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DARLING–ERDŐS THEOREMS FOR NORMALIZED SUMS OF I.I.D. VARIABLES CLOSE TO A STABLE LAW

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Let $\xi, \xi_1, \dots$ be i.i.d. real-valued random variables and $S_n = \xi_1 + \cdots + \xi_n$. In the case when the distribution of ξ is close to a stable (α) law for some $\alpha \in (0,1) \cup (1,2)$, we investigate the asymptotic behavior in distribution of the maximum of normalized sums, $\max_{k=1,\dots,n} k^{-1/\alpha} S_k$. This completes the Darling–Erdős limit theorem for the case $\alpha = 2$.

1. Introduction. Let $\xi, \xi_1, \dots$ be i.i.d. real-valued random variables; we write $S_n = \xi_1 + \cdots + \xi_n$ for the partial sums. Darling and Erdős [6] have proven the following limit theorem for the maximum of normalized sums: if $\mathbb{E}(\xi) = 0$, $\mathbb{E}(\xi^2) = 1$ and $\mathbb{E}(|\xi|^3) < \infty$, then, for every $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\Big(\max_{k=1,\dots,n} k^{-1/2} S_k \le x a_2(n) + b_2(n)\Big) = \exp\{-e^{-x}\}, \tag{1.1}$$

with $a_2(n) = (2 \log\log n)^{-1/2}$ and

$$b_2(n) = \big(2\log\log n + \tfrac{1}{2}\log\log\log n - \tfrac{1}{2}\log(4\pi)\big) a_2(n).$$

Several extensions to cases when the assumption on the third moment is weakened have been obtained since, culminating with Einmahl [9] who proved that when ξ is a centered variable with variance 1, (1.1) holds if and only if

$$\mathbb{E}(\xi^2, |\xi| > t) = o(\log\log t) \quad \text{as } t \to \infty.$$

In the case when the foregoing condition fails, Einmahl also proved a general Darling–Erdős type theorem by slightly changing the normalization. We refer to [10] for an extension of (1.1) to martingales and further references.

The finite-variance case being completely treated, it is interesting to consider the infinite-variance case. Typically, we should like to obtain an analogue of (1.1) in the situation where the variable ξ belongs to the normal domain of attraction of some (strictly) stable law of index $\alpha \in (0,2)$, that is, when the normalized partial sums $n^{-1/\alpha} S_n$ converge in distribution towards a nondegenerate law which is then a strictly stable law of index α. Here is our first result in this vein.

Theorem 1. *Suppose that, for some $\alpha \in (0,1) \cup (1,2)$ and some real number $c > 0$, the distribution function $F(x) = \mathbb{P}(\xi \le x)$ fulfils*

$$1 - F(x) \sim c x^{-\alpha} \quad \textit{and} \quad F(-x) = O(x^{-\alpha}), \qquad x \to \infty,$$

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and that $\mathbb{E}(\xi) = 0$ *if* $\alpha > 1$. *Then, for every* $x \geq 0$, *we have*

$$\lim_{n\to\infty} \mathbb{P}\Big(\max_{k=1,\dots,n} k^{-1/\alpha} S_k \leq x(\log n)^{1/\alpha} \Big) = \exp\{-cx^{-\alpha}\}.$$

Note that the hypotheses of Theorem 1 are strictly weaker than the requirement that ξ belongs to the normal domain of attraction of a strictly stable law of parameters (α, β) with $\alpha \in (0,1) \cup (1,2)$ and $\beta \in (-1, 1]$. Theorem 1 relies on a large deviation result à la Nagaev (see [13]). More precisely, the hypotheses of Theorem 1 ensure that the tail distribution of the normalized sum $n^{-1/\alpha} S_n$ is equivalent to that of the normalized extreme of the n first steps, $n^{-1/\alpha} \max_{k=1,\dots,n} \xi_k$, uniformly as $n \to \infty$. Loosely speaking, this entails that, for large n, the maximum of the normalized sums $\max_{k=1,\dots,n} k^{-1/\alpha} S_k$ is close to the extreme of the normalized n first steps, $\max_{k=1,\dots,n} k^{-1/\alpha} \xi_k$. The asymptotic behavior in distribution of the latter quantity is easily determined, and this yields Theorem 1.

In comparison with (1.1), the asymptotic behavior of the maximum of normalized sums stated in Theorem 1 is thus quite crude; and intuitively, this is mainly due to the prevalence of the contribution of the large increments. The next purpose of this work is to show that when the upper tail distribution of the step is much smaller than the lower tail, the maximum of the normalized sums has again a smooth asymptotic behavior as in (1.1). Here is the precise statement.

THEOREM 2. *Suppose that* $\mathbb{E}(\xi) = 0$ *and that there are* $\alpha \in (1,2)$ *and* $c > 0$ *such that the distribution function* $F(x) = \mathbb{P}(\xi \leq x)$ *fulfills*

$$F(-x) = cx^{-\alpha} + O(x^{-\alpha-\varepsilon}) \quad \text{and} \quad 1 - F(x) = O(x^{-\alpha-\varepsilon}), \qquad x \to \infty$$

for some $\varepsilon > 0$. *Then, for every* $x \in \mathbb{R}$, *we have*

$$\lim_{n\to\infty} \mathbb{P}\Big(\max_{k=1,\dots,n} k^{-1/\alpha} S_k \leq x a_{\alpha,c}(n) + b_{\alpha,c}(n) \Big) = \exp\{-e^{-x}\},$$

with

$$a_{\alpha,c}(n) = \left(\frac{c\Gamma(2-\alpha)}{\log\log n} \right)^{1/\alpha}$$

and

$$b_{\alpha,c}(n) = a_{\alpha,c}(n) \left(\frac{\alpha}{\alpha - 1} \log\log n + \frac{1}{2} \log\log\log n - \frac{1}{2} \log(2\alpha\pi) \right).$$

The hypotheses of Theorem 2 ensure that $n^{-1/\alpha} S_n$ converges in distribution towards a completely asymmetric stable law of index α, but they are more restrictive than this. Presumably, they can be weakened; I have made no attempt at obtaining the most general result.

Just as for the Darling–Erdős theorem, the proof of Theorem 2 is divided into two parts. First, one establishes an analogue of Theorem 2 when the normalized sum is replaced by some completely asymmetric stable Ornstein–Uhlenbeck process; see Theorem 3 below. Then one deduces Theorem 2 by an argument involving strong approximation. More precisely, recall that a (strictly) stable Lévy process is a càdlàg process $X = (X_t\colon t \geq 0)$ with stationary and independent increments, which has the scaling property: for some real number $\alpha \in (0, 2]$ known as the index, the variables X_t and $t^{1/\alpha} X_1$ have the same law for every $t \geq 0$. The scaling property enables us to introduce the stationary process $Y = (Y_t\colon -\infty < t < \infty)$:

$$Y_t = e^{-t} X_{e^{\alpha t}},$$

which will be referred to as the stable Ornstein–Uhlenbeck process in the sequel. One says that X (or Y) is completely asymmetric if it has either no negative jumps or no positive jumps. More precisely, X (or Y) is called spectrally positive in the first case, and spectrally negative in the second case. Our main task in this work will be to prove the following limit theorem for extremes of spectrally negative stable Ornstein–Uhlenbeck processes.

THEOREM 3. *Suppose that X is a spectrally negative stable Lévy process of index $\alpha \in (1, 2]$, with Laplace transform*

$$\mathbb{E}(\exp(qX_1)) = \exp\{\lambda q^\alpha\}, \qquad q > 0$$

for some $\lambda > 0$. Then, for every $x \in \mathbb{R}$, we have

$$\lim_{t\to\infty} \mathbb{P}\Big(\sup_{0\leq s\leq t} Y_s \leq x a(t) + b(t)\Big) = \exp\left(-\sqrt{\frac{\alpha}{2\pi}} e^{-x}\right)$$

with

$$a(t) = \left(\frac{\lambda(\alpha-1)}{\log t}\right)^{1/\alpha}$$

and

$$b(t) = \left(\frac{\alpha}{\alpha-1}\log t + \frac{1}{2}\log\log t\right) a(t).$$

When one specializes Theorem 3 to the Brownian case $\alpha = 2$ and $\lambda = 1/2$, one of course recovers the original result of Darling and Erdős [6]. There are two natural routes for trying to establish Theorem 3. First, one might use directly extreme value theory for stationary processes (see [1]). Second, one might invoke the expression of the stable Ornstein–Uhlenbeck process in terms of moving averages and then use results of Rootzén [16, 17]. However, neither of these approaches seems easy when working details, and we will follow here an alternative path.

Recall that a spectrally positive stable Lévy process of index less than 1 is monotone increasing, and is simply called a subordinator. It is well known that the first-passage process of a spectrally negative stable Lévy process of index $\alpha \in (1, 2]$ is a stable subordinator of index $1/\alpha$, and this enables us to reduce Theorem 3 to the following companion for stable subordinators.

THEOREM 4. *Let X be a stable subordinator with Laplace transform*

$$\mathbb{E}\big(\exp(-qX_1)\big) = \exp\{-\lambda q^\alpha\}, \qquad q > 0$$

for some $\alpha \in (0,1)$ and $\lambda > 0$. For every $x \in \mathbb{R}$, we have

$$\lim_{t\to\infty} \mathbb{P}\Big(\inf_{0\le s\le t} Y_s \ge xa(t) + b(t)\Big) = \exp\bigg(-\sqrt{\frac{\alpha}{2\pi}} e^x\bigg),$$

with

$$a(t) = \left(\frac{(1-\alpha)\lambda}{\log t}\right)^{1/\alpha}$$

and

$$b(t) = \left(\frac{\alpha}{1-\alpha}\log t - \frac{1}{2}\log\log t\right)a(t).$$

The method for proving Theorem 4 can be viewed as a Markovian counterpart of that of Berman [2] for extremes of stationary Gaussian processes. Rootzén [18] has also used a closely related idea for stationary Markov chains. More precisely, we exploit the fact that Y is a stationary Markov process which hits points. Roughly speaking, we then decompose the path of the Ornstein–Uhlenbeck process into i.i.d. excursions away a point, say 1. This reduces the study to a usual extreme values problem involving the depth of each excursion. Thus, the main technical issue is the estimation of the tail distribution of the depth of the excursion of Y. This is done by investigating the occupation time of the excursion near 0.

The rest of this paper is organized as follows. Theorem 1 is proven in Section 2. Section 3 is divided into four subsections: the first is devoted to elementary properties of the Ornstein–Uhlenbeck process associated with a stable subordinator, which are used in the second subsection for studying the excursions. Theorem 4 is proven in the third, and Theorem 3 is deduced from Theorem 4 in the fourth subsection. Finally, Section 4 presents the strong approximation result which enables us to derive Theorem 2 from Theorem 3; and some comments and complements are presented in Section 5.

2. Proof of Theorem 1.

Throughout this section, we suppose that the hypotheses of Theorem 1 hold. To start with, we recall that they ensure that the normalized partial sums are stochastically bounded, that is,

$$\lim_{x\to\infty} \max_{n\ge 1} \mathbb{P}\big(n^{-1/\alpha}|S_n| > x\big) = 0. \tag{2.2}$$

See, for instance, the compactness lemma ([12], page 309). We also observe that

$$\mathbb{P}\Big(\max_{j=1,\dots,k} \xi_j > yk^{1/\alpha}\Big) \sim cy^{-\alpha} \quad \text{as } y\to\infty, \text{ uniformly in } k\in\mathbb{N}. \tag{2.3}$$

The proof of Theorem 1 relies on a standard large deviation result for random walks.

LEMMA 1. *We have, as* $(k, y) \to \infty$,

$$cy^{-\alpha} \sim \mathbb{P}\big(S_k > yk^{1/\alpha}\big) \sim \mathbb{P}\Big(\max_{j=1,\dots,k} S_j > yk^{1/\alpha}\Big).$$

PROOF. The first equivalence is mainly due to Tkachuk [21]; see [22], Corollary 1.1.1, page 36, for an accessible reference, and also [8] for a recent extension. The second is doubtless well known; we sketch its derivation from the first for the sake of completeness. As $S_k \le \max_{j=1,\dots,k} S_j$, we only need to consider the upper bound. Write $T = \min\{j\colon S_j > yk^{1/\alpha}\}$, so

$$\mathbb{P}\Big(\max_{j=1,\dots,k} S_j > yk^{1/\alpha}\Big) = \mathbb{P}(T \le k).$$

For every fixed $\varepsilon > 0$, we have, by an application of the Markov property,

$$\mathbb{P}\big(S_{[(1+\varepsilon)^\alpha k]} > (1-\varepsilon)yk^{1/\alpha}\big) \ge \sum_{j=1}^{k} \mathbb{P}(T = j)\mathbb{P}\big(S_{[(1+\varepsilon)^\alpha k]-j} > -\varepsilon yk^{1/\alpha}\big).$$

Making use of (2.2), it is readily seen that, for every $\eta > 0$, provided that k and y are large enough, one has

$$\mathbb{P}\big(S_{[(1+\varepsilon)^\alpha k]-j} \ge -\varepsilon yk^{1/\alpha}\big) \ge 1-\eta \quad \text{for } j = 1, \dots, k.$$

It now follows from the first part of the lemma that

$$\limsup_{k,y\to\infty} \mathbb{P}(T \le k)y^\alpha \le (1-\eta)^{-1}c(1+\varepsilon)^\alpha(1-\varepsilon)^{-\alpha},$$

and the proof is now complete. □

To state the next consequence of Lemma 1, it will be convenient to use the following notation. Given two families $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ of events, we write $A_i \sim B_i$ if $\mathbb{P}(A_i \Delta B_i) = o(\mathbb{P}(A_i))$ (as i goes to some limit point of I), where $A\Delta B$ stands for the symmetric difference of A and B, namely, the set of points lying in A or in B but not both.

COROLLARY 1. *One has*

$$\Big\{\max_{j=1,\dots,k} \xi_j > yk^{1/\alpha}\Big\} \sim \Big\{\max_{j=1,\dots,k} S_j > yk^{1/\alpha}\Big\}$$

as $(y, k) \to \infty$.

PROOF. Because for every fixed $\varepsilon > 0$, $S_j > yk^{1/\alpha}$ as soon as both $\xi_j > (1+\varepsilon)yk^{1/\alpha}$ and $S_{j-1} > -\varepsilon yk^{1/\alpha}$, we have

$$\begin{aligned}
&\mathbb{P}\Big(\max_{j=1,\dots,k} \xi_j > (1+\varepsilon)yk^{1/\alpha},\ \max_{j=1,\dots,k} S_j \le yk^{1/\alpha}\Big)\\
&\quad\le \sum_{j=1}^{k} \mathbb{P}\big(\xi_j > (1+\varepsilon)yk^{1/\alpha}, S_{j-1} \le -\varepsilon yk^{1/\alpha}\big)\\
&\quad\le \big(1 - F((1+\varepsilon)yk^{1/\alpha})\big) \max_{j=1,\dots,k} \mathbb{P}\big(S_j \le -\varepsilon yk^{1/\alpha}\big).
\end{aligned}$$

Since

$$k\big(1-F\big((1+\varepsilon)yk^{1/\alpha}\big)\big)\sim c(1+\varepsilon)^{-\alpha}y^{-\alpha}\quad \text{as } y\to\infty\text{, uniformly in } k\geq 1,$$

we can use (2.2) to conclude that

$$\mathbb{P}\Big(\max_{j=1,\dots,k}\xi_j>(1+\varepsilon)yk^{1/\alpha},\ \max_{j=1,\dots,k}S_j\leq yk^{1/\alpha}\Big)=o(y^{-\alpha})\quad \text{as } (k,y)\to\infty.$$

We now see from (2.3) that as $(k,y)\to\infty$,

$$\mathbb{P}\Big(\max_{j=1,\dots,k}\xi_j>(1+\varepsilon)yk^{1/\alpha},\ \max_{j=1,\dots,k}S_j>yk^{1/\alpha}\Big)\sim c(1+\varepsilon)^{-\alpha}y^{-\alpha}.$$

Letting $\varepsilon\to 0+$, we deduce, using (2.3) again, that

$$\mathbb{P}\Big(\max_{j=1,\dots,k}\xi_j>yk^{1/\alpha},\ \max_{j=1,\dots,k}S_j>yk^{1/\alpha}\Big)\sim cy^{-\alpha},$$

which entails our claim by an application of Lemma 1 and (2.3). □

LEMMA 2. *For every $x>0$ and $r>1$,*

$$\mathbb{P}\Big(\max_{k=1,\dots,n}k^{-1/\alpha}S_k>rx(\log n)^{1/\alpha},\ \max_{k=1,\dots,n}k^{-1/\alpha}\xi_k\leq x(\log n)^{1/\alpha}\Big)$$

tends to 0 *as* $n\to\infty$.

PROOF. Considering the partition of $\{1,\dots,n\}$ induced by the real numbers $r^{\alpha}, r^{2\alpha},\dots,r^{p\alpha}$, where $p=[\log n/\alpha\log r]$, we see that the probability in the statement is bounded from above by

$$\mathbb{P}\Big(\bigcup_{m=0}^{p}\Lambda_m\Big)\leq\sum_{m=0}^{p}\mathbb{P}(\Lambda_m),$$

where

$$\Lambda_m=\Big\{\max_{k=1,\dots,[r^{(m+1)\alpha}]}S_k>rx(\log n)^{1/\alpha}r^m,\ \max_{k=1,\dots,[r^{(m+1)\alpha}]}\xi_k\leq x(\log n)^{1/\alpha}r^{m+1}\Big\}.$$

Then fix $\varepsilon>0$ arbitrarily small. By Corollary 1, there are integers n_ε and m_ε such that

$$\mathbb{P}(\Lambda_m)\leq\varepsilon\mathbb{P}\Big(\max_{k=1,\dots,[r^{(m+1)\alpha}]}\xi_k>x(\log n)^{1/\alpha}r^{m+1}\Big),$$

provided that $m\geq m_\varepsilon$ and $n\geq n_\varepsilon$. On the one hand, we have, according to (2.3),

$$\mathbb{P}\Big(\max_{k=1,\dots,[r^{(m+1)\alpha}]}\xi_k>x(\log n)^{1/\alpha}r^{m+1}\Big)\sim cx^{-\alpha}(\log n)^{-1}$$

as $n\to\infty$, uniformly in $m\in\mathbb{N}$. On the other hand, it is plain that

$$\lim_{n\to\infty}\sum_{m=0}^{m_\varepsilon}\mathbb{P}\Big(\max_{k=1,\dots,[r^{(m+1)\alpha}]}S_k>r^{1/\alpha}x(\log n)^{1/\alpha}r^m\Big)=0.$$

Putting the pieces together, we find that

$$\limsup_{n\to\infty} \mathbb{P}\Big(\max_{k=1,\dots,n} k^{-1/\alpha} S_k > rx(\log n)^{1/\alpha},\ \max_{k=1,\dots,n} k^{-1/\alpha}\xi_k > x(\log n)^{1/\alpha}\Big)$$
$$\le \varepsilon c x^{-\alpha}(\alpha \log r)^{-1},$$

and our claim is proven. □

We are now able to proceed to the proof of Theorem 1. The final key lies in the following elementary result on extreme values: for every $x > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}\Big(\max_{k=1,\dots,n} k^{-1/\alpha}\xi_k \le x(\log n)^{1/\alpha}\Big) = \exp\{-cx^{-\alpha}\}. \tag{2.4}$$

More precisely, it is plain that

$$\begin{aligned}\mathbb{P}\Big(\max_{k=1,\dots,n} k^{-1/\alpha}\xi_k \le x(\log n)^{1/\alpha}\Big) &= \prod_{k=1}^{n}\big(1 - F\big((\log n)^{1/\alpha}k^{1/\alpha}x\big)\big)\\ &\sim \exp\bigg\{-\sum_{k=1}^{n} c\big((\log n)^{1/\alpha}k^{1/\alpha}x\big)^{-\alpha}\bigg\}\\ &\sim \exp\{-cx^{-\alpha}\}.\end{aligned}$$

Then, using Lemma 2 and (2.4), we get, for every $r > 1$,

$$\limsup_{n\to\infty} \mathbb{P}\Big(\max_{k=1,\dots,n} k^{-1/\alpha} S_k > rx(\log n)^{1/\alpha}\Big) \le 1 - \exp\{-cx^{-\alpha}\},$$

and hence,

$$\liminf_{n\to\infty} \mathbb{P}\Big(\max_{k=1,\dots,n} k^{-1/\alpha} S_k \le x(\log n)^{1/\alpha}\Big) \ge \exp\{-cx^{-\alpha}\}.$$

To establish the converse upper bound, we use a slight variation of the argument that proves Corollary 1 to show that, for every $\varepsilon > 0$,

$$\mathbb{P}\Big(\max_{k=1,\dots,n} k^{-1/\alpha}\xi_k > (1+\varepsilon)x(\log n)^{1/\alpha},\ \max_{k=1,\dots,n} k^{-1/\alpha} S_k \le x(\log n)^{1/\alpha}\Big)$$

tends to 0 as $n \to \infty$. We then conclude from (2.4) that

$$\liminf_{n\to\infty} \mathbb{P}\Big(\max_{k=1,\dots,n} k^{-1/\alpha} S_k > x(\log n)^{1/\alpha}\Big) \ge 1 - \exp\big\{-c(1+\varepsilon)^{-\alpha}x^{-\alpha}\big\},$$

and finally,

$$\limsup_{n\to\infty} \mathbb{P}\Big(\max_{k=1,\dots,n} k^{-1/\alpha} S_k \le x(\log n)^{1/\alpha}\Big) \le \exp\{-cx^{-\alpha}\}. \qquad \square$$

3. Extremes of stable Ornstein–Uhlenbeck processes.

3.1. *Preliminaries.* The framework throughout the first three subsections is that of Theorem 4. The distribution of X_1 is absolutely continuous with a continuous density denoted by p:

$$\mathbb{P}(X_1 \in dx) = p(x)\,dx, \qquad x > 0.$$

We will make use of the following estimate at the origin:

$$p(x) \sim c_2 x^{-(2-\alpha)/(2-2\alpha)} \exp\{-c_3 x^{-\alpha/(1-\alpha)}\} \quad (x \to 0+) \tag{3.5}$$

with

$$c_2 = \frac{(\lambda\alpha)^{1/(2-2\alpha)}}{\sqrt{2\pi(1-\alpha)}}, \qquad c_3 = \lambda^{1/(1-\alpha)}(1-\alpha)\alpha^{\alpha/(1-\alpha)}. \tag{3.6}$$

See [23], equation (2.5.18).

Recall that the Ornstein–Uhlenbeck process has been defined as $Y_t = e^{-t} X_{e^{\alpha t}}$. Breiman [5] proved that Y is a time-homogeneous Markov process. For every $y > 0$, we write

$$\mathbb{P}^y = \mathbb{P}(\cdot \mid Y_0 = y) = \mathbb{P}(\cdot \mid X_1 = y)$$

for the law of Y started from y at time 0. The semigroup of Y is specified by the kernel $\mathbb{P}^x(Y_t \in dy) = p_t(x, y)\,dy$ $(t > 0)$ with

$$p_t(x, y) = e^t (e^{\alpha t} - 1)^{-1/\alpha} p\big((e^{\alpha t} - 1)^{-1/\alpha}(e^t y - x)\big), \qquad x, y > 0. \tag{3.7}$$

Observe that $\lim_{t\to\infty} p_t(x, y) = p(y)$ for every $x > 0$, which entails the strong mixing property, and a fortiori the ergodicity, of the semigroup. The reversed process $(Y_{-t}\colon -\infty < t < \infty)$ is also a time-homogeneous Markov process; its semigroup is given by

$$\mathbb{P}^x(Y_{-t} \in dy) = p_t(y, x)\big(p(x)/p(y)\big)\,dy.$$

It is immediately seen from (3.7) that both the direct and the reversed Ornstein–Uhlenbeck processes are Feller processes; in particular, they enjoy the strong Markov property.

We next recall the simple structure of the jumps of Y, which is immediately seen from the Itô decomposition of the stable subordinator into a Poisson point process and the scaling property.

LEMMA 3. *The process* $\Delta Y = (\Delta Y_t\colon t \ge 0)$ *given by*

$$\Delta Y_t = Y_t - Y_{t-}, \qquad t \ge 0,$$

is independent of Y_0. *It has the law of a Poisson point process valued in* $(0, \infty)$ *with characteristic measure* $kx^{-\alpha-1}\,dx$ *for some* $k > 0$.

We finally mention an elementary local property of the sample paths of Y.

LEMMA 4. *We have* $\lim_{t\to 0}(Y_t - Y_0)/t = -Y_0$, $\mathbb{P}$-*a.s.*

PROOF. According to a well-known result of Khintchine on the rate of growth of stable processes, we have $\lim_{s\to 0+}(X_{1+s} - X_1)/s = 0$ a.s. (see, e.g., [14]). This yields our statement in the case $t \to 0+$. As a time-reversed Lévy process has the same distribution as its negative, the case $t \to 0-$ follows. □

3.2. *Excursions.* We use the notation

$$H_y = \inf\{t > 0: Y_t = y\}$$

for the first-hitting time of $y > 0$. It is plain from the ergodicity and the absence of negative jumps that $H_y < \infty$ a.s., and we first consider the sample path behavior of Y in the neighborhood of H_y. We know from Lemma 4 that $Y_t < Y_0$ and $Y_{-t} > Y_0$ for every small enough $t > 0$. As both the direct and reversed processes have the strong Markov property, this shows that Y remains above y immediately before time H_y and below y immediately after time H_y. In other words, the Ornstein–Uhlenbeck process hits y from above and leaves y from below.

It will be convenient to write

$$\zeta = H_{Y_0} = \inf\{t > 0: Y_t = Y_0\}$$

for the first return time to the starting point; the piece of path $(Y_t: t \in [0, \zeta])$ will be referred to as the excursion of Y away from Y_0, and ζ as the duration of the excursion. It should be clear from the foregoing that the excursion remains strictly below Y_0 until the instant when it jumps above Y_0, and then stays above Y_0 until time ζ. Note also that the range of the excursion is necessarily a compact interval whose interior contains Y_0.

Our first task is to obtain precise information on the duration ζ of the excursion. In the classical theory of Markov processes, it is well known that the distribution of ζ can be characterized in terms of the adequate version of resolvent density of the Markov process; see, for example, [4] or [7]. In our setting, the main difficulty when one uses this technique is to find the correct version of the resolvent density. In order to circumvent this problem, we will rather use an elementary approach which is based on the fact that, as the Ornstein–Uhlenbeck process is ergodic, $\mathbb{E}^{y_0}(\zeta) < \infty$ for every $y_0 > 0$, and more importantly,

$$\mathbb{E}^{y_0}\left(\int_0^\zeta f(Y_t)\,dt\right) = \mathbb{E}^{y_0}(\zeta)\left(\int_{-\infty}^\infty f(x)p(x)\,dx\right) \tag{3.8}$$

for every measurable function $f \geq 0$.

We now observe the following.

LEMMA 5. *For every $x, y > 0$, we have*

$$\lim_{\varepsilon\to 0+} \varepsilon^{-1} \int_0^\zeta 1_{\{Y_s \in [x-\varepsilon, x]\}}\,ds = x^{-1}\,\mathrm{Card}\{t \in [0, \zeta): Y_t = x\}$$

both $\mathbb{P}^y$-a.s. and in $L^1(\mathbb{P}^y)$.

PROOF. The almost sure convergence should be clear from Lemma 4, the above description of the sample path behavior near a first-hitting time and the strong Markov property. By an extension of Lebesgue's theorem of dominated convergence, the L^1-convergence follows, provided that we can check that

$$\mathbb{E}^y\left(\left(\int_0^\zeta 1_{\{Y_s\in[x-\varepsilon,\,x]\}}\,ds\right)^2\right) = O(\varepsilon^2). \tag{3.9}$$

Using the Markov property, we rewrite the left-hand side in (3.9) as

$$2\mathbb{E}^y\left(\int_0^\zeta 1_{\{Y_s\in[x-\varepsilon,\,x]\}}\mathbb{E}^{Y_s}\left(\int_0^{H_y} 1_{\{Y_t\in[x-\varepsilon,\,x]\}}\,dt\right)ds\right).$$

On the other hand, the strong Markov property shows that

$$\mathbb{P}^y(H_{x'}<\zeta)\mathbb{E}^{x'}\left(\int_0^{H_y} 1_{\{Y_t\in[x-\varepsilon,\,x]\}}\,dt\right) \le \mathbb{E}^y\left(\int_0^\zeta 1_{\{Y_t\in[x-\varepsilon,\,x]\}}\,dt\right),$$

and since Y has no negative jumps,

$$\inf_{x'\in[x-\varepsilon,\,x]}\mathbb{P}^y(H_{x'}<\zeta) = \mathbb{P}^y(H_{x-\varepsilon}<\zeta)\wedge\mathbb{P}^y(H_x<\zeta).$$

We thus see that, provided that $\varepsilon>0$ is small enough, the left-hand side in (3.9) is bounded from above by

$$k\left(\mathbb{E}^y\left(\int_0^\zeta 1_{\{Y_s\in[x-\varepsilon,\,x]\}}\,ds\right)\right)^2$$

for some constant $k<\infty$. Invoking (3.8) and the boundedness of the stable density p, we conclude that (3.9) holds. □

Lemma 5 enables us to determine the distribution of the duration of the excursion in terms of the stable density p.

COROLLARY 2. (i) *For every $y>0$, we have*

$$\frac{1}{\mathbb{E}^y(\zeta)} = yp(y).$$

(ii) *For every $y>0$ and $q>0$, we have*

$$\frac{\mathbb{E}^y(e^{-q\zeta})}{1-\mathbb{E}^y(e^{-q\zeta})} = y\int_0^\infty e^{-qt}p_t(y,y)\,dt,$$

where the kernel $p_t(y,z)$ is given by (3.7).

PROOF. (i) By the continuity of the stable density and (3.8), we have

$$\begin{aligned} p(y) &= \lim_{\varepsilon\to0+}\varepsilon^{-1}\int_{y-\varepsilon}^y p(z)\,dz \\ &= \frac{1}{\mathbb{E}^y(\zeta)}\lim_{\varepsilon\to0+}\varepsilon^{-1}\mathbb{E}^y\left(\int_0^\zeta 1_{\{Y_s\in[y-\varepsilon,\,y]\}}\,ds\right). \end{aligned}$$

According to Lemma 5, the right-hand side equals $1/(y\mathbb{E}^y(\zeta))$.

(ii) Fix $s > 0$ and write $d(s) = \inf\{t \geq s\colon Y_t = Y_0\}$ for the first return to Y_0 after time s. An argument closed to that in Lemma 5 shows that

$$\frac{\mathbb{E}^y(e^{-qd(s)})}{1 - \mathbb{E}^y(e^{-q\zeta})} = \mathbb{E}^y(e^{-qd(s)})\big(1 + \mathbb{E}^y(e^{-q\zeta}) + \mathbb{E}^y(e^{-q\zeta})^2 + \cdots\big)$$
$$= \lim_{\varepsilon \to 0+} y\varepsilon^{-1}\mathbb{E}^y\left(\int_s^\infty e^{-qt}1_{\{Y_t \in [y-\varepsilon,\, y]\}}\,dt\right)$$
$$= \lim_{\varepsilon \to 0+} y\int_s^\infty e^{-qt}\left(\varepsilon^{-1}\int_{y-\varepsilon}^y p_t(y, z)\,dz\right)dt.$$

From the continuity of the stable density, we know that

$$\lim_{\varepsilon \to 0+} \varepsilon^{-1}\int_{y-\varepsilon}^y p_t(y, z)\,dz = p_t(y, y).$$

On the other hand, it is straightforward to check that $\sup_{t \geq s,\, z > 0} p_t(y, z) < \infty$. Hence Lebesgue's theorem of dominated convergence applies and gives

$$\frac{\mathbb{E}^y(e^{-qd(s)})}{1 - \mathbb{E}^y(e^{-q\zeta})} = y\int_s^\infty e^{-qt}p_t(y, y)\,dt.$$

Plainly, $d(s)$ tends to ζ as $s \to 0+$, and we get the desired formula. □

The asymptotic behavior as $y \to 0+$ of the quantity that appears in the first part of Corollary 2 is given by (3.5). For the second part, we need the following elementary result.

LEMMA 6. *For every $q > 1$, we have*

$$\lim_{y \to 0+} y\int_0^\infty e^{-qt}p_t(y, y)\,dt = \frac{\alpha}{1-\alpha}.$$

PROOF. According to (3.7), we have

$$\int_0^\infty e^{-qt}p_t(y, y)\,dt = \int_0^\infty e^{-qt}e^t(e^{\alpha t} - 1)^{-1/\alpha}p\big((e^{\alpha t} - 1)^{-1/\alpha}(e^t - 1)y\big)\,dt.$$

Introduce the change of variables $s(t) = (e^{\alpha t} - 1)^{-1/\alpha}(e^t - 1)$ and write $t(s)$ for the inverse function. Observe that $s(t)$ decreases from ∞ to 1 as t increases from 0 to ∞. Because

$$ds(t) = e^t(e^{\alpha t} - 1)^{-1/\alpha}\left(\frac{e^{\alpha t} - e^t}{e^{(\alpha+1)t} - e^t}\right)dt,$$

the preceding integral can be expressed as

$$\int_1^\infty e^{-qt(s)}\left(\frac{e^{(\alpha+1)t(s)} - e^{t(s)}}{e^{t(s)} - e^{\alpha t(s)}}\right)p(ys)\,ds.$$

We conclude the proof by recalling that p is a probability density and by observing that the function

$$s \to e^{-qt(s)}\left(\frac{e^{(\alpha+1)t(s)} - e^{t(s)}}{e^{t(s)} - e^{\alpha t(s)}}\right)$$

is bounded on $(1, \infty)$ and has limit $\alpha/(1-\alpha)$ at ∞. □

An intuitively obvious consequence of Corollary 2 and Lemma 6 is the following asymptotic result on the height of the excursion started from a small point.

COROLLARY 3. *One has*

$$\lim_{y\to 0+} \mathbb{P}^y(H_1 \le \zeta) = 1 - \alpha.$$

PROOF. Note first from a combination of Corollary 2 and Lemma 6 that $\mathbb{E}^y(e^{-q\zeta})$ tends to α as y goes to $0+$, which shows that, for every $t > 0$,

$$\lim_{y\to 0+} \mathbb{P}^y(\zeta \le t) = \alpha. \tag{3.10}$$

Then, observe that we must have $H_1 < \zeta$ whenever Y has a jump of length greater than 1 before time ζ. Hence, for every $t > 0$,

$$\mathbb{P}^y(H_1 \le \zeta) \ge \mathbb{P}^y(\Delta Y_s > 1 \text{ for some } s \in [0, t]) - \mathbb{P}^y(\zeta \le t).$$

Since t can be chosen arbitrarily large, Lemma 3 and (3.10) give the lower bound

$$\liminf_{y\to 0+} \mathbb{P}^y(H_1 \le \zeta) \ge 1 - \alpha.$$

To establish the upper bound, we simply note that, for every $t > 0$,

$$\begin{aligned}\mathbb{P}^y(H_1 > \zeta) &\ge \mathbb{P}^y(\zeta \le t) - \mathbb{P}\big(e^{-s} X_{e^{\alpha}s} > 1 \text{ for some } s \in [0, t] \,\big|\, X_1 = y\big)\\ &\ge \mathbb{P}^y(\zeta \le t) - \mathbb{P}(X_{e^{\alpha t}-1} > 1 - y).\end{aligned}$$

Since t can be chosen arbitrarily small, the upper bound follows from (3.10). □

We are now able to estimate the distribution of the depth of the excursion started at 1, which is the crucial step to the proof of Theorem 4.

LEMMA 7. *We have*

$$\mathbb{P}^1(H_y < \zeta) \sim \frac{c_2(1-\alpha)}{p(1)} y^{-\alpha/(2-2\alpha)} \exp\{-c_3 y^{-\alpha/(1-\alpha)}\} \qquad (y \to 0+),$$

where c_2 and c_3 are given in (3.6).

PROOF. For every $y > 0$, consider the number of visits to the point y made by Y before its first return to its starting point:

$$N_y = \text{Card}\{t \in [0, \zeta]: Y_t = y\}.$$

A standard application of the Markov property at the successive passage times at y gives

$$\begin{aligned}\mathbb{E}^1(N_y) &= \mathbb{P}^1(H_y < \zeta)\big(1 + \mathbb{P}^y(H_1 > \zeta) + \mathbb{P}^y(H_1 > \zeta)^2 + \cdots\big)\\ &= \mathbb{P}^1(H_y < \zeta)/\mathbb{P}^y(H_1 \le \zeta),\end{aligned}$$

so we deduce from Lemma 3 that

$$\mathbb{P}^1\left(H_y < \zeta\right) \sim (1-\alpha)\mathbb{E}^1(N_y) \quad (y \to 0+).$$

To calculate the expectation in the right-hand side, we combine Lemma 5, (3.8) and the continuity of the stable density to obtain

$$\mathbb{E}^1(N_y) = yp(y)\mathbb{E}^1(\zeta) = yp(y)/p(1),$$

where the ultimate equality stems from Corollary 2(i). We conclude with (3.5). □

REMARK. We point out that essentially the same result appears in [5]. More precisely, Theorem 3 in [5] states that

$$\mathbb{P}^1(H_y < \zeta) \sim y^{-\alpha/(2-2\alpha)} \exp\{-c_3 y^{-\alpha/(1-\alpha)}\},$$

but a perusal of Breiman's proof shows that what is really proven there is

$$\mathbb{P}^1(H_y < \zeta) \sim ky^{-\alpha/(2-2\alpha)} \exp\{-c_3 y^{-\alpha/(1-\alpha)}\}$$

for some constant $k > 0$. Presumably, it should be possible to refine Breiman's calculation and get $k = c_2(1-\alpha)/p(1)$, but this does not seem easy and the purely probabilistic argument of the proof of Lemma 7 looks simpler.

PROOF OF THEOREM 4. The last key to Theorem 4 is a simple lemma of extreme values theory (see, e.g., [15], exercise 1.1.4).

LEMMA 8. *Let $\xi_1, \xi_2, \ldots$ be a sequence of i.i.d. nonnegative random variables with distribution function F: $[0,\infty) \to [0,\infty)$ such that, for some positive real numbers k_1, μ, k_2, ν,*

$$F(t) \sim k_1 t^{-\mu} \exp\{-k_2 t^{-\nu}\} \qquad (t \to 0+).$$

Put, for every integer $n \ge 3$,

$$a(n) = \frac{1}{k_2\nu}\left(\frac{1}{k_2}\log n\right)^{-1-1/\nu}, \qquad b(n) = \left(\frac{1}{k_2}\log n\right)^{-1/\nu}\left(1 - \frac{\mu \log\log n}{\nu^2 \log n}\right).$$

For every $x \in \mathbb{R}$ and $r > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}\Big(\min_{1\le \ell \le rn} \xi_\ell \le xa(n) + b(n)\Big) = \exp\{-rk_1 k_2^{-\mu/\nu} e^x\}.$$

PROOF. Note first that

$$xa(n)+b(n)=\left(\frac{1}{k_2}\log n\right)^{-1/\nu}\left(1+\frac{\nu x-\mu\log\log n}{\nu^2\log n}\right),$$

which entails

$$\begin{aligned}\big(xa(n)+b(n)\big)^{-\nu}&=\frac{\log n}{k_2}\left(1-\frac{\nu x-\mu\log\log n}{\nu\log n}\right)+o(1)\\&=\frac{1}{k_2}\left(\log n-x+\frac{\mu}{\nu}\log\log n\right)+o(1).\end{aligned}$$

This yields

$$\exp\{-k_2\big(xa(n)+b(n)\big)^{-\nu}\}\sim n^{-1}e^x(\log n)^{-\mu/\nu},$$

and as

$$\big(xa(n)+b(n)\big)^{-\mu}\sim\left(\frac{1}{k_2}\log n\right)^{\mu/\nu},$$

we conclude that

$$\lim_{n\to\infty}nF\big(xa(n)+b(n)\big)=k_1k_2^{-\mu/\nu}e^x.$$

Our claim now follows from the fact that

$$\mathbb{P}\Big(\min_{1\le\ell\le rn}\xi_\ell<xa(n)+b(n)\Big)=\big(1-F(xa(n)+b(n))\big)^{[rn]}.\qquad\square$$

We are now able to establish Theorem 4.

PROOF. Consider the succession of the excursions away from 1 made by Y after time H_1, and denote by ξ_n the minimum of the nth excursion. The regenerative property entails that the sequence $\xi_1,\xi_2,\dots$ is i.i.d., and Lemma 7 that it satisfies the condition of Lemma 8 with

$$k_1=\frac{c_2(1-\alpha)}{p(1)},\qquad k_2=c_3,\qquad \mu=\frac{\alpha}{2-2\alpha},\qquad \nu=\frac{\alpha}{1-\alpha}.$$

This incites us to introduce the functions

$$\begin{aligned}a(t)&=\frac{1-\alpha}{\alpha c_3}\left(\frac{1}{c_3}\log t\right)^{-1/\alpha},\\ b(t)&=\left(\frac{1}{c_3}\log t\right)^{1-1/\alpha}\left(1-\frac{(1-\alpha)\log\log t}{2\alpha\log t}\right).\end{aligned}$$

Using (3.6), we get $p(1)k_1k_2^{-\mu/\nu}=\sqrt{\alpha/2\pi}$,

$$\begin{aligned}a(t)&=\left(\frac{(1-\alpha)\lambda}{\log t}\right)^{1/\alpha},\\ b(t)&=\left(\frac{(1-\alpha)^{1-\alpha}\lambda}{\log t}\right)^{1/\alpha}\left(\alpha\log t-\frac{1-\alpha}{2}\log\log t\right).\end{aligned}$$

Recall from Corollary 2(i) that the expected duration of an excursion is $1/p(1)$, so by the law of large numbers, the number of excursions accomplished by Y up to time t is equivalent to $p(1)t$ as $t \to \infty$. It is then easily checked that

$$\mathbb{P}(I_t \le xa(t) + b(t)) \sim \mathbb{P}\Big(\min_{1 \le \ell \le p(1)t} \xi_\ell \le xa(t) + b(t)\Big),$$

so Theorem 4 follows from Lemma 8. □

PROOF OF THEOREM 3. Theorem 3 is essentially a consequence of Theorem 4 and the well-known fact that the first-passage process of a stable Lévy process with no positive jumps is a stable subordinator.

Let $(X_t\colon t \ge 0)$ be as in Theorem 3 and introduce the first passage of X,

$$\check{X}_t = \inf\{s \ge 0\colon X_s > t\}, \qquad t \ge 0.$$

Then $\check{X} = (\check{X}_t\colon t \ge 0)$ is a stable subordinator, and more precisely, one has

$$\mathbb{E}\big(\exp(-q\check{X}_1)\big) = \exp\{-\check{\lambda} q^{\check{\alpha}}\},$$

with $\check{\lambda} = \lambda^{-1/\alpha}$ and $\check{\alpha} = 1/\alpha$. See, for example, [3], equation (8.1). We write $\check{Y}_s = e^{-s}\check{X}_{\exp\{\check{\alpha}s\}}$ for the Ornstein–Uhlenbeck process corresponding to $\check{X}$.

Aiming at applying Theorem 4 to $\check{X}$, we introduce the functions

$$\check{a}(t) = \left(\frac{\check{\lambda}(1-\check{\alpha})}{\log t}\right)^{1/\check{\alpha}} = \lambda^{-1}\alpha^{-\alpha}(\alpha-1)^{\alpha}(\log t)^{-\alpha}$$

and

$$\begin{aligned}\check{b}(t) &= \left(\frac{\check{\lambda}(1-\check{\alpha})^{1-\check{\alpha}}}{\log t}\right)^{1/\check{\alpha}}\left(\check{\alpha}\log t - \frac{1-\check{\alpha}}{2}\log\log t\right)\\ &= \lambda^{-1}\alpha^{-\alpha}(\alpha-1)^{\alpha-1}(\log t)^{-\alpha}\left(\log t - \frac{\alpha-1}{2}\log\log t\right).\end{aligned}$$

An easy calculation shows that when the functions $a(t)$ and $b(t)$ are given as in Theorem 3, then for every $x \in \mathbb{R}$,

$$\big(xa(t) + b(t)\big)^{-\alpha} = -x\check{a}(t) + \check{b}(t) + o(1)\check{a}(t) \qquad (t \to \infty).$$

It then follows from Theorem 4 that

$$\begin{aligned}\exp\left(-\sqrt{\frac{\check{\alpha}}{2\pi}}e^{-x}\right) &= \lim_{t\to\infty}\mathbb{P}\Big(\inf_{0\le s\le t}\check{Y}_s \ge -x\check{a}(t) + \check{b}(t)\Big)\\ &= \lim_{t\to\infty}\mathbb{P}\Big(\inf_{0\le s\le t}\check{Y}_s \ge \big(xa(t)+b(t)\big)^{-\alpha}\Big).\end{aligned}$$

Using the stationarity of $\check{Y}$, we note that

$$\begin{aligned}&\mathbb{P}\big(\check{Y}_s \geq \big(xa(t)+b(t)\big)^{-\alpha} \text{ for all } 0 \leq s \leq t\big)\\&\quad = \mathbb{P}\big(\check{Y}_{s+\alpha \log(xa(t)+b(t))} \geq \big(xa(t)+b(t)\big)^{-\alpha} \text{ for all } 0 \leq s \leq t\big)\\&\quad = \mathbb{P}\big(\check{X}_{\exp\{\check{\alpha} s+\log(xa(t)+b(t))\}} \geq e^s \text{ for all } 0 \leq s \leq t\big).\end{aligned}$$

Then, using the fact that $\check{X}$ is the first-passage process of X, we deduce that the latter quantity is equivalent to

$$\begin{aligned}&\mathbb{P}\big(X_{e^s} \leq e^{s/\alpha}\big(xa(t)+b(t)\big) \text{ for all } 0 \leq s \leq t\big)\\&\quad = \mathbb{P}\Big(\sup_{0\leq s\leq t/\alpha} Y_s \leq xa(t)+b(t)\Big).\end{aligned}$$

(Beware that equality does not hold, as we have to take into account the event that the supremum of X on the time interval $[0, e^t]$ occurs before time 1. Since the probability of this event tends to 0 as $t \to \infty$, the stated equivalence does hold.) Putting the pieces together, we have thus proven that

$$\lim_{t\to\infty} \mathbb{P}\Big(\sup_{0\leq s\leq t/\alpha} Y_s \leq xa(t)+b(t)\Big) = \exp\bigg(-\sqrt{\frac{1}{2\alpha\pi}}e^{-x}\bigg). \tag{3.11}$$

It is immediately checked that

$$xa(\alpha t)+b(\alpha t) = \big(x+\log\alpha+o(1)\big)a(t)+b(t),$$

so that we can rephrase (3.11) as

$$\lim_{t\to\infty} \mathbb{P}\Big(\sup_{0\leq s\leq t} Y_s \leq xa(t)+b(t)\Big) = \exp\bigg(-\sqrt{\frac{\alpha}{2\pi}}e^{-x}\bigg),$$

which is our statement. □

4. Proof of Theorem 2. We now suppose that the distribution function F fulfils the hypotheses of Theorem 2, that is, that for some $\varepsilon > 0$,

$$F(-x) = cx^{-\alpha} + O(x^{-\alpha-\varepsilon}) \quad \text{and} \quad 1-F(x) = O(x^{-\alpha-\varepsilon}), \qquad x \to \infty.$$

Let $X = (X_t\colon t \geq 0)$ be a spectrally negative stable Lévy process with index $\alpha \in (1,2)$ as in Theorem 3, that is, $\mathbb{E}(\exp(qX_1)) = \exp(\lambda q^\alpha)$ for $q > 0$. We write F_X for the distribution function of X_1 and choose the parameter λ in such a way that F and F_X have the same asymptotic behavior at $-\infty$. Specifically,

$$F_X(-x) \sim cx^{-\alpha} \quad \text{as } x \to \infty, \text{ with } c = \lambda(\alpha - 1)/\Gamma(2-\alpha). \tag{4.12}$$

See, for instance, [19], Property 1.2.15.

Write $\varphi(y) = \inf\{x: F(x) > y\}$ for the right-continuous inverse of F. Then the variables

$$\xi_n = \varphi \circ F_X(X_n - X_{n-1}), \qquad n = 1, 2, \dots$$

are independent and have all the distribution F. We put, as usual, $S_n = \xi_1 + \cdots + \xi_n$ and $R_n = S_n - X_n$. Introduce for every $t \geq 1$,

$$M_t^X = \max_{k=1,\dots,[t]} X_k, \qquad M_t^S = \max_{k=1,\dots,[t]} S_k, \qquad \overline{X}_t = \sup_{1 \leq s \leq t} X_s.$$

LEMMA 9. *For every $\beta > 1/(\alpha + \varepsilon)$, we have almost surely*

$$\lim_{t\to\infty} t^{-\beta} \left| M_t^S - \overline{X}_t \right| = 0.$$

PROOF. Note that $R = (R_n: n \in \mathbb{N})$ is a centered random walk, and because

$$\varphi \circ F_X(-x) = -x + O\big(x^{-\alpha-\varepsilon}\big), \qquad x \to \infty,$$

it is immediately seen that

$$\mathbb{P}\big(|R_1| > x\big) = O(x^{-\alpha-\varepsilon}), \qquad x \to \infty.$$

According to [11] (see also [14]), we have, for every $\beta > 1/(\alpha + \varepsilon)$,

$$\lim_{n\to\infty} n^{-\beta} |R_n| = 0 \quad \text{a.s.} \tag{4.13}$$

This entails

$$\left| M_t^X - M_t^S \right| \leq \max_{k=1,\dots,[t]} |R_k| = o(t^\beta) \quad \text{a.s.}$$

On the other hand, we have

$$0 \leq \overline{X}_t - M_t^X \leq \max_{k=1,\dots,[t]} Z_k, \qquad \text{where } Z_k = \sup_{0\leq s\leq 1} \big(X_{k+s} - X_k\big).$$

The variables $Z_1, \dots$ are independent and have all a Mittag–Leffler distribution; in particular,

$$\mathbb{P}(Z_k > x) = o(e^{-x}), \qquad x \to \infty$$

[see, e.g., [3], equation (8.8)]. A straightforward application of the Borel–Cantelli lemma shows that $\max_{k=1,\dots,[t]} Z_k = o(t^\beta)$ a.s., which completes the proof. □

LEMMA 10. *We have both*

$$\lim_{n\to\infty} \mathbb{P}\Big(\sup_{1\leq s\leq \log n} s^{-1/\alpha} \overline{X}_s > (\log\log n)^{1-1/\alpha} \Big) = 0$$

and

$$\lim_{n\to\infty} \mathbb{P}\Big(\sup_{1\le s\le \log n} s^{-1/\alpha} M_s^S > (\log\log n)^{1-1/\alpha}\Big) = 0.$$

PROOF. Recall that there is some $\kappa > 0$ such that, for all x large enough and all $t > 0$,

$$\mathbb{P}\big(\overline{X}_t \ge t^{1/\alpha} x\big) \le \exp\big\{-\kappa x^{\alpha/(\alpha-1)}\big\}. \tag{4.14}$$

See again [3], equation (8.8). As a consequence, we have, for every $\eta > 0$, provided that the integer m is large enough,

$$\begin{aligned}\mathbb{P}\Big(\sup_{1\le s\le 2^m} s^{-1/\alpha}\overline{X}_s > \eta m^{1-1/\alpha}\Big) &\le \sum_{k=0}^{m-1} \mathbb{P}\big(\overline{X}_{2^{k+1}} > \eta 2^{k/\alpha} m^{1-1/\alpha}\big)\\ &\le m \exp\big\{-\kappa(\eta 2^{-1/\alpha} m^{1-1/\alpha})^{\alpha/(\alpha-1)}\big\}\\ &= m\exp\{-\kappa' m\},\end{aligned}$$

for some $\kappa' > 0$. This entails our first assertion; the second follows, thanks to (4.13) and the inequality

$$M_s^S \le \overline{X}_s + \max_{k=1,\dots,[s]} |R_k|.$$

This completes the proof of the lemma. □

We are now able to proceed to the derivation of Theorem 2 from Theorem 3. It is clear that we may replace X_s by $\overline{X}_s$ in Theorem 3. By a slight variation of the argument of [20], Lemmas 9 and 10 then imply that $\sup_{1\le s\le t} s^{-1/\alpha} M_s^S$ and $\sup_{1\le s\le t} s^{-1/\alpha}\overline{X}_s$ have the same asymptotic behavior, so that by Theorem 3,

$$\lim_{t\to\infty} \mathbb{P}\Big(\sup_{0\le s\le t} e^{-s} M^S_{e^{\alpha s}} \le x a(t) + b(t)\Big) = \exp\Big(-\sqrt{\frac{\alpha}{2\pi}} e^{-x}\Big)$$

with

$$a(t) = \Big(\frac{\lambda(\alpha-1)}{\log t}\Big)^{1/\alpha}$$

and

$$b(t) = \Big(\frac{\alpha}{\alpha-1}\log t + \frac{1}{2}\log\log t\Big) a(t).$$

Plainly, we may replace $M^S_{e^{\alpha s}}$ by $S_{[e^{\alpha s}]}$ in the foregoing, which yields

$$\lim_{n\to\infty} \mathbb{P}\Big(\max_{k=1,\dots,n} k^{-1/\alpha} S_k \le x a(\alpha^{-1}\log n) + b(\alpha^{-1}\log n)\Big) = \exp\Big(-\sqrt{\frac{\alpha}{2\pi}} e^{-x}\Big).$$

Recall (4.12). A few lines of calculations shows that, in the notation of Theorem 2,

$$xa(\alpha^{-1}\log n)+b(\alpha^{-1}\log n)=\big(x-\log\alpha+\tfrac{1}{2}\log(2\alpha\pi)+\eta(n)\big)a_{\alpha,c}(n)+b_{\alpha,c}(n),$$

where $\eta(n)$ tends to 0 as n goes to ∞. This completes the proof of Theorem 2. □

5. Complements. (i) Darling and Erdős have also proven an analogue of (1.1) for the absolute maximum of normalized sums, namely,

$$\lim_{n\to\infty}\mathbb{P}\Big(\max_{k=1,\dots,n}\big|k^{-1/2}S_k\big|\le xa_2(n)+b_2(n)\Big)=\exp\{-2e^{-x}\}, \tag{5.15}$$

provided that $\mathbb{E}(\xi)=0$, $\mathbb{E}(\xi^2)=1$ and $\mathbb{E}(|\xi|^3)<\infty$; see [9] for the definitive result in the finite-variance case. Here we state the companion of Theorem 1 for the absolute maximum: suppose that, for some $\alpha\in(0,1)\cup(1,2)$ and some real numbers c_+, $c_-\ge 0$ with $c_++c_->0$, the distribution function $F(x)=\mathbb{P}(\xi\le x)$ fulfils

$$1-F(x)\sim c_+x^{-\alpha}\quad\text{and}\quad F(-x)\sim c_-x^{-\alpha},\qquad x\to\infty,$$

and that $\mathbb{E}(\xi)=0$ if $\alpha>1$. (It is well known that these hypotheses are equivalent to the convergence in distribution of $n^{-1/\alpha}S_n$ towards a nondegenerate law.) Then, for every $x\ge 0$, we have

$$\lim_{n\to\infty}\mathbb{P}\Big(\max_{k=1,\dots,n}\big|k^{-1/\alpha}S_k\big|\le x(\log n)^{1/\alpha}\Big)=\exp\big\{-(c_++c_-)x^{-\alpha}\big\}.$$

The proof is similar to that of Theorem 1 and therefore is omitted.

(ii) In order to complete Theorems 3 and 4, we mention the following limit theorem for the extremes of non-completely asymmetric stable Ornstein–Uhlenbeck processes, which can be viewed as a continuous analogue of Theorem 1. We refer to [16] for a closely related result.

THEOREM 5. *Let X be a stable Lévy process with characteristic function*

$$\mathbb{E}\big(\exp(iqX_1)\big)=\exp\big\{-\lambda|q|^\alpha\big(1-i\beta\,\mathrm{sgn}(q)\tan(\pi\alpha/2)\big)\big\}$$

for some $\lambda>0$, $\alpha\in(0,1)\cup(1,2)$ and $\beta\in(-1,1]$. Then, for every $x>0$,

$$\lim_{t\to\infty}\mathbb{P}\Big(\sup_{0\le s\le t}\big(e^{-s}X_{e^{\alpha s}}\big)\le t^{1/\alpha}x\Big)=\exp\bigg\{-\frac{\lambda\alpha(1-\alpha)(1+\beta)}{2\Gamma(2-\alpha)\cos(\pi\alpha/2)}x^{-\alpha}\bigg\}.$$

(iii) Theorem 4 also has a counterpart for the minima of the normalized partial sums of nonnegative i.i.d. variables, which we now state without proof (the arguments are close to those developed in Section 4). Suppose that $\xi\ge 0$ a.s. and that there are $\alpha\in(0,1)$ and $c>0$ such that the distribution function $F(x)=\mathbb{P}(\xi\le x)$ fulfils $1-F(x)=cx^{-\alpha}+O(x^{-\alpha-\varepsilon})$ as $x\to\infty$, for some $\varepsilon>0$.

Then, for every $x \in \mathbb{R}$, we have

$$\lim_{n\to\infty} \mathbb{P}\Big(\min_{k=1,\dots,n} k^{-1/\alpha} S_k \geq x a_{\alpha,c}(n) + b_{\alpha,c}(n)\Big) = \exp\{-e^x\},$$

with

$$a_{\alpha,c}(n) = \left(\frac{c\Gamma(2-\alpha)}{\log\log n}\right)^{1/\alpha}$$

and

$$b_{\alpha,c}(n) = a_{\alpha,c}(n)\left(\frac{\alpha}{1-\alpha}\log\log n - \frac{1}{2}\log\log\log n - \frac{1}{2}\log(2\alpha\pi)\right).$$

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