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THE COMPLETE CONVERGENCE THEOREM FOR COEXISTENT THRESHOLD VOTER MODELS

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We consider the d-dimensional threshold voter model. It is known that, except in the one-dimensional nearest-neighbor case, coexistence occurs (nontrivial invariant measures exist). In fact, there is a nontrivial limit $\eta_\infty^{1/2}$ obtained by starting from the product measure with density 1/2. We show that in these coexistent cases,

$$\eta_t \Rightarrow \alpha\delta_0 + \beta\delta_1 + (1-\alpha-\beta)\eta_\infty^{1/2} \quad \text{as } t \to \infty,$$

where $\alpha = P(\tau_0 < \infty)$, $\beta = P(\tau_1 < \infty)$, τ_0 and τ_1 are the first hitting times of the all-zero and all-one configurations, respectively, and $\Rightarrow$ denotes weak convergence.

1. Introduction. A continuous-time Markov process η_t on $\{0,1\}^{\mathbf{Z}^d}$ is called a d-dimensional spin system if it evolves as follows: at rate $c(x,\eta)$ the value at site $x \in \mathbf{Z}^d$ of the configuration η changes from $\eta(x)$ to $1-\eta(x)$. When $c(\cdot,\eta) \equiv 0$ for $\eta \equiv 0$ and for $\eta \equiv 1$, the point masses on these two configurations, δ_0 and δ_1, respectively, are invariant for the process. In this case, if there are any nontrivial invariant measures, ones which are not a linear combination of these two, we say that coexistence occurs, because there is a limiting distribution which contains zeros and ones together. When coexistence occurs, it is natural to ask what the nontrivial, invariant distributions are and to try to determine the limiting behavior of the process.

The threshold voter model is a d-dimensional spin system that was introduced by Cox and Durrett and has become the subject of much recent study. The transition rates are given by

$$c(x,\eta) = \begin{cases} 1, & \text{if there is a } y \text{ with } \|x-y\| \le M \text{ and } \eta(x) \ne \eta(y), \\ 0, & \text{otherwise.} \end{cases}$$

(We can use any norm such that $\inf\{\|x\|: x \in \mathbf{Z}^d,\ x \ne 0\} = 1$; M is a positive integer.) Cox and Durrett (1991) showed that if $M = d = 1$, the threshold voter model has only the trivial invariant measures δ_0 and δ_1, and if M and d are large enough, coexistence occurs. In particular, they proved that the threshold voter model coexists in one dimension if $M \ge 4$, in two dimensions if $M \ge 3$ (when $\|\cdot\|$ is the l_1 norm), and in three or more dimensions if $M \ge 1$. It has also been shown that for $M = 1$, the one-dimensional process converges weakly to a convex combination of δ_0 and δ_1, provided the initial distribution is translation invariant. [See Andjel, Liggett and Mountford (1992).]

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Liggett (1994) has proved that for any $(M, d) \neq (1, 1)$ the threshold voter model on $\mathbf{Z}^d$ with parameter M coexists. We will see below that starting from product measure with density 1/2, the threshold voter model always converges to a stationary distribution which we will call $\eta_\infty^{1/2}$. In fact, Liggett proved coexistence by showing that this limit is nontrivial. Neither Liggett nor Cox and Durrett have characterized all the invariant measures for the $(M, d) \neq (1, 1)$ process. However, Cox and Durrett (1991) conjectured that for any threshold voter model in which coexistence occurs, if the initial distribution is translation invariant and puts no mass on the all-zero or all-one configurations, then the process converges to $\eta_\infty^{1/2}$. When we refer to convergence of a process or measures we will mean weak convergence (convergence of finite-dimensional distributions), denoted by $\Rightarrow$. The purpose of this paper is to prove a generalization of this conjecture, the complete convergence theorem.

THEOREM 1.1. *Consider the threshold voter model with parameters* $(M, d) \neq (1, 1)$. *Let* $\tau_0 = \inf\{t \geq 0: \eta_t \equiv 0\}$ *and* $\tau_1 = \inf\{t \geq 0: \eta_t \equiv 1\}$, *and set* $\alpha = P(\tau_0 < \infty)$ *and* $\beta = P(\tau_1 < \infty)$. *Then*

$$\eta_t \Rightarrow \alpha\delta_0 + \beta\delta_1 + (1 - \alpha - \beta)\eta_\infty^{1/2} \quad \text{as } t \to \infty.$$

To prove this theorem we look at the dual of the threshold voter model. The dual is an annihilating branching process ζ_t defined on the finite subsets of $\mathbf{Z}^d$. The first part of the proof involves showing

$$P(0 < |\zeta_t| \leq k) \to 0 \quad \text{as } t \to \infty \tag{1.1}$$

for all $k < \infty$; this is done in Section 2. To obtain (1.1) we use the duality equation and a comparison with the threshold contact process to get that ζ_t is not positive recurrent if we identify configurations which differ by a translation. Then, after proving some technical lemmas, (1.1) follows from an induction argument on the number of particles in the dual. The second part of the proof is to show that (1.1) implies the theorem, which is done in Section 3. This is similar to other proofs of convergence theorems for processes with annihilating duals, when the initial distribution is translation invariant. [See Bramson, Ding and Durrett (1991).] To get the more general result we use a comparison with the threshold contact process and its complete convergence theorem.

2. The dual process. The d-dimensional threshold voter model has a dual which is a continuous-time Markov chain on the set of finite subsets of $\mathbf{Z}^d$; it is constructed in Cox and Durrett (1991). Let $\mathscr{N} = \{z \in \mathbf{Z}^d: \|z\| \leq M\}$. The dual is an annihilating branching process with transition rates $q(B, C)$ defined as follows: each $x \in B$ is removed from B at rate 2 and is replaced uniformly by an odd subset of $x + \mathscr{N}$, but when an attempt is made to put a point at a site which is already occupied, the points annihilate one another. Throughout this paper we will identify a configuration in $\{0, 1\}^{\mathbf{Z}^d}$ with the subset of $\mathbf{Z}^d$ which consists of the set of ones. If η_t^A is the threshold voter model at time

t with initial configuration A, and ζ_t^B is the annihilating branching process at time t with initial configuration B, the two are related by the following duality equation:

$$P\big(|\eta_t^A \cap B| \text{ is odd}\big) = P\big(|\zeta_t^B \cap A| \text{ is odd}\big) \tag{2.1}$$

for all finite B. [See Cox and Durrett (1991), Section 2.]

The goal of this section is to prove (1.1), but we first need some results about the threshold contact process. The threshold contact process with parameters (M, d, λ) is a d-dimensional spin system ξ_t that has the following flip rates:

$$c(x, \xi) = \begin{cases} \lambda, & \text{if } \xi(x) = 0 \text{ and } \xi(y) = 1 \text{ for some } \|x - y\| \le M, \\ 1, & \text{if } \xi(x) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The threshold voter model η_t dominates the $\lambda \le 1$ threshold contact process ξ_t in the sense that if $\xi_0 \le \eta_0$, then

$$P(\xi_t \le \eta_t) = 1$$

for all $t \ge 0$. [Here $\xi_t \le \eta_t$ iff $\xi_t(x) \le \eta_t(x)$ for all $x \in \mathbf{Z}^d$.] The proof of this result is obtained using the standard coupling and appears in Liggett (1985), Theorem 1.5 of Chapter III. It is based on the fact that both processes are attractive, meaning that the presence of ones in the configuration makes transitions to ones more likely, and the presence of zeros in the configuration makes transitions to zero more likely. It can be shown that the threshold contact process has a (coalescing) dual. [See Liggett (1985), Section 4 of Chapter III.] The dual process is a Markov chain γ_t on the set of finite subsets of $\mathbf{Z}^d$ which has transitions as follows:

$$B \to B \backslash \{x\} \text{ at rate } 1 \quad \text{for each } x \in B$$

and

$$B \to B \cup (x + \mathscr{N}) \text{ at rate } \lambda \quad \text{for each } x \in B.$$

The two processes are related by the following duality equation:

$$P(\xi_t^A \cap B \ne \varnothing) = P(\gamma_t^B \cap A \ne \varnothing)$$

for B finite.

If the pointmass on the all-zero configuration is invariant for a particle system X_t on $\{0, 1\}^{\mathbf{Z}^d}$, we say that the process survives if it has a nontrivial invariant measure. We say there is finite survival of the process if

$$\lim_{t \to \infty} P(X_t \ne \varnothing) > 0$$

for some finite initial set X_0. Liggett (1994) has shown that for $(M, d) \ne (1, 1)$, the $\lambda \ge 1$ threshold contact process survives, but we will need finite survival of the process. This will be proved in Lemma 2.1. The proof follows from the fact that the threshold contact process has a dual with many nice properties.

LEMMA 2.1. *For $(M, d) \neq (1, 1)$ and $\lambda \geq 1$, the threshold contact process has finite survival.*

PROOF. Denoting the configuration $\xi \equiv 1$ by the symbol 1, the duality equation gives

$$P(\xi_t^1 \cap \{0\} \neq \varnothing) = P(\gamma_t^{\{0\}} \neq \varnothing). \tag{2.2}$$

By taking the limit as t goes to infinity in (2.2) we see that survival of the threshold contact process is equivalent to finite survival of its dual. So by Liggett's result we have finite survival of the dual process. The dual γ_t can be considered as a Markov process on $\{0, 1\}^{\mathbf{Z}^d}$ with transition rates dependent on the local configurations. Viewed in this manner, it can be extended to an infinite particle system. [See Chapter I of Liggett (1985).] For any positive integer N, let $B(N)$ denote the set $[-N, N]^d \cap \mathbf{Z}^d$. Then using the duality equation again, we get

$$P(\gamma_t^{B(N)} \cap \{0\} \neq \varnothing) = P(\xi_t^{\{0\}} \cap B(N) \neq \varnothing),$$

and taking the limit as $N \to \infty$ gives us this dual analogue of (2.2):

$$P(\gamma_t^1 \cap \{0\} \neq \varnothing) = P(\xi_t^{\{0\}} \neq \varnothing).$$

So we also have that survival of the infinite dual is equivalent to finite survival of the threshold contact process. Thus our problem can be reduced to showing that finite survival of the dual implies survival of the infinite dual. However, this follows from the arguments in Bezuidenhout and Gray (1994). The dual has all the essential properties of the processes considered in that paper. Perhaps the most important of these properties is that of additivity. This means that versions of the process can be coupled so that

$$\gamma_t^{A\cup B} = \gamma_t^A \cup \gamma_t^B$$

for any finite sets A and B. It is also important that the dual is spatially translation invariant. These features allow a comparison with oriented percolation, a process for which the two notions of survival are equivalent. Thus, by the methods of Bezuidenhout and Gray, we can conclude that finite survival of the dual implies survival of the infinite dual. Hence we have finite survival of the threshold contact process. $\square$

Throughout this paper we will be using the complete convergence theorem for the threshold contact process; it appears in Cox and Durrett (1991). We denote the upper stationary distribution for the process (the limit obtained by starting from the all-one configuration) by ξ_∞^1.

THEOREM. *Let ξ_t be the threshold contact process at time t. Then*

$$\xi_t \Rightarrow \alpha\delta_0 + (1-\alpha)\xi_\infty^1 \quad \text{as } t \to \infty,$$

where $\alpha = P(\tau < \infty)$ and τ is the time it takes for the process to die out (hit the empty set).

Thus by Lemma 2.1 we see that, starting from any initial distribution except δ_0, the $(M, d) \neq (1, 1)$, $\lambda \geq 1$ threshold contact process converges to a nontrivial distribution (something other than δ_0).

By looking at the transition rates or duality equation (2.1) with $A = \mathbf{Z}^d$, we see that the dual of the threshold voter model has the property that if B is odd, ζ_t^B is odd for all t, while if B is even, ζ_t^B is even for all t. Let ζ_t^o denote the dual process concentrating on the odd subsets of $\mathbf{Z}^d$ and ζ_t^e be the process concentrating on the even subsets of $\mathbf{Z}^d$. We will write $\tilde{\zeta}_t$ to represent the process where we identify configurations which differ by a translation. We can make $\tilde{\zeta}_t^e$ into an irreducible Markov chain by adding the transitions that at rate 2, $\varnothing$ is replaced with a uniformly chosen even subset of $\mathcal{N}$. The process then has no traps and can reach any element in its state space, which consists of the collection of finite even subsets of $\mathbf{Z}^d$ (modded out by translation). Let $\eta_t^{1/2}$ be the threshold voter model at time t, where $\eta_0^{1/2}$ is distributed as the product measure with density 1/2. Equation (2.1) implies that

$$P(|\eta_t^{1/2} \cap B| \text{ is odd}) = P(|\zeta_t^B \cap \eta_0^{1/2}| \text{ is odd}) = \tfrac{1}{2} P(\zeta_t^B \neq \varnothing) \tag{2.3}$$

for any finite set B. The distribution of η_t, the process at time t, is determined by knowing

$$P(|\eta_t \cap B| \text{ is odd})$$

for all finite B. So (2.3) shows that (the distribution of) $\eta_t^{1/2}$ converges to some stationary distribution, and Liggett (1994) has proved that the limit is nontrivial for $(M, d) \neq (1, 1)$. Thus

$$\lim_{t\to\infty} P(|\eta_t^{1/2} \cap \{x, y\}| \text{ is odd}) > 0$$

for some $x, y \in \mathbf{Z}^d$, and by setting $B = \{x, y\}$ in (2.3) we get

$$P(\tilde{\zeta}_t^{\{x, y\}} \neq \varnothing \text{ for all } t) = \lim_{t\to\infty} P(\zeta_t^{\{x, y\}} \neq \varnothing) > 0, \tag{2.4}$$

implying that the Markov chain $\tilde{\zeta}_t^e$ is transient for $(M, d) \neq (1, 1)$. We might expect the Markov chain $\tilde{\zeta}_t^o$ to also be transient. Although this may be true, it seems like a nontrivial problem to resolve. However, the following result is fairly easy to obtain and turns out to be all that we need.

LEMMA 2.2. *If $(M, d) \neq (1, 1)$, then $\tilde{\zeta}_t^o$ is not positive recurrent.*

PROOF. Now ζ_t^o is not positive recurrent. Otherwise this process would have an invariant measure μ which concentrates on finite configurations. However, because ζ_t^o is irreducible, μ would also have to be translation invariant.

Since the threshold voter model η_t dominates the $\lambda = 1$ threshold contact process ξ_t, we have

$$P(\zeta_t^{\{0\}} \cap \{0\} \neq \varnothing) = P(\eta_t^{\{0\}} \cap \{0\} \neq \varnothing) \geq P(\xi_t^{\{0\}} \cap \{0\} \neq \varnothing).$$

Thus, finite survival and the complete convergence theorem of the threshold contact process imply

$$\liminf_{t\to\infty} P(\zeta_t^{\{0\}} \cap \{0\} \neq \varnothing) = \varepsilon$$

for some $\varepsilon > 0$.

Assume that $\tilde{\zeta}_t^o$ is positive recurrent. Let ν be an invariant measure for the process $\tilde{\zeta}_t^o$ which concentrates on finite configurations. Choose k so that

$$\nu\{A\text{: diameter } A > k\} < \varepsilon/4.$$

Since ζ_t^o is not positive recurrent, there exists t_1 so large that

$$P(\zeta_t^{\{0\}} \subset [-k, k]^d) < \varepsilon/4$$

for all $t > t_1$. Also, there exists t_2 so large that

$$P(\text{diameter } \tilde{\zeta}_t^{\{0\}} > k) < \varepsilon/2$$

for all $t \geq t_2$. Let $T = \max\{t_1, t_2\}$. Then for all $t \geq T$,

$$P(\zeta_t^{\{0\}} \cap \{0\} \neq \varnothing) \leq P(\zeta_t^{\{0\}} \subset [-k, k]^d) + P(\text{diameter } \tilde{\zeta}_t^{\{0\}} > k) < 3\varepsilon/4.$$

Since this gives a contradiction, $\tilde{\zeta}_t^o$ must not be positive recurrent. □

We will use Lemma 2.2 and induction to obtain (1.1), but we first need some preliminary results.

LEMMA 2.3. *Let Y_j be a sequence of independent and identically distributed exponential random variables with parameter θ, and set $S_k = \sum_{j=1}^k Y_j$. Let f_k be the density function of S_k. Suppose $W > 0$. Then there exists K such that for $k \geq K$, for all $t \geq 0$ and $0 \leq w \leq W$,*

$$f_k(t) \leq f_k(t-w) + f_k(t+w). \tag{2.5}$$

PROOF. We will actually verify (2.5) for $k+1$. The density function for S_{k+1} is

$$f_{k+1}(t) = \begin{cases} ct^k e^{-\theta t}, & \text{if } t \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

for some positive constant c, and so

$$\frac{d}{dt} f_{k+1}(t) = ckt^{k-1}e^{-\theta t} - \theta c t^k e^{-\theta t} = ct^{k-1}e^{-\theta t}(k - \theta t), \qquad t \geq 0.$$

Thus f_{k+1} is increasing on $(-\infty, k/\theta]$ and decreasing on $[k/\theta, \infty)$. So (2.5) holds if $t + w \leq k/\theta$ or $t - w \geq k/\theta$. Assume that $t - w \leq k/\theta \leq t + w$. Then by monotonicity,

$$f_{k+1}(t - W) + f_{k+1}(t + W) \leq f_{k+1}(t - w) + f_{k+1}(t + w).$$

Hence it suffices to show that there exists K such that for all $k \geq K$ and $t - W \leq k/\theta \leq t + W$,

$$f_{k+1}(t) \leq f_{k+1}(t-W) + f_{k+1}(t+W). \tag{2.6}$$

For large k, since θ and W are fixed, we can assume that $k/\theta > 2W$. So when $t - W \leq k/\theta \leq t + W$, we have $t > W$, and (2.6) is equivalent to

$$ct^k e^{-\theta t} \leq c(t-W)^k e^{-\theta(t-W)} + c(t+W)^k e^{-\theta(t+W)},$$

which is equivalent to

$$1 \leq \left(1 - \frac{W}{t}\right)^k e^{\theta W} + \left(1 + \frac{W}{t}\right)^k e^{-\theta W}.$$

Since $k/\theta - W \leq t \leq k/\theta + W$, we have

$$\begin{aligned}&\left(1 - \frac{W}{t}\right)^k e^{\theta W} + \left(1 + \frac{W}{t}\right)^k e^{-\theta W} \\ &\qquad \geq \left(1 + \frac{W}{W - k/\theta}\right)^k e^{\theta W} + \left(1 + \frac{W}{W + k/\theta}\right)^k e^{-\theta W} \to 2\end{aligned}$$

as $k \to \infty$, because

$$\left(1 + \frac{W}{W - k/\theta}\right)^k \to e^{-\theta W}$$

and

$$\left(1 + \frac{W}{W + k/\theta}\right)^k \to e^{\theta W}.$$

Thus (2.6) is satisfied, and the conclusion of Lemma 2.3 follows. □

For the remainder of this section, we will be working with $\tilde{\zeta}_t$, the irreducible process obtained from the dual by identifying configurations which differ by a translation. Each transition of $\tilde{\zeta}_t$ occurs by replacement of a particle with a subset of its neighborhood. Consider this to be a transition for the embedded discrete time chain, even if the particle gets replaced by itself. Thus, this discrete time chain can make transitions from a state to itself. Now fix a finite initial configuration B. Throughout the rest of this section, when we refer to the process we will mean $\tilde{\zeta}_t^B$, and when we refer to the discrete time chain we will mean the embedded chain for $\tilde{\zeta}_t^B$ described above. Often we will write something like, "we had i particles n times." By this we will mean at n different times of this discrete time chain.

LEMMA 2.4. *Fix $i > 0$. Suppose σ is a sequence of transitions by the discrete time chain with the property that it starts with the initial configuration B, ends with i particles and there are i particles at least n times. Let $F_\sigma(t) = P($between* *times* 0 *and* t *our sequence of transitions by the process is* σ). *Then for all* $W > 0$, *there exists* n *such that, for* $t \geq W$ *and all* $0 \leq w \leq W$,

$$F_\sigma(t) \leq F_\sigma(t-w) + F_\sigma(t+w)$$

for all σ *as above.*

PROOF. Let g_σ be the density function for the time to complete the sequence σ, given our transitions begin with σ. Then

$$g_\sigma(t) = \begin{cases} \int_0^t f_k(t-u)\, dG(u), & \text{if } t \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $k+1$ is the number of times we have had i particles in the sequence σ, f_k is as in Lemma 2.3 (where $\theta = 2i$), and G is the conditional distribution function for the time spent not having i particles during the sequence σ. Fix $W > 0$. Now by Lemma 2.3, for large n,

$$\begin{aligned} g_\sigma(t) &= \int_0^t f_k(t-u)\, dG(u) \\ &\leq \int_0^t f_k(t-w-u)\, dG(u) + \int_0^t f_k(t+w-u)\, dG(u) \\ &\leq \int_0^{t-w} f_k(t-w-u)\, dG(u) + \int_0^{t+w} f_k(t+w-u)\, dG(u) \\ &= g_\sigma(t-w) + g_\sigma(t+w) \end{aligned}$$

for $t \geq 0$ and all $0 \leq w \leq W$. Also, if $t \geq 0$,

$$F_\sigma(t) = \int_0^t g_\sigma(u) \exp(-2i(t-u))\, du\ p(\sigma),$$

where $p(\sigma)$ is the probability that the discrete time chain starts with the sequence σ. Thus, for large n,

$$\begin{aligned} F_\sigma(t) &= \int_0^t g_\sigma(t-u) e^{-2iu}\, du\ \ p(\sigma) \\ &\leq \int_0^t g_\sigma(t-w-u) e^{-2iu}\, du\ \ p(\sigma) + \int_0^t g_\sigma(t+w-u) e^{-2iu}\, du\ \ p(\sigma) \\ &\leq \int_0^{t-w} g_\sigma(t-w-u) e^{-2iu}\, du\ \ p(\sigma) + \int_0^{t+w} g_\sigma(t+w-u) e^{-2iu}\, du\ \ p(\sigma) \\ &= F_\sigma(t-w) + F_\sigma(t+w) \end{aligned}$$

for $t \geq W$ and $0 \leq w \leq W$. $\square$

COROLLARY 2.5. *For* $i > 0$, *let* $G_n^i(t) = P($*at time* t *the process has* i *particles for at least the* n*th time*$)$. *Then, given* $W > 0$, *there exists* N *such that, for* $t \geq W$ *and all* $0 \leq w \leq W$,

$$G_N^i(t) \leq G_N^i(t-w) + G_N^i(t+w).$$

PROOF. Let $S(n)$ be the set of finite sequences of transitions by the discrete time chain with the property that they start with the initial configuration B, end with i particles, and have i particles at least n times. For each n, $S(n)$ is a countable set, so

$$G_n^i(t) = \sum_{\sigma \in S(n)} F_\sigma(t).$$

Thus by Lemma 2.4, there exists N so that if $t \geq W$,

$$\begin{aligned} G_N^i(t) &= \sum_{\sigma \in S(N)} F_\sigma(t) \leq \sum_{\sigma \in S(N)} F_\sigma(t-w) + \sum_{\sigma \in S(N)} F_\sigma(t+w) \\ &= G_N^i(t-w) + G_N^i(t+w) \end{aligned}$$

for $0 \leq w \leq W$. $\square$

PROPOSITION 2.6. *Let ζ_t^B be the dual for the threshold voter model, where $(M, d) \neq (1, 1)$ and B is any finite set. Then*

$$\lim_{t \to \infty} P(0 < |\zeta_t^B| \leq k) = 0$$

for all $0 < k < \infty$.

PROOF. It suffices to show that

$$\lim_{t \to \infty} P(|\tilde{\zeta}_t^B| = i) = 0 \tag{2.7}$$

for any $0 \leq i < \infty$. By (2.4) and Lemma 2.2 we know that both $\tilde{\zeta}_t^e$ and $\tilde{\zeta}_t^o$ are not positive recurrent, so (2.7) must be true for $i = 0$ and $i = 1$. We will use induction to prove it for all i. Assume (2.7) is true for $i-2$, but not for i. We will show that this leads to a contradiction. The idea is the following: if we keep having i particles, where two are close together, then there is a good chance that the two will annihilate each other, leaving us with $i - 2$ particles which contradicts the induction hypothesis. However, if we keep having i particles that are all very far apart, they will behave like i independent particles for long periods of time, contradicting the fact that (2.7) is true for $B = \{0\}$ and $i = 1$.

Since (2.7) is not true for i, there exists $\varepsilon > 0$ such that

$$P(|\tilde{\zeta}_{t_j}^B| = i) > 5\varepsilon, \tag{2.8}$$

for some sequence $\{t_j\}$ converging to infinity. Choose an integer L such that $L > 2/\varepsilon$. Let T be so large that

$$P(|\tilde{\zeta}_t^{\{0\}}| = 1) < \frac{1}{2L^2} \tag{2.9}$$

for $t \geq T$. Choose $W > LT$. By Corollary 2.5 there exists N such that, for $t \geq W$ and all $0 \leq w \leq W$,

$$G_N^i(t) \leq G_N^i(t-w) + G_N^i(t+w). \tag{2.10}$$

We will say the particles of the dual at time s *interact* by time t, denote this event by $I_s(t)$, if there exist $x_1, y_1 \in \zeta_s$ ($x_1 \neq y_1$) and $u \in (s, t]$ such that $\|x_2 - y_2\| \le M$ for some $x_2 \in \zeta_u^{\{x_1\}}$ and $y_2 \in \zeta_u^{\{y_1\}}$. Let D be so large that

$$P\big(I_0(2W)\big||\zeta_0| = i,\ \|x - y\| \ge D \text{ for all } x, y \in \zeta_0\big) < \frac{1}{2L^2}. \tag{2.11}$$

Suppose that

$$P(|\tilde{\zeta}_t^B| = i,\ 0 < \|x - y\| \le D \text{ for some } x, y \in \tilde{\zeta}_t^B) \ge \varepsilon$$

for a sequence of times converging to infinity. Each time $|\tilde{\zeta}_t^B| = i$, where two particles are within distance D, there is positive probability at least $p > 0$ that in one unit of time the two close particles annihilate each other while all other particles remain fixed. So we would have

$$P(|\tilde{\zeta}_t^B| = i - 2) \ge \varepsilon p$$

for a sequence of times converging to infinity. Since this contradicts the induction hypothesis, we can choose S to be so large that

$$P(|\tilde{\zeta}_t^B| = i,\ 0 < \|x - y\| \le D \text{ for some } x, y \in \tilde{\zeta}_t^B) < \varepsilon \tag{2.12}$$

for $t \ge S$.

Now let $p(m, n) = P$(at time m the discrete time chain has i particles for the nth time). Then

$$\sum_{m=0}^{\infty} p(m, n) = P(\text{the process has } i \text{ particles at least } n \text{ times}) \le 1,$$

and so

$$\lim_{K \to \infty} \sum_{m \ge K} p(m, n) = 0. \tag{2.13}$$

Let $N(t)$ be the number of times the process has had i particles between time 0 and t. Then by (2.13), for fixed n,

$$\lim_{t \to \infty} P(|\tilde{\zeta}_t^B| = i,\ N(t) = n) = 0.$$

So

$$P(|\tilde{\zeta}_t^B| = i,\ N(t) < N) < \varepsilon \tag{2.14}$$

for t sufficiently large.

By (2.8) and (2.14) we can find t_J, a time in the sequence $\{t_j\}$, such that $t_J > W + S$ and

$$G_N^i(t_J) > 4\varepsilon.$$

Since $W > LT$, (2.10) implies that there exist L times $\{s_1, s_2, \dots, s_L\}$ in the interval $[t_J - W, t_J + W]$ and all at least distance T apart, so that

$$G_N^i(s_j) > 2\varepsilon$$

for $1 \leq j \leq L$. Let $E^i_{N,D}(t)$ be the event that $|\tilde{\zeta}^B_t| = i$, this be at least the Nth time this has occurred, and all particles be at least distance D apart. Then, by (2.12),

$$P(E^i_{N,D}(s_j)) > \varepsilon$$

for $1 \leq j \leq L$. Let $s_{j(1)}$ be a time in the set $\{s_1, s_2, \dots, s_L\}$ such that

$$P\Bigg(E^i_{N,D}(s_{j(1)}) \Bigg\backslash \bigcup_{\substack{1\leq j\leq L\\ j\neq j(1)}} E^i_{N,D}(s_j)\Bigg) \leq 1/L.$$

Then

$$P\Bigg(E^i_{N,D}(s_{j(1)}) \cap \bigcup_{\substack{1\leq j\leq L\\ j\neq j(1)}} E^i_{N,D}(s_j)\Bigg) > \varepsilon - 1/L > 1/L.$$

So there exists a time $s_{j(2)} \neq s_{j(1)}$ in $\{s_1, s_2, \dots, s_L\}$ such that

$$P\big(E^i_{N,D}(s_{j(1)}) \cap E^i_{N,D}(s_{j(2)})\big) > 1/L^2. \tag{2.15}$$

Assume, without loss of generality, that $s_{j(2)} > s_{j(1)}$. Now if

$$E^i_{N,D}(s_{j(1)}) \cap E^i_{N,D}(s_{j(2)})$$

occurs, then either the particles of the dual at time $s_{j(1)}$ interact by time $s_{j(2)}$ or any given particle from $\tilde{\zeta}^B_{s_{j(1)}}$ is again a single particle after time $s_{j(2)} - s_{j(1)}$. So

$$\begin{aligned} &P\big(E^i_{N,D}(s_{j(1)}) \cap E^i_{N,D}(s_{j(2)})\big)\\ &\qquad \leq P\big(I_{s_{j(1)}}(s_{j(2)}) \big| E^i_{N,D}(s_{j(1)})\big) + P\big(|\tilde{\zeta}^{\{0\}}_{s_{j(2)}-s_{j(1)}}| = 1\big) < 1/L^2, \end{aligned}$$

where the last inequality follows from (2.11) and (2.9). Thus we have a contradiction to (2.15), which implies the desired result. □

3. The complete convergence theorem. In this section we will use the duality result from Section 2 to prove the complete convergence theorem. Theorem 1.1 will be proved in three steps: first for translation invariant initial distributions, then for any initial distribution which concentrates on configurations with infinitely many zeros and ones and finally for distributions which put positive mass on finite configurations. Recall (2.3), which implies

$$P(|\eta^{1/2}_t \cap B| \text{ is odd}) \to \tfrac{1}{2} P(\zeta^B_t \neq \varnothing \text{ for all } t)$$

for any finite set B. So to prove that η_t converges to the limit for product measure with density 1/2, it will suffice to show that

$$P(|\eta_t \cap B| \text{ is odd}) \to \tfrac{1}{2} P(\zeta^B_t \neq \varnothing \text{ for all } t) \tag{3.1}$$

for any finite B. We will use (1.1) to obtain (3.1), but first we will need some preliminaries.

The graphical representation for the threshold voter model can be found in Cox and Durrett (1991). We will need this construction in the proof of Proposition 3.2 below. For $x \in \mathbf{Z}^d$ and odd $S \subset \mathscr{N}$, let $\{T_n^{x,S},\ n \geq 1\}$ be independent Poisson processes with rate $2^{2-|\mathscr{N}|}$. The times $T_n^{x,S}$ will be the only possible times at which a flip can occur at the site x. At time $T_n^{x,S}$ draw arrows from $x + y$ to x for each $y \in S\backslash\{0\}$. If $0 \notin S$ we write a δ at x. We say there is a path from $(x, 0)$ to (y, t) if there is a pair of sequences $x_0 = x, \dots, x_n = y$ and $s_0 = 0 < s_1 < \cdots s_n < s_{n+1} = t$ so that:

1. For $1 \leq m \leq n$ there is an arrow from x_{m-1} to x_m at time s_m.
2. For $1 \leq m \leq n+1$ there are no δ's in (s_{m-1}, s_m).

Now let

$$N_t^x(y) = \text{ the number of paths from } (x,0) \text{ to } (y,t)$$

and

$$N_t^A(y) = \sum_{x \in A} N_t^x(y).$$

Then set

$$\eta_t^A(y) = \begin{cases} 1, & \text{if } N_t^A(y) \text{ is odd,} \\ 0, & \text{if } N_t^A(y) \text{ is even.} \end{cases} \tag{3.2}$$

We see that η_t^A is the threshold voter model at time t with initial configuration A, since straightforward parity calculations using (3.2) give the following flip rates at 0:

$$\frac{1}{2^{|\mathscr{N}|-2}} \sum_{\substack{S \subset \mathscr{N} \\ S \text{ odd}}} c_S(0, \eta),$$

where

$$c_S(0, \eta) = \tfrac{1}{2}\biggl(1 - \prod_{y \in S \triangle \{0\}} [1 - 2\eta(y)]\biggr).$$

The graphical representation also allows us to describe the dual process; this is done by reversing time. For further explanations and details consult Cox and Durrett (1991).

We will also need the following lemma, which appears in Bramson, Ding and Durrett (1991).

LEMMA 3.1. *Let $X_1, X_2, \dots$ be independent r.v.'s with $P(X_m = 1) = 1 - P(X_m = 0) = \theta_m$, where $0 < \beta \leq \theta_m \leq 1 - \beta < 1$, and let $S_n = X_1 + \cdots + X_n$. Then*

$$\bigl|P(S_n \text{ is odd}) - \tfrac{1}{2}\bigr| \leq \tfrac{1}{2}(1 - 2\beta)^n.$$

PROPOSITION 3.2. *Consider the threshold voter model with* $(M,d) \neq (1,1)$. *Suppose the initial distribution* μ *is translation invariant and puts no mass on the all-zero and all-one configurations. Then*

$$\eta_t^\mu \Rightarrow \eta_\infty^{1/2} \quad \text{as } t \to \infty,$$

the (*nontrivial*) *limit starting from product measure with density* 1/2.

PROOF. Let $\overline{\eta}_1^\mu$ denote the set $\{x \in \mathbf{Z}^d\colon\ \eta_1^\mu(x) = 1 \text{ and } \eta_1^\mu(x+e_1) = 0\}$, where e_1 is the d-dimensional unit vector $(1,0,\dots,0)$. Our first step will be to show that, given $\varepsilon > 0$, there exists K so that if $|A| \geq K$, then

$$P(\overline{\eta}_1^\mu \cap A = \varnothing) < \varepsilon. \tag{3.3}$$

For any finite set B

$$\begin{aligned} P(\overline{\eta}_1^\mu \cap B = \varnothing) &= \int \left[\prod_{x\in B} [1-\eta(x)+\eta(x)\eta(x+e_1)] \right] \mu_1(d\eta) \\ &= \int \left[E^\eta \prod_{x\in B} [1-\eta_1(x)+\eta_1(x)\eta_1(x+e_1)] \right] \mu(d\eta), \end{aligned} \tag{3.4}$$

where μ_1 is the distribution of η_1^μ. Given $\delta > 0$, by Theorem 4.6 of Chapter I of Liggett (1985), there exists L so that for finite B satisfying

$$\min\{\|x-y\|\colon x, y \in B,\ x \neq y\} \geq L, \tag{3.5}$$

we have

$$\begin{aligned} &E^\eta \prod_{x\in B} [1-\eta_1(x)+\eta_1(x)\eta_1(x+e_1)] \\ &\qquad \leq \prod_{x\in B} E^\eta [1-\eta_1(x)+\eta_1(x)\eta_1(x+e_1)] + \delta|B|. \end{aligned} \tag{3.6}$$

[The reasoning behind (3.6) is basically that, at finite times, distant coordinates evolve almost independently.] Hölder's inequality and the translation invariance of μ imply

$$\begin{aligned} &\int \prod_{x\in B} E^\eta [1-\eta_1(x)+\eta_1(x)\eta_1(x+e_1)]\ \mu(d\eta) \\ &\qquad \leq \int [E^\eta[1-\eta_1(0)+\eta_1(0)\eta_1(e_1)]]^{|B|}\ \mu(d\eta). \end{aligned} \tag{3.7}$$

Hence, using (3.4), (3.6) and (3.7), we see that if B satisfies (3.5), then

$$P(\overline{\eta}_1^\mu \cap B = \varnothing) \leq \delta|B| + \int [E^\eta[1-\eta_1(0)+\eta_1(0)\eta_1(e_1)]]^{|B|}\ \mu(d\eta). \tag{3.8}$$

Starting from any configuration which is not identically zero or identically one, there is a positive chance that after one unit of time, the threshold voter model will have a one at the origin and a zero at e_1. Thus for μ a.e. η, we have

$$E^\eta[1-\eta_1(0)+\eta_1(0)\eta_1(e_1)] = 1 - P^\eta(\eta_1(0) = 1 \text{ and } \eta_1(e_1) = 0) < 1.$$

So by the bounded convergence theorem

$$\lim_{k\to\infty}\int [E^{\eta}[1-\eta_1(0)+\eta_1(0)\eta_1(e_1)]]^k\ \mu(d\eta)=0. \tag{3.9}$$

Fix $\varepsilon>0$. Use (3.9) to choose k' so large that

$$\int [E^{\eta}[1-\eta_1(0)+\eta_1(0)\eta_1(e_1)]]^{k'}\ \mu(d\eta)<\frac{\varepsilon}{2}.$$

Now set $\delta=\varepsilon/2k'$. Let L be such that (3.6) is satisfied for B satisfying (3.5). If $|A|$ is sufficiently large, there is a $B\subset A$ such that $|B|=k'$ and B satisfies (3.5) for that L. Hence (3.8) implies

$$\begin{aligned} P(\overline{\eta}_1^{\mu}\cap A=\varnothing) &\le P(\overline{\eta}_1^{\mu}\cap B=\varnothing)\\ &\le \delta k'+\int [E^{\eta}[1-\eta_1(0)+\eta_1(0)\eta_1(e_1)]]^{k'}\ \mu(d\eta)<\varepsilon, \end{aligned}$$

and so (3.3) is satisfied.

Now we want to strengthen (3.3) to the following: given $\varepsilon>0$ and any positive integer N, there exists K' such that if $|A|\ge K'$, then

$$P(|\overline{\eta}_1^{\mu}\cap A|<N)<\varepsilon. \tag{3.10}$$

By (3.3) we can choose K so that if $|A|\ge K$, then

$$P(\overline{\eta}_1^{\mu}\cap A=\varnothing)<\varepsilon/N.$$

Let K' be so large that if $|A|\ge K'$, then A contains N disjoint sets $A_1,A_2,\dots,A_N$ with $|A_j|\ge K$ for all j. Then, for $|A|\ge K'$, we have

$$P(|\overline{\eta}_1^{\mu}\cap A|<N)\le\sum_{j=1}^{N}P(\overline{\eta}_1^{\mu}\cap A_j=\varnothing)<\varepsilon.$$

The rest of the proof is similar to the approach found in Bramson, Ding, and Durrett (1991). Let $\Omega_\infty^B=\{\zeta_t^B\ne\varnothing \text{ for all } t\}$, where $B\subset \mathbf{Z}^d$ is a fixed finite set. Then Proposition 2.6 and inequality (3.10) imply

$$|\overline{\eta}_1^{\mu}\cap\zeta_t^B|\to\infty 1_{\Omega_\infty^B}\quad\text{in probability,} \tag{3.11}$$

where the right-hand side is ∞ on Ω_∞^B and 0 on its compliment. By (3.1), the proposition will be proved if we can show that

$$P(|\eta_2^{\mu}\cap\zeta_t^B| \text{ is odd})\to\tfrac12 P(\Omega_\infty^B)\quad\text{as } t\to\infty, \tag{3.12}$$

since, by duality,

$$P(|\eta_{2+t}^{\mu}\cap B| \text{ is odd})=P(|\eta_2^{\mu}\cap\zeta_t^B| \text{ is odd}).$$

To get (3.12) we will find a lot of independent events which change the parity.

Construct the processes η_s^{μ} and ζ_s^B on independent graphical representations. Recall that at times $T_n^{x,S}$, $n\ge 1$, a flip can occur at the site x for the threshold voter model; these are times at which the value of the process on the set S can influence the site x to change. Let $U_t = \overline{\eta}_1^{\mu} \cap \zeta_t^B$. We say that $x \in U_t$ is *almost isolated* if in the graphical representation of η_s^{μ},

$$\{T_n^{x,S}: n \geq 1\} \cap [1,2] = \varnothing \quad \text{for all odd } S \subset \mathscr{N}, \text{ with } S \neq \{e_1\},$$

$$\{T_n^{x+e_1,S}: n \geq 1\} \cap [1,2] = \varnothing \quad \text{for all odd } S \subset \mathscr{N}$$

and for $y \in \mathscr{N}$,

$$\{T_n^{x+y,S}: n \geq 1\} \cap [1,2] = \varnothing \quad \text{for all odd } S \subset \mathscr{N}, \text{ with } -y \in S$$

(i.e., only $x+e_1$ influences x, nothing influences $x+e_1$, and x influences nothing from time 1 to time 2). For any set $A \subset U_t$, with $\|x_1 - x_2\| > 2M$ for all $x_1, x_2 \in A$, $\{x \text{ is almost isolated}: x \in A\}$ are i.i.d. events that are independent of $\overline{\eta}_1^{\mu}$ and $\{\zeta_s^B: s \geq 0\}$. So letting V_t be the set of almost isolated $x \in U_t$, it follows from (3.11) that

$$|V_t| \to \infty 1_{\Omega_\infty^B} \quad \text{in probability.} \tag{3.13}$$

Now let $\mathscr{G}_t$ be the σ-field generated by ζ_t^B, η_1^{μ}, V_t and all the Poisson points in the graphical representation of η_s^{μ} in $\mathbf{Z}^d \times [1,2]$ except those coming from $\{T_n^{x,S}: n \geq 1\}$, $x \in V_t$. Then

$$P\big(|\eta_2^{\mu} \cap \zeta_t^B| \text{ is odd} \big| \mathscr{G}_t\big) = P\bigg(\sum_{x \in V_t} g_x = h \bmod 2 \bigg| \mathscr{G}_t\bigg) \quad \text{a.s.,} \tag{3.14}$$

where

$$g_x = 1_{\{\text{there is no flip at } x \text{ in } [1,2]\}}$$

and

$$h = 1 - \{|\eta_2^{\mu} \cap \zeta_t^B \cap V_t^c| \bmod 2\}.$$

Observe that, given $\mathscr{G}_t$, V_t and h are constant, and g_x, $x \in V_t$, are independent. So it follows from (3.13) and Lemma 3.1 that

$$P\bigg(\sum_{x \in V_t} g_x = h \bmod 2 \bigg| \mathscr{G}_t\bigg) \to \tfrac{1}{2} 1_{\Omega_\infty^B} \quad \text{in probability.} \tag{3.15}$$

Combining (3.14) and (3.15), taking expected values and using the bounded convergence theorem, we get

$$P(|\eta_2^{\mu} \cap \zeta_t^B| \text{ is odd}) \to \tfrac{1}{2} P(\Omega_\infty^B) \quad \text{as } t \to \infty$$

and thus our proof is complete. □

It is fairly easy to generalize the previous proposition if we use duality and the complete convergence theorem for the threshold contact process.

PROPOSITION 3.3. *Consider the threshold voter model with $(M, d) \neq (1, 1)$. Suppose that the initial distribution μ puts all its mass on configurations with infinitely many zeros and ones. Then*

$$\eta_t^{\mu} \Rightarrow \eta_\infty^{1/2} \quad \textit{as } t \to \infty.$$

PROOF. Fix a finite set $B \subset \mathbf{Z}^d$. Let $\varepsilon > 0$ and $\{t_n\}$ be any sequence tending to infinity. We will show that there exists a subsequence $\{s_n\}$ of $\{t_n\}$, such that

$$\lim_{n\to\infty}\left|P(|\eta^{\mu}_{s_n} \cap B| \text{ is odd}) - \tfrac{1}{2}P(\Omega^B_\infty)\right| < \varepsilon, \tag{3.16}$$

which will verify (3.1) and so complete the proof. □

If μ_1 and μ_2 are two probability measures, the inequality $\mu_1 \le \mu_2$ means

$$\int f\,d\mu_1 \le \int f\,d\mu_2$$

for all monotone functions f. Let $S_b(t)$ be the semigroup for the $\lambda = 1$ threshold contact process and $S_a(t)$ be the semigroup for the spin system whose rates are those of this threshold contact process with the roles of zeros and ones reversed. If $S(t)$ is the semigroup for the threshold voter model, then since all processes are attractive,

$$\mu S_b(t) \le \mu S(t) \le \mu S_a(t) \tag{3.17}$$

for all $t \ge 0$. [See Corollary 1.7 of Chapter III in Liggett (1985).] Let ν_b be the upper invariant measure for the ($\lambda = 1$) threshold contact process (the limit obtained by starting from the all-one configuration), and define ν_a to be the measure obtained by applying ν_b to configurations with zeros and ones interchanged. Then by the complete convergence theorem for the threshold contact process $\mu S_b(t) \Rightarrow \nu_b$ and $\mu S_a(t) \Rightarrow \nu_a$ as $t \to \infty$, so (3.17) implies $\nu_b \le \nu_a$. Suppose ρ is any measure such that $\nu_b \le \rho \le \nu_a$. Then by attractiveness again, we have

$$\nu_b S(t) \le \rho S(t) \le \nu_a S(t)$$

for all $t \ge 0$.

The measure ν_b satisfies the conditions of Proposition 3.2, so

$$\nu_b S(t) \Rightarrow \nu \quad \text{as } t \to \infty$$

and

$$\nu_a S(t) \Rightarrow \nu \quad \text{as } t \to \infty,$$

where ν is the invariant measure obtained in the limit when starting the voter model from product measure with density 1/2. Since any function which depends on a finite number of coordinates can be written as a linear combination of monotone functions, there exists T such that

$$\left|P(|\eta^{\rho}_T \cap B| \text{ is odd }) - \tfrac{1}{2}P(\Omega^B_\infty)\right| < \varepsilon \tag{3.18}$$

for all $\nu_b \le \rho \le \nu_a$. Now, since the space of all probability measures is compact, we can choose a subsequence $\{s_n\}$ of $\{t_n\}$ such that $\mu S(s_n - T)$ converges to some measure ρ. By (3.17), we have $\nu_b \le \rho \le \nu_a$. Thus

$$\begin{aligned}
&\lim_{n\to\infty}\left|P(|\eta^{\mu}_{s_n} \cap B| \text{ is odd}) - \tfrac{1}{2}P(\Omega^B_\infty)\right| \\
&\quad = \lim_{n\to\infty}\left|P(|\eta^{\mu}_{s_n - T} \cap \zeta^B_T| \text{ is odd}) - \tfrac{1}{2}P(\Omega^B_\infty)\right| \\
&\quad = \left|P(|\eta^{\rho}_{0} \cap \zeta^B_T| \text{ is odd}) - \tfrac{1}{2}P(\Omega^B_\infty)\right| \\
&\quad = \left|P(|\eta^{\rho}_{T} \cap B| \text{ is odd}) - \tfrac{1}{2}P(\Omega^B_\infty)\right| < \varepsilon.
\end{aligned}$$

To get the convergence theorem in the case of finite initial configurations, we will again use a comparison with the threshold contact process, but we will need some preliminary results.

LEMMA 3.4. *Let ξ_t be the threshold contact process, where $(M, d) \neq (1, 1)$ and $\lambda \ge 1$. Set $\tau^A = \inf\{t \ge 0\colon \xi^A_t = \varnothing\}$. Then*

$$\lim_{|A|\to\infty} P(\tau^A = \infty) = 1.$$

PROOF. We will first consider the $(2, 1)$ case. The proof here is the same as that for the ordinary contact process. [See Liggett (1985), Chapter VI, Theorems 1.9 and 1.10.] Now the $(2, 1)$ case implies the cases $(M, 1)$, for $M > 2$. Thus we have the lemma for $d = 1$.

Now we will deal with the $d = 2$ cases. It suffices to prove the $(1, 2)$ case. Given $\varepsilon > 0$, choose k so large that for the $(2, 1)$ process,

$$P(\tau^A = \infty) > 1 - \varepsilon$$

for all A such that $|A| \ge k$. Suppose $B \subset \mathbf{Z}^2$ and $|B| \ge k^2$. There must be at least k points in the orthogonal projection of B to the line $y = 2x$ or to the line $y = -\frac{1}{2}x$. Assume, without loss of generality, the former situation.

We will use the coupling found in Liggett (1994) (proof of Proposition 2) to compare the (1,2) process to the (2,1) process. Define the mapping $\pi\colon \mathbf{Z}^2 \to \mathbf{Z}$ by $\pi(m, n) = m+2n$. Then the four neighbors of (m, n) in $\mathbf{Z}^2$ map onto the four neighbors of $\pi(m, n)$ in $\mathbf{Z}$, and $|\pi(B)| \ge k$. Denoting for the moment the (2,1) process by ξ_t and the (1,2) process by γ_t, we will construct a coupling which maintains the relation $\xi_t \le \pi(\gamma_t)$, showing that survival of ξ_t implies survival of γ_t. Associate each $j \in \mathbf{Z}$ such that $\xi(j) = 1$ with any of the $(m, n) \in \mathbf{Z}^2$ such that $\pi(m, n) = j$ and $\gamma(m, n) = 1$, letting the $1 \to 0$ transition at associated sites occur together. For sites j with $\xi(j) = 0$, such that some neighbor i satisfies $\xi(i) = 1$, let (m, n) be the site associated with i, and then associate j with any neighbor of (m, n). Again, couple the $0 \to 1$ transitions at the associated sites. Thus, for the (1,2) process,

$$P(\tau^A = \infty) > 1 - \varepsilon$$

for all A such that $|A| \ge k^2$.

We will use the two-dimensional case to get the higher-dimensional cases. Suppose $d > 2$. Let $B \subset \mathbf{Z}^d$ satisfy $|B| \ge k^{2d}$. Then there are at least k^2 particles in the projection $\pi(B)$ of B to some coordinate plane. Consider now a coupling of the two-dimensional process and the d-dimensional process. Construct the coupling using the projection map as above, to show that survival of the two-dimensional process implies survival of the d-dimensional process. Hence for the $d > 2$ process,

$$P(\tau^A = \infty) > 1 - \varepsilon$$

for all A such that $|A| \geq k^{2d}$. $\square$

REMARK. Both Lemma 3.4 and Proposition 3.3 generalize to some extent to other attractive spin systems. The key ingredients in the proof of Proposition 3.3 are the complete convergence theorem for the threshold contact process and annihilating duality for the threshold voter model. The complete convergence theorem for the contact process generalizes to many other additive processes. [See Durrett (1995).] Thus our ideas could be applied to give more general results for attractive spin systems which satisfy duality equation (2.1) and dominate an appropriate additive process which survives.

Now we have what we need to prove the remaining part of the theorem.

PROPOSITION 3.5. *Let η_t^A be the threshold voter model where $(M, d) \neq (1, 1)$, and the initial state A is finite. Set $\tau^A = \inf\{t \geq 0\colon \eta_t^A = \varnothing\}$. Then*

$$\eta_t^A \Rightarrow \alpha\delta_0 + (1-\alpha)\eta_\infty^{1/2} \quad \textit{as } t \to \infty,$$

where $\alpha = P(\tau^A < \infty)$.

PROOF. We will follow the proof of Proposition 3.3 to show that the process $\eta_t^A|_{\tau^A=\infty}$ converges to $\eta_\infty^{1/2}$ as $t \to \infty$. Fix an arbitrary finite set $B \subset \mathbf{Z}^d$. Let $\{t_n\}$ be any sequence tending to infinity and $0 < \varepsilon < 1$. It will suffice to show that there exists a subsequence $\{s_n\}$ of $\{t_n\}$ such that

$$\limsup_n \left| P\left(|\eta_{s_n}^A \cap B| \text{ is odd} \,\middle|\, \tau^A = \infty\right) - \tfrac{1}{2} P(\Omega_\infty^B)\right| < \varepsilon.$$

[This is the analogue of (3.16).]

If ξ_t is the $\lambda = 1$ threshold contact process, by Lemma 3.4 we can choose N so large that

$$P(\xi_t^C \neq \varnothing \text{ for all } t) > 1 - \varepsilon/5 \tag{3.19}$$

for all sets C such that $|C| \geq N$.

We claim that there exists U such that

$$P\left(|\eta_t^A| \geq N \,\middle|\, \tau^A = \infty\right) > 1 - \varepsilon/10 \tag{3.20}$$

for $t \geq U$. Suppose not, that is,

$$P\left(|\eta_{t_k}^A| < N \,\middle|\, \tau^A = \infty\right) \geq \varepsilon/10$$

for some sequence $\{t_k\}$ tending to infinity. Each time the cardinality of the threshold voter model drops below N, there is probability at least $p > 0$, independent of the configuration, that the process dies out. Now, by comparison with the threshold contact process, $1 - \alpha = P(\tau^A = \infty) > 0$. Thus

$$\begin{aligned} P(\tau^A > t_k) &> P(0 < |\eta^A_{t_k}| < N)p \geq P(|\eta^A_{t_k}| < N,\ \tau^A = \infty)p \\ &= P(|\eta^A_{t_k}| < N \,|\, \tau^A = \infty)(1-\alpha)p \geq \frac{\varepsilon}{10}(1-\alpha)p > 0 \end{aligned} \tag{3.21}$$

for all t_k. Since $\sum_{n=0}^{\infty} P(n \leq \tau^A < n+1) < 1$, we know $P(\tau^A \geq n) \to 0$ as $n \to \infty$. So this contradiction of (3.21) verifies (3.20).

There is a collection $\mathscr{D}$ which consists of a finite number of finite sets $D \subset \mathbf{Z}^d$, with $|D| \geq N$, such that

$$P(\eta^A_U \in \mathscr{D} | \tau^A = \infty) > 1 - \varepsilon/5. \tag{3.22}$$

Fix a set $D \in \mathscr{D}$. We will use the notation and arguments of Proposition 3.3; however, μ will be replaced by the pointmass measure on the set D, and ν_b will be replaced by $\tilde{\nu}_b = \varepsilon'\delta_0 + (1-\varepsilon')\nu_b$, where $\varepsilon' < \varepsilon/5$. Proposition 3.2 implies $\tilde{\nu}_b S(t) \Rightarrow \varepsilon'\delta_0 + (1-\varepsilon')\nu$ as $t \to \infty$. So, following the reasoning leading to (3.18), there exists T such that

$$\left| P(|\eta^\rho_T \cap B| \text{ is odd}) - \tfrac{1}{2} P(\Omega^B_\infty) \right| < 2\varepsilon/5$$

for all $\tilde{\nu}_b \leq \rho \leq \nu_a$. Again using the ideas from Proposition 3.3, we see that there exists a subsequence $\{r_n\}$ of $\{t_n - U - T\}$ such that $\eta^D_{r_n}$ converges to a distribution ρ, and $\tilde{\nu}_b \leq \rho \leq \nu_a$ by contact process comparisons. So

$$\begin{aligned} &\lim_{n\to\infty} \left| P(|\eta^D_{r_n+T} \cap B| \text{ is odd}, \tau^D = \infty) - \tfrac{1}{2} P(\Omega^B_\infty) \right| \\ &\qquad = \left| P(|\eta^\rho_T \cap B| \text{ is odd}) - \tfrac{1}{2} P(\Omega^B_\infty) \right| < 2\varepsilon/5. \end{aligned} \tag{3.23}$$

Since $\mathscr{D}$ contains only a finite number of sets, we can choose the sequence $\{r_n\}$ so that (3.23) is satisfied for all $D \in \mathscr{D}$. Thus, because (3.19) shows that $P(\tau^D = \infty) > 1 - \varepsilon/5$, we have

$$\lim_{n\to\infty} \left| P\big(|\eta^D_{r_n+T} \cap B| \text{ is odd} \,\big|\, \tau^D = \infty\big) - \tfrac{1}{2} P(\Omega^B_\infty) \right| < 4\varepsilon/5 \tag{3.24}$$

for any $D \in \mathscr{D}$. Therefore, using (3.22) and (3.24), we get

$$\limsup_n \left| P\big(|\eta^A_{r_n+U+T} \cap B| \text{ is odd} \,\big|\, \tau^A = \infty\big) - \tfrac{1}{2} P(\Omega^B_\infty) \right| < \varepsilon.$$

So, setting $s_n = r_n + U + T$, the proposition is proved. □

Now using the symmetry of the process in zeros and ones, Theorem 1.1 follows from Proposition 3.3 and Proposition 3.5.

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