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THE MAXIMUM OF THE PERIODOGRAM OF A NON-GAUSSIAN SEQUENCE[1]

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It is a well-known fact that the periodogram ordinates of an iid mean-zero Gaussian sequence at the Fourier frequencies constitute an iid exponential vector, hence the maximum of these periodogram ordinates has a limiting Gumbel distribution. We show for a non-Gaussian iid mean-zero, finite variance sequence that this statement remains valid. We also prove that the point process constructed from the periodogram ordinates converges to a Poisson process. This implies the joint weak convergence of the upper order statistics of the periodogram ordinates. These results are in agreement with the empirically observed phenomenon that various functionals of the periodogram ordinates of an iid finite variance sequence have very much the same asymptotic behavior as the same functionals applied to an iid exponential sample.

1. Introduction. Let $(Z_t)_{t\in\mathbb{Z}}$, be a sequence of iid random variables with $EZ = 0$ and $EZ^2 < \infty$, where $Z =_d Z_1$. We assume for convenience that $EZ^2 = 1$. Let

$$I_{n,A}(\lambda) = n^{-1}\left|\sum_{t=1}^{n}\exp(-i\lambda t)A_t\right|^2, \qquad \lambda\in[0,\pi],$$

denote the *periodogram* of any sequence of random variables A_t. We are interested in the limit distribution of

$$M_n(Z) = \max_{i=1,\dots,q} I_{n,Z}(\omega_i), \qquad \omega_j = 2\pi j/n,$$

where $q = q_n = \max\{j\colon 0 < \omega_j < \pi\}$, that is, $q \sim n/2$.

If the Z_t are iid $N(0,1)$, $(I_{n,Z}(\omega_i))_{i=1,\dots,q}$ is a sequence of iid standard exponential random variables, and the extreme value theory for these periodogram ordinates is contained in the well-known theory for extremes of an iid exponential sequence. In particular,

$$M_n(Z) - \ln q_n \to_d Y, \tag{1.1}$$

where $\to_d$ denotes convergence in distribution and Y has the standard Gumbel distribution $\Lambda(x) = \exp\{-e^{-x}\}$, $x\in\mathbb{R}$.

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A key to the surprising "almost iid" behavior of the periodogram ordinates of an iid sequence is the representation

$$\begin{aligned} I_{n,Z}(\lambda) &= C_{n,Z}^2(\lambda) + S_{n,Z}^2(\lambda) \\ &= \left(n^{-1/2}\sum_{t=1}^{n}\cos(\lambda t)Z_t\right)^2 + \left(n^{-1/2}\sum_{t=1}^{n}\sin(\lambda t)Z_t\right)^2. \end{aligned}$$

For an iid mean-zero, finite variance sequence (Z_t), $((C_n(\omega_i), S_n(\omega_i)))_{i=1,\dots,q}$ constitutes a sequence of mean-zero, uncorrelated random variables with variance 0.5. In the Gaussian case, this argument gives an iid standard exponential sample $(I_{n,Z}(\omega_i))_{i=1,\dots,q}$. In the non-Gaussian case, the periodogram ordinates $I_{n,Z}(\omega_i)$ are neither independent nor uncorrelated, although their autocovariances are $O(1/n)$, whatever the Fourier frequencies. However, many of their properties are very similar to an iid sequence. For example, an application of the Lindeberg–Feller central limit theorem yields that every vector $(I_{n,Z}(\omega_{j_i}))_{i=1,\dots,k}$ is asymptotically iid standard exponential whatever the choice of the k distinct Fourier frequencies ω_{j_i}. This limit result remains valid if one considers a finite vector $(I_{n,Z}(\lambda_i))_{i=1,\dots,k}$ at fixed frequencies $\lambda_i \in (0, \pi)$, $i = 1,\dots,k$; compare, for example, [2].

The asymptotic independence property of the periodogram ordinates $I_{n,Z}(\omega_i)$ has some more surprising consequences. For example, [8, 9] shows that the empirical distribution function based on the $I_{n,Z}(\omega_i)$ has the Glivenko–Cantelli property with exponential limit, that is,

$$\sup_x \left| \frac{1}{q}\sum_{i=1}^{q} I_{(-\infty,x]}(I_{n,Z}(\omega_i)) - e^{-x} \right| \to_P 0.$$

Assuming certain smoothness conditions on the characteristic function of Z, which in particular imply the existence of a density of Z, [3] extends this result from convergence in probability to a.s. convergence. A further indication of the asymptotic independence of the periodogram ordinates at the Fourier frequencies is given by the validity of resampling methods in the frequency domain. Bootstrap resampling techniques based on the empirical distribution function of the $I_{n,Z}(\omega_i)$ (or tapered versions of them) have recently been proposed by various authors; see [7], [4].

The asymptotic independence of the $I_{n,Z}(\omega_i)$ is also the basis for a series of results on the maximum of the periodogram. One important application, for example, is the construction of large sample tests for detection of a jump in the spectral distribution function of a time series. In this setting, the process under the null hypothesis is assumed to be a linear process,

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \qquad t \in \mathbb{Z}, \tag{1.2}$$

while under the alternative hypothesis, $\{X_t\}$ is the sum of a linear process and a random sinusoid, that is, $X_t = \sum_{j=-\infty}^{\infty}\psi_j Z_{t-j} + a\cos(\lambda t) + b\sin(\lambda t)$, where a, b and λ are unknown parameters. Hannan and coauthors proved various results about the a.s. order of magnitude of the maximum of the periodogram ordinates $I_{n,X}(\lambda)$ of the X-sequences satisfying (1.2). Under conditions on the smoothness of the characteristic function of Z, [1] proves that

$$\lim_{n\to\infty} \frac{1}{\ln n} \max_{\lambda\in[0,\pi]} \frac{I_{n,X}(\lambda)}{f_X(\lambda)} = 1 \quad \text{a.s.,} \tag{1.3}$$

where $f_X(\lambda)/(2\pi)$ is the spectral density of the X-sequence. If one specifies $X = Z$, then $f_X \equiv 1$, and (1.3) is the analogue to the corresponding a.s. limit result for the maxima of an iid standard exponential sample. Refinements of this result were obtained by Turkman and Walker [17] in the case of an iid Gaussian sequence. Hannan and Mackisack [10] proved a law of the iterated logarithm for the estimate of the frequency λ_0 which maximizes $f_X(\lambda)$.

As mentioned before, the distributional limit of $(M_n(Z) - \ln q)$ is the Gumbel distribution Λ, provided the Z_t are iid Gaussian. The limit distribution of $(M_n(Z) - \ln q)$ for a non-Gaussian iid sequence seems an open problem. In [18], page 122, it is mentioned that "It seems reasonable to expect (1.1) to hold provided that moments of Z_t up to some sufficiently high order exist..., but no proof is known and the problem of constructing one is undoubtedly extremely difficult." In this paper we show that (1.1) indeed holds for an iid Z-sequence having finite sth moment for some $s > 2$. The main tool for proving this result for a non-Gaussian sequence is a very powerful Gaussian approximation technique for sums of independent random vectors due to Einmahl [5]. See Lemma 3.2 below.

We also extend (1.1) to weak convergence of the point processes based on the points $(\omega_i, I_{n,Z}(\omega_i))$. The limiting point process is Poisson, thus yielding the same result as for an iid exponential sample. In turn, point process convergence yields the joint limit distribution of the upper order statistics of the sample of the periodogram ordinates $(I_{n,Z}(\omega_i))_{i=1,\dots,q}$. By a standard approach, the results for the periodogram of an iid Z-sequence can be shown to hold for a linear X-process, provided one replaces $I_{n,Z}(\omega_i)$ with $I_{n,X}(\omega_i)/f_X(\omega_i)$.

Our main results are formulated in Section 2. In Section 3 some technical results are collected. Finally, the proofs of the main results are given in Section 4.

2. Main results.

We commence with the distributional limit of the sequence of maxima $(M_n(Z))$, where (Z_t) is an iid mean-zero, unit variance sequence.

THEOREM 2.1. *If $E|Z|^s < \infty$ for some $s > 2$, then*

$$M_n(Z) - \ln q \to_d Y,$$

where Y has the standard Gumbel distribution $\Lambda(x) = \exp\{-\exp\{-x\}\}$, $x \in \mathbb{R}$.

REMARK 2.2. The main ingredient for the proof of this and the following results is Lemma 3.2 proved in [5]. It yields the necessary joint Gaussian approximation to the sine and cosine transforms of the Z-sequence at different frequencies in a suitable domain of the distribution. This approximation turns out to be a very powerful tool for approximating the joint distribution of a finite number of periodogram ordinates $I_{n,Z}(\omega_i)$ by iid exponential random variables. An alternative approach (but in the context of a.s. convergence of the maximum of the periodogram) was suggested in various papers [1, 3, 10]. The authors of these papers preferred asymptotic expansions for the joint distribution of vectors of periodogram ordinates which yield a uniform approximation to the whole distribution. The price one has to pay for the uniformity are moment restrictions and conditions on the smoothness of the characteristic function of Z, which are not easy to verify and imply the existence of a density of Z.

REMARK 2.3. The same arguments as used for the proof of Theorem 2.1 give the joint limit distribution of the maxima of the cosine and sine transform of $Z_1, \dots, Z_n$ evaluated at the Fourier frequencies. The limit distribution is the product distribution of standard Gumble's and the centering and normalizing constants are chosen in the same way as if the cosine and sine transformed data were iid mean-zero Gaussian with variance 0.5.

REMARK 2.4. The limit distribution of $\max_{\lambda \in [0,\pi]} I_{n,Z}(\lambda)$ for an iid Gaussian sequence is considered in [16]. The normalizing and centering constants in this limit result are slightly different from those needed for the convergence of $(M_n(Z))$. This shows that adjustments are necessary if one replaces the maximum of the periodogram ordinates at the Fourier frequencies by the maximum at a continuum of frequencies.

REMARK 2.5. Theorem 2.1 can be proved under weaker moment assumptions. For example, it remains valid under $EZ^2 \ln^+|Z| h^2(|Z|) < \infty$ where $h(x)$ is the k times iterated logarithm for any $k \geq 1$. We conjecture that the condition $EZ^2 \ln^+|Z| < \infty$ is sufficient for Theorem 2.1 to hold. At the end of the proofs of Lemma 3.4 and Theorem 2.1, we indicate the necessary changes required in the proof under the weaker moment conditions.

Now let (X_t) be a linear process (1.2) with coefficients ψ_j satisfying

$$\sum_{j=-\infty}^{\infty} |\psi_j|\,|j|^{1/2} < \infty \quad \text{and} \quad f_X(\lambda) > 0 \quad \text{for all } \lambda \in [0,\pi]. \tag{2.1}$$

Under this assumption it is known that (see [18], Theorem 3)

$$\max_{\lambda \in [0,\pi]} \left| \frac{I_{n,X}(\lambda)}{f_X(\lambda)} - I_{n,Z}(\lambda) \right| \to_P 0,$$

where $f_X(\lambda)/(2\pi)$ is the spectral density of the X-sequence. In [18] it is also proved that (2.1) can be relaxed to $\sum_{j=-\infty}^{\infty} |\psi_j|\,|j|^{1/4+\delta} < \infty$ for some $\delta = \delta(\varepsilon) >$

0 if $E|Z|^{4+\varepsilon} < \infty$ for some $\varepsilon > 0$. From Walker's result, we immediately obtain the following corollary.

COROLLARY 2.6. *Let (X_t) be the linear process* (1.2) *where* $EZ = 0$, $EZ^2 = 1$, $E|Z|^s < \infty$ *for some* $s > 2$ *and* (ψ_j) *obeys* (2.1). *Then*

$$\max_{j=1,\dots,q} I_{n,X}(\omega_j)/f_X(\omega_j) - \ln q \to_d Y,$$

where Y has the standard Gumbel distribution Λ.

REMARK 2.7. Corollary 2.6 remains valid for the self-normalized periodogram, that is,

$$\max_{j=1,\dots,q} \frac{I_{n,X}(\omega_j)/f_X(\omega_j)}{q^{-1}\sum_{i=1}^{q} I_{n,X}(\omega_i)/f_X(\omega_i)} - \ln q, \tag{2.2}$$

provided $EZ^4 < \infty$. Indeed, under the given assumptions, it follows from Theorem 10.3.1 of [2] that

$$q^{-1}\sum_{i=1}^{q} I_{n,X}(\omega_i)/f_X(\omega_i) - q^{-1}\sum_{i=1}^{q} I_{n,Z}(\omega_i) = O_p(n^{-1/2}).$$

Since

$$q^{-1}\sum_{i=1}^{q} I_{n,Z}(\omega_i) = n^{-1}\sum_{t=1}^{n} Z_t^2 + O_p(n^{-1}) = 1 + O_p(n^{-1/2}),$$

we have

$$q^{-1}\sum_{i=1}^{q} I_{n,X}(\omega_i)/f_X(\omega_i) = 1 + O_p(n^{-1/2}).$$

Also from Corollary 2.6,

$$\max_{j=1,\dots,q} I_{n,X}(\omega_j)/f_X(\omega_j) = o_p(n^{1/4}),$$

so that

$$\max_{j=1,\dots,q} \frac{I_{n,X}(\omega_j)/f_X(\omega_j)}{q^{-1}\sum_{i=1}^{q} I_{n,X}(\omega_i)/f_X(\omega_i)} - \max_{j=1,\dots,q} I_{n,X}(\omega_j)/f_X(\omega_j) = o_p(1),$$

from which (2.2) is easily deduced.

Theorem 2.1 can be extended to cover convergence for the sequence of point processes based on the points $(\omega_j, I_{n,Z}(\omega_j))$. Specifically, define the sequence of point processes η_n on the space $E = [0, \pi] \times (-\infty, \infty]$ by

$$\eta_n(\cdot) = \sum_{j=1}^{q} \varepsilon_{(\omega_j, I_{n,Z}(\omega_j) - \ln q)}(\cdot), \tag{2.3}$$

where $\varepsilon_x(\cdot)$ is the point measure which assigns unit mass to a set containing x and 0 otherwise. Let $\mathscr{M}_p(E)$ be the set of Radon point measures on the set E endowed with the vague topology. See [11] and [15] for further details and properties of $\mathscr{M}_p(E)$ and the vague topology.

THEOREM 2.8. *For the sequence of point processes* η_n *defined in* (2.3), *we have*

$$\eta_n \to_d \eta,$$

where η *is a Poisson process on* E *with intensity measure* $\pi^{-1}e^{-x}(dt \times dx)$ *and* $\to_d$ *denotes convergence in distribution on the space* $\mathscr{M}_p(E)$ *relative to the vague topology.*

REMARK 2.9. If one replaces in the definition of η_n the points $(\omega_j, I_{n,Z}(\omega_j) - \ln q)$ by $(\omega_j, E_j - \ln q)$, where (E_j) is an iid standard exponential sequence, then the conclusion of Theorem 2.8 remains valid; compare, for example, [15]. Moreover, the same arguments as given for the replacement of the $I_{n,Z}(\omega_j)$ in $M_n(Z)$ with the $I_{n,X}(\omega_j)/f_X(\omega_j)$ also apply here, that is, the conclusion of the theorem remains valid for the point process based on the points $(\omega_j, I_{n,X}(\omega_j)/f_X(\omega_j) - \ln q)$ provided that the conditions of Corollary 2.6 are met.

One of the great advantages of the point process approach to extreme value theory is that the relation $\eta_n \to_d \eta$ immediately yields the joint weak convergence of a finite vector of upper order statistics in a sample. To be precise, we introduce for every n the ordered version of the sample $I_{n,Z}(\omega_i)$, $i = 1, \dots, q$,

$$I_{n,(q)} \le \cdots \le I_{n,(1)}.$$

Note that in particular, $M_n(Z) = I_{n,(1)}$. Let $x_k < \cdots < x_1$ be any real numbers, and write $N_i = \eta_n([0, \pi] \times (x_i, \infty))$ for the number of exceedances of x_i by $I_{n,Z}(\omega_j) - \ln q$, $j = 1, \dots, q$. Then

$$\begin{aligned}\{I_{n,(1)} - \ln q \le x_1, \dots, I_{n,(k)} - \ln q \le x_k\}\\ = \{N_1 = 0,\ N_2 \le 1, \dots, N_k \le k-1\}.\end{aligned}$$

Thus the joint limit distribution of the vector of the k upper order statistics of the periodogram ordinates $I_{n,Z}(\omega_j)$ can be derived from Theorem 2.8. The following is an immediate consequence of this fact combined with standard arguments from extreme value theory; see, for example, [12], Section 5.6, or [14].

COROLLARY 2.10. *Under the assumptions of Theorem* 2.8, *for any real numbers* $x_k < \cdots < x_1$,

$$P\big(I_{n,(1)} - \ln q \le x_1, \dots, I_{n,(k)} - \ln q \le x_k\big) \to P\big(Y^{(1)} \le x_1, \dots, Y^{(k)} \le x_k\big),$$

where $(Y^{(1)}, \dots, Y^{(k)})$ *has density* $\exp\{-\exp(-x_k) - (x_1 + \cdots + x_k)\}$.

Additional limiting results for the spacings and related quantities may be obtained directly from this corollary. For example (see [6], Corollary 4.2.1),

$$\big(I_{n,(i)} - I_{n,(i+1)}\big)_{i=1,\dots,k} \to_d \big(i^{-1}E_i\big)_{i=1,\dots,k},$$

$$\sum_{i=1}^{k} I_{n,(i)} - kI_{n,(k+1)} \to_d \sum_{i=1}^{d} E_i,$$

where (E_i) is an iid standard exponential sequence. More generally, every asymptotic result which only depends on the distribution of a finite number of upper order statistics of an exponential sample remains valid if one replaces these order statistics by the corresponding $I_{n,(i)}$.

We conclude with an a.s. convergence result which generalizes a theorem in [1] in which (2.4) below was proved under the existence of a sixth moment of Z and certain smoothness conditions on the characteristic function of Z. In particular these conditions imply that Z has a density; see Remark 2.2.

PROPOSITION 2.11. *Let (X_t) be the linear process* (1.2) *with noise satisfying $EZ = 0$, $EZ^2 = 1$ and $E|Z|^s < \infty$ for some $s > 2$ and $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$. Also assume that $f_X(\lambda) > 0$ on $[0, \pi]$. Then*

$$\lim_{n\to\infty} \frac{1}{\ln n} \max_{\lambda\in[0,\pi]} \frac{I_{n,X}(\lambda)}{f_X(\lambda)} = 1 \quad \text{a.s.} \tag{2.4}$$

3. Tools. We start with a well-known result which we recall here for convenience.

LEMMA 3.1 (Bonferroni inequality). *Let $A_1, \dots, A_n$ be events from a σ-field $\mathscr{F}$ and $(\Omega, \mathscr{F}, P)$ be a probability space. Then for every integer $k \ge 1$,*

$$\sum_{j=1}^{2k} (-1)^{j-1} S_j \le P(A_1 \cup \cdots \cup A_n) \le \sum_{j=1}^{2k-1} (-1)^{j-1} S_j, \tag{3.1}$$

where

$$S_j = \sum_{1\le i_1 < \cdots < i_j \le n} P\big(A_{i_1} \cap \cdots \cap A_{i_j}\big).$$

The statement and the notation of Lemma 3.2 below are taken from [5]. The lemma is Einmahl's Corollary 1(b), page 31, in combination with the Remark on page 32.

LEMMA 3.2. *Let $\xi_1, \dots, \xi_n$ be independent random vectors with mean zero and values in $\mathbb{R}^d$. Assume that the moment generating function of ξ_i, $i = 1, \dots, n$, exists in a neighborhood of the origin and that*

$$\operatorname{cov}(\xi_1 + \cdots + \xi_n) = B_n I_d,$$

where $B_n > 0$ and I_d denotes the d-dimensional identity matrix. Let η_k be independent $N(0, \sigma^2 \operatorname{cov}(\xi_k))$ random vectors, $k = 1, \dots, n$, independent of (ξ_k), and $\sigma^2 \in (0, 1]$. Let $\xi_k^* = \xi_k + \eta_k$, $k = 1, \dots, n$, and write p_n^* for the density of $B_n^{-1/2} \sum_{k=1}^n \xi_k^*$. Choose $\alpha \in (0, 0.5)$ such that

$$\alpha \sum_{k=1}^{n} E|\xi_k|^3 \exp(\alpha|\xi_k|) \le B_n, \tag{3.2}$$

where $|x|$ denotes the Euclidean norm in $\mathbb{R}^d$. Let

$$\beta_n = \beta_n(\alpha) = B_n^{-3/2} \sum_{k=1}^{n} E|\xi_k|^3 \exp(\alpha|\xi_k|). \tag{3.3}$$

If

$$|x| \le c_1 \alpha B_n^{1/2}, \qquad \sigma^2 \ge -c_2 \beta_n^2 \ln \beta_n \quad \textit{and} \quad B_n \ge c_3 \alpha^{-2}, \tag{3.4}$$

where c_1, c_2, *and* c_3 *are constants depending only on* d, *then*

$$p_n^*(x) = \varphi_{(1+\sigma^2)I_d}(x)\exp\big(\overline{T}_n(x)\big) \quad \textit{with } |\overline{T}_n(x)| \le c_4 \beta_n\big(|x|^3 + 1\big), \tag{3.5}$$

where φ_C *is the density of a* d*-dimensional centered Gaussian vector with covariance matrix* C *and* c_4 *is a constant only depending on* d.

LEMMA 3.3. *Assume* $E|Z|^s < \infty$ *for some* $s > 2$ *and for* $n \ge 1$ *define the array of random variables* $(\overline{Z}_t^{(n)})_{t=1,\dots,n}$ *by*

$$\overline{Z}_t = \overline{Z}_t^{(n)} = Z_t I_{\{|Z_t| \le n^{1/s}\}} - EZI_{\{|Z| \le n^{1/s}\}}. \tag{3.6}$$

Then

$$\max_{j=1,\dots,q} I_{n,Z}(\omega_j) - \max_{j=1,\dots,q} I_{n,\overline{Z}}(\omega_j) = o(1) \quad a.s.,$$

where $I_{n,\overline{Z}}(\lambda)$ *is the periodogram based on* $\overline{Z}_1^{(n)}, \dots, \overline{Z}_n^{(n)}$.

PROOF. Since $\sum_{t=1}^n \exp(i\omega_j t) = 0$, it follows that $I_{n,\overline{Z}}(\omega_j) = I_{n,\tilde{Z}}(\omega_j)$, where

$$\tilde{Z}_t = \overline{Z}_t + EZI_{\{|Z| \le n^{1/s}\}}.$$

By the Borel–Cantelli lemma, $\sum_{t=1}^\infty |Z_t| I_{\{|Z_t| > t^{1/s}\}}$ is bounded with probability 1 and consists of only a finite number of nonzero terms. Thus, there exists a positive integer $N(\omega)$ such that

$$\begin{aligned}\sum_{t=1}^n |Z_t - \tilde{Z}_t| & \\ &= \sum_{t=1}^n |Z_t| I_{\{|Z_t| > n^{1/s}\}} \le \sum_{t=1}^\infty |Z_t| I_{\{|Z_t| > t^{1/s}\}} = \sum_{t=1}^{N(\omega)} |Z_t| I_{\{|Z_t| > t^{1/s}\}}.\end{aligned} \tag{3.7}$$

It follows that for $n \ge \max\{N(\omega), |Z_1|^s, \dots, |Z_{N(\omega)}|^s\}$ the left-hand side of (3.7) is zero. Consequently, the periodograms for the Z and $\tilde{Z}$ sequences have to be identical a.s. for all n sufficiently large, which proves the result. □

For $d \ge 1$, define

$$v_d(t) = \big(\cos(\omega_{i_1} t), \sin(\omega_{i_1} t), \dots, \cos(\omega_{i_d} t), \sin(\omega_{i_d} t)\big)', \tag{3.8}$$

where $\omega_{i_1}, \dots, \omega_{i_d}$ are any distinct Fourier frequencies. For ease of notation we suppress the dependence of v_d on the particular frequencies.

The following approximation is crucial for the proof of Theorem 2.1.

LEMMA 3.4. *Fix* $d \ge 1$ *and let* $\tilde{p}_n$ *be the density function of*

$$2^{1/2} n^{-1/2} \sum_{t=1}^n \big(\overline{Z}_t + \sigma_n N_t\big) v_d(t),$$

where (N_t) is a sequence of iid $N(0,1)$ random variables, independent of (Z_t), and $\sigma_n^2 = \mathrm{var}(\overline{Z})s_n^2$. If $n^{-2c_6}\ln n \le s_n^2 \le 1$ with $c_6 = 1/2 - (1-\delta)/s$ for arbitrarily small $\delta > 0$, then the relation

$$\tilde{p}_n(x) = \varphi_{(1+\sigma_n^2)I_{2d}}(x)(1+o(1))$$

holds uniformly for $|x|^3 = o_d(\min(n^{c_6}, n^{1/2-1/s}))$.

PROOF. We apply Lemma 3.2 to $\overline{Z}_1 v_d(1), \dots, \overline{Z}_n v_d(n)$. Notice that $E\overline{Z} = 0$ and

$$\mathrm{cov}\big(\overline{Z}_1 v_d(1) + \cdots + \overline{Z}_n v_d(n)\big) = B_n I_{2d},$$

where $B_n = \mathrm{var}(\overline{Z})q \sim q$. Choose for some fixed constant $c_5 > 0$

$$\tilde{\alpha} = c_5 n^{-1/s}/d^{1/2}.$$

Then

$$\begin{aligned}
&\tilde{\alpha}\sum_{t=1}^{n} E|\overline{Z}_t v_d(t)|^3 \exp\big\{\tilde{\alpha}|\overline{Z}_t v_d(t)|\big\}\\
&\qquad \le d^{3/2}\tilde{\alpha} n E|\overline{Z}|^3 \exp\{\tilde{\alpha}|\overline{Z}|d^{1/2}\}\\
&\qquad \le d c_5 n^{1-1/s} E|\overline{Z}|^3 \exp(2c_5)\\
&\qquad \le 8 d c_5 \exp(2c_5) n^{1-\delta/s} E|Z|^{2+\delta},
\end{aligned}$$

where $\delta \in (0,1)$ is chosen such that $E|Z|^{2+\delta} < \infty$. Hence (3.2) is satisfied with $\alpha = \tilde{\alpha}$ for sufficiently small c_5. Next choose

$$\begin{aligned}
\tilde{\beta}_n &= B_n^{-3/2}\sum_{t=1}^{n} E|\overline{Z}_t v_d(t)|^3 \exp\big(\tilde{\alpha}|\overline{Z}_t v_d(t)|\big)\\
&\le d^{3/2} B_n^{-3/2} n E|\overline{Z}|^3 \exp(\tilde{\alpha}|\overline{Z}|d^{1/2}).
\end{aligned}$$

Notice that

$$\tilde{\beta}_n \le \text{const}\, B_n^{-3/2} n^{1+(1-\delta)/s} E|\overline{Z}|^{2+\delta} \le \text{const}\, n^{-c_6}, \tag{3.9}$$

where δ is chosen as above and $c_6 = 1/2 - (1-\delta)/s > 0$. Next we consider condition (3.4). We can choose x according to the restriction

$$|x| \le c_1 \tilde{\alpha} B_n^{1/2} \sim \text{const}\, n^{1/2-1/s}. \tag{3.10}$$

By (3.4) and (3.9) we can choose $\sigma^2 = \sigma_n^2$ according to

$$1 \ge \sigma_n^2 \ge \text{const}\,(\ln n)\, n^{-2c_6},$$

and B_n also satisfies

$$B_n \ge c_3 \tilde{\alpha}^{-2}.$$

An application of (3.5) yields

$$\tilde{p}_n(x) = \varphi_{(1+\sigma_n^2)I_{2d}}(x)\exp\big(\overline{T}_n(x)\big) \quad \text{with } |\overline{T}_n(x)| \le c_4 \tilde{\beta}_n(|x|^3 + 1).$$

But by (3.9) and (3.10),

$$\tilde{\beta}_n(|x|^3 + 1) = o(1)$$

uniformly for $|x|^3 = o(\min(n^{c_6}, n^{1/2-1/s}))$ for arbitrarily small $\delta > 0$; that is, the remainder term $|\overline{T}_n(x)|$ converges to zero, uniformly for the x considered.

Finally, we want to indicate the changes in the proof of Lemma 3.4 if one assumes that $EZ^2 \ln^+|Z|h^2(|Z|) < \infty$, where h is the k times iterated logarithm. First one has to truncate the random variables Z_t at the level $a_n = \sqrt{n}/(\sqrt{\ln n}\, h(n))$. Then choose $\tilde{\alpha} = 1/a_n$, which implies that $\tilde{\beta}_n \le$ const $1/(\ln^{3/2} n h^3(n))$. Straightforward calculation shows that the statement of the lemma remains valid for $|x| = o(\sqrt{n}\, h(n))$. □

4. Proofs of the main results.

PROOF OF THEOREM 2.1. In view of Lemma 3.3, it suffices to prove the statement with $I_{n,Z}(\lambda)$ replaced by $I_{n,\overline{Z}}(\lambda)$ defined in Lemma 3.3. We first consider the periodogram based on $\overline{Z}_t + \sigma_n N_t$, where $\sigma_n^2 = n^{-c_6}$ and c_6 is given in the statement of the lemma. In particular, we show that $M_n(\overline{Z} + \sigma_n N) - \ln q \to_d Y$, where Y is standard Gumbel.

By virtue of the Bonferroni inequalities (3.1) and the identity

$$\exp(-e^{-x}) = \sum_{j=0}^{\infty} (-1)^j \frac{\exp(-jx)}{j!},$$

it suffices to show that

$$\begin{aligned} &P\big(I_{n,\overline{Z}+\sigma_n N}(\omega_{i_1}) > x + \ln q, \dots, I_{n,\overline{Z}+\sigma_n N}(\omega_{i_d}) > x + \ln q\big) \\ &\quad = \big(\exp\big((-x - \ln q)\big(1 + \sigma_n^2\big)\big)\big)^d (1 + o(1)) \\ &\quad = q^{-d} \exp(-dx)(1 + o(1)), \end{aligned} \tag{4.1}$$

for every fixed $d \ge 1$ and uniformly over all d-tuples $1 \le i_1 < \cdots < i_d \le q$ as $n \to \infty$. Notice that the left-hand side of (4.1) is equal to the probability of the event

$$\left\{ 2^{1/2} n^{-1/2} \sum_{t=1}^{n} \big(\overline{Z}_t + \sigma_n N_t\big) v_d(t) \in A_n^{(d)} \right\},$$

where $v_d(t)$ is defined in (3.8), and

$$A_n^{(d)} = \big\{ (x_1, y_1, \dots, x_d, y_d)' \colon x_i^2 + y_i^2 > 2(x + \ln q),\ i = 1, \dots, d \big\},$$

whose distance to the origin is not less than $2(x + \ln q)^{1/2}$. Since

$$\int_{A_n^{(d)}} \varphi_{(1+\sigma_n^2) I_d}(x)\, dx = \big(\exp\big((-x - \ln q)\big(1 + \sigma_n^2\big)\big)\big)^d,$$

and $\sigma_n^2 \ln q \to 0$ by the choice of σ_n^2, we conclude from Lemma 3.4, which does not depend on the choice of the d distinct frequencies in the definition of $v_d(t)$, that (4.1) holds and hence

$$M_n\big(\overline{Z} + \sigma_n N\big) - \ln q \to_d Y. \tag{4.2}$$

To complete the proof of the theorem we need to transfer the convergence in (4.2) onto the $\overline{Z}_t$. To this end, write $\overline{Z}_t = \overline{Z}_t + \sigma_n N_t - \sigma_n N_t$ so that

$$
\begin{aligned}
&\left| M_n\left(\overline{Z} + \sigma_n N\right) - M_n(\overline{Z})\right| \\
&\qquad \le M_n(\sigma_n N) + 4 M_n^{1/2}\left(\overline{Z} + \sigma_n N\right) M_n^{1/2}(\sigma_n N).
\end{aligned} \tag{4.3}
$$

Since $M_n(N)$ is the maximum of iid exponential random variables, it follows that

$$M_n(\sigma_n N) = O_p(\sigma_n \ \ln n)$$

and, moreover, from (4.2),

$$M_n\left(\overline{Z} + \sigma_n N\right) = O_p(\ln n).$$

These two relations together with $\sigma_n^2 = n^{-c_6}$ imply that the bound in (4.3) is $o_p(1)$, from which we conclude that

$$M_n(\overline{Z}) - \ln q \to_d Y,$$

as claimed.

Finally, we want to indicate the changes in the proof which are necessary under the assumption $EZ^2 \ln^+|Z| h^2(|Z|) < \infty$ where h is the k times iterated logarithm. Choose $\sigma_n^2 = \ln^{-4} n h^{-2}(n)$ and use the remarks at the end of the proof of Lemma 3.4. □

PROOF OF THEOREM 2.8. We first establish convergence in distribution for the point process corresponding to the periodogram based on the $\overline{Z}_t + \sigma_n N_t$ for $\sigma_n^2 = n^{-c_6}$. Let

$$\eta_n^*(\cdot) = \sum_{j=1}^{q} \varepsilon_{(\omega_j, I_{n, \overline{Z} + \sigma_n^2 N}(\omega_j) - \ln q)}(\cdot).$$

Since the limit point process η is simple, it suffices to show that (see Theorem 4.7 of [11])

$$E\eta_n^*((a, b] \times (x, y]) \to E\eta((a, b] \times (x, y]) = \frac{b-a}{\pi}(e^{-x} - e^{-y}) \tag{4.4}$$

for all $0 \le a < b \le \pi$ and $x < y$, and for all $k \ge 1$,

$$
\begin{aligned}
&P\left(\eta_n^*((a_1, b_1] \times R_1) = 0, \dots, \eta_n^*((a_k, b_k] \times R_k) = 0\right) \\
&\qquad \to P\left(\eta((a_1, b_1] \times R_1) = 0, \dots, \eta((a_k, b_k] \times R_k) = 0\right),
\end{aligned} \tag{4.5}
$$

where $0 \le a_1 < b_1 < \cdots < b_k \le \pi$ and $R_1, \dots, R_k$ are bounded Borel sets, each consisting of a finite union of intervals on $(-\infty, \infty]$.

To prove (4.4), note that

$$E\eta_n^*((a, b] \times (x, y]) = \sum_{\omega_j \in (a, b]} P\left(x + \ln q < I_{n, \overline{Z} + \sigma_n N}(\omega_j) \le y + \ln q\right),$$

which by (4.1),

$$\begin{aligned} &\sim \frac{(b-a)n}{2\pi} q^{-1}(e^{-x} - e^{-y}) \\ &\to \frac{(b-a)}{\pi}(e^{-x} - e^{-y}). \end{aligned}$$

Now turning to (4.5), set $n_j = \#\{i\colon \omega_i \in (a_j, b_j]\} \sim n(b_j - a_j)/(2\pi)$. Then the complement of the event in (4.5) is the union of $m = n_1 + \cdots + n_k$ events, that is,

$$\begin{aligned} &1 - P\big(\eta_n^*((a_1, b_1] \times R_1) = 0, \dots, \eta_n^*((a_k, b_k] \times R_k) = 0\big) \\ &\quad = P\left(\bigcup_{j=1}^{k} \bigcup_{\omega_i \in (a_j, b_j]} \left\{ I_{n, \bar{Z} + \sigma_n N}(\omega_i) - \ln q \in R_j \right\} \right). \end{aligned} \tag{4.6}$$

For any choice of d distinct integers $i_1, \dots, i_d \in \{1, \dots, q\}$ and integers $j_1, \dots, j_d \in \{1, \dots, k\}$ we have from (4.1) that

$$P\left(\bigcap_{r=1}^{d} \left\{ I_{n, \bar{Z} + \sigma_n N}(\omega_{i_r}) - \ln q \in R_{j_r} \right\} \right) = q^{-d} \prod_{r=1}^{d} \lambda(R_{j_r})(1 + o(1)), \tag{4.7}$$

where λ is the measure on $(-\infty, \infty]$ given by $e^{-x}\,dx$ and the relation is uniform over all distinct d-tuples $i_1, \dots, i_d$. Using an elementary counting argument and (4.7), the sum of the probabilities of all collections of d distinct sets from the m that comprise the union in (4.6) is given by

$$\begin{aligned} S_d &= \sum_{\substack{(u_1, \dots, u_k) \\ u_1 + \cdots + u_k = d}} \binom{n_1}{u_1} \cdots \binom{n_k}{u_k} q^{-u_1} \lambda^{u_1}(R_1) \cdots q^{-u_k} \lambda^{u_k}(R_k)(1 + o(1)) \\ &= \sum_{\substack{(u_1, \dots, u_k) \\ u_1 + \cdots + u_k = d}} \frac{1}{u_1! u_2! \cdots u_k! \pi^d} \\ &\qquad \times \big((b_1 - a_1)\lambda(R_1)\big)^{u_1} \cdots \big((b_k - a_k)\lambda(R_k)\big)^{u_k} (1 + o(1)) \\ &\to (d!)^{-1} \pi^{-d} \big((b_1 - a_1)\lambda(R_1) + \cdots + (b_k - a_k)\lambda(R_k)\big)^d. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{j=1}^{2s} (-1)^{j-1} S_j &\to \sum_{j=1}^{2s} (-1)^{j-1} (j!)^{-1} \pi^{-j} \\ &\qquad \times \big((b_1 - a_1)\lambda(R_1) + \cdots + (b_k - a_k)\lambda(R_k)\big)^j \text{ as } n \to \infty \\ &\to 1 - \exp\left\{ -\sum_{j=1}^{k} (b_j - a_j)\pi^{-1}\lambda(R_j) \right\}, \end{aligned}$$

as $s \to \infty$ which, by the Bonferroni inequality (3.1) and (4.6), proves (4.5).

It remains to transfer the convergence of the η_n^* onto η. First define the point process

$$\overline{\eta}_n(\cdot) = \sum_{j=1}^{q} \varepsilon_{(\omega_j, I_{n,\overline{Z}}(\omega_j) - \ln q)}(\cdot).$$

It then suffices to show that (see Theorem 4.2 of [11])

$$\overline{\eta}_n - \eta_n^* \to_P 0 \tag{4.8}$$

and

$$\eta_n - \overline{\eta}_n \to_P 0 \tag{4.9}$$

or, equivalently, that for any continuous function f on E with compact support,

$$\overline{\eta}_n(f) - \eta_n^*(f) \to_P 0 \quad \text{and} \quad \eta_n(f) - \overline{\eta}_n(f) \to_P 0,$$

where the notation $\eta(f)$ is shorthand for $\int f\,d\eta$. Suppose the compact support of f is contained in the set $[0, \pi] \times [K + \gamma_0, \infty)$ for some $\gamma_0 > 0$ and $K \in \mathbb{R}$. Since f is uniformly continuous, $\omega(\gamma) := \sup\{|f(t,x) - f(t,y)|;\ t \in [0,1],\ |x - y| \le \gamma\} \to 0$ as $\gamma \to 0$. On the set $A_n = \{\max_{j=1,\dots,q} |I_{n,\overline{Z}+\sigma_n N}(\omega_j) - I_{n,\overline{Z}}(\omega_j)| \le \gamma\}$, we have for $\gamma < \gamma_0$,

$$\begin{aligned} &\left| f\big(\omega_j, I_{n,\overline{Z}+\sigma_n N}(\omega_j) - \ln q\big) - f\big(\omega_j, I_{n,\overline{Z}}(\omega_j) - \ln q\big)\right| \\ &\quad \le \begin{cases} \omega(\gamma), & \text{if } I_{n,\overline{Z}+\sigma_n N}(\omega_j) - \ln q > K, \\ 0, & \text{if } I_{n,\overline{Z}+\sigma_n N}(\omega_j) - \ln q \le K. \end{cases} \end{aligned} \tag{4.10}$$

Also, $P(A_n^c) \to 0$ which follows from the argument used to show that (4.3) is $o_p(1)$. Now, for any $\varepsilon > 0$, choose γ sufficiently small that $\gamma < \gamma_0$. Then, by intersecting the event $\{|\overline{\eta}_n(f) - \eta_n^*(f)| > \varepsilon\}$ with A_n and A_n^c, respectively, and using (4.10) and (4.4), we obtain

$$\begin{aligned} &\limsup_{n\to\infty} P\big(|\overline{\eta}_n(f) - \eta_n^*(f)| > \varepsilon\big) \\ &\quad \le \limsup_{n\to\infty} \big(P\big(\omega(\gamma)\eta_n^*([0,1] \times [K,\infty)) > \varepsilon\big) + P(A_n^c)\big) \\ &\quad \le \limsup_{n\to\infty} E\eta_n^*([0,1] \times [K,\infty))\,\omega(\gamma)/\varepsilon \\ &\quad \le e^{-K}\omega(\gamma)/\varepsilon. \end{aligned}$$

Since $\omega(\gamma) \to 0$ as $\gamma \to 0$, (4.8) follows.

The proof of (4.9) is essentially identical to the argument given for (4.8) with the conclusion of Lemma 3.3 playing the key role. □

PROOF OF PROPOSITION 2.11. The lim sup part of (2.4) is proved in [1] under a second moment second condition. Under this condition, they also proved that

$$\lim_{n\to\infty} \frac{1}{\ln n} \max_{\lambda \in [0,\pi]} \left| \frac{I_{n,X}(\lambda)}{f_X(\lambda)} - I_{n,Z}(\lambda) \right| = 0 \quad \text{a.s.} \tag{4.11}$$

The lim inf part is similar to the proof in [1], pages 395 and 396. By (4.11), it suffices to show the result for the Z-sequence. Instead of applying the asymptotic expansion in the beginning of their proof, one can use arguments similar to the ones for the proof of Theorem 2.1. Utilizing Lemma 3.4 for $I_{n,\bar{Z}+\sigma_n N}(\omega_j)$, one obtains the same asymptotic order of the probabilities in (2.9) and (2.10) of [1] for the periodogram of the $(\bar{Z}+\sigma_n N)$-sequence. Then one can proceed in the same way as on pages 395 and 396 in [1] in order to prove the lim inf part for $I_{n,\bar{Z}+\sigma_n N}$. Now, choosing $\sigma_n^2 = n^{-c_6}$ as before,

$$\max_{j=1,\ldots,q} I_{n,\bar{Z}+\sigma_n N}(\omega_j) - \max_{j=1,\ldots,q} I_{n,\bar{Z}}(\omega_j) \to 0 \quad \text{a.s.}$$

and, under the condition $E|Z|^s < \infty$, one may replace the $\bar{Z}_t$ with Z_t; see Lemma 3.3. □

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