i \geq 1\}$ be an orthonormal basis in $L_2(\mathbb{R}^d)$. Then there exist independent Wiener processes $w^i_t,\ i \geq 1$, such that*

$$d\mu_t = \Delta\mu_t\,dt + \sum_{i=1}^{\infty}(1-\Delta)^{\delta/2}Q(\mu_t)\varphi_i\,dw^i_t. \tag{2.5}$$

This theorem follows from the next one in the same way in which Theorem 1.4 follows from Theorem 1.6.

THEOREM 2.3. *Let the assumptions of Theorem 1.6 be satisfied and let $\{\varphi_i,\ i \geq 1\}$ be an orthonormal basis in $L_2(\mathbb{R}^d)$. Then there exist independent Wiener processes $w^i_t,\ i \geq 1$, such that for any $\psi \in C_0^\infty(\mathbb{R}^d)$,*

$$m_t(\psi) = \sum_{i=1}^{\infty}\int_0^t\int_{\mathbb{R}^d}\{(1-\Delta)^{\delta/2}\psi\}Q(\mu_t)\varphi_i\,dx\,dw^i_t \quad \textit{a.s.}$$

PROOF. Take $J^*(\mu)$ from Lemma 2.1 if $\mu \neq 0$ and let $J^*(0) := 0$. Notice that for any $\psi \in L_2(\mathbb{R}^d, \mu)$ we have

$$||\psi||^2_{L_2(\mathbb{R}^d,\,\mu)} = ||J(\mu)\psi||^2_{L_2} = \sum_{i=1}^{\infty}|(J(\mu)\psi, \varphi_i)_{L_2}|^2 = \sum_{i=1}^{\infty}\left|(\psi, J^*(\mu)\varphi_i)_{L_2(\mathbb{R}^d,\,\mu)}\right|^2.$$

Also $J^*(\mu)\varphi_i$ are appropriately measurable. By Theorem 1.6 for certain Wiener processes,

$$m_t(\psi) = \sum_{i=1}^{\infty}\int_0^t\int_{\mathbb{R}^d}\psi J^*(\mu_t)\varphi_i\,\mu_t(dx)\,dw^i_t \quad \text{a.s.}$$

To prove (2.5) it only remains to observe that for any $\phi \in L_2(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d}\psi J^*(\mu_t)\phi\,\mu_t(dx) &= \int_{\mathbb{R}^d}\phi J(\mu_t)\psi\,dx\\ &= \int_{\mathbb{R}^d}\phi Q(\mu_t)(1-\Delta)^{\delta/2}\psi\,dx\\ &= \int_{\mathbb{R}^d}[Q(\mu_t)\phi](1-\Delta)^{\delta/2}\psi\,dx.\end{aligned}$$

The theorem is proved. □

REMARK 2.4. As we have seen in the above proof, $\{J^*(\mu)\varphi_i,\ i \geq 1\}$ is a frame if $\{\varphi_i\}$ is any orthonormal basis in $L_2(\mathbb{R}^d)$. However, generally speaking, $\{J^*(\mu)\varphi_i,\ i \geq 1\}$ is *not* an orthonormal basis in $L_2(\mathbb{R}^d, \mu)$ for $\mu \neq 0$. For instance, if μ is a unit measure concentrated at the origin, then one can see that $J(\mu)\psi(x) = \alpha R_\delta(x)\psi(0)$, where $\alpha^{-1} := ||R_\delta||_{L_2}$. Therefore

$$J^*(\mu)\phi(x) = \alpha\int_{\mathbb{R}^d}\phi R_\delta\,dx, \quad \mu\text{-a.s.},$$

and $\{J^*(\mu)\varphi_i,\ i \geq 1\}$ is just a set of constants (μ-a.s.) $\{\gamma_i\}$ such that $\sum_i \gamma_i^2 = 1$.

REMARK 2.5. If $\delta > d$, the frame $\{J^*(\mu)\varphi_i,\ i \ge 1\}$ has an additional property of continuity. It turns out that for any $\psi \in C_0^\infty(\mathbb{R}^d)$ and $\varphi \in L_2(\mathbb{R}^d)$ the function $(\psi J^*(\mu)\varphi, \mu)$ is continuous with respect to μ (we write $\mu_n \to \mu$ if $(f, \mu_n) \to (f, \mu)$ for any bounded and continuous f).

Owing to the equality

$$(\psi J^*(\mu)\varphi, \mu) = (J(\mu)\psi, \varphi)_{L_2} = (Q(\mu)(1-\Delta)^{\delta/2}\psi, \varphi)_{L_2},$$

it suffices to prove the continuity of $(Q(\mu)\psi, \varphi)_{L_2}$ for any $\psi, \varphi \in L_2(\mathbb{R}^d)$. Taking into account (2.2) and Taylor's series for $\exp\{-tK(\mu)\}$, we see that it suffices to prove that $K^m(\mu)f$ is a continuous $L_2(\mathbb{R}^d)$-valued function for any $f \in L_2(\mathbb{R}^d)$ and integer $m \ge 1$. Formulas like $K^m(\mu)f - K^m(\mu_n)f = K(\mu_n)[K^{m-1}(\mu)f - K^{m-1}(\mu_n)f] + [K(\mu) - K(\mu_n)]K^{m-1}(\mu)f$ and estimate (2.4) show that we may take $m = 1$. In this case, we notice that $K(\mu)f = R_\delta * [(R_\delta * f)\mu]$, where R_δ is a bounded and continuous function ($\delta > d$) and $R_\delta * f \in H_2^\delta(\mathbb{R}^d)$ which implies (even for $\delta > d/2$) that $R_\delta * f$ also is a bounded and continuous function. Therefore, $K(\mu_n)f(x) \to K(\mu)f(x)$ at any x if $\mu_n \to \mu$. If in addition all measures μ_n have supports in the same ball, then it is easy to see that there exist constants N, λ independent of n, x such that $|K(\mu_n)f(x)| = |R_\delta * [(R_\delta * f)\mu_n](x)| \le Ne^{-\lambda|x|}$. Hence $||K(\mu)f - K(\mu_n)f||_{L_2} \to 0$.

The case of general μ_n can be reduced to the particular one on the basis of (2.4) and the observation (the Helly–Bray theorem) that for any $\varepsilon > 0$ there exist measures μ_n^ε, ν_n^ε, μ^ε ν^ε such that $\mu_n = \mu_n^\varepsilon + \nu_n^\varepsilon$, $\mu = \mu^\varepsilon + \nu^\varepsilon$, $\mu_n^\varepsilon \to \mu^\varepsilon$ and all μ_n^ε are supported in the same ball with total variations of ν_n^ε, ν^ε less than ε.

Finally let us note that if $\{\varphi_i,\ i \ge 1\}$ is an orthonormal basis in $L_2(\mathbb{R}^d)$ and $\psi \in C_0^\infty(\mathbb{R}^d)$, then the l_2-valued function $r(\mu) := \{(\psi J^*(\mu)\varphi_i, \mu),\ i \ge 1\}$ is continuous in μ. Indeed, by definition, $||r(\mu)||_{l_2}^2 = (\psi^2, \mu)$, which is continuous and also $r(\mu)$ is weakly continuous by what has been proved above.

3. Other superdiffusions.

In order to state our result we need the following lemma, which, roughly speaking, provides a procedure of constructing random vectors on the same probability space for any given distribution in such a way that if the distributions converge weakly, the random vectors converge almost surely.

LEMMA 3.1. *On $\mathscr{M}$ there is a $\mathscr{B}(\mathbb{R}^d)$-valued function $\Theta(\mu)$ and for any $\mu \in \mathscr{M}$ on $\Theta(\mu)$, there is a Borel $\mathbb{R}^d$-valued function $\xi(\mu, y)$ such that we have the following.*

(i) *For any Borel positive ψ and $\mu \in \mathscr{M}$,*

$$\int_{\Theta(\mu)} \psi(\xi(\mu, y))\, dy = \int_{\mathbb{R}^d} \psi(x)\, \mu(dx). \tag{3.1}$$

(ii) *For any $\psi \in C_0^\infty(\mathbb{R}^d)$ the function*

$$I_{\Theta(\mu)}\psi(\xi(\mu, \cdot))$$

is an $L_2(\mathbb{R}^d)$-valued continuous function on $\mathscr{M}$.

PROOF. One can prove the lemma in many different ways. The way we choose relies on some properties of solutions of the simplest Monge–Ampère equation, which were discovered by A. D. Aleksandrov (see, for instance, [1, 2]).

First we fix a smooth one-to-one mapping η: $B_1 = \{|x| < 1\} \to \mathbb{R}^d$ with smooth inverse. For any $\mu \in \mathscr{M}$, define a measure $\nu = \mu\eta$ on Borel subsets Γ of B_1 by the formula

$$\nu(\Gamma) := \mu\eta(\Gamma) := \mu(\eta(\Gamma)).$$

Furthermore, we can find a real-valued continuous convex function $v = v(\mu, x) = v(x)$ on $\bar{B}_1$ with $v = 0$ on ∂B_1, which is a unique convex generalized solution of the following Monge–Ampère equation:

$$\det(v_{xx}) = \mu\eta. \tag{3.2}$$

To explain the precise meaning of (3.2), we recall that for any $x_0 \in B_1$ one can define the so-called normal mapping $\nabla v(x_0)$ as the set of all $p \in \mathbb{R}^d$ such that $z = p\cdot x + f$ is a supporting plane to the graph of v at the point $(x_0, v(x_0))$. Then by definition, (3.2) means that for any Borel $\Gamma \subset B_1$ we have ($|A|$ stands for the Euclidean volume of the set A)

$$|\nabla v(\Gamma)| = \mu\eta(\Gamma).$$

It is easy to see that for any $r \in (0, 1)$ the set $\nabla v(\bar{B}_r)$ is closed, so that their union $\Theta(\mu) := \nabla v(B_1)$ is Borel. Furthermore, it turns out that for almost every (Lebesgue) $y \in \Theta(\mu)$, there is only one point $x \in B_1$ such that $y \in \nabla v(x)$. We set $\xi(y) = \xi(\mu, y) := \eta(x)$. Thus we regard $\xi(\mu, y)$ as a function uniquely defined only almost everywhere in $\Theta(\mu)$. It is known that for any positive Borel ψ,

$$\int_{\Theta(\mu)} \psi(\xi(\mu, y))\, dy = \int_{B_1} \psi(\eta(x))\, \mu\eta(dx)$$

so that (3.1) holds.

Let us use some more facts from [1] and [2]. The function $\xi(\mu, y)$ possesses certain continuity properties following from the fact that if μ_n converge to μ weakly $[\int_{\mathbb{R}^d} \phi\, \mu_n(dx) \to \int_{\mathbb{R}^d} \phi\, \mu(dx)$ for any bounded continuous $\phi]$, then $v(\mu(n))$ converges to $v(\mu)$ uniformly. This easily implies that, if for a $y \in \Theta(\mu)$, the point $\xi(\mu, y)$ is uniquely defined, then $y \in \Theta(\mu_n)$ for all large n and if in addition $\xi(\mu_n, y)$ are well defined for all large n, then $\xi(\mu_n, y) \to \xi(\mu, y)$, so that $\xi(\mu_n, y) \to \xi(\mu, y)$ at least almost everywhere (Lebesgue) in $\Theta(\mu)$. In particular,

$$I_{\Theta(\mu)} \le \liminf_{n\to\infty} I_{\Theta(\mu_n)} \quad \text{a.e.},$$

and generally for $\Theta(\mu, r) := \{y\colon\, y \in \Theta(\mu),\ |\eta^{-1}\xi(\mu, y)| < r\}$ and $r \in (0, 1]$, we have

$$I_{\Theta(\mu, r)} \le \liminf_{n\to\infty} I_{\Theta(\mu_n, r)} \quad \text{a.e.}$$

On the other hand, one knows that $\sup_x |v(\mu, x)| \le N(d)\mu(\mathbb{R}^d)$, which implies that for any $r \in (0,1)$, the sets $\Theta(\mu_n, r)$ are uniformly bounded with respect to n. Moreover, from (3.1) we get $|\Theta(\mu) \setminus \Theta(\mu, r)| = \mu(x\colon |\eta^{-1}(x)| \ge r)$. Therefore the weak convergence of μ_n to μ implies that for any $\varepsilon > 0$, there is an $r \in (0,1)$ such that

$$\limsup_{n\to\infty} |\Theta(\mu_n) \setminus \Theta(\mu_n, r)| \le \varepsilon, \qquad |\Theta(\mu) \setminus \Theta(\mu, r)| \le \varepsilon.$$

In addition, (3.1) implies that $|\Theta(\mu_n)| \to |\Theta(\mu)|$. Therefore, by Fatou's lemma

$$\begin{aligned}\limsup_{n\to\infty} \int_{\mathbb{R}^d} |I_{\Theta(\mu)} - I_{\Theta(\mu_n)}|\, dx &= 2 \limsup_{n\to\infty} \int_{\mathbb{R}^d} [I_{\Theta(\mu)} - I_{\Theta(\mu_n)}]_+ \, dx \\ &\le 4\varepsilon + 2 \limsup_{n\to\infty} \int_{\mathbb{R}^d} [I_{\Theta(\mu, r)} - I_{\Theta(\mu_n, r)}]_+ \, dx \le 4\varepsilon \\ &\quad + 2 \int_{\mathbb{R}^d} \limsup_{n\to\infty} [I_{\Theta(\mu, r)} - I_{\Theta(\mu_n, r)}]_+ \, dx = 4\varepsilon.\end{aligned}$$

Now we see that

$$I_{\Theta(\mu_n)} \psi(\xi(\mu_n, \cdot)) \to I_{\Theta(\mu)} \psi(\xi(\mu, \cdot)) \tag{3.3}$$

in measure. From (3.1) it follows that integrals of the left-hand sides in (3.3) converge to the integral of its right-hand side. By Scheffé's theorem, the convergence in (3.3) is in L_1 if $\psi \ge 0$ (above we actually reproduce the proof of this theorem). The last restriction can be easily removed. Finally, from (3.1) it follows that the L_2-norms of left-hand sides in (3.3) converge to the L_2-norm of its right-hand side. This implies that the convergence in (3.3) is in $L_2(\mathbb{R}^d)$, which completes the proof of the lemma. □

Now let $(\Omega, \mathscr{F}, P)$ be a probability space with a filtration $\{\mathscr{F}_t\}$. Assume that on $(\Omega, \mathscr{F}, P)$ we are given a Poisson random measure $p(\Gamma) = \int_\Gamma p(dx, du, dt)$, $\Gamma \subset \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dx\pi(du)dt$, where $\pi(du)$ is a measure such that

$$c_1 := \int_1^\infty u\pi(du) < \infty, \qquad c_2 := \int_0^1 u^2 \pi(du) < \infty, \qquad \int_a^b := \int_{(a,b]}.$$

Also assume that p is $\{\mathscr{F}_t\}$-adapted. For $\lambda \ge 0$ define

$$\gamma(\lambda) := \int_0^\infty (1 - e^{-\lambda u} - \lambda u)\, \pi(du)$$

and define a martingale measure $q(dx, du, dt) = p(dx, du, dt) - dx\pi(du)dt$.

THEOREM 3.2. *Assume that we have a measure-valued process μ_t which is $\mathscr{F}_t$-predictable, satisfies*

$$\int_0^T \mu_t(\mathbb{R}^d)\, dt < \infty \quad \forall\, T < \infty \text{ a.s.} \tag{3.4}$$

and satisfies the equation: for any $\psi \in C_0^\infty(\mathbb{R}^d)$,

$$
\begin{aligned}
(\psi, \mu_t) = (\psi, \mu_0) + \int_0^t (\Delta\psi - c_1\psi, \mu_s)\, ds \\
+ \int_0^t \int_0^\infty \int_{\Theta(\mu_s)} u\psi(\xi(\mu_s, x)) \\
\times [I_{u\le 1} q(dx, du, ds) + I_{u>1} p(dx, du, ds)].
\end{aligned} \tag{3.5}
$$

Then μ_t *is a superdiffusion corresponding to* Δ, γ *in the sense that for any* $T \ge 0$ *and any solution of class* $C_b^{1,2}([0, T] \times \mathbb{R}^d)$ *of the parabolic equation*

$$
\frac{\partial v}{\partial t} + \Delta v + \gamma(v) = 0, \qquad t \in [0, T] \tag{3.6}
$$

we have

$$
Ee^{-(v(T), \mu_T)} = Ee^{-(v(0), \mu_0)}. \tag{3.7}
$$

PROOF. First we claim that if $\psi = \psi(t) = \psi(t, x)$ and $\psi \in C_0^\infty(\mathbb{R}^{d+1})$ then

$$
\begin{aligned}
(\psi(t), \mu_t) = (\psi(0), \mu_0) + \int_0^t \left(\frac{\partial \psi}{\partial s} + \Delta\psi - c_1\psi, \mu_s \right) ds \\
+ \int_0^t \int_0^\infty \int_{\Theta(\mu_s)} u\psi(s, \xi(\mu_s, x)) \\
\times [I_{u\le 1} q(dx, du, ds) + I_{u>1} p(dx, du, ds)].
\end{aligned} \tag{3.8}
$$

Indeed, if $\psi(t, x) = \psi(x)\eta(\tau)\exp(i\tau t)$, where $\psi \in C_0^\infty(\mathbb{R}^d)$ and τ, $\eta(\tau)$ are some numbers, then one obtains (3.8) from (3.5) by Itô's formula. Next one notices that

$$
\begin{aligned}
\left\langle \int_0^{\cdot} \int_0^1 \int_{\Theta(\mu_s)} u\psi(s, \xi(\mu_s, x))\, q(dx, du, ds) \right\rangle_t \\
= \int_0^t \int_0^1 \int_{\Theta(\mu_s)} u^2\psi^2(s, \xi(\mu_s, x))\, dx\, \pi(du)\, ds \\
= c_2 \int_0^t \int_{\mathbb{R}^d} \psi^2(s, x)\, \mu_s(dx)\, ds.
\end{aligned} \tag{3.9}
$$

Now the general case of (3.8) follows from the above particular case after integrating with respect to τ, applying the Fourier transform and relying on (3.4) while passing to the limit. Furthermore, by using (3.4) and (3.9) and obvious approximations, one can easily see that (3.8) holds true for $\psi = v$ and $t \in [0, T]$.

This gives us the stochastic differential of $(v(t), \mu_t)$ and by Itô's formula allows us to conclude that for any stopping time $\tau \le T$,

$$
\begin{aligned}
&\exp(-(v(\tau), \mu_\tau)) \\
&\quad = \exp(-(v(0), \mu_0)) - \int_0^\tau \exp(-(v(t), \mu_t))\left(\frac{\partial v}{\partial t} + \Delta v - c_1 v, \mu_t\right) dt \\
&\qquad + \int_0^\tau \exp(-(v(t), \mu_t)) \\
&\qquad\quad \times \int_0^1 \int_{\Theta(\mu_t)} \big[\exp(-uv(t, \xi(\mu_t, x))) - 1 + uv(t, \xi(\mu_t, x))\big]\, dx\,\pi(du)\, dt \\
&\qquad + \int_0^\tau \exp(-(v(t), \mu_t)) \\
&\qquad\quad \times \int_1^\infty \int_{\Theta(\mu_t)} [\exp(-uv(t, \xi(\mu_t, x))) - 1]\, p(dx, du, dt) + m_t,
\end{aligned}
\tag{3.10}
$$

where m_t is a certain local martingale.

Next

$$
\begin{aligned}
&E \int_0^\tau \exp(-(v(t), \mu_t)) \int_1^\infty \int_{\Theta(\mu_t)} [\exp(-uv(t, \xi(\mu_t, x))) - 1]\, p(dx, du, dt) \\
&\quad = E \int_0^\tau \exp(-(v(t), \mu_t)) \int_1^\infty \int_{\Theta(\mu_t)} [\exp(-uv(t, \xi(\mu_t, x))) - 1]\, dx\,\pi(du)\, dt,
\end{aligned}
$$

$$
(c_1 v, \mu_t) = \int_1^\infty \int_{\Theta(\mu_t)} uv(t, \xi(\mu_t, x))\, dx\,\pi(du).
$$

Therefore, if $\tau \le T$ and $m_{\tau\wedge t}$ is a martingale, then by (3.10),

$$
\begin{aligned}
E\exp(-(v(\tau), \mu_\tau)) &= E\exp(-(v(0), \mu_0)) \\
&\quad - E\int_0^\tau \exp(-(v(t), \mu_t))\left(\frac{\partial v}{\partial t} + \Delta v, \mu_t\right) dt \\
&\quad + E\int_0^\tau \exp(-(v(t), \mu_t)) \\
&\quad \times \int_0^\infty \int_{\Theta(\mu_t)} \big[\exp(-uv(t, \xi(\mu_t, x))) - 1 + uv(t, \xi(\mu_t, x))\big] \\
&\qquad \times dx\,\pi(du)\, dt.
\end{aligned}
\tag{3.11}
$$

Here

$$
\int_0^\infty [\exp(-uv(t, \xi(\mu_t, x))) - 1 + uv(t, \xi(\mu_t, x))]\pi(du) = -\gamma(v(t, \xi(\mu_t, x))),
$$

and by (3.1),

$$
\int_{\Theta(\mu_t)} \gamma(v(t, \xi(\mu_t, x)))\, dx = \int_{\mathbb{R}^d} \gamma(v(t, y))\, \mu_t(dy).
$$

From (3.6) and (3.11) we finally see that (3.7) is true with τ instead of T. The arbitrariness of τ, actually, shows that (3.7) holds as well. The theorem is proved. □

REMARK 3.3. For Borel $\Gamma \subset \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$ define

$$p(\mu_\cdot, \Gamma) = \int_{\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+} I_{\Theta(\mu_t)}(x) I_\Gamma(\xi(\mu_t, x), u, t)\, p(dx, du, dt),$$

and similarly define $q(\mu_\cdot, \Gamma)$. Then by the formula for change of variables, the last integral in (3.5) takes the form

$$\int_0^t \int_0^\infty \int_{\mathbb{R}^d} u\psi(y)\, [I_{u\le 1} q(\mu_\cdot, dy, du, ds) + I_{u>1} p(\mu_\cdot, dy, du, ds)]$$

and one can rewrite (3.5) in the following symbolic form:

$$d\mu_t = (\Delta\mu_t - c_1\mu_t)\, dt + \int_0^1 u\, q(\mu_\cdot, \cdot, du, dt) + \int_1^\infty u\, p(\mu_\cdot, \cdot, du, dt).$$

REMARK 3.4. Yet another form of (3.5) can be obtained by using the idea from [6] mentioned in Remark 1.8. Let $\{\varphi_i\}$ be an orthonormal basis in $L_2(\mathbb{R}^d)$. Define

$$m_t^i = \int_0^t \int_0^1 \int_{\mathbb{R}^d} \varphi_i(x) u\, q(dx, du, ds),$$

so that m_t^i are square-integrable martingales (defined without using μ_t). Then

$$\begin{aligned} &E\left|\int_0^\tau \int_0^1 \int_{\Theta(\mu_s)} u\psi(\xi(\mu_s, x))\, q(dx, du, ds) - \sum_{i\le n} \int_0^\tau (I_{\Theta(\mu_s)}\psi(\xi(\mu_s, \cdot)), \varphi_i)\, dm_s^i\right|^2 \\ &\quad = c_2 E\int_0^\tau \int_{\mathbb{R}^d} \left| I_{\Theta(\mu_s)}\psi(\xi(\mu_s, x)) - \sum_{i\le n} (I_{\Theta(\mu_s)}\psi(\xi(\mu_s, \cdot)), \varphi_i)\varphi_i(x)\right|^2 dx\, ds \\ &\quad \to 0 \end{aligned}$$

as $n \to \infty$ if the stopping time τ is such that

$$\begin{aligned} E\int_0^\tau \int_{\Theta(\mu_s)} \psi^2(\xi(\mu_s, x))\, dx\, ds &= E\int_0^\tau \int_{\mathbb{R}^d} \psi^2(x)\, \mu_s(dx)\, ds \\ &\le \sup \psi^2 E\int_0^\tau \mu_s(\mathbb{R}^d) ds < \infty. \end{aligned}$$

Also

$$\begin{aligned} (I_{\Theta(\mu_s)}\psi(\xi(\mu_s, \cdot)), \varphi_i) &= \int_{\mathbb{R}^d} I_{\Theta(\mu_s)}(x)\psi(\xi(\mu_s, x))\varphi_i(x)\, dx \\ &= \int_{\mathbb{R}^d} I_{\Theta(\mu_s)}(x)\psi(\xi(\mu_s, x))\hat\varphi_i(\mu_s, \xi(\mu_s, x))\, dx \\ &= \int_{\mathbb{R}^d} \psi(x)\hat\varphi_i(\mu_s, x)\, \mu_s(dx), \end{aligned}$$

where for any $\mu \in \mathscr{M}$, $\hat{\varphi}_i(\mu, \xi(\mu, \cdot))$ is the projection in $L_2(\mathbb{R}^d)$ of φ_i on the subspace spanned by all functions $I_{\Theta(\mu)}(x)\psi(\xi(\mu, x))$ [so to speak, the conditional expectation of φ_i, given $\xi(\mu, x)$].

Therefore, in (3.5),

$$\int_0^t \int_0^1 \int_{\Theta(\mu_s)} u\psi(\xi(\mu_s, x))\, q(dx, du, ds) = \sum_i \int_0^t \int_{\mathbb{R}^d} \psi(x)\hat{\varphi}_i(\mu_s, x)\, \mu_s(dx)\, dm_s^i,$$

where the series converges in probability uniformly on every finite interval of time. Now (3.5) takes the form

$$d\mu_t = (\Delta\mu_t - c_1\mu_t)\, dt + \sum_i \hat{\varphi}_i(\mu_t)\mu_t\, dm_t^i + \int_1^\infty u\; p(\mu_\cdot, \cdot, du, dt).$$

As is easy to see, $\{\hat{\varphi}_i(\mu)\}$ is again a frame function.

REMARK 3.5. As in [7] or, in the case of equations with coefficients depending on ω, t, x, as in [10] (cf., Corollary 1.9) one can prove that under the conditions of Theorem 3.2, if $c_1 = 0$ and $d = 1$, then for any t, μ_t is absolutely continuous with respect to Lebesgue measure (a.s.).

REMARK 3.6. One can consider the case in which the function $\gamma(\lambda)$ contains the term $a\lambda^2$ by combining the methods from this and the previous sections.

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