The Annals of Probability
1998, Vol. 26, No. 4, 1874–1912

FIRST ORDER CORRECTION FOR THE HYDRODYNAMIC LIMIT OF SYMMETRIC SIMPLE EXCLUSION PROCESSES WITH SPEED CHANGE IN DIMENSION $d \geq 3$

By Elise Janvresse

Université de Rouen

The hydrodynamic limit of the symmetric simple exclusion process with speed change is given by a diffusive equation in the appropriate scale. Following the nongradient method introduced by Varadhan and the Navier–Stokes methods developed by Yau, we prove that in the same scale, the next order correction is given by a third order equation for dimension $d \geq 3$.

1. Introduction. We consider the configuration space consisting of particles in $\mathbb{Z}^d$ and we suppose that there is a hard core interaction so that two particles cannot occupy the same site. By imposing some probability distribution on the configuration space we obtain a model of lattice gases in equilibrium. This lattice gas model can be turned into a dynamical one if some rules for the motion of particles are given.

We consider the symmetric simple exclusion process with speed change. This model can be informally described as follows: A particle at site x jumps to a neighboring site $x+y$ chosen with equiprobability at a rate which depends on the presence of particles at sites $x-y$ and $x+2y$. If the site $x+y$ is occupied, the jump is suppressed. The generator of this process is given, for $\alpha > -1/2$ fixed, by

$$Lf(\eta) = \sum_{x,\,|y|=1} r_{x,\,x+y}(\eta)\,\big(f(\eta^{x,\,x+y}) - f(\eta)\big), \tag{1.1}$$

where $r_{x,\,x+y}(\eta) = \eta(x)(1-\eta(x+y))(1+\alpha\eta(x-y)+\alpha\eta(x+2y))$ and $\eta^{x,\,x+y}$ is the configuration η after a particle jumped from site x to site $x+y$.

The total number of particles is the unique conserved quantity and for each $0 \leq \rho \leq 1$, there exists a translation invariant product probability measure denoted by ν_ρ that is invariant for the dynamics and for which the density of particles is ρ: ν_ρ is the product Bernoulli measure with parameter ρ. Moreover, since the process is symmetric, the measures are reversible.

We are interested in the evolution of the local density of particles. Let us fix the scale of the lattice to be N^{-1}. In order to have the particle moving a distance of order 1 in the macroscopic scale, we have to wait for a time of order N^2. So introducing a diffusive time scaling ($t' = tN^2$), we can check that the density follows the hydrodynamic equation

$$\partial_t \rho = \sum_i \partial^2_{u_i}(\rho(1+\alpha\rho)). \tag{1.2}$$

Received July 1997.

AMS 1991 *subject classifications*. Primary 60K35; secondary 82C22.

Key words and phrases. Infinite interacting particle systems, hydrodynamic limit, nongradient methods, symmetric simple exclusion process.

A fundamental question in mathematical physics is the derivation and interpretation of equations which are not scaling invariant and thus cannot be obtained from a scaling limit.

For example, in the context of asymmetric interacting particle systems, several interpretations have been proposed for the Navier–Stokes (NS) equation.

It is well known that the macroscopic evolution of the strictly asymmetric simple exclusion process (ASEP) under Euler rescaling is described (cf. [9]) by the first order quasilinear hyperbolic equation

$$\partial_t \rho + \sum_i \partial_{u_i}[\rho(1-\rho)] = 0. \tag{1.3}$$

In the context of ASEP, the equation that corresponds to the NS equation is

$$\partial_t \rho + \sum_i \partial_{u_i}[\rho(1-\rho)] = N^{-1} \sum_{i,j} \partial_{u_i}\big(a^{i,j}(\rho)\partial_{u_j}\rho\big), \tag{1.4}$$

where $a^{i,j}(\rho)$ is the diffusion coefficient depending on ρ.

This leads to the following three interpretations of the NS equation:

1. The NS equation (1.4) is the first order correction to the hydrodynamic equation (1.3): It describes the evolution of the density up to order N^{-1} (see [1, 7]).
2. Long time behavior: Equation (1.4) describes the evolution of the density up to time scale N^2. In diffusive scale, we obtain from (1.4),
$$\partial_t \rho + N \sum_i \partial_{u_i}[\rho(1-\rho)] = \sum_{i,j} \partial_{u_i}\big(a^{i,j}(\rho)\partial_{u_j}\rho\big).$$
Assuming that the initial data are constant along the drift direction, the diverging term vanishes.
3. Incompressible limit (see [2, 3]): Consider a small perturbation of a constant profile ($\rho = \rho_0 + N^{-1}u$) in the diffusive scale. For simplicity, take $\rho_0 = 1/2$. Otherwise a uniform motion has to be taken out. We find that u satisfies the equation
$$\partial_t u + 2\sum_e \partial_e u^2 = \sum_{i,j} \partial_i(a^{i,j}(1/2)\,\partial_j u).$$
Notice that the diffusion coefficient is computed at the equilibrium density 1/2 and is a constant in the equation.

Note that the second and third interpretations concern the behavior of the system under diffusive rescaling. The first interpretation is a statement on the process under Euler rescaling.

Let us turn to the symmetric process we introduced above. Consider the equation

$$\partial_t \rho^N = \Delta_u(\rho^N(1-\rho^N)) - \frac{\alpha}{N} \sum_{i,j=1}^{d} \partial^2_{u_i}\big(R_{ij}(\rho^N)\,\partial_{u_j}\rho^N\big), \tag{1.5}$$

where R_{ij} is a continuous function on $(0, 1)$. Since our process is symmetric, we already used diffusive rescaling to get the hydrodynamic limit, so the second and third interpretations may not be applied in our case. In this paper, we shall prove the first interpretation holds in dimension $d \geq 3$: Equation (1.5) describes the evolution of the density up to order N^{-1}. It is the first order correction to (1.2).

Our method is based on a nongradient system method (see [10]), a multi-scale analysis and a relative entropy argument (see [11]). The paper is organized as follows: In Section 2, we first recall the rigorous definitions of the symmetric simple exclusion process with speed change and state our main result. Its proof is outlined in Sections 3 and 8. Beside other problems, we need to prove a very strong control of the specific relative entropy: For the usual hydrodynamic limit, following the arguments in [3], we need to bound the specific relative entropy by $O(N^{-1})$ (Section 7). Unfortunately, this is not sufficient to identify the correction term and we need to bound the specific relative entropy by $O(N^{-2})$. Details will be described in Section 4 and the proofs can be found in Sections 7 and 9. In Section 10, we state the properties of the diffusion coefficient R and prove lemmas we assumed in Sections 8 and 9.

2. Statement of the result.

We consider the symmetric simple exclusion process with speed change on a lattice of size N in dimension $d \geq 3$ with periodic boundary conditions. We shall denote by T_N^d the d-dimensional torus with length N. The configurations of this process are given by

$$\eta = \{\eta(x);\ x \in \mathbb{Z}^d;\ \eta(x + Ne_i) = \eta(x)\},$$

where $(e_i)^j = \delta_{ij}$ and $\eta(x) = 1$ or 0, indicating if the site x is occupied or not.

The generator of this process is given, for $\alpha > -1/2$ fixed, by

$$L_N f(\eta) = \sum_{x,\, |y|=1} r_{x,\, x+y}(\eta)\, \big(f(\eta^{x,\, x+y}) - f(\eta)\big), \tag{2.1}$$

where $r_{x,\, x+y}(\eta) = \eta(x)(1 - \eta(x + y))(1 + \alpha\eta(x - y) + \alpha\eta(x + 2y))$ and

$$\eta^{x,\, x+y}(z) = \begin{cases} \eta(x + y), & z = x, \\ \eta(x), & z = x + y, \\ \eta(z), & \text{otherwise.} \end{cases}$$

The Bernoulli measures $\{\nu_\rho,\ 0 \leq \rho \leq 1\}$ are the invariant measures for this process.

Let $W_{0,\, e_i} = r_{0,\, e_i} - r_{e_i,\, 0}$ be the current between 0 and e_i. A simple computation gives that

$$W_{0,\, e_i} = (h_i - \tau_{e_i} h_i) - (v_i - 2\tau_{e_i} v_i + \tau_{2e_i} v_i)$$

with $h_i(\eta) = \eta_0 - \alpha\eta_{-e_i}\eta_{e_i} + 2\alpha\eta_{-e_i}\eta_0$ and $v_i(\eta) = \alpha\eta_{-e_i}\eta_0$.

We start the process with the inhomogeneous product measure

$$\mu_0^N(\eta) = Z_N^{-1}(\lambda_0) \exp\left[\sum_x \lambda_0\left(\frac{x}{N}\right)\eta(x)\right], \tag{2.2}$$

where $\lambda_0(u)$ is a smooth function on T^d (the d-dimensional torus of volume 1) and $Z_N(\lambda_0)$ is a normalization constant.

For a positive integer N and $T_0 > 0$, denote by P_N the probability measure on the path space $D([0, T_0], \{0, 1\}^{T_N^d})$ of the Markov process with generator L_N accelerated by N^2 and starting from μ_0^N, and denote by E_N the expectation with respect to P_N.

To the measure (2.2) corresponds an initial density profile

$$m_0(u) = \frac{\exp(\lambda_0(u))}{\exp(\lambda_0(u)) + 1}. \tag{2.3}$$

Notice that the initial profile m_0 is bounded away from 0 and 1: there exists $\delta > 0$ such that $\delta \le m_0(u) \le 1 - \delta$.

Denote by m the solution of the equation

$$\begin{aligned} &\partial_t m = \Delta_u \phi(m), \\ &m(0, \cdot) = m_0(\cdot), \end{aligned} \tag{2.4}$$

with $\phi(m) = m(1 + \alpha m)$. Since ϕ and m_0 are smooth, there exists a classical smooth solution to this equation.

This equation describes the macroscopic behavior of the symmetric simple exclusion process with speed change. More precisely, if $q^N(t, u)$ is defined as

$$q^N(t, u) := E_N[\eta_t([Nu])],$$

it follows that, for each fixed time t, $q^N(t, \cdot)$ converges weakly to $m(t, \cdot)$. In fact, for each fixed time t, $q^N(t, \cdot)$ converges pointwise to $m(t, \cdot)$ in each continuity point of $m(t, \cdot)$ (see Chapter 9 in [5]).

Our result describes the first order correction in this limit: For each integer N consider the equation

$$\begin{aligned} &\partial_t m^N = \Delta_u \phi(m^N) - \frac{\alpha}{N} \sum_{i,\, j=1}^{d} \partial_{u_i}^2 \bigl(R_{ij}(m^N)\, \partial_{u_j} m^N\bigr), \\ &m^N(0, \cdot) = m_0(\cdot), \end{aligned} \tag{2.5}$$

where R is a continuous function on $(0, 1)$ which will be defined later (cf. Section 10). We can understand (2.5) as a perturbation of (2.4) as follows.

For any fixed solution m of (2.4), consider the linear equation

$$\begin{aligned} &\partial_t S = \Delta_u[\phi'(m) S] - \alpha \sum_{i,\, j=1}^{d} \partial_{u_i}^2 (R_{ij}(m)\, \partial_{u_j} m), \\ &S(0, \cdot) = 0. \end{aligned} \tag{2.6}$$

Since $R(m)$ is only continuous, we consider the solution of (2.6) in the following weak sense: Fix $T > 0$ and consider the dual backward equation

$$\partial_s J(s, u) + \phi'(m(s, u))\, \Delta_u J(s, u) = 0, \tag{2.7}$$

with final condition $J(T, u) = J(u)$. It is easy to obtain the estimates

$$\sum_{i=1}^{d} \int_0^T ds \int_{T^d} du\, (\partial_{u_i} J(s, u))^2 \le C_1 \textstyle\int_{T^d} du\, J(u)^2 \exp(C_2 T),$$

$$(2.8)\qquad \sum_{i=1}^{d} \int_0^T ds \int_{T^d} du\, (\partial^2_{u_i, u_j} J(s, u))^2 \le C_3 \textstyle\int_{T^d} du (\partial_{u_i} J(u))^2 \exp(C_4 T)$$

for some finite constants C_1, C_2, C_3 and C_4 that depend only on the bounds on m and ϕ.

Denote by $\mathscr{H}_1$ the Hilbert space generated by $C^1(T^d)$ and the inner product $\langle \cdot, \cdot \rangle_1$ defined by

$$\langle F, G \rangle_1 = \sum_{i=1}^{d} \int_{T^d} (\partial_{u_i} F)(\partial_{u_i} G).$$

Let $\mathscr{H}_{-1}$ be the dual space of $\mathscr{H}_1$ with respect to $L^2(T^d)$. The mapping S: $[0, T] \to \mathscr{H}_{-1}$ is said to be a solution of (2.6) if for every J_0: $T^d \to \mathbb{R}$ smooth,

$$\langle S(t), J_0 \rangle = -\alpha \int_0^t ds \int du\, \partial^2_{u_i} J(s, u) \sum_{i,\, j=1}^{d} \big(R_{ij}(m_s(u))\, \partial_{u_j} m_s(u)\big),$$

where $J(s, u)$ is the solution of the backward equation (2.7) with final condition $J(t, u) = J_0(u)$. In view of estimates (2.8), it is not difficult to prove the existence of a unique solution to (2.6).

The main result of this paper is the following theorem.

THEOREM 2.1. *For any $t \le T_0$ we have*

$$N(q^N(t, \cdot) - m(t, \cdot)) \rightharpoonup S(t)$$

weakly in $\mathscr{H}_{-1}$.

3. Proof of Theorem 2.1. Denote by S: $[0, T_0] \times T^d \to \mathbb{R}$ the weak solution of the linear equation (2.6). Fix a time $0 \le T < T_0$ and a smooth function J_0: $T^d \to \mathbb{R}$. Denote by J: $\mathbb{R}_+ \times T^d \to \mathbb{R}$ the solution of the linear equation (2.7) with final condition $J(T, u) = J_0(u)$. We have to prove that

$$\lim_{N\to\infty} E_N\bigg[N^{1-d} \sum_x J\Big(T, \frac{x}{N}\Big) \Big[\eta_T(x) - m\Big(T, \frac{x}{N}\Big)\Big]\bigg] = \langle S_T, J_T \rangle.$$

For $\sigma > 0$, let R^σ be a smooth function on $[0, 1]$ converging to R uniformly on each compact subset of $(0, 1)$. This is possible since R is continuous in $(0, 1)$. Then there exists a smooth solution to the linear equation (2.6) (with the corresponding smoothed coefficient R^σ), which is denoted by S^σ: $\mathbb{R}_+ \times T^d \to \mathbb{R}$. We start computing the time derivative of

$$E_N\bigg[N^{1-d} \sum_x J\Big(T, \frac{x}{N}\Big) \Big[\eta_T(x) - m\Big(T, \frac{x}{N}\Big) - N^{-1} S^\sigma\Big(T, \frac{x}{N}\Big)\Big]\bigg].$$

A simple computation shows that it is equal to

$$
\begin{aligned}
&E_N\left[N^{1-d}\sum_x J\left(t,\frac{x}{N}\right)N^2 L_N\eta_t(x)\right]\\
&\quad - E_N\left[N^{1-d}\sum_{x,i}\phi'\left(m\left(t,\frac{x}{N}\right)\right)(\partial_{u_i}^2 J)\left(t,\frac{x}{N}\right)\eta_t(x)\right]\\
&\quad - N^{1-d}\sum_x J\left(t,\frac{x}{N}\right)\left\{\Delta_u\phi\left(m_t\left(\frac{x}{N}\right)\right)\right.\\
&\qquad\qquad\left. - \frac{\alpha}{N}\sum_{i,j}\partial_{u_i}^2\left\{R_{ij}^\sigma\left(m_t\left(\frac{x}{N}\right)\right)\partial_{u_j}m_t\left(\frac{x}{N}\right)\right\}\right\}\\
&\quad + N^{1-d}\sum_x\phi'\left(m\left(t,\frac{x}{N}\right)\right)(\partial_{u_i}^2 J)\left(t,\frac{x}{N}\right)m_t\left(\frac{x}{N}\right)+\varepsilon_N.
\end{aligned}
\tag{3.1}
$$

The last term in the above formula is the error we obtain when making a summation by parts in the sums. At the end of this section, we prove that ε_N is of $o_N(1)$. Moreover,

$$L_N\eta_t(x)=\sum_i\tau_x(W_{-e_i,0}-W_{0,e_i}),$$

where W_{0,e_i} is the current between 0 and e_i. Recall that

$$W_{0,e_i}=(h_i-\tau_{e_i}h_i)-(v_i-2\tau_{e_i}v_i+\tau_{2e_i}v_i)$$

with $h_i(\eta)=\eta_0-\alpha\eta_{-e_i}\eta_{e_i}+2\alpha\eta_{-e_i}\eta_0$ and $v_i(\eta)=\alpha\eta_{-e_i}\eta_0$. Hence, summing by parts and using Taylor expansion, we obtain that

$$
\begin{aligned}
&N^{1-d}\sum_x J\left(t,\frac{x}{N}\right)N^2L_N\eta_t(x)\\
&\quad = N^{1-d}\sum_{x,i}\partial_{u_i}^2 J_t\left(\frac{x}{N}\right)\tau_x h_i(\eta_t)-N^{-d}\sum_{x,i}\partial_{u_i}^3 J_t\left(\frac{x}{N}\right)\tau_x v_i+o_N(1).
\end{aligned}
$$

Therefore, making summations by parts, (3.1) is equal to

$$
\begin{aligned}
&E_N\left[N^{1-d}\sum_{x,i}\partial_{u_i}^2 J_t\left(\frac{x}{N}\right)\left\{\tau_x h_i(\eta_t)-\phi(m_t)-\phi'(m_t)\left(\eta_t(x)-m_t\left(\frac{x}{N}\right)\right)\right\}\right]\\
&\quad - N^{-d}\sum_{x,i}\partial_{u_i}^3 J_t\left(\frac{x}{N}\right)E_N(\tau_x v_i)\\
&\quad - N^{-d}\sum_{x,i}\alpha\,\partial_{u_i}^2 J_t\left(\frac{x}{N}\right)R_{ij}^\sigma\left(m_t\left(\frac{x}{N}\right)\right)\partial_{u_j}m_t\left(\frac{x}{N}\right)
\end{aligned}
$$

plus a small error of $o_N(1)$. The second line is the correction needed for the first line to converge to 0. By the law of large numbers (more precisely, using the hydrodynamical behavior of the exclusion process with speed change and, in particular, the classical one-block and two-block estimates), this line converges as N increases to ∞ to

$$-\alpha\sum_{i,j}\int_0^T dt\int du\,\partial^2_{u_i}J_t(u)\big[-2m_t(u)\delta_{ij}+R^{\sigma}_{ij}(m_t(u))\big]\,\partial_{u_j}m_t(u). \tag{3.2}$$

We now concentrate on the first line. Recall that $\phi(m)=m(1+\alpha m)$. Since the linear terms of the first line cancel, we may rewrite it as

$$\begin{aligned}\alpha E_N\Bigg[N^{1-d}\sum_{x,i}(\partial^2_{u_i}J)\left(t,\frac{x}{N}\right)\\ \times\Bigg\{\tau_x H_i(\eta_t)-m^2\left(t,\frac{x}{N}\right)\\ -2m\left(t,\tfrac{x}{N}\right)\left[\eta_t(x)-m\left(t,\tfrac{x}{N}\right)\right]\Bigg\}\Bigg]\end{aligned} \tag{3.3}$$

with $H_i(\eta)=2\eta(-e_i)\eta(0)-\eta(-e_i)\eta(e_i)$.

Notice that if $\alpha=0$, Theorem 2.2 is proved. Namely, there is no first order correction for the usual symmetric simple exclusion process. Hence, we suppose $\alpha\neq 0$.

At this point, following the approach of [4], we would like to replace the cylinder function $\tau_x H_i(\eta_t)$ by a function of the empirical measure. In order to do this, we need some notation.

For an integer l, denote by $\eta^l(x)$ the empirical density on a hypercube of length $2l+1$ centered at x:

$$\eta^l(x)=\frac{1}{|\Lambda_l|}\sum_{y;\,y-x\in\Lambda_l}\eta(y),$$

$$\Lambda_l=\big\{y\in\mathbb{Z}^d;\ -l\le y_i\le l,\ \text{ for } 1\le i\le d\big\}.$$

Furthermore, denote by $S_{i,l}(\eta)$ the conditional expectation given the density of particles on Λ_l of $H_i(\eta)=2\eta(-e_i)\eta(0)-\eta(-e_i)\eta(e_i)$:

$$S_{i,l}(\eta)=E\big[H_i(\eta)|\eta^l(0)\big].$$

Recall that the canonical measure on finite boxes is the uniform measure. In particular, $S_{i,l}(\eta)$ does not depend on m and can be explicitly computed. It is given by

$$S_{i,l}(\eta)=(\eta^l(0))^2-\frac{1}{|\Lambda_l|-1}\eta^l(0)(1-\eta^l(0)).$$

Since $S_{i,l}(\cdot)$ is a function of $\eta^l(0)$, for convenience we denote it sometimes by $S_l(\eta^l(0))$. Define the sequence $K(N)$ by

$$K=BN^{1/d},$$

where B is a positive integer independent of N which will increase to ∞ after N.

A summation by parts permits us to rewrite (3.3) as

$$\alpha\, E_N\biggl[N^{1-d}\sum_{x,\,i}(\partial^2_{u_i}J)\Bigl(t,\frac{x}{N}\Bigr) \times\Bigl\{\bigl[\tau_x H_i(\eta_t)-S_K(\eta_t^K(x))\bigr] + (\eta_t^K(x))^2 - m^2\Bigl(t,\frac{x}{N}\Bigr) - 2m\Bigl(t,\frac{x}{N}\Bigr)\Bigl[\eta_t^K(x)-m\Bigl(t,\frac{x}{N}\Bigr)\Bigr]\Bigr\}\biggr] + o_{N,\,B}(1).$$

Here $o_{N,\,B}(1)$ represents a term vanishing as $N\uparrow\infty$ and then $B\uparrow\infty$. The error made in the summation by parts is small since $K\ll\sqrt{N}$ and m and J are smooth functions. Moreover, the term $(|\Lambda_K|-1)^{-1}\eta^K(0)[1-\eta^K(0)]$ vanishes because $N^{1/d}\ll K$.

To keep notation simple, for each $1\le i\le d$ and each positive integer l, denote by $V_{i,\,l}(\eta)$ the cylinder function defined by

$$V_{i,\,l}(\eta)=H_i(\eta)-S_l(\eta^l(0)). \tag{3.4}$$

With this notation we may rewrite the last expectation as

$$\alpha E_N\biggl[N^{1-d}\sum_{x,\,i}(\partial^2_{u_i}J)\Bigl(t,\frac{x}{N}\Bigr)\Bigl\{\tau_x V_{i,\,K}(\eta_t)+\Bigl[\eta_t^K(x)-m\Bigl(t,\frac{x}{N}\Bigr)\Bigr]^2\Bigr\}\biggr] + o_{N,\,B}(1).$$

In view of (3.2), since m is bounded away from 0 and 1 and since $R^{\sigma}_{i,\,j}$ converges uniformly to $R_{i,\,j}$ on each compact subset of $(0,1)$, to conclude the proof of Theorem 1.2 it remains to prove the following lemmas.

LEMMA 3.1. *Recall that $K=BN^{1/d}$. For each $1\le i\le d$,*

$$\lim_{N\to\infty}\int_0^T dt\, E_N\biggl[N^{1-d}\sum_{x,\,i}(\partial^2_{u_i}J)\Bigl(t,\frac{x}{N}\Bigr)\tau_x V_{i,\,K}(\eta_t)\biggr] = \sum_{i,\,j}\int_0^T dt\int du\,(\partial^2_{u_i}J)(t,u)\bigl[R_{i,\,j}(m(t,u))-2m(t,u)\delta_{i,\,j}\bigr](\partial_{u_j}m)(t,u).$$

LEMMA 3.2. *Let K_N be a sequence increasing to ∞ faster than $N^{1/d}$ and slower than $N^{3/4}$ ($N^{1/d}\ll K_N\ll N^{3/4}$). Then*

$$\lim_{N\to\infty}\int_0^T dt\, E_N\biggl[N^{1-d}\sum_{x}\Bigl(\eta_t^{K_N}(x)-m\Bigl(t,\frac{x}{N}\Bigr)\Bigr)^2\biggr]=0.$$

It is this last lemma that forced us to choose the sequence $K(N)$ much larger than $N^{1/d}$.

CONJECTURE 3.1 (Summation by parts formula). Let $G, H\colon T^d \to \mathbb{R}$ be two smooth functions. Then

$$\lim_{N\to\infty} N^{1-d}\sum_x \left[(\partial_{u_i} G)\left(\frac{x}{N}\right)H\left(\frac{x}{N}\right) + (\partial_{u_i} H)\left(\frac{x}{N}\right)G\left(\frac{x}{N}\right)\right] = 0.$$

PROOF. Denote by F the product of the functions GH. With this notation, the sum in the statement can be rewritten as

$$N^{1-d}\sum_x \left[(\partial_{u_i} F)\left(\frac{x}{N}\right)\right] = N^{1-d}\sum_x \left[(\partial_{u_i} F)\left(\frac{x}{N}\right) - N\left\{F\left(\frac{x}{N}+\frac{e_i}{N}\right) - F\left(\frac{x}{N}\right)\right\}\right].$$

From Taylor's expansion at the second order, this expression is equal to

$$-2^{-1}N^{-d}\sum_x (\partial^2_{u_i} F)\left(\frac{x}{N}\right) + O(N^{-1}).$$

As N increases to ∞, this expression converges to

$$-2^{-1}\int du\,(\partial^2_{u_i} F)(u) = 0.$$

This concludes the proof of the conjecture. $\square$

4. Bounds on entropy and Dirichlet form. The strategy in proving the two lemmas of the previous section is based on the study of the time evolution of the relative entropy. Fix a reference product invariant measure ν_ρ on the configuration space $S_N^d = \{0,1\}^{T_N^d}$. For each time $t \ge 0$, let ψ_t^N be the density of the product measure $\nu^N_{m(t,\cdot)}$ with slowly varying density profile $m(t,\cdot)$ with respect to ν_ρ:

$$\psi_t^N(\eta) = \frac{d\nu^N_{m(t,\cdot)}}{d\nu_\rho} = \frac{1}{Z_t^N}\exp\left\{\sum_x \lambda\left(t,\frac{x}{N}\right)\eta_x\right\},$$

where

$$\lambda\left(t,\frac{x}{N}\right) = \log\left\{\frac{m(t,x/N)[1-\rho]}{\rho[1-m(t,x/N)]}\right\} \tag{4.1}$$

and Z_t^N is the normalizing constant

$$Z_t^N = \exp\left\{-\sum_x \log\frac{1-m(t,x/N)}{1-\rho}\right\}.$$

Let $f_t^N = d\nu^N_{m(0,\cdot)}P_t^N/d\nu_\rho$, where P_t^N denotes the semigroup of the Markov process with generator L_N accelerated by N^2. It is well known that f_t^N is the solution of

$$\begin{aligned} \partial_t f_t^N &= N^2 L_N^* f_t^N,\\ f_0^N &= \psi_0. \end{aligned} \tag{4.2}$$

Here L_N^* represents the adjoint of L_N in $L^2(\nu_\rho^N)$. Since the process is reversible w.r.t. ν_ρ^N, $L_N^* = L_N$. From now on we will omit the index N in f_t^N and ψ_t^N. Denote by $H_N(t)$ the entropy of $\nu_{m(0,\cdot)}^N P_t^N$ with respect to $\nu_{m(t,\cdot)}$:

$$H_N(t) = H\big(\nu_{m(0,\cdot)}^N P_t^N \big| \nu_{m(t,\cdot)}\big) = \int f_t \log \frac{f_t}{\psi_t}\, d\nu_\rho.$$

Notice that $H_N(0) = 0$. For each density $f\colon S_N^d \to \mathbb{R}_+$, denote by $D_N(f)$ the Dirichlet form, that is, the convex semicontinuous functional defined by

$$D_N(f) = -\int \sqrt{f} L_N \sqrt{f}\, d\nu_\rho.$$

PROPOSITION 4.1 (First entropy bound). *There exists a constant C such that for every $t < T_0$,*

$$\limsup_{N\to\infty} \left\{ N^{1-d} H_N(t) + \int_0^t N^{2-d} D_N(f_s)\, ds \right\} \le C.$$

The proof of Lemma 3.2 is based on this result. However, to prove Lemma 3.1 we need more: a bound on the entropy of $o(N^{d-1})$ instead of $O(N^{d-1})$. To obtain such a bound, we need to consider corrections of order $1/N^2$ in the density ψ_t.

Let us introduce some notation. Denote by $\mathscr{C}$ the space of cylinder functions. For each positive integer K and m in $\{0, 1/(\bar{K}^d), \dots, 1\}$ we denote by $\nu_{K,m}$ the canonical measure on $\{0,1\}^{\Lambda_K}$ with density m (i.e., the corresponding uniform measure). Let $\mathscr{G}$ be the linear space of cylinder functions that have mean zero with respect to all canonical measures on a sufficiently large box Λ_K:

$$\mathscr{G} = \big\{ g \in \mathscr{C};\ \nu_{K,m}[g] = 0 \text{ for some } K > 0 \text{ and all } m \big\}. \tag{4.3}$$

For functions F_i and G_i in $\mathscr{G}$, $1 \le i \le d$, for a time $0 \le t \le T_0$ and an integer N, define the density $\psi_t^{F,G}(\eta)$ with respect to the reference measure ν_ρ by

$$\begin{aligned}\psi_t^{F,G}(\eta) = \frac{1}{Z_t^{F,G}} \exp\bigg\{ &\sum_x \lambda\Big(t, \frac{x}{N}\Big)\eta_x - N^{-2} \sum_{x,i} \partial_{u_i}^2 \lambda\Big(t, \frac{x}{N}\Big) F_i(\tau_x \eta) \\ &- N^{-2} \sum_{x,i} \Big(\partial_{u_i} \lambda\Big(t, \frac{x}{N}\Big)\Big)^2 G_i(\tau_x \eta)\bigg\},\end{aligned}$$

where $Z_t^{F,G}$ is a normalizing constant. Denote by $H_N(f_t|\psi_t^{F,G})$ the entropy of $\nu_{m(0,\cdot)} P_t^N$ with respect to $\psi_t^{F,G}\, d\nu_\rho$:

$$H_N\big(f_t \big| \psi_t^{F,G}\big) = \int f_t \log \frac{f_t}{\psi_t^{F,G}}\, d\nu_\rho.$$

For each density $f\colon S_N^d \to \mathbb{R}_+$ and each $t \le 0$, denote by $D_N(f|\psi_t^{F,G})$ the positive convex lower semicontinuous functional defined by

$$D_N\big(f \big| \psi_t^{F,G}\big) = \int \psi_t^{F,G} \left\{ L_N \frac{f}{\psi_t^{F,G}} - \frac{f}{\psi_t^{F,G}} L_N \log \frac{f}{\psi_t^{F,G}} \right\} d\nu_\rho.$$

PROPOSITION 4.2 (Second entropy bound). *For every* $t < T_0$,

$$\inf_{F,\,G\in\mathscr{G}}\ \lim_{N\to\infty}\left\{N^{1-d}H_N\big(f_t\big|\psi_t^{F,G}\big)+\int_0^t N^{2-d}D_N\big(f_s\big|\psi_s^{F,G}\big)\,ds\right\}=0.$$

From Propositions 4.1 and 4.2, we can prove the following proposition:

PROPOSITION 4.3. $\lim_{N\to\infty} N^{1-d}H_N(t)=0.$

PROOF. We have

$$\begin{aligned}N^{2-d}\big(H_N(t)-H_N\big(f_t\big|\psi_t^{F,G}\big)\big)&=N^{2-d}\int f_t\left(\log\frac{f_t}{\psi_t}-\log\frac{f_t}{\psi_t^{F,G}}\right)d\nu_\rho\\&=N^{2-d}\int f_t\log\frac{\psi_t^{F,G}}{\psi_t}\,d\nu_\rho.\end{aligned}$$

Hence,

$$\begin{aligned}&N^{2-d}\big(H_N(t)-H_N\big(f_t\big|\psi_t^{F,G}\big)\big)\\&\quad=N^{2-d}E_{f_t}\bigg[-N^{-2}\sum_{x,i}\partial_{u_i}^2\lambda\Big(t,\frac{x}{N}\Big)F_i(\tau_x\eta)\\&\qquad\qquad-N^{-2}\sum_{x,i}\Big(\partial_{u_i}\lambda\Big(t,\frac{x}{N}\Big)\Big)^2G_i(\tau_x\eta)\bigg]-N^{2-d}\log\frac{Z_t^{F,G}}{Z_t}.\end{aligned}$$

The second term on the right-hand side of the previous equality is equal to

$$\begin{aligned}N^{2-d}\log E_{\psi_t}\bigg[\exp\bigg\{&-N^{-2}\sum_{x,i}\partial_{u_i}^2\lambda\Big(t,\frac{x}{N}\Big)F_i(\tau_x\eta)\\&-N^{-2}\sum_{x,i}\Big(\partial_{u_i}\lambda\Big(t,\frac{x}{N}\Big)\Big)^2G_i(\tau_x\eta)\bigg\}\bigg],\end{aligned}$$

which can be rewritten as

$$\begin{aligned}N^{2-d}\log E_{\psi_t}\bigg[\prod_{j\in\Lambda_R}&\exp\bigg\{-N^{-2}\sum_{r,\,r(2R+1)\in T_N^d}\sum_{i=1}^d\partial_{u_i}^2\lambda\Big(t,\frac{j+(2R+1)r}{N}\Big)\\&\qquad\qquad\times\tau_{j+(2R+1)}F_i\bigg\}\\&\times\exp\bigg\{-N^{-2}\sum_{r,\,r(2R+1)\in T_N^d}\sum_{i=1}^d\Big(\partial_{u_i}\lambda\Big(t,\frac{j+(2R+1)r}{N}\Big)\Big)^2\\&\qquad\qquad\times\tau_{j+(2R+1)}G_i\bigg\}\bigg],\end{aligned}$$

where Λ_R is the smaller box of size R which contains the supports of F_i and G_i. If $2R+1$ does not divide N, there is a small error in the above formula which converges to 0 as N goes to infinity.

By Hölder's inequality, since ψ_t is a product measure, the previous term is bounded above by

$$\frac{N^{2-d}}{d(2R+1)^d}\sum_{x,i}\log E_{\psi_t}\Big[\exp\Big(-\frac{d(2R+1)^d}{N^2}\partial^2_{u_i}\lambda\Big(t,\frac{x}{N}\Big)F_i(\tau_x\eta)\\ -\frac{d(2R+1)^d}{N^2}\Big(\partial_{u_i}\lambda\Big(t,\frac{x}{N}\Big)\Big)^2 G_i(\tau_x\eta)\Big)\Big].$$

Expanding the exponential, we obtain

$$\frac{N^{2-d}}{d(2R+1)^d}\sum_{x,i}\log\Big(1-\frac{d(2R+1)^d}{N^2}E_{\psi_t}\Big(\partial^2_{u_i}\lambda_t\Big(\frac{x}{N}\Big)F_i(\tau_x\eta)\Big)\\ -\frac{d(2R+1)^d}{N^2}E_{\psi_t}\Big(\Big(\partial_{u_i}\lambda_t\Big(\frac{x}{N}\Big)\Big)^2 G_i(\tau_x\eta)\Big)+O(N^{-4})\Big).$$

Hence $N^{2-d}\log(Z_t^{F,G}/Z_t)$ is bounded above by

$$N^{2-d}E_{\psi_t}\Big[-N^{-2}\sum_{x,i}\partial^2_{u_i}\lambda_t\Big(\frac{x}{N}\Big)F_i(\tau_x\eta)-N^{-2}\sum_{x,i}\Big(\partial_{u_i}\lambda_t\Big(\frac{x}{N}\Big)\Big)^2 G_i(\tau_x\eta)\Big]$$

plus a small error of $O(N^{-2})$. On the other hand, by Jensen's inequality, $N^{2-d}\log(Z_t^{F,G}Z_t^{-1})$ is bounded below by

$$N^{2-d}E_{\psi_t}\Big[-N^{-2}\sum_{x,i}\partial^2_{u_i}\lambda_t\Big(\frac{x}{N}\Big)F_i(\tau_x\eta)-N^{-2}\sum_{x,i}\Big(\partial_{u_i}\lambda_t\Big(\frac{x}{N}\Big)\Big)^2 G_i(\tau_x\eta)\Big].$$

Hence, $N^{2-d}(H_N(t)-H_N(f_t|\psi_t^{F,G}))$ is equal to

$$N^{2-d}E_{f_t}\Big[N^{-2}\sum_{x,i}\partial^2_{u_i}\lambda\Big(t,\frac{x}{N}\Big)\tilde F_i(\tau_x\eta)+N^{-2}\sum_{x,i}\Big(\partial_{u_i}\lambda\Big(t,\frac{x}{N}\Big)\Big)^2\tilde G_i(\tau_x\eta)\Big]$$

plus a small error of $O(N^{-2})$. In this expression, $\tilde F_i$ and $\tilde G_i$ denote, respectively, $F_i-E_{\psi_t}(F_i)$ and $G_i-E_{\psi_t}(G_i)$. By the entropy inequality, for each $\varepsilon>0$, this term is bounded above by

$$\varepsilon N^{1-d}H_N(t)+\varepsilon N^{1-d}\log E_{\psi_t}\Big(\exp\Big\{\frac{1}{\varepsilon N}\sum_{x,i}\partial^2_{u_i}\lambda_t\Big(\frac{x}{N}\Big)\tilde F_i(\tau_x\eta)\\ +\frac{1}{\varepsilon N}\sum_{x,i}\Big(\partial_{u_i}\lambda_t\Big(\frac{x}{N}\Big)\Big)^2\tilde G_i(\tau_x\eta)\Big\}\Big).$$

Using the same arguments, we easily check that for any $\varepsilon>0$ the second term converges to 0 as $N\uparrow\infty$. Moreover, by Proposition 4.1, $\varepsilon N^{1-d}H_N(t)\le C\varepsilon$. Letting ε go to 0, we proved that $N^{2-d}(H_N(t)-H_N(f_t|\psi_t^{F,G}))$ converges to 0 as $N\uparrow\infty$.

Hence, we may replace $H_N(f_t|\psi_t^{F,G})$ by $H_N(t)$ in Proposition 4.2, and we get Proposition 4.3. □

5. A bound from the entropy. In this section we will prove a bound that is controlled by the relative entropy. Lemma 3.2 is a simple corollary of this estimate and Proposition 4.3.

LEMMA 5.1. *Let K_N be a sequence that increases to ∞ as $N \uparrow \infty$ and is small if compared to $N^{3/4}$ ($K_N \ll N^{3/4}$). There exists $\gamma > 0$ such that*

$$E_N\left[N^{1-d}\sum_x \left(\eta_t^{K_N}(x) - m\left(t, \frac{x}{N}\right)\right)^2\right] \le \frac{H_N(t)}{\gamma N^{d-1}} + C(\gamma)\, O\left(\frac{N}{K_N^d}\right).$$

PROOF. By the entropy inequality for any positive constant γ,

$$\begin{aligned}
&E_N\left[N^{1-d}\sum_x \left(\eta_t^{K_N}(x) - m\left(t, \frac{x}{N}\right)\right)^2\right]\\
&\quad = E_{f_t}\left[N^{1-d}\sum_x \left(\eta^{K_N}(x) - m\left(t, \frac{x}{N}\right)\right)^2\right]\\
&\quad \le \frac{H_N(t)}{\gamma N^{d-1}} + \frac{1}{\gamma N^{d-1}} \log E_{\psi_t}\left[\exp\left\{\gamma \sum_x \left(\eta^{K_N}(x) - m\left(t, \frac{x}{N}\right)\right)^2\right\}\right].
\end{aligned}$$

By Hölder's inequality, since ψ_t is a product measure, the limit as N increases to ∞ of the second term on right-hand side is bounded above by

$$\limsup_{N\to\infty} \frac{1}{\gamma N^{d-1} \bar{K}_N^d} \sum_x \log E_{\psi_t}\left[\exp\left\{\gamma \bar{K}_N^d \left(\eta^{K_N}(x) - m\left(t, \frac{x}{N}\right)\right)^2\right\}\right].$$

Here $\bar{K}_N$ stands for $2K_N + 1$. Since m_t is a smooth function, this limit is equal to

$$\begin{aligned}
\limsup_{N\to\infty} \frac{N}{\bar{K}_N^d}\ \frac{1}{\gamma N^d} \sum_x \log E_{\psi_t}\Bigg[\exp\Bigg\{2\gamma \bar{K}_N^d&\\
\times \left(\frac{1}{\bar{K}_N^d} \sum_{y \in x + \Lambda_{K_N}} \left(\eta(y) - m\left(t, \frac{y}{N}\right)\right)\right)^2&\Bigg\}\Bigg]
\end{aligned}$$

plus a small error of $O(\bar{K}_N^4 N^{-3})$.

If one establishes

$$E_{\psi_t}\left[\exp\left\{2\gamma X_{K_N}^2(x)\right\}\right] \le \text{const}, \tag{5.1}$$

where $X_{K_N}(x) = (1/\sqrt{\bar{K}_N^d}) \sum_{y \in x+\Lambda_{K_N}} (\eta(y) - m(t, y/N))$, Lemma 5.1 is proved.

By logarithmic Chebychev inequality, for each $\theta > 0$,

$$\begin{aligned}
E_{\psi_t}\left[\exp\left\{2\gamma X_{K_N}^2(x)\right\}\right] &\le \sum_{a\in\mathbb{Z}} \exp(2\gamma a^2) P(X_{K_N}(x) \ge a)\\
&\le \sum_{a\in\mathbb{Z}} \exp(2\gamma a^2)\exp(-a\theta) E_{\psi_t}\left[\exp(\theta X_{K_N}(x))\right].
\end{aligned}$$

Since $|X_{K_N}(x)| \le \sqrt{\bar{K}_N^d}$, notice that the sum reduces to $|a| \le \sqrt{\bar{K}_N^d}$.

Moreover, an easy computation shows that $E_{\psi_t}[\exp(\theta X_{K_N}(x))]$ is equal to

$$\prod_{y\in x+\Lambda_{K_N}} \exp\left\{\frac{-\theta}{\sqrt{\bar{K}_N^d}} m_t\left(\frac{y}{N}\right) + \log\left(1 + m_t\left(\frac{y}{N}\right)\left(\exp\left(\frac{\theta}{\sqrt{\bar{K}_N^d}}\right) - 1\right)\right)\right\}.$$

Since $\exp\{\theta(\bar{K}_N^d)^{-1/2}\} \le 1 + \theta(\bar{K}_N^d)^{-1/2} + C\theta^2(\bar{K}_N^d)^{-1}$ for $\theta \le \sqrt{\bar{K}_N^d}$,

$$E_{\psi_t}\left[\exp\{2\gamma X_{K_N}^2(x)\}\right] \le \sum_{a\in\mathbb{Z}} \exp\left\{2\gamma a^2 + \frac{C\theta^2}{\bar{K}_N^d} \sum_{y\in x+\Lambda_{K_N}} m_t\left(\frac{y}{N}\right) - a\theta\right\}.$$

Choosing

$$\theta = \frac{a\bar{K}_N^d}{2C\sum_{y\in x+\Lambda_{K_N}} m_t(y/N)}$$

and noting that $\theta \le \sqrt{\bar{K}_N^d}$ if C is large enough, we obtain

$$E_{\psi_t}\left[\exp\{2\gamma X_{K_N}^2(x)\}\right] \le \sum_{a\in\mathbb{Z}} \exp\left\{\left(2\gamma - \frac{\bar{K}_N^d}{4C\sum_{y\in x+\Lambda_{K_N}} m_t(y/N)}\right)a^2\right\},$$

which is bounded by a constant for γ small enough. □

The following lemma will be useful in the proofs of the entropy bounds.

LEMMA 5.2. *Let K_N be a sequence that increases to ∞ as $N \uparrow \infty$ and is small if compared to $N^{3/4}$ ($K_N \ll N^{3/4}$). There exists $\gamma > 0$ such that*

$$E_N\left[N^{1-d}\sum_x \left(\eta_t^{K_N}(x) - m\left(t, \frac{x}{N}\right)\right)^3\right] \le \frac{H_N(t)}{\gamma N^{d-1}} + C(\gamma)\, O\left(\frac{N}{K_N^d}\right).$$

PROOF. Using the fact that $|\eta^{K_N}(x) - m(t, x/N)| \le \text{const}$, the left-hand side of the inequality appearing in Lemma 5.2 is bounded above by

$$\text{const}\, E_N\left[N^{1-d}\sum_x \left(\eta_t^{K_N}(x) - m\left(t, \frac{x}{N}\right)\right)^2\right].$$

Using Lemma 5.1, we get Lemma 5.2. □

6. Multiscale estimates. In this section we recall some multiscale estimates from [3] and extend them to our setting. These estimates replace the usual one-block–two-block estimates and will be used later on to prove Lemma 3.1 and Propositions 4.1 and 4.2. Our estimates are similar to [3] except some extra work is needed because our system is not near equilibrium. In particular, we need estimates uniform with respect to the density ρ. We start with some notation.

Recall from the previous sections that we denote by $\mathscr{G}$ the linear space of cylinder functions that have mean zero with respect to all canonical measures on a sufficiently large box Λ_K:

$$\mathscr{G} = \{g \in \mathscr{C};\ \nu_{K,m}[g] = 0 \text{ for some } K > 0 \text{ and all } m\}. \tag{6.1}$$

Moreover, for a density $0 \le m \le 1$, let $\mathscr{G}_m$ be the space of cylinder functions such that

$$\tilde{g}(m) = \nu_m[g] = 0 \quad \text{and} \quad \partial_\rho \nu_\rho[g]\big|_{\rho=m} = \tilde{g}'(m) = 0.$$

Note that the second condition is equivalent to imposing that the covariance, with respect to the measure ν_m, of g and the formal sum $\sum_x \eta(x)$ vanishes:

$$\sum_z \nu_m[g(\eta);\ \eta(z)] = 0.$$

Notice that $\mathscr{G} \subset \mathscr{G}_m$ for all m in $[0,1]$. The following definition is taken from [3].

DEFINITION 6.1. Let g be a cylinder function and denote by $s(g)$ its support:

$$s(g) = \min\{l \in \mathbb{N};\ \operatorname{supp} g \subset \Lambda_l\}.$$

For each $l \ge s(g)$ and m in $\{0, 1/\bar{l}^d, \dots, 1\}$, define the "variance" $V_l(g,m)$ of g with respect to $\nu_{l,m}$ by

$$\begin{aligned} V_l(g,m) &= \frac{1}{(2l_g+1)^d} \\ &\quad \times \left\langle \left[\sum_{|x|\le l_g} (\tau_x g - \tilde{g}_l(m))\right](-L_l)^{-1}\left[\sum_{|x|\le l_g} (\tau_x g - \tilde{g}_l(m))\right]\right\rangle_{\nu_{l,m}}. \end{aligned} \tag{6.2}$$

In this formula l_g denotes the integer $l - s(g)$ so that $\sum_{|x|\le l_g} \tau_x g$ is measurable with respect to $\{\eta(x);\ x \in \Lambda_l\}$. Moreover, L_l is the restriction to Λ_l of the generator L and $\tilde{g}_l(m)$ is the expected value of g with respect to the canonical measure $\nu_{l,m}$.

If $g \in \mathscr{G}_m$ we define also the "variance" of g by

$$\mathbb{V}_m(g) = \limsup_{l\to\infty} \nu_m\big[V_l(g, \eta^l(0))\big].$$

Notice that for $g \in \mathscr{G}$ the subtraction in (6.2) is unnecessary for l sufficiently large.

We need the following two results. The proof of the first lemma is the same as in [6] and the second can be found in [3].

LEMMA 6.1. *For each cylinder function h in $\mathscr{G}$,*

$$\lim_{l\to\infty} V_l(h,m) = \mathbb{V}_m(h)$$

uniformly for $m \in [0,1]$.

LEMMA 6.2 (Integration by parts formula). *Let $g \in \mathscr{G}_m$ be a cylinder function. Denote by l the smallest integer such that Λ_l contains the support of g. Then there exists a family of functions $\Phi_b(x, g)$, where $x \in \mathbb{Z}^d$ and b is an edge, such that*

$$\langle \tau_x g, u \rangle_m = \sum_{b \in x+\Lambda_l} \langle \Phi_b(x, g), \nabla_b u \rangle_m,$$

$$\sum_{b \in x+\Lambda_l} |b-x|^{d+(1/2)} \langle (\Phi_b(x, g))^2 \rangle_m \le C(g)$$

for some constant $C(g)$ depending only on g.

In the above formula $\nabla_b u(\eta) = u(\eta^b) - u(\eta)$ and η^b is the configuration η with the sites in the bond b exchanged.

The following result is a one-block estimate whose proof relies on the standard perturbation theorem on the largest eigenvalue of a symmetric operator.

LEMMA 6.3 (One-block estimate). *Let f_t be the solution of the forward equation (4.2). There exists a universal constant C_1 such that for any smooth function J, positive γ and $h \in \mathscr{G}$,*

$$\limsup_{N\to\infty} \left\{ N^{1-d} \sum_x \int J\left(\frac{x}{N}\right) \tau_x h\, f_t\, d\nu_\rho - \gamma N^{2-d} D_N(f_t) \right\} \le C_1 \gamma^{-1} \int J(u)^2 \mathbb{V}_{m(t,u)}(h)\, du. \tag{6.3}$$

In this formula $m(t, u)$ is the solution of (2.4).

PROOF. Fix a positive integer l independent of N and that will increase to ∞ after N. Since J is smooth, the summation on the left-hand side of the statement of the lemma can be rewritten as

$$N^{1-d} \sum_x \int J\left(\frac{x}{N}\right) (Av_{|y-x|\le l}\, \tau_y h)\, f_t\, d\nu_\rho + O\left(\frac{l^2}{N}\right),$$

where we denote $(2l+1)^{-d} \sum_{|y-x|\le l} \tau_y h$ by $Av_{|y-x|\le l}\, \tau_y h$.

Since ν_ρ is translation invariant, this term is equal to

$$N^{1-d} \sum_x \int J\left(\frac{x}{N}\right) (Av_{|y|\le l}\, \tau_y h)\, \tau_x f_t\, d\nu_\rho + O\left(\frac{l^2}{N}\right).$$

It can easily be rewritten as

$$N^{1-d} \sum_x \sum_K c(x, f_t, K) \int J\left(\frac{x}{N}\right) (Av_{|y|\le l}\, \tau_y h)\, (\tau_x f_t)_{l,K}\, d\nu_{l,K} + O\left(\frac{l^2}{N}\right), \tag{6.4}$$

where $(\tau_x f_t)_{l,K}$ denotes the projection of $\tau_x f_t$ on the space of configurations with K particles on Λ_l and $c(x, f_t, K)$ is given by

$$c(x, f_t, K) = \nu_\rho\left[\tau_x f_t\, 1_{\{\sum_{|y|\le l} \eta_y = K\}}\right].$$

By convexity of the Dirichlet form, we have that

$$\sum_x \sum_K c(x, f_t, K) D_{l,K}\big((\tau_x f_t)_{l,K}\big) \le (2l+1)^d D_N(f_t)$$

with

$$D_{l,K}(f) = (1/2) \sum_{\substack{|x-y|\le 1\\ x,\, y\in\Lambda_l}} \Big\langle r_{x,y}(\eta)\Big[\sqrt{f(\eta^{x,y})} - \sqrt{f(\eta)}\Big]^2\Big\rangle_{l,K}.$$

From the standard perturbation theorem on the largest eigenvalue of a symmetric operator (cf. [5, 8, 10]), we have

$$\begin{aligned}
&\limsup_{N\to\infty}\bigg\{N^{1-d}\sum_x \int J\Big(\frac{x}{N}\Big)(Av_{|y-x|\le l}\,\tau_y h) f_t\, d\nu_\rho - \gamma N^{2-d} D_N(f_t)\bigg\}\\
&\quad\le \limsup_{N\to\infty}\bigg\{N^{-d}\sum_x\sum_K c(x, f_t, K)\frac{N^2\gamma}{(2l+1)^d}\\
&\qquad\qquad \times\bigg[\frac{1}{N\gamma}\int J\Big(\frac{x}{N}\Big)\sum_{|y|\le l}\tau_y h\ (\tau_x f_t)_{l,K}\, d\nu_{l,K} - D_{l,K}\big((\tau_x f_t)_{l,K}\big)\bigg]\bigg\}\\
&\quad\le \limsup_{N\to\infty}\bigg\{N^{-d}\sum_x\sum_K c(x, f_t, K)\frac{N^2\gamma}{(2l+1)^d}\\
&\qquad\qquad \times\bigg[\frac{C_1(2l+1)^d}{N^2\gamma^2}\int J^2\Big(\frac{x}{N}\Big) V_l\Big(h, \frac{K}{(2l+1)^d}\Big)\, d\nu_{l,K}\bigg]\bigg\}\\
&\quad= C_1\gamma^{-1}\limsup_{N\to\infty} N^{-d}\sum_x J^2\Big(\frac{x}{N}\Big) E_N\big[V_l(h, \eta^l(x))\big].
\end{aligned}$$

In this formula, V_l stands for the finite volume variance defined in (6.2) and C_1 is a universal constant. By the law of large numbers, the right-hand side of the last expression is equal to

$$C_1\gamma^{-1}\int du\, J^2(u)\nu_{m(t,u)}\big[V_l(h, \eta^l(0))\big].$$

To conclude the proof, it remains to invoke Lemma 6.1, which states that the finite volume variance converges uniformly to the infinite volume variance. $\square$

Notice that if $h(\eta)$ is of the form

$$h(\eta) = w(\eta^l(0)) Av_{|y|\le l}\tau_y g,$$

with g in $\mathscr{G}$ and w smooth, the left-hand side of the inequality appearing in Lemma 6.3 is bounded above by

$$C_1\gamma^{-1}\int du\, J^2(u)\mathbb{V}_{m_t(u)}(w(m_t(u))g).$$

In this case, the term corresponding to (6.4) is

$$N^{1-d}\sum_x\sum_K c(x, f_t, K)\int J\Big(\frac{x}{N}\Big) w\Big(\frac{K}{(2l+1)^d}\Big)(Av_{|y|\le l}\,\tau_y g)\,(\tau_x f_t)_{l,K}\, d\nu_{l,K}.$$

For each K, w behaves as a constant. We obtain that this term is bounded above by

$$C_1\gamma^{-1}N^{-d}\sum_x J^2\left(\frac{x}{N}\right)E_N\big[w(\eta^l(x))^2\,V_l(g,\eta^l(x))\big],$$

which converges as $N\uparrow\infty$, then $l\uparrow\infty$, to

$$\frac{C_1}{\gamma}\int du\,J^2(u)\,w(m_t(u))^2\mathbb{V}_{m_t(u)}(g)=\frac{C_1}{\gamma}\int du\,J^2(u)\,\mathbb{V}_{m_t(u)}(w(m_t(u))\,g).$$

For any local function h and any integer $L\ge s(h)$ fixed, consider the decomposition

$$h=\{h-\nu_\rho[h\mid\eta^L(0)]\}+\nu_\rho[h\mid\eta^L(0)]:=h_{(L)}+\tilde h_L(\eta^L(0)). \tag{6.5}$$

Notice that for each L, $h_{(L)}$ belongs to $\mathscr{G}$ since it has mean zero with respect to all canonical measures on boxes of length larger than $\bar L$.

The following theorem is taken from [7].

THEOREM 6.1 (Multiscale estimates). *Fix a cylinder function h and a sequence $K=K(N)$ such that*

$$\lim_{N\to\infty}\frac{K^{2+(1/2)+a}}{N}=0$$

for some constant $a>0$. Let f_t be the solution of the forward equation (4.2). There exists an universal constant C_1 and a function $C(h,L)$ vanishing as $L\uparrow\infty$ [$\lim_{L\to\infty}C(h,L)=0$] such that for any smooth function J and any positive γ,

$$\begin{aligned}\lim_{N\to\infty}&\left\{N^{1-d}\int\sum_x J\left(\frac{x}{N}\right)\big[\tau_x h-\tilde h_K(\eta^K(x))\big]f_t\,d\nu_\rho-\gamma N^{2-d}D_N(f_t)\right\}\\&\le C_1\gamma^{-1}\int J^2(u)\mathbb{V}_{m(t,u)}(h_{(L)})\,du+C(h,L).\end{aligned}$$

Since Theorem 6.1 holds for all L, we can take the limit $L\uparrow\infty$. We would like to have that

$$\lim_{L\to\infty}\mathbb{V}_m(h_{(L)})=\mathbb{V}_m(h)$$

uniformly in m on compact subsets of $(0,1)$. Unfortunately, since h may not be in the space $\mathscr{G}_m$, $\mathbb{V}_m(h)$ may not even be defined. For this reason, we introduce the function h^m, which is the projection of h on $\mathscr{G}_m$ and is given by

$$h^m=h-\langle h\rangle_m-\tilde h'(m)[\eta(0)-m],$$

where $\tilde h(\rho)=\langle h\rangle_\rho$ and $\tilde h'(\cdot)$ is its derivative:

$$V_l(h_{(L)}-(h^m)_{(L)},m)=\tilde h'(m)^2\;V_l(\eta(0)-\eta^L(0),m).$$

From the variational formula for the finite volume variance, $V_l(\eta(0) - \eta^L(0), m)$ is equal to

$$(2l'+1)^{-d} \sup_u \left\{ 2\left\langle \sum_{|x|\le l'} (\eta_x - \eta^L(x), u \right\rangle - D_{l,m}(u) \right\}.$$

In this formula, $l' = l - L$, the supremum is carried over all functions u in $L^2(\nu_{l,m})$, $\langle \cdot, \cdot \rangle$ represents the inner product of $L^2(\nu_{l,m})$ and $D_{l,m}$ is the Dirichlet form with respect to the measure $\nu_{l,m}$.

Note that $\sum_{|x|\le l'} \eta_x - \eta^L(x)$ is of $O(l^{d-1}L)$. Hence, it is of the form $\sum_x \tau_x g$, where g is cylindrical and the sum is carried over $O(l^{d-1}L)$ terms.

We now apply the integration by parts formula stated in Lemma 6.2 to the function g to obtain that the expression inside the supremum is bounded above by

$$\left\{ \sum_x \sum_{b\in\Lambda_l} 2\langle \Phi_b(x, g), \nabla_b u\rangle - D_{l,m}(u) \right\}. \tag{6.6}$$

Using Schwarz inequality, the bound on $\Phi_b(x, g)$ given in Lemma 6.2 and the above remark, we easily obtain that

$$\lim_{l\to\infty} V_l(\eta_0 - \eta^L(0), m) = 0.$$

Hence,

$$\lim_{l\to\infty} V_l(h_{(L)} - (h^m)_{(L)}, m) = 0$$

uniformly in $[0, 1]$. So we can assume, without loss of generality, that h is in $\mathscr{G}_m$. We claim that under this assumption,

$$\lim_{L\to\infty} \mathbb{V}_m(h_{(L)}) = \mathbb{V}_m(h).$$

To prove this, it suffices to check that $\lim_{L\to\infty} \lim_{l\to\infty} V_l(\tilde{h}_L, m) = 0$. Using again the variational formula for the finite volume variance and applying the integration by parts formula stated in Lemma 6.2 to the function $\tilde{h}_L$, we obtain that $V_l(\tilde{h}_L, m)$ is bounded above by

$$\sup_u (2l'+1)^{-d} \left\{ \sum_{|x|\le l'} \sum_{b\in\Lambda_l} 2\langle \Phi_b(x, \tilde{h}_L), \nabla_b u\rangle - D_{l,m}(u) \right\}. \tag{6.7}$$

In this formula, to keep notation simple we denoted $l_{\tilde{h}_L}$ simply by l'. From the integration by parts lemma, we can derive a bound on $\Phi_b(x, \tilde{h}_L)$. This bound turns out to be insufficient for our purpose. Our function $\tilde{h}_L$ is very special and it depends only on $\eta^L(0)$.

From the proof in [3], we can check that $\Phi_b(x, g) = \tau_x \Phi_{b-x}(0, g)$. Moreover, for an integer q fixed,

$$\Phi_b(0, g) = \frac{1}{2} \sum_{n:\, b\in\Lambda_{q^{n+1}}} r_b(\eta) \nabla_b (-L_{\Lambda_{q^{n+1}}})^{-1} (g_n - g_{n+1}),$$

where

$$g_n = \nu_m\big(g \,|\, (\eta_x)_{x\in\Lambda^c_{q^n}}, \eta^{q^n}(0)\big).$$

For each n such that $\Lambda \subset \Lambda_{q^n}$, we have $(\tilde{h}_L)_n = h_n$. Hence, $\Phi_b(x, \tilde{h}_L) = \Phi_b(x, h)$ for all bonds b such that $b - x$ is not in Λ_L. If $b \in \Lambda_L$, we have $(\tilde{h}_L)_n = (\tilde{h}_L)_{n+1}$ for each n such that $q^{n+1} < L$. Hence, the sum reduces to n such that $q^{n+1} \geq L$.

Using the Schwarz inequality, the expression inside the supremum in (6.7) is bounded above by

$$(2l'+1)^{-d} \sum_{|x| \leq l'} \sum_{b \in \Lambda_l} C\, |b-x|^{d+1/2} \langle \Phi_b(x, \tilde{h}_L)^2 \rangle + (2l'+1)^{-d} \left\{ \sum_{b \in \Lambda_l} \sum_{|x| \leq l'} \frac{|b-x|^{-d-1/2}}{C} \langle r_b(\eta)(\nabla_b u)^2 \rangle - D_{l,m}(u) \right\}.$$

Since $\sum_{|x| \leq l'} |b-x|^{-d-1/2} \leq$ const, we can choose C such that the second line of the previous expression is negative. The first line can be rewritten as

$$(2l'+1)^{-d} \sum_{|x| \leq l'} \sum_{b:\, |b-x| > L} C\, |b-x|^{d+1/2} \langle \Phi_b(x, \tilde{h}_L)^2 \rangle + (2l'+1)^{-d} \sum_{|x| \leq l'} \sum_{b:\, |b-x| \leq L} C\, |b-x|^{d+1/2} \langle \Phi_b(x, \tilde{h}_L)^2 \rangle = A_1 + A_2.$$

From the previous remarks about the special feature of $\tilde{h}_L$ and the bound on the L^2 norm of $\Phi_b(x, h)$ of the above integration by parts lemma, $A_1 \leq C(h, L)$, where $C(h, L)$ is a constant depending only on h and L, which goes to 0 as L goes to ∞. Moreover, from the proof in [3] and the previous remarks, $A_2 \leq f(L)$, where $f(L)$ goes to 0 as L goes to ∞. Hence,

$$\lim_{L \to \infty} \lim_{l \to \infty} V_l(\tilde{h}_L, m) = 0$$

uniformly on $[0, 1]$. We have thus shown that for every cylinder function h,

$$\lim_{L \to \infty} \mathbb{V}_m(h_{(L)}) = \mathbb{V}_m(h^m).$$

We summarize these last conclusions in the following corollary.

COROLLARY 6.1. *Fix a cylinder function h and a sequence $K = K(N)$ such that*

$$\lim_{N \to \infty} \frac{K^{2+\alpha+1/2}}{N} = 0$$

for some constant $\alpha > 0$. Let f_t be the solution of the forward equation (4.2). There exists an universal constant C_1 such that for any positive constant γ,

$$\limsup_{N \to \infty} \left\{ N^{1-d} \int \sum_x J\left(\frac{x}{N}\right) \left[\tau_x h - \tilde{h}_K(\eta^K(x))\right] f_t\, d\nu_\rho - \gamma N^{2-d} D_N(f_t) \right\} \leq C_1 \gamma^{-1} \int J(u)^2 \mathbb{V}_{m(t,u)}(h^m)\, du.$$

Furthermore,

$$\lim_{L\to\infty} \mathbb{V}_m(h_{(L)}) = \mathbb{V}_m(h^m).$$

7. Proof of the first entropy bound. Recall the definition of the entropy $H_N(t)$ and the one of the Dirichlet form introduced before Proposition 4.2.

We start by computing the time derivative of $H_N(t)$,

$$\begin{aligned}\partial_t H_N(t) &= \int \left(N^2 L_N^* f_t \log \frac{f_t}{\psi_t} + \frac{N^2 L_N^* f_t \psi_t - \partial_t \psi_t f_t}{\psi_t} \right) d\nu_\rho \\ &= \int \left(N^2 f_t L_N \log \frac{f_t}{\psi_t} - \frac{f_t}{\psi_t} \partial_t \psi_t \right) d\nu_\rho \qquad (7.1)\\ &= -N^2 D_N(f_t|\psi_t) + \int \frac{f_t}{\psi_t}(N^2 L_N^* - \partial_t)\psi_t \, d\nu_\rho,\end{aligned}$$

where $D_N(f_t|\psi_t)$ is given by

$$\int \psi_t \left\{ L_N \frac{f_t}{\psi_t} - \frac{f_t}{\psi_t} L_N \log \frac{f_t}{\psi_t} \right\} d\nu_\rho.$$

To compute the second term on the right-hand side, we need to know the equation satisfied by λ. Recall the definition of λ given by (4.1). A simple computation shows that

$$\partial_t \lambda = \frac{\Delta\phi(m)}{m(1-m)} = \sum_{i=1}^d \phi'(m)\, \partial_{u_i}^2 \lambda + \sum_{i=1}^d \varphi'(m)(\partial_{u_i}\lambda)^2,$$

where $\phi(m) = m(1+\alpha m)$ and $\varphi(m) = m(1-m)(1+2\alpha m)$. Hence,

$$\frac{\partial_t \psi_t}{\psi_t} = \sum_x \frac{\Delta\phi(m)}{m(1-m)}(\eta_x - m).$$

We turn now to the term $N^2 \psi_t^{-1} L_N^* \psi_t$, which is equal to

$$\begin{aligned}&N^2 \sum_{x,\,|y|=1} r_{x,\,x+y}(\eta) \left[\frac{\psi_t(\eta^{x,\,x+y})}{\psi_t(\eta)} - 1 \right] \\ &\quad = N^2 \sum_{x,\,|y|=1} r_{x,\,x+y}(\eta) \left[\exp\left\{ \lambda_t\left(\frac{x+y}{N}\right) - \lambda_t\left(\frac{x}{N}\right) \right\} - 1 \right].\end{aligned}$$

Expanding the exponential up to the fourth order, we obtain

$$\begin{aligned}&N^2 L_N\left(\sum_z \lambda_t\left(\frac{z}{N}\right)\eta_t(z)\right) + \frac{1}{2} \sum_{x,\,|y|=1} r_{x,\,x+y}(\eta)\left[N\left\{ \lambda_t\left(\frac{x+y}{N}\right) - \lambda_t\left(\frac{x}{N}\right) \right\} \right]^2 \\ &\quad + \frac{1}{6N} \sum_{x,\,|y|=1} r_{x,\,x+y}(\eta)\left[N\left\{ \lambda_t\left(\frac{x+y}{N}\right) - \lambda_t\left(\frac{x}{N}\right) \right\} \right]^3 + O(N^{d-2}).\end{aligned}$$

In particular, $\psi_t^{-1}(N^2 L_N^* - \partial_t)\psi_t$ is equal to

$$
\begin{aligned}
&\sum_{x,i} \partial_{u_i}^2 \lambda\Big(t, \frac{x}{N}\Big)\big\{\tau_x h_i - \phi(m) - \phi'(m)(\eta_x - m)\big\}\\
&\quad + \sum_{x,i}\Big(\partial_{u_i}\lambda\Big(t,\frac{x}{N}\Big)\Big)^2\big\{\tau_x g_i - \varphi(m) - \varphi'(m)(\eta_x - m)\big\}\\
&\quad + \sum_{x,i} \partial_{u_i}^2 \lambda\Big(t, \frac{x}{N}\Big)\phi(m) + \Big(\partial_{u_i}\lambda\Big(t,\frac{x}{N}\Big)\Big)^2 \varphi(m)\\
&\quad - \frac{1}{N}\sum_{x,i} \partial_{u_i}^3 \lambda\Big(t, \frac{x}{N}\Big)\tau_x v_i\\
&\quad + \frac{1}{N}\sum_{x,i} \partial_{u_i}\lambda\Big(t,\frac{x}{N}\Big)\partial_{u_i}^2\lambda\Big(t,\frac{x}{N}\Big)\tau_x g_i + \frac{1}{3}\Big(\partial_{u_i}\lambda\Big(t,\frac{x}{N}\Big)\Big)^3 \tau_x w_i\\
&\quad + O(N^{d-2})
\end{aligned}
\tag{7.2}
$$

with

$$
\begin{aligned}
h_i(\eta) &= \eta_0 - \alpha\eta_{-e_i}\eta_{e_i} + 2\alpha\eta_{-e_i}\eta_0,\\
v_i(\eta) &= \alpha\eta_{-e_i}\eta_0,\\
g_i(\eta) &= \tfrac{1}{2}(r_{0,e_i} + r_{e_i,0}),\\
w_i(\eta) &= \tfrac{1}{2}(r_{0,e_i} - r_{e_i,0}) = \tfrac{1}{2}\,W_{0,e_i}.
\end{aligned}
$$

We easily check that $\partial_{u_i}^2\lambda(t,x/N)\phi(m) + (\partial_{u_i}\lambda(t,x/N))^2\varphi(m)$ is equal to $-\alpha\,\partial_{u_i}^2 m(t,x/N) - (1+\alpha)\,\partial_{u_i}^2(\log(1-m(t,x/N)))$. Hence, the third line is of $o(N^{d-1})$ by summation by parts (cf. Conjecture 3.1). By the law of large numbers, the expected value with respect to the measures $f_t\,d\nu_\rho$ of the fourth and the fifth lines divided by N^{d-1} converges to

$$
\sum_{i=1}^d \int_{T^d} du\big\{-\alpha\, m_t^2(u)\,\partial_{u_i}^3\lambda_t(u) + \partial_{u_i}\lambda_t(u)\,\partial_{u_i}^2\lambda_t(u)\varphi(m_t(u))\big\},
$$

which is bounded by a constant. In turn, to compute the expected value of the first line, recall the definition given in Section 3 of K, Λ_l and $V_{i,l}$ and the computations made just after. We showed there that the expected value with respect to $f_t\,d\nu_\rho$ of the first line divided by N^{d-1} is equal to

$$
\begin{aligned}
&\alpha\, N^{1-d}\sum_{x,i}\partial_{u_i}^2\lambda\Big(t,\frac{x}{N}\Big)\int \tau_x V_{i,K}(\eta) f_t\,d\nu_\rho + o_{N,B}(1)\\
&\qquad + \alpha\, N^{1-d}\sum_{x,i}\partial_{u_i}^2\lambda\Big(t,\frac{x}{N}\Big)\int\Big[\eta_t^K(x) - m\Big(t,\frac{x}{N}\Big)\Big]^2 f_t\,d\nu_\rho.
\end{aligned}
$$

By Lemma 5.1, there exists $\gamma_0 > 0$ such that the second line is bounded above by

$$
\gamma_0^{-1} N^{1-d} H_N(t) + o_N(1).
$$

We deal with the second line as we have done with the first:

$$\tau_x g_i - \varphi(m) - \varphi'(m)(\eta_x - m) \\ = \tfrac{1}{2}\tau_x(\eta_{e_i} - \eta_0) + \tau_x G_i - \langle G_i\rangle_m - \langle G_i\rangle'_m\ (\eta_x - m).$$

Here, $G_i = g_i - \frac{1}{2}(\eta_{e_i} + \eta_0)$ and $< G_i >_m = (2\alpha - 1)m^2 - 2\alpha m^3$ represents the expectation of G_i with respect to ν_m.

Define $V'_{iK}(\eta) = G_i(\eta) - E[G_i|\eta^K(0)]$. The term $E[G_i|\eta^K(0)]$ is easily computed and is equal to

$$(2\alpha-1)\left[\eta^K(0)^2 - \frac{1}{|\Lambda_K|-1}\eta^K(0)(1-\eta^K(0))\right] \\ - 2\alpha\left[\eta^K(0)^3 + \eta^K(0)^2(\eta^K(0)-1)\left(\frac{2}{|\Lambda_K|-2} + \frac{1}{|\Lambda_K|-1}\right)\right. \\ \left. + 2\frac{\eta^K(0)(\eta^K(0)-1)^2}{(|\Lambda_K|-1)(|\Lambda_K|-2)}\right].$$

Hence, the expected value with respect to $f_t\, d\nu_\rho$ of the second line divided by N^{d-1} is equal to

$$N^{1-d}\sum_{x,i}\left(\partial_{u_i}\lambda\left(t,\frac{x}{N}\right)\right)^2 \int \tau_x V'_{i,K}(\eta) f_t\, d\nu_\rho + o_{N,B}(1) \\ + N^{1-d}\sum_{x,i}\left(2\alpha - 1 - 6\alpha m_t\left(\frac{x}{N}\right)\right)\left(\partial_{u_i}\lambda\left(t,\frac{x}{N}\right)\right)^2 \\ \times \int\left[\eta^K(x) - m\left(t,\frac{x}{N}\right)\right]^2 f_t\, d\nu_\rho \\ - 2\alpha\, N^{1-d}\sum_{x,i}\left(\partial_{u_i}\lambda\left(t,\frac{x}{N}\right)\right)^2 \int\left[\eta^K(x) - m\left(t,\frac{x}{N}\right)\right]^3 f_t\, d\nu_\rho \\ - N^{-d}\sum_{x,i}\tfrac{1}{2}\partial_{u_i}\left(\left(\partial_{u_i}\lambda\left(t,\frac{x}{N}\right)\right)^2\right) E_N(\eta_x).$$

By the law of large numbers, the last line converges to

$$\sum_{i=1}^{d}\int_{T^d} du\ \tfrac{1}{2}(\partial_{u_i}\lambda(t,u))^2\, \partial_{u_i} m_t(u),$$

which is bounded by a constant. Moreover, by Lemmas 5.1 and 5.2 there exists $\gamma'_0 > 0$ such that the second the third and fourth lines are bounded above by

$$\gamma_0'^{-1} N^{1-d} H_N(t) + o_N(1).$$

Keep in mind that the entropy $H_N(t)$ vanishes at time 0. We prove at the end of this section that $\tilde{D}_N(f_t) = -\int f_t L_N \log f_t\, d\nu_\rho$ and $D_N(f_t|\psi_t)$ are not too far apart: In fact, there exists a constant $C_0 = C_0(m)$ such that

$$\left| \int_0^{t_0} N^{2-d} \{ \tilde{D}_N(f_t) - D_N(f_t | \psi_t) \} \, dt \right| \le C_0 t_0. \tag{7.3}$$

On the other hand, a simple computation relying on the elementary inequality $a \log(b/a) \le 2\sqrt{a}(\sqrt{b} - \sqrt{a})$ shows that $D_N(f_t) \le \tilde{D}_N(f_t)$. Therefore, from (7.1), (7.2) and the previous bounds we get that

$$\begin{aligned}
N^{1-d} H_N(t_0) &+ \int_0^{t_0} (1/2) N^{2-d} D_N(f_t) \, dt \\
&\le C t_0 + C_1 \int_0^{t_0} N^{1-d} H_N(t) \, dt \\
&\quad + \int_0^{t_0} dt \left\{ \alpha N^{1-d} \sum_{x,i} \int \partial_{u_i}^2 \lambda\left(t, \frac{x}{N}\right) \tau_x V_{i,K}(\eta) f_t \, d\nu_\rho \right. \\
&\qquad\qquad \left. - (1/4) N^{2-d} D_N(f_t) \right\} \\
&\quad + \int_0^{t_0} dt \left\{ N^{1-d} \sum_{x,i} \int \left(\partial_{u_i} \lambda\left(t, \frac{x}{N}\right)\right)^2 \tau_x V'_{i,K}(\eta) f_t \, d\nu_\rho \right. \\
&\qquad\qquad \left. - (1/4) N^{2-d} D_N(f_t) \right\}
\end{aligned}$$

for some finite constant C_1.

To conclude the proof of the proposition, it remains to show that the two last terms on the right-hand side are bounded and then apply Gronwall's lemma. This follows directly from Corollary 6.1. □

We now prove (7.3). The proof relies on the explicit formula of the Dirichlet forms. Indeed, by definition we have that

$$N^{2-d}\big(D_N(f_t|\psi_t) - \tilde{D}_N(f_t)\big) = N^{2-d} \int f_t \left\{ \frac{L_N^* \psi_t}{\psi_t} + L_N \log \psi_t \right\} d\nu_\rho.$$

The expression inside the braces multiplied by N^{2-d} is equal to

$$N^{2-d} \sum_{x, |y|=1} r_{x,x+y} \left[\exp\left\{ \lambda_t\left(\frac{x+y}{N}\right) - \lambda_t\left(\frac{x}{N}\right) \right\} - 1 + \left(\lambda_t\left(\frac{x+y}{N}\right) - \lambda_t\left(\frac{x}{N}\right)\right) \right].$$

Expanding the exponential, we obtain that this term is equal to

$$N^{2-d} \sum_{x, |y|=1} r_{x,x+y} \left[2\left(\lambda_t\left(\frac{x+y}{N}\right) - \lambda_t\left(\frac{x}{N}\right)\right) + \frac{1}{2}\left(\lambda_t\left(\frac{x+y}{N}\right) - \lambda_t\left(\frac{x}{N}\right)\right)^2 \right]$$

plus a small error of $O(N^{-1})$. Computing this term, we can rewrite it as

$$N^{-d} \sum_{x, e_i} 2 \, \partial_{u_i}^2 \lambda_t\left(\frac{x}{N}\right) \tau_x h_i + \left(\partial_{u_i} \lambda_t\left(\frac{x}{N}\right)\right)^2 \tau_x g_i + O(N^{-1}).$$

Hence, $N^{2-d} \psi_t^{-1} L_N^* \psi_t + L_N \log \psi_t$ is of order 1. □

8. Proof of Lemma 3.1. Recall that R^σ is a smooth approximation of R converging uniformly to R on each compact subset of $(0,1)$. For a positive integer l, $1 \le i \le d$ and a family $\{F_i,\ 1 \le i \le d\}$ of functions in $\mathscr{G}$, let $W_{i,l}(\eta)$ be given by

$$W_{i,l}(\eta) = V_{i,l}(\eta) + 2\eta^l(0)\,(2l'+1)^{-d} \sum_{|y|\le l'} \nabla_{e_i}\eta(y) - \sum_{j=1}^{d} R^\sigma_{i,j}(\eta^l(0))\left\{(2l'+1)^{-d}\sum_{|y|\le l'} \nabla_{e_j}\eta(y)\right\} - L_N F_i(\eta).$$

Here, as in the previous sections, $l' = l-1$ and, for $1 \le j \le d$ and $y \in \mathbb{Z}^d$, $\nabla_{e_j}\eta(y)$ stands for $\eta(y+e_j) - \eta(y)$.

The time integral on the left-hand side of the statement of Lemma 3.1 is equal to

$$\begin{aligned}
&\int_0^T dt\, E_N\left[N^{1-d}\sum_{x,i}\partial^2_{u_i} J\left(t, \frac{x}{N}\right)\tau_x W_{i,K}(\eta_t)\right] \\
&\quad + \int_0^T dt\, E_N\left[N^{1-d}\sum_{x,i}\partial^2_{u_i} J\left(t, \frac{x}{N}\right)\sum_{j=1}^{d}\left[R^\sigma_{i,j}(\eta^K_t(x)) - 2\eta^K_t(x)\,\delta_{i,j}\right]\right. \\
&\qquad\qquad \left.\times (2K'+1)^{-d}\sum_{|y-x|\le K'}\nabla_{e_j}\eta_t(y)\right] \qquad (8.1)\\
&\quad + \int_0^T dt\, E_N\left[N^{1-d}\sum_{x,i}\partial^2_{u_i} J\left(t, \frac{x}{N}\right) L_N F_i(\tau_x\eta_t)\right] = \Omega_1 + \Omega_2 + \Omega_3.
\end{aligned}$$

We claim that Ω_3 vanishes in the limit as $N \uparrow \infty$. By the martingale property we have the identity

$$\begin{aligned}
&\int_0^T dt\, E_N\left[N^{-d-1}\sum_{x,i} N^2 L_N\left\{\partial^2_{u_i} J\left(t, \frac{x}{N}\right) F_i(\tau_x\eta_t)\right\}\right] \\
&\quad = -\int_0^T dt\, E_N\left[N^{-d-1}\sum_{x,i}\partial_t\left\{\partial^2_{u_i} J\left(t, \frac{x}{N}\right) F_i(\tau_x\eta_t)\right\}\right] \\
&\qquad + E_N\left[N^{-d-1}\sum_{x,i}\partial^2_{u_i} J\left(T, \frac{x}{N}\right) F_i(\tau_x\eta_T)\right] \\
&\qquad - E_N\left[N^{-d-1}\sum_{x,i}\partial^2_{u_i} J\left(0, \frac{x}{N}\right) F_i(\tau_x\eta_0)\right].
\end{aligned}$$

As $N \uparrow \infty$, the right-hand side of the last expression converges to 0.

We turn now to the second integral of (8.1) that we denoted by Ω_2. The first step in the proof that Ω_2 vanishes as $N \uparrow \infty$ is to replace $R^\sigma_{i,j}(\eta^K_t(x))$ by $R^\sigma_{i,j}(m(t, x/N))$ and $\eta^K_t(x)$ by $m(t, x/N)$. This is the content of the next result.

LEMMA 8.1. *For every $t > 0$, every smooth function J: $[0, t] \times T^d \to \mathbb{R}$ and every smooth function a: $[0, 1] \to \mathbb{R}$,*

$$\lim_{l\to\infty}\lim_{N\to\infty}\int_0^t ds\, E_N\bigg[N^{1-d}\sum_x J_s\Big(\frac{x}{N}\Big)\Big\{a(\eta^l(x)) - a\Big(m_s\Big(\frac{x}{N}\Big)\Big)\Big\} \times \frac{1}{(2l'+1)^d}\sum_{|y-x|\le l'}\nabla_{e_i}\eta(y)\bigg] = 0.$$

PROOF. Since $\nabla_{e_i}\eta(y) = -\eta(y)(1-\eta(y+e_i)) + \eta(y+e_i)(1-\eta(y))$, the expected value appearing in the statement of the lemma may be rewritten as

$$N^{1-d}\sum_x J_s\Big(\frac{x}{N}\Big)\int\Big[a(\eta_s^l(x)) - a\Big(m_s\Big(\frac{x}{N}\Big)\Big)\Big] \times \frac{1}{(2l'+1)^d}\sum_{|y-x|\le l'}\eta(y)(1-\eta(y+e_i))\{f_s(\eta^{y,\,y+e_i}) - f_s(\eta)\}\nu_\rho(d\eta).$$

By the Schwarz inequality this expression is bounded above by

$$\begin{aligned}&\frac{N^{1-d}}{NA(2l'+1)^d}\sum_x\sum_{|y-x|\le l'}J_s^2\Big(\frac{x}{N}\Big)\int\Big[a(\eta_s^l(x)) - a\Big(m_s\Big(\frac{x}{N}\Big)\Big)\Big]^2 \\ &\qquad\times\eta(y)(1-\eta(y+e_i))\Big\{\sqrt{f_s(\eta^{y,\,y+e_i})} + \sqrt{f_s(\eta)}\Big\}^2\nu_\rho(d\eta) \\ &+\frac{N^{2-d}A}{(2l'+1)^d}\sum_x\sum_{|y-x|\le l'}\int\eta(y)(1-\eta(y+e_i)) \\ &\qquad\times\Big\{\sqrt{f_s(\eta^{y,\,y+e_i})} - \sqrt{f_s(\eta)}\Big\}^2\nu_\rho(d\eta)\end{aligned}$$

for every positive A. Since a is smooth, the first term is bounded by

$$N^{-d}C(J,a)A^{-1}\sum_x E_N\bigg[\bigg|\eta_s^l(x) - m\Big(s,\frac{x}{N}\Big)\bigg|^2\bigg],$$

which converges to 0 as $N\uparrow\infty$ and then $l\uparrow\infty$ by the law of large numbers.

The second term is bounded by A const $N^{2-d}D_N(f_s)$. We conclude the proof by letting $A\downarrow 0$ in the penultimate formula and invoking the content of Proposition 4.1. □

Notice that we may let l depend on N in the statement of the last lemma. In this case we just need to require that $l_N \ll N$.

We now return to the proof of the claim that Ω_2 vanishes as $N\uparrow\infty$. Since R^σ is smooth, we can use summation by parts so that the difference operator ∇ will act on a smooth function. Since $N\nabla$ is of order 1, by the law of large numbers, as $N\uparrow\infty$, the second integral of (8.1) converges to

$$\sum_{i,\,j}\int_0^T dt\int du\,\partial_{u_i}^2 J(t,u)\big[R_{i,\,j}^\sigma(m(t,u)) - 2m(t,u)\delta_{i,\,j}\big](\partial_{u_j}m)(t,u).$$

Letting $\sigma \downarrow 0$, since R^σ converges to R uniformly on compact subsets of $(0,1)$, we obtain that this expression converges to the expression appearing on the right-hand side of the statement of Lemma 3.1.

Finally, by Proposition 4.1 and Corollary 6.1, the limit, as $N \uparrow \infty$, of the first integral of (8.1) is bounded above by

$$\frac{C_1}{\delta}\sum_i \int_0^T ds \int du\, (\partial^2_{u_i} J(s,u))^2 \mathbb{V}_{m(s,u)}\big(W_{i,\,m(s,u),\,\sigma}\big) + C_2\delta,$$

where

$$\begin{aligned} W_{i,m,\sigma}(\eta) &= [2\eta(0)\eta(e_i) - \eta(-e_i)\eta(e_i)]^m + 2m\nabla_{e_i}\eta(0) \\ &\quad - \sum_j R^\sigma_{i,j}(m)\,\nabla_{e_j}\eta(0) - LF_i(\eta). \end{aligned}$$

Here we adopted the notation introduced in Section 6. As $\sigma \downarrow 0$, the last integral converges to

$$\frac{C_1}{\delta}\sum_i \int_0^T ds \int du\, (\partial^2_{u_i} J(s,u))^2 \mathbb{V}_{m(s,u)}\big(W_{i,\,m(s,u),\,0}\big).$$

Since the solution m of (2.4) is bounded away from 0 and 1, to conclude the proof of Lemma 3.1, it is enough to prove the following result:

LEMMA 8.2. *For every $\delta > 0$,*

$$\inf_{F\in\mathscr{G}} \sup_{\delta\le m\le 1-\delta} \mathbb{V}_m(W_{i,m,0}) = 0$$

for $1 \le i \le d$.

This concludes the proof of Lemma 3.1. □

9. Proof of Proposition 4.2.

The proof of this result follows closely the proof of Proposition 4.1. For this reason we will omit some details.

For $1 \le i \le d$, fix functions $F_i(\eta)$ and $G_i(\eta)$ in $\mathscr{G}$ and denote $(F_1,\dots,F_d)$ and $(G_1,\dots,G_d)$ by F and G. Recall the definition of $\psi_t^{F,G}$ given in Section 4. We start computing the time derivative of the entropy of f_t with respect to $\psi_t^{F,G}$, which is equal to

$$-N^2 D_N\big(f_t\,\big|\psi_t^{F,G}\big) + \int \big(\psi_t^{F,G}\big)^{-1}\big(N^2L_N^* - \partial_t\big)\psi_t^{F,G} f_t\, d\nu_\rho. \tag{9.1}$$

A careful calculation, taking into account computations already done in the proof of Proposition 4.1 (Section 7), shows that $(\psi_t^{F,G})^{-1}(N^2L_N^* - \partial_t)\psi_t^{F,G}$ is equal to

$$
\begin{aligned}
&\sum_{x,i} \partial_{u_i}^2 \lambda\left(t, \frac{x}{N}\right)\{\tau_x h_i - \phi(m) - \phi'(m)(\eta_x - m)\} \\
&\quad + \sum_{x,i} \left(\partial_{u_i}\lambda\left(t, \frac{x}{N}\right)\right)^2 \{\tau_x g_i - \varphi(m) - \varphi'(m)(\eta_x - m)\} \\
&\quad + \sum_{x,i} \partial_{u_i}^2 \lambda\left(t, \frac{x}{N}\right)\phi(m) + \left(\partial_{u_i}\lambda\left(t, \frac{x}{N}\right)\right)^2 \varphi(m) \\
&\quad - \sum_{x,i} \partial_{u_i}^2 \lambda\left(t, \frac{x}{N}\right) L_N F_i(\tau_x \eta) \\
&\quad - \sum_{x,i} \left(\partial_{u_i}\lambda\left(t, \frac{x}{N}\right)\right)^2 L_N G_i(\tau_x \eta) \\
&\quad + \partial_t \log \frac{Z_t^{F,G}}{Z_t} \qquad\qquad (9.2) \\
&\quad - \frac{1}{N}\sum_{x,i} \partial_{u_i}^3 \lambda\left(t, \frac{x}{N}\right)\tau_x v_i \\
&\quad + \frac{1}{N}\sum_{x,i} \partial_{u_i}\lambda\left(t, \frac{x}{N}\right)\partial_{u_i}^2\lambda\left(t, \frac{x}{N}\right)\tau_x g_i + \frac{1}{3}\left(\partial_{u_i}\lambda\left(t, \frac{x}{N}\right)\right)^3 \tau_x w_i \\
&\quad - \frac{1}{N}\sum_{x,i} \partial_{u_i}\lambda\left(t, \frac{x}{N}\right) W_{x,x+e_i}\nabla_{x,x+e_i}\left(\sum_{y,j}\partial_{u_j}^2\lambda\left(t, \frac{y}{N}\right)F_j(\tau_y\eta)\right) \\
&\quad - \frac{1}{N}\sum_{x,i} \partial_{u_i}\lambda\left(t, \frac{x}{N}\right) W_{x,x+e_i}\nabla_{x,x+e_i}\left(\sum_{y,j}\left(\partial_{u_j}\lambda\left(t, \frac{y}{N}\right)\right)^2 G_j(\tau_y\eta)\right) \\
&\quad + O(N^{d-2}).
\end{aligned}
$$

In this formula $Z_t^{F,G}$ and Z_t are normalizing constants associated to the densities $\psi_t^{F,G}$ and ψ_t, respectively, and $\nabla_{x,x+e_i} f(\eta)$ denotes $f(\eta^{x,x+e_i}) - f(\eta)$. We now compute separately the limit of the time integral of the expected value of each of these terms divided by N^{d-1}.

We will deal with the first and second lines of (9.2) as we have done in the proof of Proposition 4.1. Recall the definition of $V_{i,l}(\eta)$ and $V'_{i,l}(\eta)$. As shown in the proof of Proposition 4.1, there exists $\gamma_0 > 0$ such that the time integral of the expected value of the first line of (9.2) divided by N^{d-1} is bounded above by

$$
\begin{aligned}
&\alpha \int_0^t ds\, N^{1-d} \sum_{x,i} \partial_{u_i}^2 \lambda_s\left(\frac{x}{N}\right) \int \tau_x V_{i,K}(\eta) f_s \, d\nu_\rho \\
&\qquad + \gamma_0 \int_0^t ds\, N^{1-d} H_N(f_s) + o_{N,B}(1),
\end{aligned}
$$

and there exists $\gamma_0' > 0$ such that the time integral of the expected value of the second line of (9.2) divided by N^{d-1} is bounded above by

$$\int_0^t ds\, N^{1-d} \sum_{x,\, i} \left(\partial_{u_i} \lambda_s\left(\frac{x}{N}\right)\right)^2 \int \tau_x V'_{i,\, K}(\eta) f_s\, d\nu_\rho$$
$$+ \gamma_0' \int_0^t ds\, N^{1-d} H_N(f_s) + o_{N,\, B}(1)$$

plus a term which converges to

$$\sum_{i=1}^d \int_0^t ds \int_{T^d} du\ \tfrac{1}{2}(\partial_{u_i}\lambda(t, u))^2\, \partial_{u_i} m_t(u).$$

By the integration by parts formula, the third line of (9.2) divided by N^{d-1} vanishes in the limit when $N \uparrow \infty$.

The sixth line, integrated in time and divided by N^{d-1}, is equal to the difference of

$$N^{1-d} \log E_{\psi_t}\bigg[\exp\bigg\{-N^{-2} \sum_{x,\, i} \partial^2_{u_i} \lambda_t\left(\frac{x}{N}\right) F_i(\tau_x \eta) - N^{-2} \sum_{x,\, i} \left(\partial_{u_i} \lambda_t\left(\frac{x}{N}\right)\right)^2 G_i(\tau_x \eta)\bigg\}\bigg] \tag{9.3}$$

and a similar term taken at time 0. In this formula, E_{ψ_t} represents expectation with respect to the measure $\psi_t(\eta)\nu_\rho(d\eta)$. Since each F_i, G_i is a cylinder function and since $\psi_t(\eta)\nu_\rho(d\eta)$ is a product measure, by Hölder's inequality we obtain that the last expression is bounded above by

$$\frac{N^{1-d}}{d\bar{l}^d} \sum_{x,\, i} \log E_{\psi_t}\bigg[\exp\bigg\{-\frac{d\bar{l}^d}{N^2}\, \partial^2_{u_i} \lambda_t\left(\frac{x}{N}\right) F_i(\tau_x \eta) - \frac{d\bar{l}^d}{N^2} \left(\partial_{u_i} \lambda_t\left(\frac{x}{N}\right)\right)^2 G_i(\tau_x \eta)\bigg\}\bigg].$$

Here l is a positive integer such that the support of each cylinder function F_i and G_i is contained in $\{-l, \dots, l\}^d$. From the elementary inequalities $\log(1+u) \le u$ and $e^u - 1 \le u + 2^{-1}u^2 e^{|u|}$, and since $\operatorname{div} \lambda$, F_i and G_i are bounded functions, we obtain that the last sum is bounded above by

$$-N^{-d-1} \sum_{x,\, i} E_{\psi_t}\left[\partial^2_{u_i} \lambda\left(t, \frac{x}{N}\right) F_i(\tau_x \eta) + \left(\partial_{u_i} \lambda\left(t, \frac{x}{N}\right)\right)^2 G_i(\tau_x \eta)\right]$$

plus a small error of $O(N^3)$. As $N \uparrow \infty$, this expression converges to 0. On the other hand, by Jensen's inequality, (9.3) is bounded below by

$$-N^{-d-1} \sum_{x,\, i} E_{\psi_t}\left[\partial^2_{u_i} \lambda\left(t, \frac{x}{N}\right) F_i(\tau_x \eta) + \left(\partial_{u_i} \lambda\left(t, \frac{x}{N}\right)\right)^2 G_i(\tau_x \eta)\right]$$

and we have just seen that this expression converges to 0. Therefore, the sixth line of (9.2), integrated in time and divided by N^{d-1}, is equal to

$$N^{1-d}\left\{ \log \frac{Z_t^{F,G}}{Z_t} - \log \frac{Z_0^{F,G}}{Z_0} \right\},$$

which converges to 0 as $N \uparrow \infty$.

As shown in Section 7, the time integral of the expected value of the seventh and eighth lines of (9.2) divided by N^{d-1} converges as $N \uparrow \infty$ to

$$\sum_{i=1}^{d} \int_0^t ds \int_{T^d} du \big\{ -\alpha\, m_s^2(u)\, \partial_{u_i}^3 \lambda_s(u) + \partial_{u_i} \lambda_s(u)\, \partial_{u_i}^2 \lambda_s(u) \varphi(m_s(u)) \big\}.$$

Furthermore, by the law of large numbers, the time integral of the expected value of the ninth line of (9.2) divided by N^{d-1} converges as $N \uparrow \infty$ to

$$-\sum_{i,j} \int_0^t ds \int du\, \partial_{u_i} \lambda(s,u)\, \partial_{u_j}^2 \lambda(s,u) \times \nu_{m(s,u)}\big[W_{0,e_i} \nabla_{0,e_i} \Gamma_{F_j} \big],$$

where $W_{0,e_i} = r_{e_i,0} - r_{0,e_i}$ is the current between 0 and e_i and for a cylinder function Ψ and Γ_Ψ represents the formal infinite sum

$$\Gamma_\Psi(\eta) = \sum_{y \in \mathbb{Z}^d} \Psi(\tau_y \eta).$$

We have also used here the smoothness of F, λ and m. The last expression is equal to

$$2 \sum_{i,j} \int_0^t ds \int du\, \partial_{u_i} \lambda(s,u)\, \partial_{u_j}^2 \lambda(s,u) \langle W_{0,e_i}, F_j \rangle_{m_s(u),0},$$

where

$$\langle W_{0,e_i}, f \rangle_{m,0} = \sum_x \langle W_{0,e_i}, \tau_x f \rangle_m.$$

With the same arguments, the time integral of the expected value of the tenth line of (9.2) divided by N^{d-1} converges as $N \uparrow \infty$ to

$$2 \sum_{i,j} \int_0^t ds \int du\, \partial_{u_i} \lambda(s,u) (\partial_{u_j} \lambda(s,u))^2 \langle W_{0,e_i}, G_j \rangle_{m_s(u),0}.$$

Since the entropy $N^{1-d} H_N(f_s)$ is bounded by $N^{1-d} H(f_s \,|\, \psi_s^{F,G}) + o_N(1)$ (cf. proof of Proposition 4.3), up to this point we have shown that

$$N^{1-d} H\big(f_t \,|\, \psi_t^{F,G}\big) + \int_0^t ds\, N^{2-d} D_N\big(f_s \,|\, \psi_s^{F,G}\big)$$

is bounded above by

$$\begin{aligned} & N^{1-d} H\big(f_0 \,|\, \psi_0^{F,G}\big) + (\gamma_0 + \gamma_0') \int_0^t ds\, N^{1-d} H\big(f_s \,|\, \psi_s^{F,G}\big) \\ & \quad + \int_0^t ds\, E_N\bigg[N^{1-d} \sum_{x,i} \partial_{u_i}^2 \lambda\Big(s, \frac{x}{N}\Big) \big\{ \alpha \tau_x V_{i,K}(\eta_s) - L_N F_i(\tau_x \eta_s) \big\} \bigg] \end{aligned} \tag{9.4}$$

$$+\int_0^t ds\, E_N\Bigg[N^{1-d}\sum_{x,i}\left(\partial_{u_i}\lambda\left(s,\frac{x}{N}\right)\right)^2\{\tau_x V'_{i,K}(\eta_s)-L_N G_i(\tau_x\eta_s)\}\Bigg]$$

$$+\int_0^t ds\int du\sum_i \partial^2_{u_i}\lambda(s,u)\,2\alpha\, m_s(u)\,\partial_{u_i} m_s(u)$$

$$-\int_0^t ds\int du\sum_i\left(\partial_{u_i}\lambda\left(s,\frac{x}{N}\right)\right)^2\partial_{u_i}m_s(u)[(2\alpha-1)m_s(u)-3\alpha m_s(u)^2]$$

$$+2\sum_{i,j}\int_0^t ds\int du\,\partial_{u_i}\lambda(s,u)\,\partial^2_{u_j}\lambda(s,u)\,\langle W_{0,e_i},F_j\rangle_{m_s(u),0}$$

$$+2\sum_{i,j}\int_0^t ds\int du\,\partial_{u_i}\lambda(s,u)(\partial_{u_j}\lambda(s,u))^2\,\langle W_{0,e_i},G_j\rangle_{m_s(u),0}$$

plus a small error of $o_N(1)$.

We now concentrate our attention on the second, fourth and sixth lines of this sum. Recall that R^σ is a smooth approximation of the coefficient R. Let $W_{i,K}(\eta)$ be the cylinder function defined by

$$W_{i,K}(\eta)=V_{i,K}(\eta)+2\eta^K(0)(\bar K')^{-d}\sum_{|y|\le K'}\nabla_{e_i}\eta(y)$$
$$-\sum_{j=1}^d R^\sigma_{i,j}(\eta^K(0))\left\{(\bar K')^{-d}\sum_{|y|\le K'}\nabla_{e_j}\eta(y)\right\}-\frac{2}{\alpha}L_N F_i(\eta).$$

In this formula, K' stands for $K-1$. Notice also that to reduce notation we omitted the dependence of $W_{i,K}$ on σ and F.

The second line of (9.4) is equal to

$$\alpha\int_0^t ds\,E_N\Bigg[N^{1-d}\sum_{x,i}\partial^2_{u_i}\lambda_s\left(\frac{x}{N}\right)\tau_x W_{i,K}(\eta_s)\Bigg]$$
$$+\alpha\int_0^t ds\,E_N\Bigg[N^{1-d}\sum_{x,i,j}\partial^2_{u_i}\lambda_s\left(\frac{x}{N}\right)R^\sigma_{i,j}(\eta^K_s(x))(\bar K')^{-d}\sum_{|y-x|\le K'}\nabla_{e_j}\eta_s(y)\Bigg]$$
$$-2\alpha\int_0^t ds\,E_N\Bigg[N^{1-d}\sum_{x,i}\partial^2_{u_i}\lambda_s\left(\frac{x}{N}\right)\eta^K_s(x)(\bar K')^{-d}\sum_{|y-x|\le K'}\nabla_{e_j}\eta_s(y)\Bigg]$$
$$+\int_0^t ds\,E_N\Bigg[N^{1-d}\sum_{x,i}\partial^2_{u_i}\lambda_s\left(\frac{x}{N}\right)\tau_x L_N F_i(\eta_s)\Bigg].$$

We have seen in last section that the second line of this expression converges, as $N\uparrow\infty$, to

$$\alpha\sum_{i,j}\int_0^t ds\int du\,\partial^2_{u_i}\lambda(s,u)R^\sigma_{i,j}(m(s,u))\,\partial_{u_j}m(s,u).$$

By similar reasoning, the third line converges to

$$-2\alpha\int_0^t ds\int du\sum_i\partial^2_{u_i}\lambda(s,u)\;m_s(u)\,\partial_{u_i}m_s(u).$$

Notice that this term cancels the fourth line of (9.4). The last line converges, as $N \uparrow \infty$, to 0.

Furthermore, from Corollary 6.1 and Proposition 4.1, the limit as $N \uparrow \infty$ of the first line is bounded above by

$$\frac{\alpha}{\delta} \sum_i \int_0^t ds \int du \, (\partial_{u_i}^2 \lambda(s,u))^2 \mathbb{V}_{m(s,u)}(W^*_{i,m(s,u),\sigma}) + C_0 t\delta$$

for some universal constant C_0 and where $W^*_{i,m,\sigma}(\eta)$ is given by

$$\begin{aligned} W^*_{i,m,\sigma}(\eta) &= [2\eta(0)\eta(-e_i) - \eta(-e_i)\eta(e_i)]^m + 2\,m\,\nabla_{e_i}\eta(0) \\ &\quad - \sum_{j=1}^d R^\sigma_{i,j}(m)\,\nabla_{e_j}\eta(0) - \frac{2}{\alpha} LF_i(\eta). \end{aligned}$$

Define $W^*_{i,m,0}(\eta)$ to be the cylinder function $W^*_{i,m,\sigma}$ with $R_{i,j}(m)$ replacing $R^\sigma_{i,j}(m)$ in the above formula. From the variational formula for $\mathbb{V}_m$, the triangular inequality and the properties of $\mathbb{V}_m$ stated in Section 10, we obtain that for every $0 \le m \le 1$,

$$\begin{aligned} \mathbb{V}_m(W^*_{i,m,\sigma}) &\le 2\,\mathbb{V}_m(W^*_{i,m,0}) \\ &\quad + 2 \sum_{j,k=1}^d \big[R^\sigma_{i,j}(m) - R_{i,j}(m)\big] \\ &\qquad \times \big[R^\sigma_{i,k}(m) - R_{i,k}(m)\big]\mathbb{V}_m\big(\nabla_{e_j}\eta_0, \nabla_{e_k}\eta_0\big). \end{aligned}$$

Since R^σ converges pointwise to R, letting $\sigma \downarrow 0$, one obtains that the sum of the second, fourth and sixth lines of (9.4) is bounded above by

$$\begin{aligned} &\frac{\alpha}{\delta} \sum_i \int_0^t ds \int du \, (\partial_{u_i}^2 \lambda(s,u))^2 \mathbb{V}_{m(s,u)}\big(W^*_{i,m(s,u),0}\big) + C_0\,\delta t \\ &\quad + \alpha \sum_{i,j} \int_0^t ds \int du \, \partial_{u_i}^2 \lambda(s,u) R_{i,j}(m(s,u))(\partial_{u_j} m)(s,u) \\ &\quad + 2 \sum_{i,j} \int_0^t ds \int du \, \partial_{u_i} \lambda(s,u)\, \partial_{u_j}^2 \lambda(s,u)\, \langle W_{0,e_i}, F_j \rangle_{m_s(u),0} \end{aligned}$$

for every positive δ.

We turn now to the third, fifth and seventh lines of (9.4). Let $W'_{i,K}(\eta)$ be the cylinder function defined by

$$\begin{aligned} W'_{i,K}(\eta) &= V'_{i,K}(\eta) - \big[(2\alpha-1)\eta^K(0) - 3\alpha\eta^K(0)^2\big](\bar{K}')^{-d} \sum_{|y|\le K'} \nabla_{e_i}\eta(y) \\ &\quad - \sum_{j=1}^d S_{i,j}(\eta^K(0)) \left\{ (\bar{K}')^{-d} \sum_{|y|\le K'} \nabla_{e_j}\eta(y) \right\} - 2L_N G_i(\eta). \end{aligned}$$

The choice of S_{ij} will be explained later.

By similar arguments, the third, fifth and seventh lines of (9.4) are bounded by

$$\frac{1}{\delta}\sum_i \int_0^t ds \int du\, (\partial_{u_i}\lambda(s,u))^4 \mathbb{V}_{m(s,u)}\big(W'^{*}_{i,m(s,u)}\big) + C\,\delta t$$
$$+\sum_{i,j}\int_0^t ds \int du\,(\partial_{u_i}\lambda(s,u))^2 S_{i,j}(m(s,u))\,\partial_{u_j} m(s,u)$$
$$+2\sum_{i,j}\int_0^t ds \int du\,\partial_{u_i}\lambda(s,u)(\partial_{u_j}\lambda(s,u))^2\,\langle W_{0,e_i}, G_j\rangle_{m_s(u),0}$$

for every positive δ. Here

$$W'^{*}_{i,m} = \big[\tfrac{1}{2}(r_{0,e_i}(\eta) + r_{e_i,0}(\eta) - \eta_{e_i} - \eta_0)\big]^m - \big[(2\alpha-1)m - 3\alpha m^2\big]\nabla_{e_i}\eta(0)$$
$$-\sum_j S_{ij}(m)\,\nabla_{e_j}\eta(0) - 2LG_i(\eta).$$

Using the two following lemmas, Gronwall's inequality and letting $\delta\downarrow 0$, we conclude the proof of the proposition. □

LEMMA 9.1. *For each $1\le i\le d$, there exists a sequence of functions $F_k^i(\eta)$ in $\mathscr{G}$ such that*

$$\lim_{k\to\infty}\sup_{\delta_0\le m\le 1-\delta_0}\mathbb{V}_m\Big(Z_{m,i}(\eta) - \sum_j R_{i,j}(m)\,\nabla_{e_j}\eta(0) - LF_k^i(\eta)\Big) = 0, \tag{9.5}$$

where

$$Z_{m,i}(\eta) = [2\eta(0)\eta(-e_i) - \eta(-e_i)\eta(e_i)]^m + 2\,m\,\nabla_{e_i}\eta(0).$$

Moreover,

$$\lim_{k\to\infty}\langle W_{0,e_l}, F_k^i\rangle_{m,0} = -R_{il}(m)\,m\,(1-m).$$

This lemma is proved in Section 10.

LEMMA 9.2. *For each $1\le i\le d$, there exists a sequence of functions $G_k^i(\eta)$ in $\mathscr{G}$ such that*

$$\lim_{k\to\infty}\sup_{\delta_0\le m\le 1-\delta_0}\mathbb{V}_m\left(Z'_{m,i}(\eta) - \sum_j S_{ij}(m)\,\nabla_{e_j}\eta(0) - LG_k^i\right) = 0,$$

where

$$Z'_{m,i}(\eta) = \big[\tfrac{1}{2}(r_{0,e_i}(\eta) + r_{e_i,0}(\eta) - \eta_{e_i} - \eta_0)\big]^m - \big[(2\alpha-1)m - 3\alpha m^2\big]\nabla_{e_i}\eta(0).$$

Moreover,

$$\lim_{k\to\infty}\langle W_{0,e_l}, G_k^i\rangle_{m,0} = -S_{il}(m)\,m\,(1-m).$$

The proof of this lemma is omitted since it is similar to the proof of Lemma 9.1.

10. Properties of the coefficient R. In this section we state the properties of the coefficient matrix R used in the previous sections.

Recall the definitions of $\mathscr{G}$ and $\mathscr{G}_m$. For each function g in $\mathscr{G}_m$ we define $\mathbb{V}_m(g)$ by

$$\mathbb{V}_m(g) = \sup_{\substack{\beta\in\mathbb{R}^d\\ h\in\mathscr{G}}}\bigg\{2\langle g, h\rangle_{m,0} + 2\sum_i \beta_i t_i(g) - \|\beta\|^2\langle r_{0,e}(m)\rangle_m - \sum_i \langle r_{0,e_i}(m)(\nabla_{0e_i}\Gamma_h)^2\rangle_m + 2\sum_i \beta_i\langle W_{0,e_i}, h\rangle_{m,0}\bigg\}.$$

In this formula, for $1 \le i \le d$, $g \in \mathscr{G}_m$ and $h \in \mathscr{C}$, $t_i(g)$ and $\langle g, h\rangle_{m,0}$ are given by

$$t_i(g) = \sum_x \langle g, x_i\eta(x)\rangle_m, \qquad \langle g, h\rangle_{m,0} = \sum_x \langle g, \tau_x h\rangle_m,$$

and $\langle\cdot,\cdot\rangle_m$ denotes the inner product in $L^2(\nu_m)$. By Lemma 6.1, the finite volume variance converges to the infinite volume variance uniformly in $[0,1]$: $\mathbb{V}_m(g) = \lim_{l\to\infty} V_l(g,m)$, where $V_l(g,m)$ is defined by (6.2).

From the definition of $\mathbb{V}_m$ we may introduce the bilinear form $\mathbb{V}_m(\cdot,\cdot)$ on $\mathscr{G}_m$ by polarization

$$\mathbb{V}_m(g,h) = \tfrac{1}{4}\big\{\mathbb{V}_m(g+h) - \mathbb{V}_m(g-h)\big\}.$$

Denote by $\overline{\mathscr{G}}_m$ the closure of $\mathscr{G}_m$ with respect to $\mathbb{V}_m$ and by $\mathscr{N}_m$ the kernel of $\mathbb{V}_m$. Then $(\overline{\mathscr{G}}_m/_{\mathscr{N}_m}, \mathbb{V}_m)$, which we denote by $\mathscr{H}_m$, is a Hilbert space.

We easily prove that for $1\le i\le d$ and $g, h\in\mathscr{G}_m$:

1. $\mathbb{V}_m(h, Lg) = -\langle g, h\rangle_{m,0}$.
2. $\mathbb{V}_m(h, W_{0,e_i}) = -t_i(h)$.
3. $\mathbb{V}_m(\nabla_{e_i}\eta_0, Lg) = 0$.
4. $\mathbb{V}_m(\nabla_{e_i}\eta_0, W_{0,e_j}) = -m(1-m)\delta_{ij}$.
5. $\mathbb{V}_m(W_{0,e_i}, W_{0,e_j}) = m(1-m)(1+2\alpha m)\delta_{ij}$.
6. $\mathbb{V}_m\Big(\sum_i \beta_i W_{0,e_i} + Lh\Big) = \|\beta\|^2\langle r_{0,e}\rangle_m + \sum_i\langle r_{0,e_i}(\nabla_{0,e_i}\Gamma_h)^2\rangle_m - 2\sum_i \beta_i\langle W_{0,e_i}, h\rangle_{m,0}$.

Hence,

$$\mathbb{V}_m(g) = \sup_{\substack{\beta\in\mathbb{R}^d\\ h\in\mathscr{G}}}\bigg\{-2\mathbb{V}_m(g, Lh) - 2\sum_i\beta_i\mathbb{V}_m(g, W_{0,e_i}) - \mathbb{V}_m\Big(\sum_i\beta_i W_{0,e_i} + Lh\Big)\bigg\}.$$

We deduce the following theorem and corollaries.

THEOREM 10.1. $\mathscr{H}_m = \overline{L\mathscr{G}\oplus(W_{0,e_i})_{1\le i\le d}}$.

COROLLARY 10.1. *For each g in $\mathscr{G}_m$, there exists a unique vector β in $\mathbb{R}^d$ such that*

$$g - \sum_i \beta_i W_{0,e_i} \in \overline{L\mathscr{G}} \quad \text{in } \mathscr{H}_m.$$

COROLLARY 10.2. *For each g in $\mathscr{G}_m$, there exists a unique vector β in $\mathbb{R}^d$ such that*

$$g - \sum_i \beta_i \nabla_{e_i} \eta_0 \in \overline{L\mathscr{G}} \quad \text{in } \mathscr{H}_m.$$

From the above corollary, there exists a unique matrix $A(m)$ such that

$$W_{0,e_i} - \sum_j A_{i,j}(m) \nabla_{e_j} \eta(0) \in \overline{L\mathscr{G}} \quad \text{for } 1 \le i \le d. \tag{10.1}$$

Hence, for all vectors $\alpha \in \mathbb{R}^d$,

$$\inf_{g \in \mathscr{G}} \mathbb{V}_m\biggl(\sum_i \alpha_i W_{0,e_i} - \sum_{i,j} \alpha_i A_{i,j} \nabla_{e_j} \eta(0) - Lg\biggr) = 0.$$

Using the fourth property of $\mathbb{V}_m$ mentioned at the beginning of the section, we can rewrite the last equality as

$$\inf_{g \in \mathscr{G}} \mathbb{V}_m\biggl(\sum_i \alpha_i W_{0,e_i} - Lg\biggr) + \alpha^t ABA^t \alpha + 2m(1-m)\alpha^t A \alpha = 0,$$

where $B_{j,k} = \mathbb{V}_m(\nabla_{e_j}\eta(0), \nabla_{e_k}\eta(0))$.

By formula (10.1), $\mathbb{V}_m(W_{0,e_i}, \nabla_{e_k}\eta(0)) = \sum_j A_{i,j} \mathbb{V}_m(\nabla_{e_j}\eta(0), \nabla_{e_k}\eta(0))$, that is, $AB = -m(1-m)I$. Thus,

$$\alpha^t A \alpha = \frac{-1}{m(1-m)} \inf_{g \in \mathscr{G}} \mathbb{V}_m\biggl(\sum_i \alpha_i W_{0,e_i} - Lg\biggr).$$

For $1 \le i \le d$, let $Z_{m,i}(\eta)$ be the cylinder function given by

$$Z_{m,i}(\eta) = [2\eta(0)\eta(-e_i) - \eta(-e_i)\eta(e_i)]^m + 2\,m\, \nabla_{e_i}\eta(0).$$

Notice that $\mathbb{V}_m(Z_{m,i}(\eta), W_{0,e_k}) = 0$. There exists a unique matrix $R(m)$ such that

$$Z_{m,i}(\eta) - \sum_j R_{i,j}(m) \nabla_{e_j}\eta(0) \in \overline{L\mathscr{G}}. \tag{10.2}$$

$R(m)$ is the coefficient of (2.5) and (2.6). Hence,

$$\mathbb{V}_m\bigl(Z_{m,i}(\eta), \nabla_{e_k}\eta(0)\bigr) = \sum_j R_{i,j}(m) \mathbb{V}_m\bigl(\nabla_{e_j}\eta(0), \nabla_{e_k}\eta(0)\bigr) = (RB)_{i,k}.$$

We obtain a formula for R:

$$R = \frac{-1}{m(1-m)} \mathbb{V}_m\bigl(Z_{m,i}(\eta), \nabla_{e_k}\eta(0)\bigr) A.$$

The following proofs are similar to those in [6].

LEMMA 10.1. *R is continuous in $(0,1)$.*

The following functional space plays a key role in the proof of the continuity of the coefficient R. Denote by $\mathscr{F}$ the space of functions F: $[0,1]\times\{0,1\}^{\mathbb{Z}^d}\to\mathbb{R}$ such that:

1. For each $\rho\in[0,1]$, $F(\rho,\cdot)$ is a cylinder function with uniform support: There exists a finite set Λ such that for each $\rho\in[0,1]$, the support of $F(\rho,\cdot)$ is contained in Λ.
2. For each configuration η, $F(\cdot,\eta)$ is a smooth function.
3. For each density ρ, the cylinder function $F(\rho,\cdot)$ has mean zero with respect to all canonical measure $\nu_{K,m}$:

$$\nu_{K,m}[F(\rho,\cdot)]=0\quad\text{for some } K>0 \text{ and all } m.$$

PROOF OF LEMMA 10.1. Fix $\sigma>0$. Since for each $m\in[0,1]$, $Z_{m,i}(\eta)-\sum_j R_{i,j}(m)\nabla_{e_j}\eta(0)\in\overline{L\mathscr{G}}$, there exist cylinder functions $H_i(m,\eta)$ in $\mathscr{G}$ such that

$$\mathbb{V}_m\Big(Z_{m,i}(\eta)-\sum_j R_{i,j}(m)\,\nabla_{e_j}\eta(0)-LH_i(m,\eta)\Big)\le\sigma.$$

Since for each $G\in\mathscr{G}_m$, $\mathbb{V}_m(G(m,\cdot))$ is continuous in m, for each $m_0\in[0,1]$, there exists a neighborhood N_{m_0} of m_0 such that for m in N_{m_0},

$$\mathbb{V}_m\Big(Z_{m,i}(\eta)-\sum_j R_{i,j}(m_0)\,\nabla_{e_j}\eta(0)-LH_i(m_0,\eta)\Big)\le 2\sigma.$$

The family $(N_{m_0})_{m_0\in[0,1]}$ constitutes an open covering of $[0,1]$. We may therefore find a finite open subcovering and, by interpolation, construct continuous functions $R^\sigma_{ij}(m)$ and a function $H^\sigma_i(m,\eta)$ in $\mathscr{F}$ such that

$$\mathbb{V}_m\Big(Z_{m,i}(\eta)-\sum_j R^\sigma_{i,j}(m)\,\nabla_{e_j}\eta(0)-LH^\sigma_i(m,\eta)\Big)\le 4\sigma.$$

From the triangle inequality, we obtain

$$\begin{aligned}
&\mathbb{V}_m\Big(-\sum_j\,[R^\sigma_{i,j}(m)-R_{i,j}(m)]\,\nabla_{e_j}\eta(0)\Big)\\
&\quad\le\mathbb{V}_m\Big(-\sum_j\,[R^\sigma_{i,j}(m)-R_{i,j}(m)]\,\nabla_{e_j}\eta(0)-L(H^\sigma_i(m,\eta)-H_i(m,\eta))\Big)\\
&\quad\le 2\mathbb{V}_m\Big(Z_{m,i}(\eta)-\sum_j R^\sigma_{i,j}(m)\,\nabla_{e_j}\eta(0)-LH^\sigma_i(m,\eta)\Big)\\
&\qquad+2\mathbb{V}_m\Big(Z_{m,i}(\eta)-\sum_j R_{i,j}(m)\,\nabla_{e_j}\eta(0)-LH_i(m,\eta)\Big)\\
&\quad\le 2(4\sigma)+2\sigma=10\sigma.
\end{aligned}$$

Since

$$\mathbb{V}_m\Big(-\sum_j [R^\sigma_{i,j}(m)-R_{i,j}(m)]\,\nabla_{e_j}\eta(0), W_{0,e_l}\Big)=[R^\sigma_{i,l}(m)-R_{i,l}(m)]m(1-m),$$

by the Schwarz inequality and the previous bound, we obtain

$$\big|R^\sigma_{i,l}(m)-R_{i,l}(m)\big|\le\frac{\sqrt{10\sigma\mathbb{V}_m(W_{0,e_l})}}{m(1-m)}=\bigg(\frac{10(1+2\alpha m)}{m(1-m)}\bigg)^{1/2}\sqrt{\sigma}.$$

This expression is uniformly bounded by $C\sqrt{\sigma}$ on each compact subset of $(0,1)$. This proves that R_{ij} can be uniformly approximated by smooth functions on each compact subset of $(0,1)$ and is therefore continuous in $(0,1)$. □

PROOF OF LEMMA 9.1. Fix $1\le i\le d$ and $\varepsilon>0$. From the proof above, we know there exist $H(m,\eta)\in\mathscr{F}$ such that

$$\sup_{\delta\le m\le 1-\delta}\mathbb{V}_m\Big(Z_{m,i}(\eta)-\sum_j R_{i,j}(m)\,\nabla_{e_j}\eta(0)-LH_i(m,\eta)\Big)\le\varepsilon.$$

Fix a positive integer l and set $h_i(\eta)=H_i(\eta^l(0),\eta)$. We defined h in such a way that $h\in\mathscr{G}$. By the triangle inequality,

$$\begin{aligned}&\sup_{\delta\le m\le 1-\delta}\mathbb{V}_m\Big(Z_{m,i}(\eta)-\sum_j R_{i,j}(m)\,\nabla_{e_j}\eta(0)-Lh_i(\eta)\Big)\\&\qquad\le 2\varepsilon+2\sup_{\delta\le m\le 1-\delta}\mathbb{V}_m\big(Lh_i(\eta)-LH_i(m,\eta)\big).\end{aligned}\tag{10.3}$$

By the sixth property of the bilinear form $\mathbb{V}_m(\cdot,\cdot)$ with $\beta=0$,

$$\begin{aligned}&\mathbb{V}_m\big(Lh_i-LH_i(m,\eta)\big)\\&\qquad=\sum_{i=1}^d\Big\langle r_{0,e_i}\Big(\nabla_{0,e_i}\sum_x\tau_x\big[H_i(\eta^l(0),\eta)-H_i(m,\eta)\big]\Big)^2\Big\rangle_m.\end{aligned}$$

Since $\nabla_{0,e_i}\tau_x=\tau_x\nabla_{-x,-x+e_i}$, the previous expression is equal to

$$\sum_{i=1}^d\Big\langle\Big(\sqrt{r_{0,e_i}}\sum_x\tau_{-x}\nabla_{x,x+e_i}\big[H_i(\eta^l(0),\eta)-H_i(m,\eta)\big]\Big)^2\Big\rangle_m,$$

which can be rewritten, since ν_m is translation invariant, as

$$\sum_{i=1}^d\Big\langle\Big(\sum_x\sqrt{r_{x,x+e_i}}\nabla_{x,x+e_i}\big[H_i(\eta^l(0),\eta)-H_i(m,\eta)\big]\Big)^2\Big\rangle_m.$$

Since H_i belongs to $\mathscr{F}$, there exists a cube Λ such that we can restrict the sum above x to $x\in\Lambda$. The error term comes from jumps of a particle from Λ_l

to Λ_l^c or from jumps in the opposite direction. Since $H \in \mathscr{F}$, the error term is of $O(l^{-1})$. By the Schwarz inequality and since for every bond b and every $L^2(\nu_m)$ function g, $\langle(\nabla_b g)^2\rangle_m \le 4\langle g^2\rangle_m$, we obtain that the second term on the right-hand side of (10.3) is bounded above by

$$\sup_{\delta\le m\le 1-\delta}\left\{C(H)\left\langle\left[H_i(\eta^l(0),\eta) - H_i(m,\eta)\right]^2\right\rangle_m + O(l^{-2})\right\},$$

which vanishes as l goes to ∞ by the law of large number.

This concludes the first part of the lemma: we proved there exists a sequence of functions $h_k^i(\eta)$ in $\mathscr{G}$ such that

$$\lim_{k\to\infty}\sup_{\delta_0\le m\le 1-\delta_0}\mathbb{V}_m\left(Z_{m,i}(\eta) - \sum_j R_{i,j}(m)\nabla_{e_j}\eta(0) - Lh_k^i(\eta)\right) = 0.$$

By the Schwarz inequality,

$$\left|\mathbb{V}_m\left(Z_{m,i}(\eta) - \sum_j R_{i,j}(m)\nabla_{e_j}\eta(0) - Lh_i^k(\eta), W_{0,e_l}\right)\right|,$$

which is equal to

$$\left|\sum_j R_{ij}(m)m(1-m)\delta_{jl} - \mathbb{V}_m\left(Lh_i^k, W_{0,e_l}\right)\right|$$

because $\mathbb{V}_m(Z_{m,i}(\eta), W_{0,e_l}) = 0$ is bounded above by $o_k(1)\sqrt{\mathbb{V}_m(W_{0,e_l})}$. Hence,

$$\lim_{k\to\infty}\langle W_{0,e_l}, h_i^k\rangle_{m,0} = -R_{il}(m)\,m\,(1-m). \qquad \square$$

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UPRESA 6085
UNIVERSITÉ DE ROUEN
76821 MONT SAINT AIGNAN
FRANCE
E-MAIL: Elise.Janvresse@univ-rouen.fr