The Annals of Probability
1998, Vol. 26, No. 2, 479–499

EXPANSIONS AND CONTRACTIONS OF ISOTROPIC STOCHASTIC FLOWS OF HOMEOMORPHISMS[1]

BY VLADIMIR V. PITERBARG

University of Southern California

A sequence of piecewise constant approximations to rescaled isotropic homeomorphic stochastic flows is shown to converge weakly in Skorohod metric to the coalescing Brownian flow. Intermittent behavior of isotropic flows is exposed, and the clustering properties of isotropic flows are studied by the means of this convergence. We obtain qualitative and quantitative description of expansions and contractions of an arbitrary isotropic homeomorphic flow on large time- and space-scales.

1. Introduction and main result. Our goal is to study long time, large space-scale behavior of isotropic stochastic flows of homeomorphisms in $\mathbb{R}^1$. A stochastic flow can be viewed as a collection of particles started at each point on the real line and evolving according to some law, such that each particle performs a Brownian motion, but the motions of two or more particles are, in general, dependent. The condition of isotropy requires that the motions of ensembles of particles do not depend on the particles' absolute positions in space, but only on the relative distances between them.

An exact definition of a stochastic isotropic flow $\{\xi_t(x),\ x \in \mathbb{R}^1,\ t \in \mathbb{R}_+\}$ is given below; see (F1)–(F5). Here $\xi_t(x)$ is the position of a particle at time t started at point x at time 0. We impose smoothness conditions on the coefficients of ξ that guarantee it to be a flow of homeomorphisms.

Another object that appears in our study is the coalescing Brownian flow in $\mathbb{R}^1$, denoted by $\{C_t(x),\ x \in \mathbb{R}^1,\ t \in \mathbb{R}_+\}$. Then $C_t(x)$ is the position of a particle at time t started at point x at time 0. This flow can be described as follows. At time 0, every point in $\mathbb{R}^1$ begins to execute a Brownian motion. The particles move independently, but if two particles meet, they stick together, or coalesce, and move as one. This flow was studied by Arratia; see [1].

It is known that for the coalescing Brownian flow, the uncountable set of particles started at time zero at every point on the real line coalesces into a discrete set of particles by time t for any $t > 0$. For fixed $t > 0$, the real line can be split into countably many intervals such that each interval coalesces into a single particle by time t, and different intervals coalesce into different particles. It is natural to call these intervals "regions of contraction," meaning that two particles started in the same contraction region get squashed

Received February 1997; revised December 1997.
[1]Supported in part by Office of Naval Research Contract N00014-96-1-0413.
AMS 1991 *subject classifications*. 60J30, 60H10, 60F05, 60F17, 60J60.
Key words and phrases. Coalescence, coalescing Brownian motion, convergence of flows, expansions, contractions, clustering.

together and become one particle by time t. The points separating the intervals of contraction are called "expansion points," meaning that two particles started arbitrarily close to the expansion point but on different sides of it get separated by time t. This picture of *intermittent behavior* will play an important role in what follows.

It is convenient to rescale a flow to study its large time- and space-scale behavior. We denote

$$\xi_t^k(x) = \frac{1}{k}\xi_{k^2 t}(kx), \qquad t \geq 0,\ x \in \mathbb{R}^1,\ k \in \mathbb{Z}_+. \tag{1.1}$$

It was proven in [11], Chapter 3 that the sequence of rescaled isotropic flows ξ^k converges weakly as diffusions to the coalescing Brownian flow C, meaning that finite-particle motions of rescaled flows converge weakly to finite-particle motions of the coalescing flow (see Section 4 for a short discussion). Under the flow ξ, particles never meet (ξ is the flow of homeomorphisms). Nevertheless, we should see the same kind of intermittent behavior for rescaled isotropic flows as we saw for the coalescing flow. For a homeomorphic flow, we should see contraction regions occupying almost all of the real line separated by small expansion regions. As the rescaling parameter goes to infinity, we would expect the clustering properties of the rescaled homeomorphic flow to approximate those of the coalescing flow. This paper seeks to quantify this observation and derive properties of expansions and contractions for arbitrary homeomorphic flows from those of the coalescing Brownian flow.

The finite-particle convergence discussed in Section 4 is not strong enough to provide information on expansions and contractions. Another type of convergence, one that takes into account all particles at once rather than only the finite number of them, is needed.

For fixed $t \geq 0$, $\xi_t^k(x)$ is a continuous function of x for any $k \in \mathbb{Z}_+$. We may ask whether the sequence $\{\xi_t^k(\cdot)\}_{k=1}^{\infty}$, $t > 0$ fixed, converges weakly to $C_t(\cdot)$ in some space of continuous functions with uniform topology. The answer is "no" because the proposed limiting random element, $C_t(\cdot)$, is not a continuous function of x (see Corollary 5). To have any hope of convergence, we must make $\xi_t^k(\cdot)$ discontinuous and consider convergence in some appropriate space of discontinuous functions.

Define

$$\hat{\xi}_t^k(x) = \frac{1}{k}\xi_{k^2 t}(\lfloor kx \rfloor), \qquad x \in \mathbb{R}^1,\ t \in \mathbb{R}_+,\ k \in \mathbb{Z}_+, \tag{1.2}$$

where $\lfloor\cdot\rfloor = \text{floor}(\cdot)$. Then $\hat{\xi}^k$ is a piecewise constant approximation to ξ^k on the scale $1/k$. Effectively, we divide $\mathbb{R}^1$ into intervals $[i/k, (i+1)/k)$ and set $\hat{\xi}^k$ to be a constant on these intervals with jumps at the endpoints $\{i/k\}_{i=-\infty}^{\infty}$. If $x \in [i/k, (i+1)/k)$, then

$$\hat{\xi}^k(x) = \hat{\xi}^k(i/k) = \xi^k(i/k).$$

In this approximation some information is lost, but the information of our primary concern, the information on expansions and contractions of the original isotropic flow, is retained, as we will see later.

Our main result is the following theorem.

THEOREM 1. *For fixed* $t > 0$,

$$\hat{\xi}_t^k(\cdot) \Rightarrow C_t(\cdot) \quad \text{as } k \to \infty$$

weakly as random elements in the space $D(\mathbb{R};\mathbb{R})$ *of right-continuous functions with left-hand limits, equipped with the Skorohod topology.*

The paper is organized in the following way. We define isotropic homeomorphic flows in Section 2. The coalescing Brownian flow is studied in Section 3. Results from [11], Chapter 3, are recapped in Section 4. Estimates on which our whole analysis is based are obtained in Section 5. Theorem 1 is proved in Section 6. The results on expansions and contractions of homeomorphic flows that follow from Theorem 1 are obtained in Section 7.

2. Isotropic homeomorphic flows. Every isotropic flow in $\mathbb{R}^1$ is uniquely determined in law by its covariance structure. So we start with a function $B: \mathbb{R} \to \mathbb{R}_+$ such that the following holds.

(B1) $B(\cdot)$ is the autocovariance function of some stationary Gaussian process on $\mathbb{R}$; that is, it is positive definite and symmetric.

The following conditions define the class of isotropic flows we will be dealing with.

(B2) Smoothness:

$$B \in C^2(\mathbb{R};\mathbb{R});$$

(B3) Locality:

$$B(x) \to 0 \text{ as } |x| \to \infty;$$

(B4) Normalization:

$$B(0) = 1.$$

Conditions (B1)–(B4) are assumed to hold throughout.

A family $\{\xi_{st},\ 0 \le s \le t < \infty\}$ of random variables with values in $\mathbb{R}^R$ (all functions $\mathbb{R} \to \mathbb{R}$) is called an isotropic, time-homogeneous pure stochastic flow with independent increments corresponding to the covariance structure $B(\cdot)$ (isotropic stochastic flow for short) if (we define $\xi_t = \xi_{0t}$) we have the following.

(F1) $\xi_{tt}(x, \omega) = x$ for any $\omega \in \Omega$, $x \in \mathbb{R}$;
(F2) $\xi_{tu}(\xi_{st}(x, \omega), \omega) = \xi_{su}(x, \omega)$ for $0 \le s \le t \le u$ and for all $\omega \in \Omega$ without any exceptions;
(F3) The processes $\{\xi_{s+h,t+h},\ 0 \le s \le t < \infty\}$ have the same law for all $h \ge 0$;
(F4) For any n, any $0 \le s_1 \le t_1 \le \cdots \le s_n \le t_n < \infty$, the increments $\xi_{s_1 t_1}, \ldots, \xi_{s_n t_n}$ are independent;

(F5) The n-particle motions $\{(\xi_t(x_1),\dots,\xi_t(x_n)),\ t\ge 0\}$, $(x_1,\dots,x_n)\in\mathbb{R}^n$, are diffusions with the infinitesimal generator

$$L_n = \frac{B(0)}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + \sum_{1\le i<j\le n} B(x_i - x_j)\frac{\partial^2}{\partial x_i \partial x_j},$$

where $B(\cdot)$ satisfies conditions (B1)–(B4).

Such flows have been extensively studied. For existence and thorough study of properties, see [7]. A wider class of "Brownian flows" is considered by Kunita in [9]; see Section 4.2 there.

One can think of a stochastic flow as being defined by a stochastic differential equation (see [9]). Consider a SDE driven by the (possibly infinite number of) independent Brownian motions $\{W_t^\alpha,\ \alpha\ge 1\}$:

$$x_t = x + \int_0^t \sum_{\alpha\ge 1} V_\alpha(x_s)\, dW_s^\alpha, \tag{2.1}$$

$x\in\mathbb{R}$, $t\ge 0$, $V_\alpha(\cdot)$ are functions $\mathbb{R}\to\mathbb{R}$. The flow $\{\xi_t(x),\ x\in\mathbb{R}^1,\ t\in\mathbb{R}_+\}$ is said to be defined by the SDE above if for any $x\in\mathbb{R}$, $\{\xi_t(x),\ t\ge 0\}$ solves (2.1) with the initial point x, and (F2) is satisfied. In this definition the covariance structure, also known as local characteristic, $a(\cdot,\cdot)$ is defined by

$$a(y,z) = \sum_{\alpha\ge 1} V_\alpha(y)V_\alpha(z),$$

the sum on the right-hand side is assumed to converge in the appropriate sense. If there exists a function $B(\cdot)$ of one variable such that

$$B(y-z) = a(y,z),$$

then the one-dimensional flow ξ is said to be isotropic. Note that this is the same $B(\cdot)$ that appears in (F5). See [2] for more on the connection between the two definitions.

For a rescaled isotropic flow ξ^k [see (1.1)] the covariance structure $B_k(\cdot)$ is given by

$$B_k(x) = B(kx), \qquad x\in\mathbb{R}, \tag{2.2}$$

where $B(\cdot)$ is the covariance structure of ξ.

LEMMA 2. *Define*

$$D_t = \xi_t^k(x) - \xi_t^k(y), \qquad x>y,\ t\ge 0,\ k\in\mathbb{Z}_+.$$

Then $\{D_t,\ t\ge 0\}$ *is a positive martingale. Moreover,* $\mathbf{E}\,D_t = x-y$.

PROOF. The process D_t satisfies a stochastic integral equation [see (F5) and (2.2)]

$$D_t = (x-y) + \int_0^t \sqrt{1 - B(kD_s)}\, dW_s;$$

$\{W_t,\ t\ge 0\}$ is a standard Brownian motion. Immediately D_t is a martingale. Also $x-y>0$, and 0 is inaccessible for D_t, therefore $D_t>0$ a.s. for any $t\ge 0$. Since $D_0 = x-y$, the second statement of the lemma is obvious. □

Recall that $\hat{\xi}_t^k(x)$ was defined in (1.2) to be the piecewise constant approximation to $\xi_t^k(x)$ in x, on the scale $1/k$.

Let us fix $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. Then finite-particle motions of the processes $\xi^k, \hat{\xi}^k$ started at $\mathbf{x}$ are defined as

$$\xi_t^{k,\mathbf{x}} = \big(\xi_t^k(x_1), \dots, \xi_t^k(x_n)\big),$$
$$\hat{\xi}_t^{k,\mathbf{x}} = \big(\hat{\xi}_t^k(x_1), \dots, \hat{\xi}_t^k(x_n)\big).$$

The following lemma shows that ξ^k and $\hat{\xi}^k$ are "close" for large k in some sense.

LEMMA 3. *Let us fix* $\mathbf{x} \in \mathbb{R}^n$, $t \geq 0$. *Then*

$$\xi_t^{k,\mathbf{x}} - \hat{\xi}_t^{k,\mathbf{x}} \to 0 \quad \text{as } k \to \infty$$

in probability.

PROOF. It is enough to consider the case $n = 1$ only. By definition of $\hat{\xi}$, for any $x \in \mathbb{R}$, for every $s \geq 0$,

$$\hat{\xi}_s^k(x) = \xi_s^k(\lfloor kx \rfloor / k).$$

In particular,

$$\left\{\xi_s^k(x) - \hat{\xi}_s^k(x),\, s \geq 0\right\} = \left\{\xi_s^k(x) - \xi_s^k(\lfloor kx \rfloor / k),\, s \geq 0\right\}$$

is a positive martingale; see Lemma 2. Moreover,

$$\xi_0^k(x) - \xi_0^k(\lfloor kx \rfloor / k) = (kx - \lfloor kx \rfloor)/k.$$

Thus, by Markov's inequality for any $\varepsilon > 0$,

$$\mathbf{P}\left(\xi_t^k(x) - \hat{\xi}_t^k(x) > \varepsilon\right) \leq \varepsilon^{-1}(kx - \lfloor kx \rfloor)/k \leq \varepsilon^{-1}/k \to 0$$

as $k \to \infty$. The lemma follows. $\square$

For fixed $t > 0$, $\xi_t(\cdot)$ [as well as $\xi_t^k(\cdot)$ for any $k \in \mathbb{Z}_+$] is a homeomorphism $\mathbb{R}^1 \to \mathbb{R}^1$:

$$\xi_t(\cdot)\colon x \mapsto \xi_t(x), \qquad x \in \mathbb{R}^1;$$

see condition (B2), Harris [7], 4.13 and Matsumoto [10], 3.5. In particular, for fixed $t > 0$, $\xi_t^k(x)$ as a function of x is (i) continuous and (ii) strictly increasing.

The random function

$$\hat{\xi}_t^k(\cdot)\colon x \mapsto \hat{\xi}_t^k(x), \qquad x \in \mathbb{R}^1,$$

for fixed $t > 0$ has the following properties:

(i) It is nondecreasing right-continuous function with left-hand limits.
(ii) It is constant on intervals $[i/k, (i+1)/k)$, $i \in \mathbb{Z}$.
(iii) It has (positive) jumps of size $\xi_t^k(i/k) - \xi_t^k((i-1)/k)$ at points i/k, $i \in \mathbb{Z}$.

In particular, for fixed $t > 0$, $\hat{\xi}_t(\cdot)$ is a random element in the space $D(\mathbb{R};\mathbb{R})$ of right-continuous functions with left-hand limits equipped with the Skorohod metric.

3. Coalescing Brownian flow. The coalescing Brownian flow in $\mathbb{R}^1$ is a process

$$C = \{C_t(x),\ x \in \mathbb{R}^1,\ t \in \mathbb{R}_+\},$$

where $C_t(x)$ is the position of the particle at time t, started at the point x at time 0, such that the following hold.

1. For fixed $x \in \mathbb{R}$, $\{C_t(x),\ t \ge 0\}$ is a (one-dimensional standard) Brownian motion started at x.
2. For any $x \ne y$,

$$\{C_t(x) = C_t(y)\} \Rightarrow \{C_s(x) = C_s(y) \text{ for any } s \ge t\}.$$

3. The motions $\{C_t(x),\ t \ge 0\}$ for different x's are independent until coalescence.

This is not a definition, but rather a general picture of a coalescing Brownian flow. The existence and properties of such a process were established by Arratia in [1]. By

$$C_t^{\mathbf{x}} \equiv (C_t(x_1), \dots, C_t(x_n)),$$

we denote the n-particle motion of the coalescing Brownian flow started at

$$\mathbf{x} = (x_1, \dots, x_n).$$

The rigorous definition of a finite-particle coalescing Brownian motion was given in [11], Lemma 2.1, Theorem 2.1. The following result is due to Arratia [1].

PROPOSITION 4. *There exists a random process*

$$\{C_t,\ t \ge 0\} = \{C_t(x),\ x \in \mathbb{R}^1,\ t \in \mathbb{R}_+\}$$

such that for any $(x_1, \dots, x_n) \in \mathbb{R}^n$ *the process*

$$\{(C_t(x_1), \dots, C_t(x_n)),\ t \ge 0\}$$

is an n-particle coalescing Brownian motion as defined in [11], *Lemma* 2.1. *This process has the following properties*: *for any* $t > 0$ *there exists a random discrete set of points* $\{E_k\}_{k=-\infty}^{\infty}$, $E_k = E_k(t, \omega)$, *such that for any* $x \in \mathbb{R}$,

$$C_t(x) = \sum_{k=-\infty}^{\infty} C_t(E_k) 1_{[E_k, E_{k+1})}(x).$$

In particular,

$$\{x, y \in [E_k(t), E_{k+1}(t))\} \Rightarrow \{C_t(x) = C_t(y)\},$$

and the set of noncoalesced by time $t > 0$ particles is discrete:

$$\{C_t(x),\ x \in \mathbb{R}\} = \{C_t(E_k),\ k \in \mathbb{Z}\} \quad \text{as sets a.s.}$$

Moreover, for fixed $t > 0$, the random set $\{E_k(t)\}_{k=-\infty}^{\infty}$ is a point process in $\mathbb{R}$ with known zero function (and, therefore, known distribution); see [1], *page* 17. *The random set $\{C_t(E_k(t)),\ k \in \mathbb{Z}\}$ is also a point process in $\mathbb{R}$ with the same zero function (and, therefore, the same distribution).*

Let us define (for fixed $t > 0$) jumps at the points E_k, $k \in \mathbb{Z}$:

$$J_k = C_t(E_k) - C_t(E_k -) = C_t(E_k) - C_t(E_{k-1}), \qquad k \in \mathbb{Z}. \tag{3.1}$$

From Proposition 4 we obtain the following.

COROLLARY 5. *Fix $t > 0$. For almost all $\omega \in \Omega$, the function $C_t(\cdot) = C_t(\cdot, \omega)$ has the following properties*:

(i) *$C_t(\cdot)$ is a nondecreasing right-continuous function with left-hand limits*;
(ii) *$C_t(\cdot)$ is constant on intervals $[E_k, E_{k+1})$*;
(iii) *$C_t(\cdot)$ has jumps of size J_k at points E_k.*

$C_t(\cdot)$ for fixed $t > 0$ is a random element in $D(\mathbb{R}; \mathbb{R})$, just as $\hat{\xi}_t^k(\cdot)$ is.

4. Weak convergence as diffusions. The following result is intuitive, even though its proof is quite technical.

THEOREM 6. *Suppose conditions* (B1)–(B4) *hold. Then for any $n \in \mathbb{Z}_+$, any $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, the sequence of rescaled n-particle motions $\{\xi_\cdot^{k,\mathbf{x}}\}_{k=1}^{\infty}$ converges weakly to the n-particle motion $C_\cdot^{\mathbf{x}}$*:

$$\xi_\cdot^{k,\mathbf{x}} \Rightarrow C_\cdot^{\mathbf{x}} \quad \text{weakly as } k \to \infty.$$

This type of convergence is called "convergence of stochastic flows weakly as diffusions."

PROOF. The idea of the proof is quite simple. The law of an isotropic stochastic flow is uniquely determined by its covariance structure. Conditions (B1)–(B4) imply [$B_k(\cdot)$'s are defined in (2.2)] that

$$B_k(\cdot) \to B_\infty(\cdot) \quad \text{as } k \to \infty, \tag{4.1}$$

pointwise and uniformly on compact subsets of $\mathbb{R}^1 \setminus \{0\}$, where

$$B_\infty(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

The definition of the coalescing Brownian motion implies that its covariance structure is exactly $B_\infty(\cdot)$:

$$\operatorname{cov}\big(C_t(y), C_t(z)\big) = \int_0^t \mathbf{E}\, B_\infty\big(C_s(y) - C_s(z)\big)\, ds.$$

If $B_\infty(\cdot)$ were a smooth function, Theorem 6 would follow immediately from (4.1). Its not being smooth creates a great many complications, which we skip altogether by referring the interested reader to [11], where a complete proof of this result is given. □

5. Motions of adjacent intervals. This section provides technical tools to be used in the proof of Theorem 1 to establish tightness; see Section 6.

THEOREM 7. *There exists* $\alpha > 1/2$ *such that, if* $h \geq 1/(2k)$ *and* $\lambda > h$, *then*

$$\mathbf{P}\Big(\sup_{t\geq 0}\big(\xi_t^k(h) - \xi_t^k(0)\big)\big(\xi_t^k(0) - \xi_t^k(-h)\big) > \lambda^2\Big) < \frac{2h^{2\alpha}}{\lambda^{2\alpha}}.$$

The theorem is proven by the series of propositions. Define

$$\begin{aligned} \mathbb{R}_+^2 &= \{(x, y) \in \mathbb{R}^2 \colon x > 0,\ y > 0\},\\ X_t^k &= \xi_t^k(x) - \xi_t^k(0), \qquad x > 0,\\ Y_t^k &= \xi_t^k(0) - \xi_t^k(-y), \qquad y > 0. \end{aligned}$$

We start with a technical but crucial proposition.

PROPOSITION 8. *For* $(x, y) \in \mathbb{R}_+^2$ *define*

$$\begin{aligned} c(x, y) &= \frac{B(x) + B(y) - B(x+y) - 1}{xy},\\ v(x, y) &= \frac{1 - B(x)}{x^2} + \frac{1 - B(y)}{y^2}. \end{aligned}$$

Then there exists $\varepsilon_0 > 0$ *such that for any* ε, $0 < \varepsilon < \varepsilon_0$,

$$\{x > \varepsilon \text{ or } y > \varepsilon\} \Rightarrow \left\{\frac{c(x, y)}{v(x, y)} < \left(\frac{1 + B(\varepsilon)}{2}\right)^{1/2}\right\}.$$

PROOF. It follows from positive-definiteness and symmetry of $B(\cdot)$ that

$$\det\begin{pmatrix} 1 & B(x) & B(x+y)\\ B(x) & 1 & B(y)\\ B(x+y) & B(y) & 1 \end{pmatrix} \geq 0.$$

Expanding the determinant, we obtain that

$$1 + 2B(x+y)B(x)B(y) - B^2(x+y) - B^2(x) - B^2(y) \geq 0.$$

From this it follows readily that

$$B(x+y) \geq B(x)B(y) - \sqrt{(1 - B^2(x))(1 - B^2(y))}. \tag{5.1}$$

Here $B(\cdot)$ is a continuous positive-definite function with purely nonatomic spectrum [conditions (B1)–(B4) in Section 2]. Therefore, there exists $\varepsilon_0 > 0$ such that if $x > \varepsilon_0$ then $B(x) < B(\varepsilon_0)$, and $B(x)$ is monotone on $[0, \varepsilon_0]$.

Choose ε such that $0 < \varepsilon < \varepsilon_0$. The choice of ε_0 guarantees that if $x > \varepsilon$ then $B(x) < B(\varepsilon)$. Set

$$\rho = \left(\frac{1 + B(\varepsilon)}{2}\right)^{1/2}.$$

If $\max(x, y) > \varepsilon$ then

$$\sqrt{(1 + B(x))(1 + B(Y))} < 2\rho.$$

Thus for $\max(x, y) > \varepsilon$,

$$\begin{aligned}
&x^2(1 - B(y)) + y^2(1 - B(x)) \\
&\quad - \frac{1}{2\rho}((1 + B(x))(1 + B(y)))^{1/2} 2xy((1 - B(x))(1 - B(y)))^{1/2} \\
&\quad \geq x^2(1 - B(y)) + y^2(1 - B(x)) - 2xy((1 - B(x))(1 - B(y)))^{1/2} \\
&\quad = \left(x(1 - B(y))^{1/2} - y(1 - B(x))^{1/2}\right)^2 \\
&\quad \geq 0 \\
&\quad \geq \frac{xy}{\rho}(-(1 - B(x))(1 - B(y))),
\end{aligned}$$

where the last inequality holds because the last term is negative. Rearranging,

$$\begin{aligned}
x^2&(1 - B(y)) + y^2(1 - B(x)) \\
&\geq \frac{xy}{\rho}\left(((1 - B^2(x))(1 - B^2(y)))^{1/2} - (1 - B(x))(1 - B(y))\right) \\
&\geq \frac{xy}{\rho}(B(x) + B(y) - B(x+y) - 1),
\end{aligned}$$

the last inequality holds thanks to (5.1).

We can rewrite the inequality above as

$$\frac{c(x, y)}{v(x, y)} \leq \rho = \left(\frac{1 + B(\varepsilon)}{2}\right)^{1/2}.$$

The proof is complete. □

Choose ε such that

$$0 < \varepsilon < \min\{\varepsilon_0, 1/8\}, \tag{5.2}$$

where ε_0 is from Proposition 8. Also choose h, ε such that

$$h \geq \frac{1}{2k}, \qquad 0 < \varepsilon < \frac{1}{2},$$

and define

$$\begin{aligned} A_\varepsilon &= A_\varepsilon^k = \left\{(x, y) \in \mathbb{R}_+^2 : 0 < x \leq \varepsilon/k, 0 < y \leq \varepsilon/k\right\}, \\ C_h &= \left\{(x, y) \in \mathbb{R}_+^2 : 0 < x \leq h, 0 < y \leq h\right\}, \\ D_h &= \left\{(0, h] \times \{h\}\right\} \cup \left\{\{h\} \times (0, h]\right\}, \\ \tau_h &= \tau_h^k = \inf\left\{t \geq 0 : \left(X_t^k, Y_t^k\right) \in (C_h)^c\right\}. \end{aligned}$$

PROPOSITION 9. *For any* ε *as in* (5.2), *any* $k \in \mathbb{Z}_+$,

$$\sup_{(x, y) \in A_\varepsilon} \mathbf{P}_{x, y}(\tau_h < \infty) < \tfrac{1}{2}.$$

PROOF. If $(x, y) \in A_\varepsilon$, then

$$\begin{aligned} \mathbf{P}_{x, y}(\tau_h < \infty) &= \lim_{T \to \infty} \mathbf{P}_{x, y}(\tau_h < T) \\ &\leq \lim_{T \to \infty} \left(\mathbf{P}_x\left(\sup_{t \leq T} X_t^k > h\right) + \mathbf{P}_y\left(\sup_{t \leq T} Y_t^k > h\right)\right). \end{aligned}$$

Since ε is less than $1/8$ by choice and $(x, y) \in A_\varepsilon$,

$$x < \frac{1}{8k}, \qquad y < \frac{1}{8k}.$$

Then X_t^k is a positive martingale and $\mathbf{E}\, X_t^k = x$; see Lemma 2. Therefore

$$\mathbf{P}_x\left(\sup_{t \leq T} X_t^k > h\right) \leq \frac{1}{h} \mathbf{E}_x\left(X_T^k\right) = \frac{x}{h} \leq \frac{1/(8k)}{1/(2k)} < \frac{1}{4}.$$

Similarly

$$\mathbf{P}_y\left(\sup_{t \leq T} Y_t^k > h\right) < \tfrac{1}{4}.$$

Hence,

$$\sup_{(x, y) \in A_\varepsilon} \mathbf{P}_{x, y}(\tau_h < \infty) < \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}.$$

□

Choose $\lambda > h$ and define

$$\begin{aligned} \tau_\varepsilon &= \tau_\varepsilon^k = \inf\left\{t > 0 : \left(X_t^k, Y_t^k\right) \in A_\varepsilon\right\}, \\ B_\lambda &= \left\{(x, y) \in \mathbb{R}_+^2 : xy \geq \lambda^2\right\}, \\ \tau_\lambda &= \tau_\lambda^k = \inf\left\{t \geq 0 : \left(X_t^k, Y_t^k\right) \in B_\lambda\right\}, \end{aligned}$$

PROPOSITION 10. *For any* ε *as in* (5.2),

$$\mathbf{P}_{h,h}(\tau_\lambda < \infty) < 2 \sup_{(x,y)\in D_h} \mathbf{P}_{x,y}(\tau_\lambda < \tau_\varepsilon < \infty).$$

PROOF. Apply the strong Markov property of (X_t^k, Y_t^k) at stopping times τ_ε and τ_h, the fact that

$$B_\lambda \subset (C_h)^c \subset (A_\varepsilon)^c$$

and Proposition 9. □

We define

$$M_t^k = \left(X_t^k Y_t^k\right)^\alpha.$$

Note that

$$\tau_\lambda = \inf\left\{t \geq 0\colon M_t^k \geq \lambda^{2\alpha}\right\}.$$

PROPOSITION 11. *For any* ε *as in* (5.2) *there exists* $\alpha > 1/2$ *such that*

$$\left\{M_{t\wedge\tau_\varepsilon}^k, t \geq 0\right\}$$

is a supermartingale.

PROOF. Set

$$f(x,y) = x^\alpha y^\alpha.$$

By (F5) and (2.2) the process (X_t^k, Y_t^k) is a diffusion in $\mathbb{R}_+^2$ with infinitesimal generator

$$\begin{aligned} L = {} & (1 - B(kx))\frac{\partial^2}{\partial x^2} + (1 - B(ky))\frac{\partial^2}{\partial y^2} \\ & + (B(kx) + B(ky) - B(k(x+y)) - 1)\frac{\partial^2}{\partial x \partial y}. \end{aligned}$$

Applying it to the function f yields

$$\begin{aligned} \frac{Lf}{f} &= k^2\alpha(\alpha - 1)\left(\frac{(1 - B(kx))}{k^2x^2} + \frac{(1 - B(ky))}{k^2y^2}\right) \\ &\quad + k^2\alpha^2\left(\frac{B(kx) + B(ky) - B(k(x+y)) - 1}{k^2xy}\right) \\ &= k^2\alpha(\alpha - 1)v(kx, ky) + k^2\alpha^2 c(kx, ky), \end{aligned}$$

where the functions $v(\cdot,\cdot)$ and $c(\cdot,\cdot)$ come from Proposition 8.

Set

$$\rho = \left(\frac{1 + B(\varepsilon)}{2}\right)^{1/2}.$$

Suppose $(x, y) \in (A_\varepsilon)^c$. Then $\max(xk, yk) > \varepsilon$. Thus, by Proposition 8,

$$c(kx, ky) < \rho v(kx, ky)$$

and

$$\begin{aligned}\frac{Lf}{f} &\le k^2\big(\alpha(\alpha-1)v(kx, ky) + \alpha^2 \rho v(kx, ky)\big)\\ &= k^2 \alpha v(kx, ky)((\alpha - 1) + \alpha\rho)\\ &= k^2 \alpha v(kx, ky)(\alpha(\rho + 1) - 1).\end{aligned}$$

Since $0 < \rho < 1$,

$$\frac{1}{\rho + 1} > \frac{1}{2},$$

and, therefore, there exists α such that

$$\frac{1}{\rho + 1} > \alpha > \frac{1}{2}.$$

With such a choice of α,

$$\alpha(\rho + 1) - 1 < 0$$

and

$$\frac{Lf}{f} < 0.$$

Summarizing the arguments, we obtain that if α is chosen so that

$$\frac{1}{2} < \alpha < \frac{1}{1 + \big((1 + B(\varepsilon))/2\big)^{1/2}}$$

then

$$L\big((xy)^\alpha\big) < 0 \quad \text{for } (x, y) \in (A_\varepsilon)^c.$$

Since the starting point (h, h) is chosen so that $(h, h) \in (A_\varepsilon)^c$ [by assumption, $h \ge 1/(2k) > \varepsilon/k$], the assertion of the proposition is proved. □

PROPOSITION 12. *Let* $\varepsilon > 0$, $\alpha > 1/2$ *be chosen as in the statement of Proposition* 11. *Then*

$$\sup_{(x, y) \in D_h} \mathbf{P}_{x, y}(\tau_\lambda < \tau_\varepsilon < \infty) \le \frac{h^{2\alpha}}{\lambda^{2\alpha}}.$$

PROOF. Choose $(x, y) \in D_h$ and $T > 0$. Set

$$\tau = \tau_\lambda \wedge \tau_\varepsilon \wedge T.$$

Then

$$\{M^k_{t \wedge \tau}, t \ge 0\}$$

is a supermartingale by Proposition 11. Therefore

$$\mathbf{E}_{x,y} M_\tau^k = \mathbf{E}_{x,y} M_{T\wedge\tau}^k \le \mathbf{E}_{x,y} M_0^k = (xy)^\alpha \le h^{2\alpha}. \tag{5.3}$$

[The last inequality holds because $\max_{(x,y)\in D_h}(xy) = h^2$]. On the other hand,

$$M_\tau^k = M_{\tau_\lambda\wedge\tau_\varepsilon\wedge T}^k = \lambda^{2\alpha} 1_{\{\tau_\lambda<\tau_\varepsilon<T\}} + \eta 1_{\{\tau_\lambda<\tau_\varepsilon<T\}^c} \ge \lambda^{2\alpha} 1_{\tau_\lambda<\tau_\varepsilon<T}, \tag{5.4}$$

where η is a nonnegative random variable because M_τ^k is. Therefore, combining (5.3) and (5.4),

$$\lambda^{2\alpha}\mathbf{P}_{x,y}(\tau_\lambda < \tau_\varepsilon < T) \le h^{2\alpha}.$$

Taking the limit $T\to\infty$, we obtain the statement of the proposition. □

COROLLARY 13. *There exists $\alpha > 1/2$ such that*

$$\mathbf{P}_{h,h}(\tau_\lambda < \infty) < \frac{2h^{2\alpha}}{\lambda^{2\alpha}}.$$

PROOF. Choose any ε as in (5.2). Then the result follows from Propositions 10 and 12. □

By the definition of τ_λ, the statement of Corollary 13 is equivalent to

$$\mathbf{P}_{h,h}\Big(\sup_{t\ge 0}(X_t^k Y_t^k) > \lambda^2\Big) < \frac{2h^{2\alpha}}{\lambda^{2\alpha}},$$

which is exactly the statement of Theorem 7. The theorem is proved. □

6. Proof of Theorem 1.

DEFINITION 14. Let $\phi(\cdot)$ be a random element in $D(\mathbb{R};\mathbb{R})$ or $C(\mathbb{R};\mathbb{R})$. For any $n\in\mathbb{Z}_+$ the *n-particle distribution* of $\phi(\cdot)$ associated with the point $(x_1,\dots,x_n)\in\mathbb{R}^n$ is the distribution of the vector

$$(\phi(x_1),\dots,\phi(x_n)).$$

The collection of n-particle distributions for all n is referred to as *the finite-particle distributions.*

In more conventional terminology, finite-particle distributions would be called finite-dimensional distributions. We would like to keep that term reserved for time-indexed processes, hence the need for this definition.

The time variable t will be fixed throughout. If no confusion is expected it will be dropped, so that we will write $\xi(\cdot)$ for $\xi_t(\cdot)$, and so on.

Theorem 1 asserts that for fixed $t>0$,

$$\hat{\xi}_t^k(\cdot) \Rightarrow C_t(\cdot) \quad \text{as } k\to\infty$$

weakly in the space $D(\mathbb{R};\mathbb{R})$. The proof consists of two parts:

1. convergence of finite-particle distributions of $\hat{\xi}_t^k(\cdot)$ to those of $C_t(\cdot)$;
2. tightness of the family $\{\hat{\xi}_t^k(\cdot)\}_{k=1}^\infty$ in the space $D(\mathbb{R};\mathbb{R})$.

The convergence of finite-particle distributions follows immediately from Theorem 6 (see also [11], Theorem 3.1), Lemma 3 and [3], Theorem 1.4.1. The estimate obtained in Theorem 7 will play a crucial role in establishing tightness.

PROPOSITION 15. *There exist $\varepsilon > 0$, $\theta > 1$ and $A > 0$ such that*

$$\mathbf{P}\big(\hat{\xi}^k(x+h) - \hat{\xi}^k(x) \geq \lambda, \hat{\xi}^k(x) - \hat{\xi}^k(x-h) \geq \lambda\big) \leq A\lambda^{-\varepsilon}h^{\theta}$$

for all $x \in \mathbb{R}$, all $\lambda > 0$, all $h > 0$ uniformly in $k \in \mathbb{Z}_+$.

PROOF. Fix $h > 0$. The result is immediate for k such that $h < 1/(2k)$,

$$h < \frac{1}{2k} \Rightarrow \mathbf{P}\big(\hat{\xi}^k(x+h) - \hat{\xi}^k(x) \geq \lambda, \hat{\xi}^k(x) - \hat{\xi}^k(x-h) \geq \lambda\big) = 0 \tag{6.1}$$

because $\hat{\xi}^k(\cdot)$ is constant on intervals of length $1/k$. Fix k such that $h \geq 1/(2k)$. Set

$$h_1 = (\lfloor kx + kh \rfloor - \lfloor kx \rfloor)/k,$$
$$h_2 = (\lfloor kx \rfloor - \lfloor kx - kh \rfloor)/k,$$
$$h' = \max\{h_1, h_2\}.$$

Then

$$\frac{1}{2k} \leq h' < h + 1/k \leq h + 2h = 3h.$$

Using space homogeneity and monotonicity of $\xi^k(\cdot)$ and applying Theorem 7 we obtain that there exists $\alpha > 1/2$ such that

$$\begin{aligned}
&\mathbf{P}\big(\hat{\xi}^k(x+h) - \hat{\xi}^k(x) \geq \lambda, \hat{\xi}^k(x) - \hat{\xi}^k(x-h) \geq \lambda\big)\\
&\quad = \mathbf{P}\big(\xi^k(h_1) - \xi^k(0) \geq \lambda, \xi^k(0) - \xi^k(-h_2) \geq \lambda\big)\\
&\quad \leq \mathbf{P}\big(\xi^k(h') - \xi^k(0) \geq \lambda, \xi^k(0) - \xi^k(-h') \geq \lambda\big)\\
&\quad \leq \mathbf{P}\big((\xi^k(h') - \xi^k(0))(\xi^k(0) - \xi^k(-h')) \geq \lambda^2\big)\\
&\quad \leq \frac{2(h')^{2\alpha}}{\lambda^{2\alpha}}\\
&\quad \leq \frac{2(3h)^{2\alpha}}{\lambda^{2\alpha}}.
\end{aligned} \tag{6.2}$$

We denote

$$\theta = \varepsilon = 2\alpha > 1. \tag{6.3}$$

Combining (6.1), (6.2), (6.3) and setting

$$A = 2 \cdot 3^{\theta},$$

we obtain the statement of the proposition. □

COROLLARY 16. *For fixed* $t > 0$, *the family of* $\{\hat{\xi}_t^k(\cdot)\}_{k=1}^{\infty}$ *is tight in* $D(\mathbb{R};\mathbb{R})$.

The corollary follows from Proposition 15 and [6], Theorem 3.8.8, Remark 3.8.9b.

Theorem 1 is proved. □

7. Expansions and contractions. Theorem 1 provides us with the means to study expansions (and contractions) of the original flow $\xi_t(\cdot)$. The idea is that expansions of $\xi_t(\cdot)$ on large time and space scales correspond to large jumps of $\hat{\xi}_t^k(\cdot)$ for large k.

DEFINITION 17. Let $\phi(\cdot) \in D(\mathbb{R};\mathbb{R})$. Define

$$\Delta\phi(x) = \phi(x) - \phi(x-), \qquad x \in \mathbb{R},$$

to be the jump-function of $\phi(\cdot)$. Define

$$\mathbf{J}(\phi) = \sup_{0 < x \le 1} \Delta\phi(x).$$

Also fix $u > 0$ and define

$$P^0(\phi, u) = 0, \qquad P^{n+1}(\phi, u) = \inf\{x > P^n(\phi, u): |\Delta\phi(x)| > u\}.$$

$\mathbf{J}(\phi)$ is the size of the largest jump of $\phi \in D(\mathbb{R};\mathbb{R})$ between zero and one. For a fixed $u > 0$, $P^1(\phi, u)$ is the first (counting from $x = 0$) jump of function $\phi(\cdot)$ of size larger than u, and similarly $P^n(\phi, u)$ is the nth jump of $\phi(\cdot)$ of size larger than u.

PROPOSITION 18. *Define* $N \subset D(\mathbb{R};\mathbb{R})$, *the subset of functions with a jump at zero, by*

$$\phi \in N \quad \Leftrightarrow \quad \Delta\phi(0) \neq 0.$$

Also define

$$U(\phi) = \{u > 0: |\Delta\phi(x)| = u \text{ for some } x \in \mathbb{R}\}.$$

The function

$$\phi \mapsto \mathbf{J}(\phi)$$

is continuous in Skorohod metric at each point $\phi \notin N$. *For* $u > 0$, *the functions*

$$\phi \mapsto P^n(\phi, u), \qquad \phi \mapsto \Delta\phi(P^n(\phi, u))$$

are continuous in Skorohod metric at each point ϕ *such that* $\phi \notin N$, $u \notin U(\phi)$ *and that* $P^n(\phi, u) < \infty$ *for the last case.*

For the continuity of $\mathbf{J}(\cdot)$ see [6], page 153. For the continuity of the other two functionals see [8], Proposition 6.2.7. Note that the condition that ϕ does not have a jump at 0 does not appear in [6] and [8] because they consider the space $D(\mathbb{R}_+;\mathbb{R})$ rather than $D(\mathbb{R};\mathbb{R})$. Modifications to the proofs are obvious.

Let us fix $t > 0$ and recall that $\{J_l, l \in \mathbb{Z}\}$ were defined in (3.1), $\{E_l, l \in \mathbb{Z}\}$ were defined in Proposition 4. Applying $\mathbf{J}$ to $C_t(\cdot)$ we get

$$\mathbf{J}(C_t(\cdot)) = \max\{J_l: E_l \in (0,1]\}.$$

Now let us apply $\mathbf{J}$ to $\hat{\xi}_t^k(\cdot)$:

$$\mathbf{J}\big(\hat{\xi}_t^k(\cdot)\big) = \max_{i=1,\dots,k}\big(\hat{\xi}_t^k(i/k) - \hat{\xi}_t^k((i-1)/k)\big)$$
$$= \frac{1}{k}\max_{i=1,\dots,k}\big(\xi_{k^2t}(i) - \xi_{k^2t}(i-1)\big).$$

Then $\xi_{k^2t}(i) - \xi_{k^2t}(i-1)$ can be interpreted as the size of the expansion of the random function $\xi_{k^2t}(\cdot)$ at the integer point i, or simply *integer expansion*. Then $\mathbf{J}(\hat{\xi}_t^k(\cdot))$ is the (rescaled) size of the largest integer expansion of $\xi_{k^2t}(\cdot)$ over the set $\{1,\dots,k\}$. Let us enumerate $\{E_l,\ l\in\mathbb{Z}\}$ (see Proposition 4 for definition) in such a way that $0\in[E_0, E_1)$. Fix $u>0$. Define (i, l are integers)

$$K(0)=0,\qquad K(n+1)=\min\{l>K(n)\colon J_l>u\},$$
$$I^k(0)=0,\qquad I^k(n+1)=\min\{i>I^k(n)\colon \xi_{k^2t}(i)-\xi_{k^2t}(i-1)>uk\}.$$

Here $K(n)$ is the number (in the sequence of *all* jumps) of nth jump of $C_t(\cdot)$ of size larger than u; $I^k(n)$ is the position of nth integer expansion of $\xi_{k^2t}(\cdot)$ of size larger than uk. Then for any $n\in\mathbb{Z}_+$

$$E_{K(n)} = P^n\big(C_t(\cdot),u\big),$$
$$J_{K(n)} \equiv \Delta C_t\big(E_{K(n)}\big) = \Delta C_t\big(P^n\big(C_t(\cdot),u\big)\big),$$
$$I^k(n)/k = P^n\big(\hat{\xi}_t^k(\cdot),u\big),$$
$$\xi_{k^2t}\big(I^k(n)\big) - \xi_{k^2t}\big(I^k(n)-1\big) = \Delta\hat{\xi}_t^k\Big(P^n\big(\hat{\xi}_t^k(\cdot),u\big)\Big).$$

Note that for any $u>0$, $\mathbf{P}(u\in U(C(\cdot)))=0$, $\mathbf{P}(E_0\neq 0)=1$. However, before we can apply Proposition 18 to the weak convergence $\hat{\xi}^k(\cdot)\Rightarrow C(\cdot)$, we need the following simple lemma.

LEMMA 19. *Fix $u>0$, $n\in\mathbb{Z}_+$. Define $\mathbf{x}=(x_0,\dots,x_n)\in\mathbb{R}^{n+1}$ by $x_i=iu$, $i=0,\dots,n$. Let $\{C_s^{\mathbf{x}},\ s\ge 0\}$ be the $n+1$-particle coalescing Brownian motion started at $\mathbf{x}$. Then for any $t\ge 0$, there exist $0<p<1$, independent of n such that*

$$\mathbf{P}\big(C_t(x_i),\ i=0,\dots,n,\ \textit{are all distinct}\big)$$
$$= \mathbf{P}\big(C_t(x_i)\neq C_t(x_j),\ i,j=0,\dots,n,\ i\neq j\big)\le p^n.$$

PROOF. Let

$$\{W_t^{\mathbf{x}},\ t\ge 0\} = \{(W_t(x_0),\dots,W_t(x_n)),\ t\ge 0\}$$

be the $n+$ one-dimensional standard Brownian motion with the starting point $\mathbf{x}$. Before coalescence, particles of C_t move independently (also see [11], Lemma 2.1(iii), so

$$\mathbf{P}\big(C_t(x_i)\neq C_t(x_j),\ i,j=0,\dots,n,\ i\neq j\big)$$
$$= \mathbf{P}\big(W_s(x_i)-W_s(x_{i-1})>0 \text{ for any } s\in[0,t],\ \text{all } i=1,\dots,n\big).$$

For simplicity of notations, assume that n is even. Then

$$\begin{aligned}
&\mathbf{P}\big(W_s(x_i) - W_s(x_{i-1}) > 0\ \forall\, s, i\colon s \in [0,t],\ i = 1,\dots,n\big)\\
&\quad\le \mathbf{P}\big(W_s(x_{2i+1}) - W_s(x_{2i}) > 0\ \forall\, s, i\colon s \in [0,t],\ i = 0,\dots,n/2-1\big)\\
&\quad= \prod_{i=0}^{n/2-1} \mathbf{P}\big(W_s(x_{2i+1}) - W_s(x_{2i}) > 0\ \forall\, s\colon s \in [0,t]\big)\\
&\quad= \big(\mathbf{P}\big(W_s^1 - W_s^2 < u\ \forall\, s\colon s \in [0,t]\big)\big)^{n/2},
\end{aligned}$$

where $W_\cdot^1$, $W_\cdot^2$ are two independent standard Brownian motions both started at 0. Set

$$p = \sqrt{\mathbf{P}\Big(\sup_{0\le s\le t}\big(W_s^1 - W_s^2\big) < u\Big)}.$$

Clearly $0 < p < 1$, and the lemma follows. □

THEOREM 20. *Let us fix $t > 0$. Then we have the following.*

(i) *The rescaled size of the largest integer expansion of the function $\xi_{k^2t}(\cdot)$ over the set $\{1,\dots,k\}$ converges in distribution to the largest jump of $C_t(\cdot)$ over the interval $(0,1]$, that is,*

$$\frac{1}{k}\max_{i=1,\dots,k}\big(\xi_{k^2t}(i) - \xi_{k^2t}(i-1)\big) \Rightarrow \max\{J_l\colon E_l \in (0,1]\}.$$

(ii) *For any $u > 0$, $n \in \mathbb{Z}_+$ and $t > 0$, the rescaled position of nth integer expansion of $\xi_{k^2t}(\cdot)$ of size larger than uk converges to the position of nth jump of $C_t(\cdot)$ larger than u, that is,*

$$\frac{1}{k}I^k(n) \Rightarrow E_{K(n)}.$$

(iii) *For any $u > 0$, $n \in \mathbb{Z}_+$ and $t > 0$, the rescaled size of nth integer expansion of $\xi_{k^2t}(\cdot)$ that is larger than uk converges to the size of nth jump of $C_t(\cdot)$ that is larger than u, that is,*

$$\frac{1}{k}\big(\xi_{k^2t}\big(I^k(n)\big) - \xi_{k^2t}\big(I^k(n)-1\big)\big) \Rightarrow J_{K(n)}.$$

The convergence in (i), (ii) *and* (iii) *is in distribution as $k \to \infty$.*

PROOF. As was mentioned before ($t > 0$, $u > 0$ are fixed),

$$\mathbf{P}\big(C_t(\cdot) \in N\big) = \mathbf{P}(E_0 = 0) = 0$$

[positions of jumps of $C_t(\cdot)$ are not fixed],

$$\mathbf{P}\big(u \in U\big(C_t(\cdot)\big)\big) = 0$$

[sizes of jumps of $C_t(\cdot)$ are not fixed]. Therefore, the statements (i) and (ii) follow from Theorem 1 and Proposition 18. Statement (iii) will also follow from Theorem 1 and Proposition 18 if we can establish that

$$\mathbf{P}\big(P^n(C_t(\cdot), u) < \infty\big) = 1 \tag{7.1}$$

for any $u > 0$, $n \in \mathbb{Z}_+$, $t > 0$. We will prove that it holds for $n = 1$. For general n, (7.1) will follow from space-homogeneity of $C_t(\cdot)$. We can rewrite the probability above for $n = 1$ as

$$\mathbf{P}(\exists\, k \geq 1\colon J_k > u) = 1.$$

As stated in Proposition 4,

$$\{J_k,\ k \in \mathbb{Z}\} = \{E_k - E_{k-1},\ k \in \mathbb{Z}\}$$

in distribution. Let A_n be the event

$$A_n = \big\{\exists\, k\colon E_k \in (0, un],\ E_k - E_{k-1} > u\big\}.$$

Then

$$\mathbf{P}(\exists\, k \geq 1\colon J_k > u) = \lim_{n\to\infty} \mathbf{P}(A_n).$$

Set $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ by $x_i = iu$, $i = 0, \dots, n$. Let $\{C_t^{\mathbf{x}},\ t \geq 0\}$ be the $n+1$-particle coalescing Brownian motion started at $\mathbf{x}$. Then

$$\big\{E_k \in (u(i-1), ui]\big\} \Rightarrow \big\{C_t(x_{i-1}) \neq C_t(x_i)\big\}.$$

Therefore, the event $\overline{A}_n$ implies that on each interval $(u(i-1), ui]$, $i = 1, \dots, n$, there is at least one E_k. In other words,

$$\overline{A}_n \Rightarrow \big\{\forall\, i = 1, \dots, n\ \exists\, k\colon E_k \in (u(i-1), ui]\big\}.$$

In particular,

$$\overline{A}_n = \big\{C_t(x_{i-1}) \neq C_t(x_i)\ \forall\, i = 1, \dots, n\big\}.$$

The probability of the event on the right-hand side was estimated in Lemma 19. Hence there exist $0 < p < 1$, such that for any $n \in \mathbb{Z}_+$,

$$\mathbf{P}\big(\overline{A}_n\big) \leq p^n.$$

Taking the limit $n \to \infty$, we get that

$$\mathbf{P}(\exists\, k \geq 1\colon J_k > u) = \lim_{n\to\infty} \mathbf{P}(A_n) \geq 1 - \lim_{n\to\infty} p^n = 1.$$

Therefore, (7.1) is established and Proposition 18 is applicable. The second statement of the theorem follows. □

Note that the realizations of $\xi(\cdot)$ were broken at *integer* points purely for notational convenience. We could have taken the piecewise constant approximation to $\xi^k(\cdot)$ on the scale ε/k instead of $1/k$ for any $\varepsilon > 0$, say, and all the results would go through. That would allow us to study expansions at points $\{\varepsilon k,\ k \in \mathbb{Z}\}$ similarly to the way proposed in Theorem 20 (and later results) for integer expansions.

Yet another insight into the clustering properties of $\xi(\cdot)$ can be gained via what we call the *expansion measure*. Let λ be the Lebesgue measure on $\mathbb{R}$.

We push it backward using the mappings $\xi_t^k(\cdot)$ and $C_t(\cdot)$:

$$\nu^k = \nu_t^k = \lambda \circ \xi_t^k(\cdot),$$
$$\gamma = \gamma_t = \lambda \circ C_t(\cdot),$$

meaning that for Borel sets $A \subset \mathbb{R}$,

$$\nu^k(A) = \lambda\big(\xi_t^k(A)\big),$$
$$\gamma(A) = \lambda\big(C_t(A)\big).$$

If $\nu^k(A)$ is large for some set A, then A is stretched by time t by the flow $\xi_t^k(\cdot)$. If $\nu^k(A)$ is small, the set A is, correspondingly, shrunk. The same comments, of course, apply to γ. This explains the name "expansion measures."

The (random) measure γ has a particularly simple structure:

$$\gamma(\cdot) = \sum_{k=-\infty}^{\infty} J_k \delta_{E_k}(\cdot),$$

where $\delta_x(\cdot)$ assigns a unit mass to the point $x \in \mathbb{R}$. Let $[a, b]$ be an interval in $\mathbb{R}$. Then

$$\nu^k([a, b]) = \xi_t^k(b) - \xi_t^k(a),$$
$$\gamma([a, b]) = C_t(b) - C_t(a).$$

Using the results on convergence of finite-particle motions of ξ^k to those of C (Theorem 6) we obtain that for any $[a, b] \subset \mathbb{R}$,

$$\nu^k([a, b]) \Rightarrow \gamma([a, b])$$

in distribution as $k \to \infty$ (t is fixed). The following theorem shows that the stronger kind of convergence in fact holds.

THEOREM 21. *Let $\phi(\cdot)$ be a $C^1(\mathbb{R}; \mathbb{R})$ function with compact support. Then*

$$\int \phi(x) \nu^k(dx) \Rightarrow \int \phi(x) \gamma(dx) = \sum_{k=-\infty}^{\infty} J_k \phi(E_k)$$

in distribution as $k \to \infty$, t is fixed.

PROOF. Let us fix $t > 0$. Using the fact that $\xi^k(\cdot)$ is continuous and strictly increasing, we integrate by parts to get

$$\int \phi(x) \nu^k(dx) = -\int \phi'(x) \xi^k(x)\, dx \quad \text{a.s.}$$

Also

$$\int \phi(x) \gamma(dx) = -\int \phi'(x) C(x)\, dx \quad \text{a.s.},$$

because $C_t(\cdot)$ is piecewise constant. So in fact we have to prove that

$$\int \psi(x)\xi^k(x)\,dx \Rightarrow \int \psi(x)C(x)\,dx \tag{7.2}$$

in distribution as $k \to \infty$ for any $\psi(\cdot)$, a continuous function with compact support.

The functional

$$f \mapsto \int f(x)\,dx$$

for $f \in D(\mathbb{R};\mathbb{R})$ with compact support is continuous in Skorohod metric; see [6], page 153. Therefore, Theorem 1 implies that

$$\int \psi(x)\hat{\xi}^k(x)\,dx \Rightarrow \int \psi(x)C(x)\,dx \tag{7.3}$$

in distribution as $k \to \infty$. We want to replace $\hat{\xi}^k$ by ξ^k in (7.3). The argument will be similar to the one used in the proof of Lemma 3, so we omit the details. Fix $\varepsilon > 0$. Then

$$\begin{aligned}
&\mathbf{P}\left(\left|\int \psi(x)\xi^k(x)\,dx - \int \psi(x)\hat{\xi}^k(x)\,dx\right| > \varepsilon\right)\\
&\quad \le \mathbf{P}\left(\int \left|\psi(x)\big(\xi^k(x) - \hat{\xi}^k(x)\big)\right| dx > \varepsilon\right)\\
&\quad = \mathbf{P}\left(\int |\psi(x)|\big(\xi^k(x) - \hat{\xi}^k(x)\big)\,dx > \varepsilon\right) \quad \big(\xi_t^k(x) \ge \hat{\xi}_t^k(x) \text{ a.s.}\big)\\
&\quad \le \varepsilon^{-1} \int |\psi(x)|\mathbf{E}\big(\xi^k(x) - \hat{\xi}^k(x)\big)\,dx \quad \text{(by Markov's inequality)}\\
&\quad \le \varepsilon^{-1}k^{-1} \int |\psi(x)|\,dx \to 0
\end{aligned} \tag{7.4}$$

as $k \to \infty$. Combining (7.3), (7.4) and [3], Theorem 1.4.1, we establish (7.2) and, therefore, the statement of the theorem. □

Acknowledgments. I thank Professor Peter Baxendale for his immeasurable help with this research. I am also grateful to an anonymous referee for helpful comments.

REFERENCES

[1] ARRATIA, R. (1979). Coalescing Brownian motion on the line. Ph.D. dissertation, Univ. Wisconsin–Madison.

[2] BAXENDALE, P. H. and HARRIS, T. E. (1986). Isotropic stochastic flows. *Ann. Probab.* **14** 1155–1179.

[3] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

[4] DARLING, R. W. R. (1987). *Constructing Nonhomeomorphic Stochastic Flows*. Amer. Math. Soc., Providence, RI.

[5] DARLING, R. W. R. (1988). Rate of growth of the coalescent set in a coalescing stochastic flow. *Stochastics* **23** 465–508.

[6] ETHIER, S. and KURTZ, T. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York.

[7] HARRIS, T. (1984). Coalescing and noncoalescing stochastic flows in R^1. *Stochastic Process. Appl.* **17** 188–210.

[8] JACOD, J. and SHIRYAEV, A. N. (1987). *Limit Theorems for Stochastic Processes*. Springer, New York.

[9] KUNITA, H. (1990). *Stochastic Flows and Stochastic Differential Equations*. Cambridge Univ. Press.

[10] MATSUMOTO, H. (1989). Coalescing stochastic flows on the real line. *Osaka J. Math.* **26** 139–158.

[11] PITERBARG, V. V. (1997). Expansions and contractions of stochastic flows. Ph.D. dissertation, Univ. Southern California.

NATIONS BANC–CRT
233 SOUTH WACKER DRIVE, #2800
CHICAGO, ILLINOIS 60606
E-MAIL: vladimip@crt.com