The Annals of Probability
1997, Vol. 25, No. 4, 1712–1734

LOCAL ASYMPTOTIC CLASSES FOR THE SUCCESSIVE PRIMITIVES OF BROWNIAN MOTION

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Let $(B(t))_{t\geq 0}$ be the linear Brownian motion starting at 0, and set $X_n(t) = (1/n!)\int_0^t (t-s)^n\,dB(s)$. Watanabe stated a law of the iterated logarithm for the process $(X_1(t))_{t\geq 0}$, among other things. This paper proposes an elementary proof of this fact, which can be extended to the general case $n \geq 1$. Next, we study the local asymptotic classes (upper and lower) of the $(n+1)$-dimensional process $U_n = (B, X_1, \dots, X_n)$ near zero and infinity, and the results obtained are extended to the case where B is the d-dimensional Brownian motion.

1. Introduction.

Let $(B(t))_{t\geq 0}$ be the linear Brownian motion starting at 0. Denote by

$$X_n(t) = \frac{1}{n!}\int_0^t (t-s)^n\,dB(s)$$

its n-fold primitive, and

$$U_n = (B, X_1, \dots, X_n).$$

Upon integration by parts, we obtain the integral of Brownian motion as a special case: $X_1(t) = \int_0^t B(s)\,ds$ (cf. Lemma 3 for a generalization to $n > 1$).

The Gaussian process X_n was first mentioned by Shepp [18]. Later, Wahba used this process to derive a correspondence between smoothing by splines and Bayesian estimation in certain stochastic models ([22], [23]). See also [2] where X_n is equally introduced in describing some degenerate Gaussian diffusions. Let us point out that the process X_1 has been studied at length by several authors (see [11] for further references related to this particular case).

This work is concerned with the asymptotic behavior of the real process X_n, as well as that of the $(n+1)$-dimensional process U_n.

In [15], McKean studied the asymptotic behavior of the successive hitting times at level 0 for the process X_1, as well as the corresponding hitting locations of the Brownian motion B. More precisely, setting

$$\mathrm{t}_0 = 1,$$
$$\mathrm{t}_n = \inf\{t > \mathrm{t}_{n-1}\colon X_1(t) = 0\},$$
$$\mathrm{b}_n = B(\mathrm{t}_n), \qquad n \geq 1,$$

Received February 1996; revised October 1996.

AMS 1991 *subject classifications*. Primary 60J65, 60F15, 60G15; secondary 60G17, 60J25.

Key words and phrases. Law of the iterated logarithm, local asymptotic classes, integral tests.

he derived from the law of large numbers the following asymptotics:

$$\log \mathrm{b}_n \sim \frac{4\pi}{\sqrt{3}}\, n \quad \text{as } n \to \infty,$$
$$\log \mathrm{t}_n \sim \frac{8\pi}{\sqrt{3}}\, n \quad \text{as } n \to \infty.$$

So, it may be asked how the process U_1 grows. In fact, U_1 will be proven to be transient; that is, $||U_1(t)|| \to +\infty$ a.s. as $t \to +\infty$. By the way, note that each component of U_1 is recurrent. We now ask the question of how to derive the escape speed of $||U_1(t)||$. For instance, two particular results will be obtained:

$$\limsup_{t\to+\infty} \frac{||U_1(t)||}{((2t^3/3)\log\log t)^{1/2}} = 1 \quad \text{a.s.},$$

$$\limsup_{t\to+\infty} \frac{(\log t)^{1/2+\varepsilon}}{t^{1/2}}||U_1(t)|| = \begin{cases} 0, & \text{if } \varepsilon \le 0, \\ +\infty, & \text{if } \varepsilon > 0. \end{cases}$$

On the other hand, in [24], Watanabe established a law of the iterated logarithm for X_1. In fact, he derived this formula from a general result concerning a large class of Gaussian processes (see also [17]), and so his proof is quite difficult. The aim of this paper is first to give a short and elementary proof of the law of the iterated logarithm which can be extended to the process X_n for any $n \ge 1$, and second to characterize the local asymptotic classes of the $(n+1)$-dimensional process U_n, which will be shown to be a Markov process.

This article is divided into three parts. In Section 2 we write three laws of the iterated logarithm corresponding to the following cases: $t \to 0^+$, $t \to +\infty$ and $t \to t_0$ for a fixed time $t_0 \in]0, +\infty[$. The study of the case $t \to +\infty$ will follow from the case $t \to 0^+$ thanks to a time inversion, and the case $t \to t_0$ will be obtained by using the Markov property of the process U_n. The law of the iterated logarithm related to the situation $t \to 0^+$ is based on an estimate about the tail of the distribution of $\max_{0\le s\le t} X_n(s)$, which will be deduced from a large deviations principle concerning Gaussian processes stated by Marcus and Shepp [14]. We will also need an independence property of some particular increments of X_n.

Sections 3 and 4 are devoted to the local asymptotic classes of the process U_n. Several integral tests for certain classes of Gaussian processes are well known (see [9], [10], [25], [26]). Unfortunately, none of them can be applied to the process U_n, because it does not have the required conditions. In fact, in Section 3, we will derive, directly from the integral tests of Kolmogorov for Brownian motion and Watanabe for some stationary Gaussian processes, a necessary and sufficient condition to decide whether a given function belongs to the local asymptotic upper class of U_n—that is, to say exactly when the event $\{\exists\ \varepsilon > 0$ (resp. $T > 0$), $\forall\ t < \varepsilon$ (resp. $t > T$): $||U_n(t)|| < f(t)\}$ occurs with probability 1. Here, $||\cdot||$ denotes the Euclidean norm in $\mathbb{R}^{n+1}$. It will be seen that the upper classes near $+\infty$ of both processes U_n and X_n coincide, while the upper class near 0^+ of U_n is the same as the one of Brownian motion B.

In Section 4 we give some new integral tests characterizing the asymptotic lower classes of U_n (i.e., related to the events $\{\exists\ \varepsilon > 0$ (resp. $T > 0$), $\forall\ t < \varepsilon$ (resp. $t > T$) $\|U_n(t)\| > f(t)\}$). Our results are analogous to the classical tests successively written by Dvoretsky and Erdős in the case of space-valued Brownian motion [3], and by Hendricks [4], Takeuchi [20], [19] and Taylor [21] for multidimensional processes with stable and independent components. Here, the method consists in obtaining several estimates of some hitting probabilities.

Finally, in Section 5 we discuss the case where B is the d-dimensional Brownian motion, and exhibit more general results.

2. Laws of the iterated logarithm for X_n.

Let us introduce some notation. Set

$$\varphi_n(t) = \sqrt{2\gamma_n}\, t^{n+1/2}(\log\log(1/t))^{1/2},$$
$$\psi_n(t) = \sqrt{2\gamma_n}\, t^{n+1/2}(\log\log t)^{1/2},$$

where

$$\gamma_n = \frac{1}{(2n+1)(n!)^2}$$

and

$$Y_n(s,t) = X_n(t) - \sum_{k=0}^{n} \frac{(t-s)^k}{k!} X_{n-k}(s). \tag{1}$$

THEOREM 1. *The following laws of the iterated logarithm near 0^+, $+\infty$ and any $t_0 > 0$ hold:*

$$\limsup_{t\to 0^+} \frac{X_n(t)}{\varphi_n(t)} = 1 \quad \text{a.s.}, \tag{2}$$

$$\limsup_{t\to +\infty} \frac{X_n(t)}{\psi_n(t)} = 1 \quad \text{a.s.}, \tag{3}$$

$$\limsup_{t\to t_0} \frac{Y_n(t_0,t)}{\varphi_n(|t-t_0|)} = 1 \quad \text{a.s.} \tag{4}$$

Suppose that assertion (2) is true and let us prove (3) and (4). Assertion (3) obviously hinges on the following lemma.

LEMMA 2. *Set $\hat{X}_n(0) = 0$ and for $t > 0$, $\hat{X}_n(t) = t^{2n+1}X_n(1/t)$. Then the processes $(X_n(t))_{t\ge 0}$ and $(\hat{X}_n(t))_{t\ge 0}$ are identical in law.*

PROOF. It is easy to verify that the above Gaussian processes have the same covariance functions and the proof is complete. □

In order to prove assertion (4) we need some properties of the quantity $Y_n(s,t)$.

LEMMA 3. (i) *If $s<t$ then*

$$\begin{aligned} Y_n(s,t) &= \int_s^t \frac{(t-\sigma)^n}{n!}\,dB(\sigma) \\ &= \int_s^t \frac{(t-\sigma)^{n-1}}{(n-1)!}B(\sigma)\,d\sigma - \frac{(t-s)^n}{n!}B(s). \end{aligned} \tag{5}$$

(ii) *Fix an instant $s\geq 0$. Then the processes $(Y_n(s,s+t))_{t\geq 0}$ and $(X_n(t))_{t\geq 0}$ have identical laws, and $(Y_n(s,s+t))_{t\geq 0}$ is independent of $\sigma\{B(r),\ 0\leq r\leq t\}$.*

PROOF. (i) We have

$$\int_s^t \frac{(t-\sigma)^n}{n!}\,dB(\sigma) = X_n(t) - \int_0^s \frac{(s-\sigma+t-s)^n}{n!}\,dB(\sigma).$$

Then, by using the binomial theorem,

$$(s-\sigma+t-s)^n = \sum_{k=0}^n \binom{n}{j}(s-\sigma)^{n-k}(t-s)^k,$$

we get (5). For the second portion of (i), use (5) and integration by parts.

(ii) Part (ii) is a consequence of the iid increments of Brownian motion. □

As a result,

$$\limsup_{t\to t_0^+} \frac{Y_n(t_0,t)}{\varphi_n(|t-t_0|)} = 1 \quad \text{a.s.},$$

which implies

$$\limsup_{t\to t_0} \frac{Y_n(t_0,t)}{\varphi_n(|t-t_0|)} \geq 1 \quad \text{a.s.}$$

The corresponding upper bound is obtained in exactly the same manner as the one for $X_n(t)$ via the analogue of Lemma 5.

PROOF OF THEOREM 1. We imitate the original proof of Khintchine (see, e.g., [13], page 242) *mutatis mutandis*.

We first prove the relation

$$\mathbb{P}\left\{\limsup_{t\to 0^+} \frac{X_n(t)}{\varphi_n(t)} \leq \delta\right\} = 1 \tag{6}$$

for every $\delta>1$. To do this, we need an estimate on the tail of the law of the random variable $\max_{0\leq s\leq t} X_n(s)$. This one is contained in the following lemma.

LEMMA 4. *For every $t > 0$ and $\varepsilon > 0$, there exists $A = A(t, \varepsilon) > 0$ such that for all $a > A$,*

$$\mathbb{P}\Big\{\max_{0 \le s \le t} X_n(s) > a\Big\} \le \exp\biggl(-(1-\varepsilon)\frac{a^2}{2\gamma_n t^{2n+1}}\biggr). \tag{7}$$

PROOF. This inequality is the result of a large deviations principle stated in [14] as follows: if $(X_t)_{t \in I}$ is a real-bounded Gaussian process indexed by a linear interval I, then

$$\lim_{a \to +\infty} \frac{1}{a^2}\mathbb{P}\Big\{\sup_{t \in I} X_t > a\Big\} = -\Big(2\sup_{t \in I} \mathbb{E}(X_t^2)\Big)^{-1}. \qquad \square$$

Pick now $\theta \in (0, 1)$, and put

$$\begin{aligned} t_j &= \theta^j, \\ a_j &= \delta\varphi_n(t_{j+1}), \\ A_j &= \Big\{\max_{t_{j+1} \le s \le t_j} X_n(s) > a_j\Big\}, \\ p_j &= \mathbb{P}(A_j). \end{aligned}$$

We have

$$\frac{a_j^2}{2\gamma_n t_j^{2n+1}} = \delta^2\theta^{2n+1}\log(j \log\theta).$$

Therefore, it comes from (7) that

$$p_j \le (j\log\theta)^{-(1-\varepsilon)\delta^2\theta^{2n+1}}.$$

Choose now $\theta < 1$ and $\varepsilon \in (0, 1)$ such that $(1-\varepsilon)\delta^2\theta^{2n+1} > 1$. This yields

$$\sum p_j < +\infty$$

and hence, by the Borel–Cantelli lemma we get

$$\mathbb{P}\Big(\limsup_{j \ge 0} A_j\Big) = 0.$$

This can be written as follows: there exists a.s. an integer j_0 such that for all $j > j_0$ and all $t \in [t_{j+1}, t_j]$,

$$X_n(t) \le a_j \le \delta\varphi_n(t),$$

and then for every $\delta > 1$,

$$\limsup_{t \to 0^+} \frac{X_n(t)}{\varphi_n(t)} \le \delta \quad \text{a.s.}$$

Let $\delta \downarrow 1^+$ through the rational numbers and (6) ensues.

Now, let us prove the converse inequality

$$\limsup_{t\to 0^+} \frac{X_n(t)}{\varphi_n(t)} \geq \delta \quad \text{a.s.,} \tag{8}$$

for each $\delta < 1$. Pick again $\theta \in (0,1)$, and set

$$t_j = \theta^j,$$
$$B_j = \{Y_n(t_{j+1}, t_j) \geq (1-\theta)^{n+1/2}\varphi_n(t_j)\},$$
$$q_j = \mathbb{P}(B_j).$$

We will minorize the probability q_j as follows. By scaling and Lemma 3(ii),

$$q_j \geq \mathbb{P}\Big\{X_n(1) \geq t_j^{-(n+1/2)}\varphi_n(t_j)\Big\} \geq \mathbb{P}\Big\{N \geq \sqrt{2\log\log t_j}\Big\}$$

where N is a normal Gaussian variable.

Using the classical inequality

$$\int_A^{+\infty} \exp\Big(-\frac{x^2}{2}\Big)\,dx \geq \frac{\exp(-A^2/2)}{A+1/A},$$

a lower bound of q_j is now obtained:

$$q_j \geq \frac{1}{\sqrt{2\pi}} \frac{\exp(-\log\log(1/t_j))}{\sqrt{2\log\log(1/t_j)} + 1/\sqrt{2\log\log(1/t_j)}}.$$

The right-hand side is equivalent to const.$(j\sqrt{\log j})^{-1}$ when $j \to +\infty$, so that

$$\sum q_j = +\infty.$$

The Borel–Cantelli lemma together with the independence of the events B_j then yield

$$\mathbb{P}\Big(\limsup_{j\geq 0} B_j\Big) = 1.$$

Thus a.s.,

$$X_n(t_j) \geq (1-\theta)^{n+1/2}\varphi_n(t_j) + \sum_{k=0}^{n} \frac{1}{k!} t_j^k (1-\theta)^k X_{n-k}(t_{j+1})$$

for infinitely many j. Invoking the first part, we are able to minimize each term of the sum arising in the above right-hand side as follows: a.s., for each $k \in \{0,\dots,n\}$, there exists an integer j_k such that for all $j \geq j_k$ the following inequality holds:

$$X_{n-k}(t_{j+1}) \geq -2\varphi_{n-k}(t_{j+1}).$$

But we have for large enough j,

$$\log((j+1)\log(1/\theta)) \leq 2\log(j\log(1/\theta)).$$

Hence we can choose j_k such that, a.s., for all $j \geq j_k$,

$$X_{n-k}(t_{j+1}) \geq -4\varphi_{n-k}(t_j)\theta^{n-k+1/2}.$$

Thus, setting $l_0 = \max_{0\leq k\leq n} j_k$, it follows, a.s., for all $j \geq l_0$,

$$\sum_{k=0}^{n} \frac{1}{k!} t_j^k (1-\theta)^k X_{n-k}(t_{j+1}) \geq -4\sqrt{\theta} \sum_{k=0}^{n} \theta^{n-k}(1-\theta)^k t_j^k \varphi_{n-k}(t_j).$$

Finally, there exist some positive constants η_{kn}, $k \in \{0,\dots,n\}$, depending only on n, such that a.s.,

$$\frac{X_n(t_j)}{\varphi_n(t_j)} \geq (1-\theta)^{n+1/2} - \sqrt{\theta} \sum_{k=0}^{n} \eta_{kn}\theta^{n-k}(1-\theta)^k$$

for infinitely many j.

Since the right-hand side tends to 1 as $\theta \downarrow 0^+$, it can be chosen larger than δ, which is less than 1. Hence relation (8) is checked and the proof of (2) is easily completed. □

3. The asymptotic upper classes of U_n. In [24], Watanabe asserts that if $(\xi_t)_{t\geq 0}$ is a centered Gaussian process with an autocorrelation function ρ defined by $\rho(s,t) = \mathbb{E}(\xi_s\xi_t)/\sqrt{\mathbb{E}(\xi_s^2)\mathbb{E}(\xi_t^2)}$ which satisfies

$$\rho(t,t+h) \geq 1 - \alpha_1|h|^{\alpha_3} \quad \text{as } h \to 0^+ \text{ and } t \to +\infty, \tag{9}$$

$$\rho(t,t+h) \leq (1-\alpha_2|h|^{\alpha_3}) \vee \alpha_4 \quad \text{as } t \to +\infty, \tag{10}$$

$$\lim_{s\to+\infty} s\rho(t,t+s) = 0 \quad \text{uniformly with respect to } t \tag{11}$$

for some constants $\alpha_1, \alpha_2 > 0$, $0 < \alpha_3 < 2$, $\alpha_4 < 1$, then for any nondecreasing function f: $[0,+\infty) \to [0,+\infty)$ we have

$$\int^{+\infty} f(t)^{(2/\alpha_2)-1} \exp(-f(t)^2/2)\,dt \begin{Bmatrix} < \\ = \end{Bmatrix} +\infty$$
$$\Rightarrow \quad \mathbb{P}\{\exists\, T > 0\colon \forall\, t > T,\ \xi_t < f(t)\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}.$$

We will apply this result to a centered Gaussian process built on X_n. Set

$$V_n(t) = \sqrt{\gamma_n}\, e^{-(2n+1)t} X_n(e^{2t}) \quad \text{and} \quad r(s,t) = \mathbb{E}(V_n(s)V_n(t)).$$

It can be checked that V_n is a centered stationary Gaussian process satisfying (9)–(11).

Put $t = e^{2s}$ and $f(t) = g(\frac{1}{2}\log t)$. We have

$$V_n(s) \leq g(s) \quad \Leftrightarrow \quad X_n(t) \leq f(t).$$

and then $\int^{+\infty} \exp(-g(s)^2/2)\,ds$ and $\int^{+\infty} \exp(-f(t)^2/2)\,(dt/t)$ simultaneously converge or diverge. Hence, we obtain the following integral test, which was written by Watanabe in the case $n = 1$ [24].

THEOREM 5. *Let f: $[0,+\infty) \to [0,+\infty)$ be a nondecreasing function. We have*

$$\mathbb{P}\{\exists\, T > 0\colon \forall\, t > T,\ |X_n(t)| \le \sqrt{\gamma_n}\, t^{n+1/2} f(t)\} = 1 \quad \text{or} \quad 0 \tag{12}$$

according to the integral $\int^{+\infty} \exp(-f(t)^2/2)\,(dt/t)$ converges or diverges.

Due to Lemma 2, the same test holds for small t with $\int_{0^+}$ under the assumption that f is nonincreasing.

Now we are able to state a similar test related to the process U_n.

THEOREM 6. (i) *Let f: $[0,+\infty) \to [0,+\infty)$ be a nondecreasing function. Then*

> *$\|U_n(t)\| < \sqrt{\gamma_n}\, t^{n+1/2} f(t)$ for any large t with probability 1 or 0 according as the integral $\int^{+\infty} \exp(-f(t)^2/2)\,(dt/t)$ converges or diverges.*

(ii) *Assume f does not increase. Then*

> *$\|U_n(t)\| < t^{1/2} f(t)$ for any small t with probability 1 or 0 according as the integral $\int_{0^+} f(t) \exp(-f(t)^2/2)\,(dt/t)$ converges or diverges.*

PROOF. (i) The divergence part of the first assertion can be easily deduced from the one related to X_n thanks to the obvious inequality:

$$|X_n(t)| \le \|U_n(t)\|.$$

In order to prove the convergence part, write

$$\frac{\|U_n(t)\|}{\sqrt{\gamma_n}\, t^{n+1/2}} = \left[\left(\frac{X_n(t)}{\sqrt{\gamma_n}\, t^{n+1/2}}\right)^2 + \sum_{k=1}^{n} \frac{\alpha_{kn}}{t^{2k}} \left(\frac{X_{n-k}(t)}{\sqrt{\gamma_{n-k}}\, t^{n-k+1/2}}\right)^2\right]^{1/2} \tag{13}$$

with $\alpha_{kn} = \gamma_{n-k}/\gamma_n$.

Suppose $\int^{+\infty} \exp(-f(t)^2/2)\,(dt/t) < +\infty$. As in [24] we can limit ourselves to the case $\sqrt{\log\log t} \le f(t) \le \sqrt{3\log\log t}$. Put $g(t) = (1+\varepsilon(t))^{-1/2} f(t)$ where

$$\varepsilon(t) = \sum_{k=1}^{n-1} \frac{\alpha_{kn}}{t^{2k}} + \frac{\beta_n}{t^{2n}};$$

the positive constant β_n will be chosen later. Since

$$\begin{aligned} \exp\left(-\frac{g(t)^2}{2}\right) &= \exp\left(\frac{1}{2}\frac{\varepsilon(t)}{1+\varepsilon(t)} f(t)^2\right) \exp\left(\frac{-f(t)^2}{2}\right) \\ &\sim \exp\left(\frac{-f(t)^2}{2}\right) \quad \text{as } t \to +\infty, \end{aligned}$$

we get

$$\int^{+\infty} \exp\left(\frac{-g(t)^2}{2}\right) \frac{dt}{t} < +\infty,$$

which implies in regard to (12) that a.s. for large t and $k \in \{1, \dots, n\}$:

$$|X_k(t)| < \sqrt{\gamma_k}\, t^{k+1/2} g(t). \tag{14}$$

By another way, the well-known law of the iterated logarithm of Brownian motion yields the almost sure majorization

$$|B(t)| < \sqrt{3t \log\log t} < \kappa\sqrt{t}\, g(t) \tag{15}$$

for large t and some appropriate constant $\kappa > 0$. Consequently, from (13), (14) and (15) comes the following inequality, which is valid a.s. for all large t:

$$\frac{||U_n(t)||}{\sqrt{\gamma_n}\, t^{n+1/2} f(t)} \le g(t)\left[1 + \sum_{k=1}^{n-1} \frac{\alpha_{kn}}{t^{2k}} + \frac{\kappa^2 \alpha_{nn}}{t^{2n}}\right]^{1/2} = f(t)$$

for $\beta_n = \kappa^2 \alpha_{nn}$. This proves the first assertion.

(ii) The proof of the second assertion is quite similar to the proof of (i). In order to check the convergence part, write

$$\frac{||U_n(t)||}{t^{1/2}} = \left[\frac{B(t)}{t^{1/2}} + \sum_{k=1}^{n} \gamma_k t^{2k} \left(\frac{X_k(t)}{\sqrt{\gamma_k}\, t^{k+1/2}}\right)^2\right]^{1/2},$$

and suppose that $\sqrt{\log\log(1/t)} \le f(t) \le \sqrt{3\log\log(1/t)}$. We set now $h(t) = (1+\varepsilon(t))^{-1/2} f(t)$ with

$$\varepsilon(t) = \sum_{k=1}^{n} \gamma_k t^{2k}.$$

We have

$$h(t) \sim f(t) \quad \text{and} \quad \exp(-h(t)^2/2) \sim \exp(-f(t)^2/2) \quad \text{as } t \to 0^+.$$

Thus, if the integral $\int_{0^+} f(t) \exp(-f(t)^2/2)\,(dt/t)$ is convergent, so are the following:

$$\int_{0^+} h(t) \exp(-h(t)^2/2)\, \frac{dt}{t} \quad \text{and} \quad \int_{0^+} \exp(-h(t)^2/2)\, \frac{dt}{t}.$$

Hence, we get by the classical Kolmogorov test (see, e.g., [5], page 33)

$$|B(t)| < t^{1/2} h(t) \quad \text{when } t \to 0^+,$$

as well as, with the aid of (12),

$$|X_k(t)| < \sqrt{\gamma_k}\, t^{k+1/2} h(t) \quad \text{for any small } t \text{ and } k \in \{1, \dots, n\} \text{ a.s.},$$

so that

$$\frac{||U_n(t)||}{t^{1/2}} < h(t)\left(1 + \sum_{k=1}^{n} \gamma_k t^{2k}\right)^{1/2} = f(t) \quad \text{as } t \to 0^+ \text{ a.s.} \qquad \square$$

4. The asymptotic lower classes of U_n. The results we are going to state below are some integral tests which allow us to decide whether or not a function f eventually minorizes $||U_n||$ either near 0 or ∞. Our method is similar to the one described by Dvoretsky and Erdős in the case of space-valued Brownian motion [3], and in [4], [19], [20] and [21] for multidimensional processes with stable and independent components. We will need some estimates concerning the potential associated with the process U_n.

We start by stating some properties of the process U_n.

4.1. *Preliminaries.*

PROPOSITION 7. *U_n is a strong Markov process.*

PROOF. Since U_n satisfies the following stochastic differential system:

$$\begin{aligned} dX_0(t) &= \quad dB(t)\\ dX_1(t) &= \quad X_0(t)\,dt\\ &\vdots \qquad\qquad \vdots\\ dX_n(t) &= X_{n-1}(t)\,dt, \end{aligned}$$

it is easy to see that U_n is a Gaussian diffusion with generator (see, e.g., [5]):

$$\mathscr{D} = \frac{1}{2}\frac{\partial^2}{\partial x_0^2} + \sum_{k=1}^{n} x_{k-1}\frac{\partial}{\partial x_k}.$$

By Itô's formula, for all $f \in C_B^2$,

$$f(U_n(t)) = f(U_n(0)) + \int_0^t \mathscr{D} f(U_n(s))\,ds + \text{martingale}.$$

Therefore, $(U_n, \mathscr{D})$ solves the martingale problem and is a strong Markov process by the Stroock–Varadhan theorem. □

Now write

$$p_t(x; y)\,dy = \mathbb{P}_x\{U_n(t) \in dy\}, \qquad x = (x_0, \dots, x_n),\ \ y = (y_0, \dots, y_n)$$

for the transition densities of the Markov process U_n.

Since U_n is a Gaussian process, we get an explicit formula for $p_t(x; y)$:

$$p_t(x; y) = \frac{\gamma}{t^{\nu+1}} \exp\left[-\sum_{0\le i,\, j\le n} \frac{a_{ij}}{t^{i+j+1}}\left(y_i - \sum_{k=0}^{i}\frac{t^k}{k!}x_{i-k}\right)\left(y_j - \sum_{k=0}^{j}\frac{t^k}{k!}x_{j-k}\right)\right], \tag{16}$$

where the double of the matrix $(a_{ij})_{0\le i,\, j\le n}$ is the inverse of the covariance matrix of the random vector U_1, namely

$$\Gamma = \left(\frac{1}{(i+j+1)i!\,j!}\right)_{0\le i,\, j\le n} \quad \text{and} \quad \gamma = \frac{1}{(2\pi)^{(n+1)/2}\sqrt{\det\Gamma}},\ \ \nu = \tfrac{1}{2}(n+1)^2 - 1.$$

The density (16) has the following matrix representation:

$$p_t(x;y)=\frac{\gamma}{t^{\nu+1}}\exp\bigl[-(y-xJ_t)A_t(y-xJ_t)^T\bigr], \tag{17}$$

where we put for all $x=(x_0,\dots,x_n)\in\mathbb{R}^{n+1}$:

$$x^T=\begin{pmatrix}x_0\\ \vdots\\ x_n\end{pmatrix},\qquad A_t=\left(\frac{a_{ij}}{t^{i+j+1}}\right)_{0\le i,\,j\le n},$$

$$J_t=\begin{pmatrix}1 & t & \frac{t^2}{2} & \dots & \frac{t^n}{n!}\\ 0 & 1 & t & \dots & \frac{t^{n-1}}{(n-1)!}\\ 0 & 0 & 1 & \dots & \frac{t^{n-2}}{(n-2)!}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 1\end{pmatrix}.$$

Now set

$$\Phi(x)=\int_0^{+\infty}p_t(x;0)\,dt,$$

the 0-potential related to U_n (0 denotes the origin $(0,\dots,0)$). In order to make this expression more explicit, we require the duality relationship stated below.

LEMMA 8. *For all* $x=(x_0,\dots,x_n)\in\mathbb{R}^{n+1}$ *put* $x^*=(x_0,-x_1,x_2,\dots,(-1)^n x_n)$. *For all* $x,y\in\mathbb{R}^{n+1}$ *we have*

$$p_t(x;y)=p_t(y^*;x^*). \tag{18}$$

PROOF. In view of (17) we have to verify that the following equality holds:

$$(y-xJ_t)A_t(y-xJ_t)^T=(x^*-y^*J_t)A_t(x^*-y^*J_t)^T.$$

This one is the result of an elementary computation that leads to the following relation:

$$J_tA_tJ_t^T=-A_{-t}. \qquad \square$$

Therefore,

$$\Phi(x)=\int_0^{+\infty}\frac{\gamma}{t^{\nu+1}}\exp\left(-\sum_{0\le i,\,j\le n}(-1)^{i+j}\frac{a_{ij}}{t^{i+j+1}}x_ix_j\right)dt. \tag{19}$$

It is well known that

$$\mathscr{D}\Phi = -\delta_0. \tag{20}$$

Finally, notice that each component of U_n has a scaling property such that for any $\mu > 0$,

$$(X_k(\mu\,\cdot))_{0\le k\le n} \overset{\text{law}}{=} (\mu^{k+1/2} X_k(\cdot))_{0\le k\le n}.$$

This property implies the following identity:

$$p_t(x_0, x_1, \dots, x_n; 0) = \frac{1}{t^{\nu+1}} p_t\left(\frac{x_0}{t^{1/2}}, \frac{x_1}{t^{3/2}}, \dots, \frac{x_n}{t^{n+1/2}}; 0\right).$$

All scaling factors are different so this leads us to change the Euclidean norm on $\mathbb{R}^{n+1}$ into the application N defined by

$$N(x) = \max_{0\le i\le n} |x_i|^{1/(2i+1)}.$$

4.2. *Hitting probabilities.* We wish to estimate the following hitting probabilities of a ball for the process U_n:

$$p(x, T, R) = \mathbb{P}_x\{\exists\ t > T\colon \|U_n(t)\| \le R\},$$
$$p(x, T_1, T_2, R) = \mathbb{P}_x\{\exists\ t \in [T_1, T_2]\colon \|U_n(t)\| \le R\}.$$

To do this, we shall need some estimates concerning the potential Φ.

PROPOSITION 9. (i) *The potential Φ is a C^∞-function over $\mathbb{R}^{n+1} \setminus \{0\}$.*

(ii) *There exist some positive constants a, b such that for any $x \in \mathbb{R}^{n+1} \setminus \{0\}$ we have*

$$\frac{a}{N(x)^{2\nu}} \le \Phi(x) \le \frac{b}{N(x)^{2\nu}}. \tag{21}$$

(iii) *There exist some positive constants c, λ such that for any $x \in \mathbb{R}^{n+1} \setminus \{0\}$ and any $T > 0$ we have*

$$\int_T^{+\infty} p_t(x; 0)\, dt \ge \frac{c}{T^\nu} \exp\left[-\lambda\left(\frac{N(x)^2}{T} \vee \left(\frac{N(x)^2}{T}\right)^{2n+1}\right)\right]. \tag{22}$$

PROOF. (i) The first assertion of Proposition 9 is an easy fact, and its proof is left to the reader.

(ii) Since the matrix A_1 is positive definite, it is clear that there exist some positive constants λ_1 and λ_2 such that

$$\lambda_1 \sum_{i=0}^n x_i^2 \le \sum_{0\le i,j\le n} a_{ij} x_i x_j \le \lambda_2 \sum_{i=0}^n x_i^2.$$

Hence by (19),

$$\begin{aligned}\Phi(x) &\le \int_0^{+\infty} \frac{\gamma}{t^{\nu+1}} \exp\left(-\lambda_1 \sum_{i=0}^{n} \frac{x_i^2}{t^{2i+1}}\right) dt\\ &\le \min_{0\le i\le n} \int_0^{+\infty} \frac{\gamma}{t^{\nu+1}} \exp\left(-\lambda_1 \frac{x_i^2}{t^{2i+1}}\right) dt\\ &\le b \min_{0\le i\le n} |x_i|^{-2\nu/(2i+1)} = \frac{b}{N(x)^{2\nu}}\end{aligned}$$

with

$$b = \max_{0\le i\le n} \int_0^{+\infty} \frac{\gamma}{t^{\nu+1}} \exp\left(-\frac{\lambda_1}{t^{2i+1}}\right) dt.$$

On the other hand,

$$\begin{aligned}\Phi(x) &\ge \int_0^{+\infty} \frac{\gamma}{t^{\nu+1}} \exp\left(-\lambda_2 \sum_{i=0}^{n} \frac{x_i^2}{t^{2i+1}}\right) dt\\ &\ge \int_0^{+\infty} \frac{\gamma}{t^{\nu+1}} \exp\left(-\lambda_2 \sum_{i=0}^{n} \left(\frac{N(x)^2}{t}\right)^{2i+1}\right) dt\\ &\ge \frac{a}{N(x)^{2\nu}}\end{aligned}$$

with

$$a = \int_0^{+\infty} \frac{\gamma}{t^{\nu+1}} \exp\left(-\lambda_2 \sum_{i=0}^{n} \frac{1}{t^{2i+1}}\right) dt.$$

This proves fact (21).

(iii) Checking (22) is quite similar. Indeed, we have

$$\begin{aligned}\int_T^{+\infty} p_t(x;0)\,dt &\ge \int_T^{+\infty} \frac{\gamma}{t^{\nu+1}} \exp\left(-\lambda_2 \sum_{i=0}^{n} \left(\frac{N(x)^2}{t}\right)^{2i+1}\right) dt\\ &\ge \exp\left(-\lambda_2 \sum_{i=0}^{n} \left(\frac{N(x)^2}{T}\right)^{2i+1}\right) \int_T^{+\infty} \frac{\gamma}{t^{\nu+1}}\,dt.\end{aligned}$$

Finally, observing that $\sum_{i=0}^{n} X^{2i+1} \le (n+1)\max(X, X^{2n+1})$ for any $X \ge 0$, prove (22) with the choices $\lambda = (n+1)\lambda_2$ and $c = \gamma/\nu$. $\square$

The following proposition is an easy consequence of Itô's rule with the aid of (20), as well as (21).

PROPOSITION 10. *Put* $\tau_R = \inf\{t > 0\colon \|U_n(t)\| < R\}$ *with* $\inf\varnothing = +\infty$ *and let* $\|x\| > R$. *Then, under* $\mathbb{P}_x$, $(\Phi(U_n(t\wedge\tau_R)))_{t\ge0}$ *is a continuous bounded martingale with respect to the Brownian filtration.*

Doob's optional sampling theorem yields immediately

$$\Phi(x) = \mathbb{E}_x[\Phi(U_n(\tau_R)),\ \tau_R < +\infty]$$

and as a result, the corollary.

COROLLARY 11. *We have for any x such that $||x|| > R$*

$$\frac{\Phi(x)}{\sup_{||y||=R} \Phi(y)} \le \mathbb{P}_x\{\tau_R < +\infty\} \le \frac{\Phi(x)}{\inf_{||y||=R} \Phi(y)}. \tag{23}$$

Now, we are going to deal with the probability $p(x, T, R)$. This one has the following properties.

PROPOSITION 12. *There exist some constants $\alpha, \beta > 0$, $\eta \in (0, 1)$ such that if $R \le T^{1/2} \wedge T^{n+1/2}$,*

$$p(x, T, R) \le \beta\left(\frac{R^2 \vee R^{2/(2n+1)}}{T}\right)^{\nu} \tag{24}$$

and if $||x|| \le R \le \eta(T^{1/2} \wedge T^{n+1/2})$,

$$p(x, T, R) \ge \alpha\left(\frac{R^2 \vee R^{2/(2n+1)}}{T}\right)^{\nu}. \tag{25}$$

PROOF. Due to the Markov property, we get

$$\begin{aligned} p(x, T, R) &= \mathbb{E}_x[\mathbb{P}_{U_n(T)}\{\tau_R < +\infty\}] \\ &= \mathbb{P}_x\{||U_n(T)|| \le R\} + \mathbb{E}_x[\mathbb{P}_{U_n(T)}\{\tau_R < +\infty\},\ ||U_n(T)|| > R]. \end{aligned}$$

From this we obtain

$$p(x, T, R) \le P_x\{||U_n(T)|| \le R\} + \frac{\mathbb{E}_x[\Phi(U_n(T))]}{\inf_{||y||=R} \Phi(y)}, \tag{26}$$

$$\begin{aligned} p(x, T, R) \ge{}& \frac{1}{\sup_{||y||=R} \Phi(y)} \mathbb{E}_x[\Phi(U_n(T))] \\ &- \mathbb{E}_x[\Phi(U_n(T)), ||U_n(T)|| \le R]. \end{aligned} \tag{27}$$

So we have to evaluate three terms, namely $\mathbb{P}_x\{||U_n(T)|| \le R\}$, $\mathbb{E}_x[\Phi(U_n(T))]$ and $\mathbb{E}_x[\Phi(U_n(T)), ||U_n(T)|| \le R]$.

In the sequel the c_i, $i \in \{1, \dots, 18\}$ will denote some positive constants that we shall not explicitly write.

The first term is easy to estimate. By (16),

$$\mathbb{P}_x\{||U_n(T)|| \le R\} = \int_{||y||\le R} p_T(x;y)\,dy \le \int_{||y||\le R} p_T(0;0)\,dy = c_1 \frac{R^{n+1}}{T^{\nu+1}}. \tag{28}$$

The second term can be evaluated. From (16) and the Chapman–Kolmogorov equation,

$$\begin{aligned}
\mathbb{E}_x[\Phi(U_n(T))] &= \int_{\mathbb{R}^{n+1}} p_T(x;y)\,dy \int_0^{+\infty} p_t(y;0)\,dt \\
&= \int_0^{+\infty} dt \int_{\mathbb{R}^{n+1}} p_T(x;y) p_t(y;0)\,dy \\
&= \int_0^{+\infty} p_{t+T}(x;0)\,dt \\
&\le \gamma \int_T^{+\infty} \frac{dt}{t^{\nu+1}} = \frac{c_2}{T^\nu}.
\end{aligned} \tag{29}$$

On the other hand, referring to (22),

$$\mathbb{E}_x[\Phi(U_n(T))] \ge \frac{c}{T^\nu} \exp\left[-\lambda\left(\frac{N(x)^2}{T} \vee \left(\frac{N(x)^2}{T}\right)^{2n+1}\right)\right],$$

so that if $||x|| \le R \le T^{1/2} \wedge T^{n+1/2}$, then

$$N(x) \le R \vee R^{1/(2n+1)} \le \sqrt{T},$$

which implies

$$\mathbb{E}_x[\Phi(U_n(T))] \ge \frac{c_3}{T^\nu}. \tag{30}$$

Bounding from above, the third expectation needs more work. Indeed, by (9),

$$\begin{aligned}
&\mathbb{E}_x[\Phi(U_n(T)), ||U_n(T)|| \le R] \\
&\quad = \int_{||y||\le R} p_T(x;y)\Phi(y)\,dy \\
&\quad \le \frac{\gamma}{T^{\nu+1}} \int_{\bigcap_{i=0}^n \{y:\, |y_i| \le R\}} \Phi(y)\,dy \\
&\quad = \gamma \int_{\bigcap_{i=0}^n \{y:\, |y_i| \le R/(T^{i+1/2})\}} \Phi(y_0 T^{1/2}, y_1 T^{3/2}, \dots, y_n T^{n+1/2})\,dy \\
&\quad \le \gamma \int_{\bigcap_{i=0}^n \{y:\, |y_i| \le R/(T^{i+1/2})\}} b N(y_0 T^{1/2}, y_1 T^{3/2}, \dots, y_n T^{n+1/2})^{-2\nu}\,dy \\
&\quad = \frac{c_4}{T^d} \int_{\bigcap_{i=0}^n \{y:\, |y_i| \le R/(T^{i+1/2})\}} \frac{dy}{N(y)^{2\nu}}.
\end{aligned}$$

Carrying out the change of variable defined as $y_i = u_i^{2i+1},\ i \in \{0, \dots, n\}$, we get

$$\begin{aligned}
\text{(31)}\qquad & \mathbb{E}_x[\Phi(U_n(T)), \|U_n(T)\| \le R] \\
& \le \frac{c_5}{T^\nu} \int_{\bigcap_{i=0}^n \{u : |u_i| \le (R^{1/(2i+1)})/\sqrt{T}\}} \frac{\prod_{i=1}^n u_i^{2i}}{\|u\|^{2\nu}}\, du \\
& \le \frac{c_5}{T^\nu} \int_{\bigcap_{i=1}^n \{u : |u_i| \le (R^{1/(2i+1)})/\sqrt{T}\}} \frac{du}{\|u\|^{n-1}} \quad \text{since } \forall\, i, |u_i| \le \|u\| \\
& \le \frac{c_6}{T^\nu} \int_0^{\max_{0 \le i \le n} (R^{1/(2i+1)})/\sqrt{T}} r\, dr \quad \text{with } r = \|u\| \\
\text{(32)}\qquad & = \frac{c_7}{T^\nu} \frac{R^2 \vee R^{2/(2n+1)}}{T}.
\end{aligned}$$

Now we are able to derive the inequalities (24) and (25). Indeed, invoking (26) together with (21), (28) and (29) yields

$$p(x, T, R) \le c_1 \frac{R^{n+1}}{T^{\nu+1}} + \frac{c_2}{\alpha} \left(\frac{R^2 \vee R^{2/(2n+1)}}{T} \right)^\nu.$$

Since $(2(\nu+1)/(2n+1)) < n+1 < 2(\nu+1)$, we get

$$R^{n+1} \le \left(R^2 \vee R^{2/(2n+1)}\right)^{\nu+1}.$$

Therefore, $R \le T^{1/2} \wedge T^{n+1/2}$ implies

$$\frac{R^{n+1}}{T^{\nu+1}} \le \left(\frac{R^2 \vee R^{2/(2n+1)}}{T} \right)^{\nu+1} \le \left(\frac{R^2 \vee R^{2/(2n+1)}}{T} \right)^{\nu}$$

so that

$$p(x, T, R) \le \beta \left(\frac{R^2 \vee R^{2/(2n+1)}}{T} \right)^\nu$$

with $\beta = c_1 + c_2/\alpha$.

The inequality (25) can be checked by using (21), (27), (30) and (32) taken altogether. In fact, if $\|x\| \le R \le T^{1/2} \wedge T^{n+1/2}$, then

$$\mathbb{E}_x[\Phi(U_n(T)), \|U_n(T)\| > R] \ge \frac{c_3}{T^\nu} \left(1 - \frac{c_7}{c_3} \frac{R^2 \vee R^{2/(2n+1)}}{T} \right).$$

This last expression is greater than α/T^ν provided that η and α are chosen such that $\eta^2 \vee \eta^{2/(2n+1)} < c_3/(2c_7)$ and $\alpha = c_3/2$. The proof of (25) is complete. □

Now, let us consider the quantity $p(x, T_1, T_2, R)$. It is easy to deduce from proposition (12) the following one.

PROPOSITION 13. *There exist some constants* $\alpha', \eta' \in (0,1)$, $\delta' \in (1,+\infty)$ *such that if* $\|x\| \le R \le \eta'(T_1^{1/2} \wedge T_1^{n+1/2})$ *and* $T_2 \ge \delta' T_1$, *then*

$$p(x,T_1,T_2,R) \ge \alpha'\left(\frac{R^2 \vee R^{2/(2n+1)}}{T}\right)^{\nu} = \begin{cases} \alpha'\left(\dfrac{R}{\sqrt{T_1}}\right)^{2\nu}, & \text{if } R \ge 1, \\ \alpha'\left(\dfrac{R^{1/(2n+1)}}{\sqrt{T_1}}\right)^{2\nu}, & \text{if } R \le 1. \end{cases} \tag{33}$$

PROOF. The following lower bounds are obvious:

$$\begin{aligned} p(x,T_1,T_2,R) &\ge p(x,T_1,R) - p(x,T_2,R) \\ &\ge \alpha\left(\frac{R^2 \vee R^{2/(2n+1)}}{T_1}\right)^{\nu} - \beta\left(\frac{R^2 \vee R^{2/(2n+1)}}{T_2}\right)^{\nu} \\ &\ge \alpha\left(\frac{R^2 \vee R^{2/(2n+1)}}{T_1}\right)^{\nu}\left[1 - \frac{\beta}{\alpha}\left(\frac{T_1}{T_2}\right)^{\nu}\right], \end{aligned}$$

which is greater than $\alpha'((R^2 \vee R^{2/(2n+1)})/T_1)^{\nu}$ if $\alpha', \delta', T_1, T_2$ are chosen such that $\delta' \ge (2\beta/\alpha)^{1/\nu}$, $\alpha' = \alpha/2$ and $T_2 \ge \delta' T_1$. □

4.3. *Integral tests.* Here we give a characterization of the functions for which the event $\{U_n(t) > f(t)\}$ is asymptotically realized as $t \to 0^+$ or $t \to +\infty$ with probability 1.

THEOREM 14. *Let* $f\colon [0,+\infty) \to [0,+\infty)$.

(i) *Suppose the function* $t \mapsto t^{1/2} f(t)$ *is nondecreasing and greater than* 1 *for large* t. *Then*

> $\|U_n(t)\| > t^{1/2} f(t)$ *for any large* t *with probability 1 or 0 according to whether the integral* $\int^{+\infty} (f(t))^{2\nu}\,(dt/t)$ *converges or diverges.*

(ii) *Suppose the function* $t \mapsto t^{n+1/2} f(t)$ *is nondecreasing and less than* 1 *for small* t. *Then*

> $\|U_n(t)\| > t^{n+1/2} f(t)$ *for any small* t *with probability 1 or 0 according to whether the integral* $\int_{0^+} (f(t))^{2\nu/(2n+1)}\,(dt/t)$ *converges or diverges.*

Before sketching the proof, we give an immediate corollary.

COROLLARY 15.

$$\liminf_{t \to +\infty} \left[\log t \log_2 t \cdots \log_{k-1} t (\log_k t)^{1+\varepsilon}\right]^{1/2\nu} \frac{\|U_n(t)\|}{t^{1/2}} = \begin{cases} 0, & \text{if } \varepsilon \le 0 \\ +\infty, & \text{if } \varepsilon > 0, \end{cases}$$

$$\liminf_{t\to 0^+}\left[\log\frac{1}{t}\log_2\frac{1}{t}\cdots\log_{k-1}\frac{1}{t}\left(\log_k\frac{1}{t}\right)^{1+\varepsilon}\right]^{(2n+1)/2\nu}\frac{\|U_n(t)\|}{t^{n+1/2}}$$
$$=\begin{cases}0, & \text{if } \varepsilon \le 0,\\ +\infty, & \text{if } \varepsilon > 0,\end{cases}$$

with the notation $\log_k = \log\log_{k-1}$.

In particular the process U_n *is transient.*

PROOF OF THEOREM 14. The proof of this theorem is classical. We will only check the first assertion. The proof of the second one is quite similar. Here, we write $\mathbb{P}$ for $\mathbb{P}_0$.

Suppose at first the integral $\int^{+\infty}(f(t))^{2\nu}(dt/t)$ is convergent. We get, by virtue of (24),

$$\begin{aligned}
&\mathbb{P}\{\exists\ (t_k)_{k\ge 0}\nearrow +\infty\colon \|U_n(t_k)\| \le t_k^{1/2}f(t_k)\}\\
&\quad= \mathbb{P}\{\forall\ N\ge 1,\ \exists\ k\ge N,\ \exists\ t\in[2^{k-1},2^k]\colon \|U_n(t)\|\le t^{1/2}f(t)\}\\
&\quad\le \mathbb{P}\{\forall\ N\ge 1,\ \exists\ k\ge N,\ \exists\ t\ge 2^{k-1}\colon \|U_n(t)\|\le 2^{k/2}f(2^k)\}\\
&\quad\le c_8\inf_{N\ge 1}\sum_{k\ge N}(f(2^k))^{2\nu}.
\end{aligned}$$

Since the function $t\mapsto t^{1/2}f(t)$ is nondecreasing for large t, we get for large k,

$$\begin{aligned}
(f(2^k))^{2\nu} &\le \frac{1}{(2^k)^{d+1}}\int_{2^k}^{2^{k+1}}(t^{1/2}f(t))^{2\nu}\,dt\\
&\le c_9\int_{2^k}^{2^{k+1}}(t^{1/2}f(t))^{2\nu}\frac{dt}{t^{d+1}}\\
&= c_9\int_{2^k}^{2^{k+1}}(f(t))^{2\nu}\frac{dt}{t},
\end{aligned}$$

which proves that

$$\mathbb{P}\{\exists\ (t_k)_{k\ge 0}\nearrow +\infty\colon \|U_n(t_k)\|\le t_k^{1/2}f(t_k)\} = 0.$$

Hence

$$\mathbb{P}\{\|U_n(t)\| > t^{1/2}f(t)\quad\text{for any large } t\} = 1.$$

To check the divergence part, suppose the integral $\int^{+\infty}(f(t))^{2\nu}(dt/t)$ diverges and introduce the event

$$\begin{aligned}
T_k &= \{t\in[\delta'^k,\delta'^{k+1}]\colon \|U_n(t)\|\le \delta'^{k/2}f(\delta'^k)\},\\
D_k &= \{\exists\ t\colon t\in T_k\},
\end{aligned}$$

where δ' is defined in Proposition 13.

We have by virtue of (33),

$$\begin{aligned}\mathbb{P}(D_k) &\geq c_{10} f(\delta'^k)^{2\nu} \\ &\geq c_{11} \int_{\delta'^{k-1}}^{\delta'^k} (t^{1/2} f(t))^{2\nu}\, dt \\ &\geq c_{12} \int_{\delta'^{k-1}}^{\delta'^k} (f(t))^{2\nu} \frac{dt}{t},\end{aligned}$$

so that

$$\sum_{k\geq 1} \mathbb{P}(D_k) = +\infty.$$

We wish to use the Borel–Cantelli lemma. Since the events D_k are not independent, we refer to the more general following version (see, e.g., [16], page 65): *if*

$$\sum_{k\geq 1} \mathbb{P}(D_k) = +\infty$$

and if there is a constant $C > 0$ *such that for all integers* j, k *such that* $|k - j| \geq 2$,

$$\mathbb{P}(D_j \cap D_k) \leq C\mathbb{P}(D_j)\mathbb{P}(D_k),$$

then

$$\mathbb{P}\Bigl(\limsup_{k\geq 1} D_k\Bigr) > 0.$$

Let us study $\mathbb{P}(D_j \cap D_k)$ for $k \geq j + 2$. Define the following stopping times:

$$\tau_j = \begin{cases} \inf T_j, & \text{if } T_j \neq \varnothing, \\ +\infty, & \text{if } T_j = \varnothing. \end{cases}$$

Using the strong Markov property of the process U_n and remarking that $\tau_j \leq \delta'^{j+1}$ as well as $||U_n(\tau_j)|| \leq \delta'^{j/2} f(\delta'^j) < \delta'^{k/2} f(\delta'^k)$ when $\tau_j < +\infty$, we get

$$\begin{aligned}\mathbb{P}(D_j \cap D_k) &= \mathbb{P}\{\tau_j < +\infty, \tau_k < +\infty\} \\ &= \mathbb{E}\bigl[\mathbb{1}_{\{\tau_j<+\infty\}} \mathbb{P}_{U_n(\tau_j)}\bigl\{\exists\, t \in [\delta'^k - \tau_j, \delta'^{k+1} - \tau_j]: \\ &\qquad\qquad ||U_n(t)|| < \delta'^{k/2} f(\delta'^k)\bigr\}\bigr] \\ &\leq \mathbb{P}\{\tau_j < +\infty\} \sup_{||x||\leq \delta'^{j/2} f(\delta'^j)} \mathbb{P}_x\bigl\{\exists\, t \geq \delta'^k - \delta'^{j+1}: \\ &\qquad\qquad ||U_n(t)|| < \delta'^{k/2} f(\delta'^k)\bigr\}.\end{aligned}$$

From (24) we deduce that

$$\mathbb{P}(D_j \cap D_k) \leq \beta \mathbb{P}(D_j) \biggl(\frac{\delta'^k f(\delta'^k)^2}{\delta'^k - \delta'^{j+1}}\biggr)^{\nu} \leq \beta \biggl(\frac{\delta'}{\delta' - 1}\biggr)^{\nu} \mathbb{P}(D_j) f(\delta'^k)^{2\nu}.$$

On the other hand, we know from (33) that

$$f(\delta'^k)^{2\nu} \leq \frac{1}{\alpha'} \mathbb{P}(D_k).$$

Hence if $k \geq j+2$,

$$\mathbb{P}(D_j \cap D_k) \leq c_{13}\mathbb{P}(D_j)\mathbb{P}(D_k).$$

This last inequality allows us to assert that

$$\begin{aligned}
0 &< \mathbb{P}\{\forall\ N \geq 1,\ \exists\ k \geq N,\ \exists\ t \in [\delta'^k, \delta'^{k+1}]\colon \|U_n(t)\| \leq \delta'^{k/2} f(\delta'^k)\} \\
&\leq \mathbb{P}\{\forall\ N \geq 1,\ \exists\ k \geq N,\ \exists\ t \in [\delta'^k, \delta'^{k+1}]\colon \|U_n(t)\| \leq t^{1/2} f(t)\} \\
&\leq \mathbb{P}\{\exists\ (t_k)_{k\geq 0} \nearrow +\infty\colon \|U_n(t_k)\| \leq t_k^{1/2} f(t_k)\}.
\end{aligned}$$

Finally, since the tail σ-field of Brownian motion is trivial, the last probability in fact equals one. □

5. Addendum. In this section, we state some results about the multidimensional analogues of the results of the previous sections.

Let $(B^d(t))_{t\geq 0}$ be the d-dimensional Brownian motion starting at 0, that is,

$$B^d = (B^{(1)}, \ldots, B^{(d)}),$$

where $B^{(i)}$, $1 \leq i \leq d$, are independent linear Brownian motions, and set

$$X_k^d(t) = \int_0^t \frac{(t-s)^k}{k!}\, dB^d(s),$$
$$U^d = (B^d, X_1^d, \ldots, X_n^d).$$

All the previous results may be extended to the Markov process U^d. We will briefly state the corresponding results. We will also write out some integral tests concerning the non-Markov process X_n^d.

Let us introduce some other notation. Recall that

$$\gamma_n = \frac{1}{(2n+1)(n!)^2}, \qquad \nu = \tfrac{1}{2}(d+1)^2 - 1.$$

Set

$$Y_n^d(t) = \frac{1}{\sqrt{\gamma_n}\, t^{n+1/2}}\, X_n^d(t),$$
$$\Phi_d(x) = \int_{\{\|y\|>x\}} \exp\left(-\frac{1}{2}\|y\|^2\right) \frac{dy}{(2\pi)^{d/2}}.$$

The Gaussian process Y_n^d has independent components with unit variances. Elementary computations show that the increments of each component of Y_n^d satisfy the following property (denote by Y_n any of these components).

There exists some positive constants c_{14} *and* c_{15} *such that for all* $0 < s < t$,

$$c_{14}\frac{t-s}{t} \leq [\mathbb{E}(Y_n(t) - Y_n(s))^2]^{1/2} \leq c_{15}\frac{t-s}{s}. \tag{34}$$

In the next theorem the local asymptotic lower class for the process U^d is characterized.

THEOREM 16. *Let* $f\colon(0,+\infty)\to(0,+\infty)$ *be a function.*

(i) *If the function* $t\mapsto t^{1/2}f(t)$ *is nondecreasing near* $+\infty$, *then*

$$\int^{+\infty} f(t)^{\nu d-1}\,\frac{dt}{t}\begin{Bmatrix}<\\=\end{Bmatrix}+\infty \quad\Rightarrow\quad \mathbb{P}\{||U^d(t)||\geq t^{1/2}f(t)\ \textit{for large}\ t\}=\begin{Bmatrix}1\\0\end{Bmatrix}.$$

(ii) *If the function* $t\mapsto t^{n+1/2}f(t)$ *is nondecreasing near* 0, *then*

$$\int_{0^+} f(t)^{(\nu d-1)/(2n+1)}\,\frac{dt}{t}\begin{Bmatrix}<\\=\end{Bmatrix}+\infty$$
$$\Rightarrow\quad \mathbb{P}\{||U^d(t)||\geq t^{n+1/2}f(t)\ \textit{for small}\ t\}=\begin{Bmatrix}1\\0\end{Bmatrix}.$$

The proof of this characterization is quite similar to the one associated with the case $d=1$.

Let us now consider the Gaussian process X_n^d in the case $d\geq 2$. Kolokoltsov [8] recently obtained by an elementary method the following particular result:

$$\textit{for any}\ \beta<\frac{3}{2}-\frac{1}{d},\ \liminf_{t\to+\infty}\frac{||X_1^d(t)||}{t^\beta}=+\infty.$$

The reader will also find in [8] some other physical motivations of this study. In fact, we have the following integral test.

THEOREM 17. *Suppose* $d\geq 2$. *Let* $f\colon(0,+\infty)\to(0,+\infty)$ *be a function that does not increase near* $+\infty$. *Then*

$$\int^{+\infty} f(t)^{d-1}\,\frac{dt}{t}\begin{Bmatrix}<\\=\end{Bmatrix}+\infty \quad\Rightarrow\quad \mathbb{P}\{||X_n^d(t)||\geq t^{n+1/2}f(t)\ \textit{for large}\ t\}=\begin{Bmatrix}1\\0\end{Bmatrix}.$$

The same test holds near 0 *with a nondecreasing function* f.

Since the process X_n^d is not Markovian, the previous techniques cannot be used. However, X_n^d is a Gaussian process, and the general results of [9] can be applied. Indeed, due to (34), the following estimate may for instance be written as in [9], Lemma 4.

There exists a constant c_{16} *such that for small positive* x,

$$\mathbb{P}\Big\{\inf_{t\in[a,b]}\|Y^d(t)\|<x\Big\}\leq c_{16}\frac{b-a}{a}x^{d-1}.$$

After that, all the techniques described in [9] may work in the present situation. We omit the details.

REMARK. Independently, Khoshnevisan and Shi have recently derived the same test in the case $n=1$. Their method is quite different from ours and hinges on some specifical properties of the primitive of B^d [6]. They have also extended this test to integrated Brownian sheet [7]. See also [1].

We conclude this work by considering the question of regular points for the process X_n^d.

Let $f: (0, +\infty) \to (0, +\infty)$ be a function such that $x \mapsto x^{-1} f(x)$ is increasing near 0 and set

$$T = \left\{(x_1, \dots, x_d) \in \mathbb{R}^d \colon x_d \geq 0, \sum_{i=1}^{d-1} x_i^2 \leq f(x_d)^2\right\}.$$

We wish to provide a necessary and sufficient condition on f which allows us to decide whether or not the origin is a regular point for T. That is to say, we wish to decide whether

$$\mathbb{P}\{X_n^d(t) \in T \text{ for infinitly many small } t\} = 1 \text{ or } 0.$$

For Brownian motion, the relationship between the local asymptotic lower classes and regular points for T is well known; see, for example, [5], page 261. This is the result of the independence of the components of B^d. The same is true for the process X_n^d and the result can be stated as follows.

THEOREM 18. *Suppose $d \geq 3$. Then*

> 0 *is a regular point for T if and only if the integral $\int_{0^+} (f(x)/x)^{d-2} (dx/x)$ is divergent. If $d = 2$,* 0 *is a regular point for T.*

We refer the reader to [12] for some further results about regular points for the process U_n.

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