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INVARIANT PROBABILITY DISTRIBUTIONS FOR MEASURE-VALUED DIFFUSIONS[1]

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Technion

We investigate the set of invariant probability distributions for measure-valued diffusion processes corresponding to semilinear operators of the form $u_t = L_0 u + \beta u - \alpha u^2$, where $L_0 = \frac{1}{2}\sum_{i,j=1}^d a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}$.

1. Introduction and statement of results. In this article, we initiate a study of invariant probability distributions for spatially dependent measure-valued diffusions. Let

$$L_0 = \frac{1}{2}\sum_{i,j=1}^d a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i}$$

be an elliptic operator on an arbitrary domain $D \subset R^d$. It will be convenient to assume that $a_{i,j} \in C^2(D)$ and $b_i \in C^1(D)$ so that there will be no problem defining the adjoint operator. The underlying motion for the measure-valued process is the diffusion process on D generated by L_0 and killed at the boundary of D; that is, it is the solution to the generalized martingale problem [12] for L_0 on D. The branching mechanism is of the form $\Phi(x,z) = \beta(x)z - \alpha(x)z^2$. where β is bounded from above, $\alpha > 0$, and $\alpha, \beta \in C^\kappa(D)$ for some $\kappa \in (0,1]$. The coefficients β and α should be thought of respectively as the *mass creation* and *variance* parameters for the measure-valued process. A σ-finite measure-valued diffusion $X(t) = X(t,\cdot)$ is then uniquely defined via the following log-Laplace equation:

$$E\big(\exp(-\langle f, X(t)\rangle) \mid X(0) = \mu\big) = \exp\big(-\langle u_f(\cdot,t), \mu\rangle\big), \tag{1.1}$$

for $f \in C_c^+(D)$, the space of compactly supported, nonnegative, continuous functions on D, and for σ-finite initial measures μ satisfying an appropriate growth condition (see Proposition 1 below), where u_f is the minimal positive solution to the evolution equation

$$\begin{aligned} u_t &= L_0 u + \beta u - \alpha u^2 \quad \text{in } D \times (0,\infty), \\ u(x,0) &= f(x) \quad \text{in } D. \end{aligned} \tag{1.2}$$

In [8, 13], we studied the global behavior of these processes starting from a finite initial measure $\mu = X(0)$, in which case $X(t)$ is a finite measure for all t. An invariant distribution is never supported on the set of finite measures, (see

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Remark 9); thus, for the present study, we must consider the measure-valued process starting from σ-finite measures rather than finite ones. We require that the process $X(t)$ take values in the space $\mathscr{M}(D)$ of σ-finite measures on $D \subset R^d$. Let ζ_t denote the distribution of $X(t)$. Then $\zeta_t \in \mathscr{M}_1(\mathscr{M}(D))$, the space of probability measures on $\mathscr{M}(D)$. The following proposition determines how large the initial distribution ζ_0 may be in light of the σ-finiteness requirement.

Proposition 1. *For any $t \geq 0$, $X(t)$ will almost surely take values in the space $\mathscr{M}(D)$ of σ-finite measures on D if and only if for each $f \in C_c^+(D)$, the initial distribution $\zeta_0 \in \mathscr{M}_1(\mathscr{M}(D))$ satisfies the following two equivalent conditions*:

$$\zeta_0\big(\big\{\eta \in \mathscr{M}(D)\colon \langle u_{\lambda f}(\cdot, t), \eta\rangle < \infty, \text{ for sufficiently small } \lambda > 0\big\}\big) = 1; \tag{1.3a}$$

$$\zeta_0\left(\left\{\eta\colon \lim_{\lambda\to 0}\langle u_{\lambda f}(\cdot, t), \eta\rangle = 0\right\}\right) = 1. \tag{1.3b}$$

In particular, (1.3) *will hold if* $\zeta_0(\{\eta \in \mathscr{M}(D) : \langle T_t f, \eta\rangle < \infty\}) = 1$, *where* T_t *is the linear semigroup defined below in* (1.4).

For the rest of the paper, we will assume that the measure-valued process is σ-finite; thus, we will assume that (1.3) holds.

Given a probability measure $\nu \in \mathscr{M}_1(\mathscr{M}(D))$, we define its *mean measure* π_ν by

$$\pi_\nu(\cdot) = \int_{\mathscr{M}(D)} \eta(\cdot)\, d\nu(\eta).$$

A probability measure $\nu \in \mathscr{M}_1(\mathscr{M}(D))$ is an *invariant distribution* if $\zeta_0 = \nu$ implies that $\zeta_t = \nu$ for all $t > 0$. Invariant distributions are well understood in the case of *critical super-Brownian motion*; that is, the case where $L_0 = \frac{1}{2}\Delta$, $\Phi(x, z) = -cz^2$, $c > 0$ and $D = R^d$. The first results were obtained in [4] and more refined results appear in [1, 2]. When $d = 1, 2$, there is no invariant measure. More specifically, the following dichotomy holds: depending on ζ_0, either $w - \lim_{t\to\infty} \zeta_t = \delta_0$, the distribution concentrated on the 0-measure in $\mathscr{M}(R^d)$, or else ζ_t is unstable; that is, for any open set $B \subset R^d$, $\lim_{M\to\infty} \limsup_{t\to\infty} \zeta_t(\eta\colon \eta(B) > M) > 0$. On the other hand, when $d \geq 3$ there is a one-parameter family ν_θ, $\theta \in (0, \infty)$, of translation invariant, shift ergodic invariant measures, with mean measures given by $\pi_{\nu_\theta} = \theta l$, where l is Lebesgue measure. Furthermore, if ζ_0 is ergodic and translation invariant and $\pi_{\zeta_0} = \theta l$, then $w - \lim_{t\to\infty} \zeta_t = \nu_\theta$. Finally, every invariant measure ν is of the form $\nu = \int_0^\infty \nu_\theta F(d\theta)$, where F is a probability distribution on $(0, \infty)$. Note that if $\int_0^\infty \theta F(d\theta) = \infty$, then the mean measure will be ∞; that is, $\pi_\nu(B) = \infty$ for all open $B \subset R^d$.

In order to discuss invariant measures for more general measure-valued diffusions, we need some additional notation. Let $\mathscr{P}_\cdot$ denote the solution to the generalized martingale problem for L_0 on D, let $Y(t)$ denote a canonical diffusion path in $C([0, t), \hat{D})$, where $\hat{D}$ is the one-point compactification of D

obtained by the addition of a cemetery state, and let $\tau_D = \inf\{t \ge 0: Y(t) \notin D\}$ denote the lifetime of the process. Let

$$L = L_0 + \beta$$

denote the linear part of the elliptic operator on the right-hand side of (1.2), and let T_t denote the semigroup corresponding to the operator L on D. The Feynman–Kac formula gives

$$T_t f(x) = \mathscr{E}_x\left(\exp\left(\int_0^t \beta(Y(s))\,ds\right) f(Y(t)); \tau_D > t\right), \tag{1.4}$$

where $\mathscr{E}_x$ is the expectation corresponding to $\mathscr{P}_x$.

It will be useful to recall a basic definition from the criticality theory of elliptic operators [12].

DEFINITION 1. The operator L on D is called *subcritical* if it possesses a positive Green's function, in which case the cone $C_L(D) \equiv \{u \in C^2(D): u > 0$ and $Lu = 0\}$ of positive harmonic functions is nonempty. It is called *critical* if it does not possess a positive Green's function but $C_L(D)$ is not empty, and it is called *supercritical* if $C_L(D)$ is empty.

If $\beta = 0$, then $L = L_0$ is subcritical or critical according to whether it corresponds to a transient or a recurrent diffusion. We also note that the criticality classification of L is inherited by the adjoint operator $\tilde{L}$.

A measure $\mu \in \mathscr{M}(D)$ is invariant for T_t if $\mu T_t = \mu$. If μ is invariant, then it is absolutely continuous with respect to Lebesgue measure on D and, with an abuse of notation, we will call this invariant density $\mu = \mu(x)$. The density μ is a positive harmonic function for the adjoint operator $\tilde{L}$; that is, $\mu \in C_{\tilde{L}}(D)$.

In the sequel, $\delta_\mu \in \mathscr{M}_1(\mathscr{M}(D))$ denotes the probability measure which is concentrated on $\mu \in \mathscr{M}(D)$, and $\mathrm{Poiss}_\mu \in \mathscr{M}_1(\mathscr{M}(D))$ denotes the Poisson random measure with intensity μ. Note that if μ is invariant for the semigroup T_t, then since $\int_{\mathscr{M}(D)} \langle T_t f, \eta\rangle d\,\mathrm{Poiss}_\mu(\eta) = \langle T_t f, \mu\rangle = \langle f, \mu\rangle$, it follows from the final statement of Proposition 1 that (1.3) holds for δ_μ and Poiss_μ. Thus, the super-diffusion $X(t)$ obtained by starting from δ_μ or Poiss_μ will take values in $\mathscr{M}(D)$ as required.

We now present a proposition which includes a number of basic results that will set the framework for our study. Variations of these results, sometimes in other settings, may be found in the literature (see, e.g., [10, 7]). The proposition is proved in the next section.

PROPOSITION 2. (i) *If $\nu \in \mathscr{M}_1(\mathscr{M}(D))$ is an invariant distribution for the measure-valued diffusion, and its mean measure π_ν is σ-finite, then π_ν is an invariant density for the semigroup T_t.*

(ii) *Let $\mu \in \mathscr{M}(D)$ be invariant for the semigroup T_t. If $\zeta_0 = \delta_\mu$, then*

$$\zeta_\infty^{(\mu)} \equiv w - \lim_{t\to\infty} \zeta_t \text{ exists.}$$

Either $\zeta_\infty^{(\mu)} = \delta_0$, the trivial invariant distribution concentrated on the 0-*measure, or else $\zeta_\infty^{(\mu)}$ is a nontrivial invariant distribution for the meas-*

ure-valued process. The distribution $\zeta_\infty^{(\mu)}$ *is uniquely specified by its Laplace transform*),

$$
\begin{aligned}
(1.5)\qquad &\int_{\mathscr{M}(D)} \exp\bigl(-\langle f, \eta\rangle\bigr)\, d\zeta_\infty^{(\mu)}(\eta) \\
&\qquad = \exp\biggl(-\langle f, \mu\rangle + \int_0^\infty \langle \alpha u_f^2(\cdot, t), \mu\rangle\, dt\biggr) \quad \text{for } f \in C_c^+(D).
\end{aligned}
$$

Under the assumption that $\lim_{t\to\infty}\sup_{x\in D} T_t f(x) = 0$, *for all* $f \in C_c^+(D)$, *the above result is also true when the initial distribution is* Poiss_μ.

(iii) *Let* $\zeta_\infty^{(\mu)}$ *be the invariant distribution obtained in* (ii). *Then* $\pi_{\zeta_\infty^{(\mu)}}$, *the mean measure of* $\zeta_\infty^{(\mu)}$, *satisfies* $\pi_{\zeta_\infty^{(\mu)}} \le \mu$ [*that is,* $\pi_{\zeta_\infty^{(\mu)}}(A) \le \mu(A)$, *for all* $A \subset D$] *and equality holds if and only if*

$$
(1.6)\qquad \lim_{\lambda\to 0+} \frac{1}{\lambda}\int_0^\infty \bigl\langle \alpha u_{\lambda f}^2(\cdot, t), \mu\bigr\rangle\, dt = 0,
$$

for all $f \in C_c^+(D)$. *A sufficient condition for* (1.6) *to hold is that*

$$
(1.7)\qquad \int_0^\infty \langle \alpha (T_t f)^2, \mu\rangle\, dt < \infty,
$$

for all $f \in C_c^+(D)$.

(iv) *Let* $\zeta_\infty^{(\mu)}$ *be the invariant distribution obtained in* (ii). *Then* $\zeta_\infty^{(\mu)} = \delta_0$ *if and only if*

$$
(1.8)\qquad \lim_{t\to\infty} \langle u_f(\cdot, t), \mu\rangle = 0
$$

for all $f \in C_c^+(D)$.

For any measure $\mu \in \mathscr{M}(D)$ which is invariant for the semigroup T_t of the underlying motion of the superdiffusion, Proposition 2 gives a recipe for obtaining an invariant distribution $\zeta_\infty^{(\mu)}$ for the superprocess. That invariant distribution will always have mean measure less than or equal to μ. The mean measure will be equal to 0 and hence the invariant distribution will be equal to the trivial 0 measure, if and only if condition (1.8) holds, or equivalently by (1.5), if and only if $\langle f, \mu\rangle = \int_0^\infty \langle \alpha u_f^2(\cdot, t), \mu\rangle\, dt$. The mean measure will be equal to μ if and only if condition (1.6) holds. In general, it is quite difficult to verify condition (1.6) or (1.8), in particular because these conditions are in terms of the behavior of a solution to the nonlinear equation.

The aim of the rest of this paper is to study the question of when the mean measure will be equal to 0, when it will be equal to μ, and when it will be strictly in between 0 and μ. However, before we embark on this route, we present in Theorem 1 below a result which describes the precise role of condition (1.7) in terms of the variance of the invariant distribution $\zeta_\infty^{(\mu)}$. The proof of Theorem 1 leads directly to an interesting asymptotic property of $\zeta_\infty^{(\mu)}$ which is presented in Theorem 2.

We define the second moment operator $M_\nu^{(2)}(\cdot)$ and the variance operator $\mathrm{Var}_\nu(\cdot)$ of a probability measure $\nu \in \mathscr{M}_1(\mathscr{M}(D))$ as follows:

$$M_\nu^{(2)}(f) = \int_{\mathscr{M}(D)} \langle f, \eta\rangle^2 \, d\nu(\eta) \quad \text{for } f \in C_c^+(D);$$

$$\mathrm{Var}_\nu(f) = M_\nu^{(2)}(f) - \langle f, \pi_\nu\rangle^2 \quad \text{for } f \in C_c^+(D).$$

THEOREM 1. *Let μ be an invariant density of T_t and let $\zeta_\infty^{(\mu)}$ be the corresponding invariant distribution with mean measure denoted by $\pi_{\zeta_\infty^{(\mu)}}$. Let $f \in C_c^+(D)$. If $\int_0^\infty \langle \alpha(T_t f)^2, \mu\rangle \, dt < \infty$, then*

$$\mathrm{Var}_{\zeta_\infty^{(\mu)}}(f) = 2\int_0^\infty \langle \alpha(T_t f)^2, \mu\rangle \, dt.$$

If $\int_0^\infty \langle \alpha(T_t f)^2, \pi_{\zeta_\infty^{(\mu)}}\rangle \, dt = \infty$, then $\mathrm{Var}_{\zeta_\infty^{(\mu)}}(f) = \infty$.

REMARK 1. With regard to Theorem 1, see Problem 3 below.

We need to introduce a little notation for Theorem 2. Let $p(t, x, y)$ denote the kernel of the semigroup T_t. Although we have used the notation μT_t above, where μ is a density or a measure, for Theorem 2 it will be convenient to consider the dual T_t^* of T_t which operates on $\mathscr{M}(D)$: for $\eta \in \mathscr{M}(D)$, one defines $T_t^*\eta(dy) = (\int_D p(t, x, y) d\eta(x))\, dy$. Of course, $T_t^*\eta \in \mathscr{M}(D)$ if and only if $\int_D p(t, x, y) d\eta(x)$ is in $L^1_{loc}(D)$. We can also consider T_t^* operating on $\mathscr{M}_1(\mathscr{M}(D))$: for $\nu \in \mathscr{M}_1(\mathscr{M}(D))$, we define $T_t^*\nu(A) = \nu(T_t^* A)$, for measurable sets $A \in \mathscr{M}(D)$. Note that $T_t^*\nu$ is a subprobability measure on $\mathscr{M}(D)$. It will be a probability measure, that is, it will belong to $\mathscr{M}_1(\mathscr{M}(D))$, if and only if $T_t^*(\mathscr{M}(D))$ contains the support of ν.

THEOREM 2. *Assume that $\int_0^\infty \langle \alpha(T_t f)^2, \mu\rangle \, dt < \infty$, for all $f \in C_c^+(D)$. Then*

$$w - \lim_{t\to\infty} T_t^* \zeta_\infty^{(\mu)} = \delta_\mu.$$

Equivalently, for each $f \in C_c^+(D)$, the random variables $\{\langle T_t f, \eta\rangle\}_{t\ge 0}$ on $(\mathscr{M}(D), \zeta_\infty^{(\mu)})$ satisfy

$$\lim_{t\to\infty} \langle T_t f, \eta\rangle = \langle f, \mu\rangle \quad \textit{in} \quad \zeta_\infty^{(\mu)}\textit{-probability}.$$

REMARK 2. Let S_t denote the semigroup corresponding to the measure-valued diffusion $X(t)$. Its dual, S_t^* operates on $\mathscr{M}_1(\mathscr{M}(D))$. Recalling that ζ_t denotes the distribution of $X(t)$, we have by definition, $S_t^*\zeta_0 = \zeta_t$. Thus, under the condition $\int_0^\infty \langle \alpha(T_t f)^2, \mu\rangle \, dt < \infty$, for all $f \in C_c^+(D)$, we obtain from Proposition 2 and Theorem 2 the duality

$$w - \lim_{t\to\infty} S_t^* \delta_\mu = \zeta_\infty^{(\mu)} \quad \text{and } w - \lim_{t\to\infty} T_t^* \zeta_\infty^{(\mu)} = \delta_\mu.$$

We note that Theorem 2 holds for all the invariant distributions that arise in this paper when considering particular classes of operators—namely, those in Theorems 4, 5 and 7. In particular, it holds for the invariant measures associated with the standard, critical, d-dimensional super Brownian motion, for $d \geq 3$. These measures appear as a particular case of Theorem 4.

REMARK 3. It may well be true in complete generality, and it is certainly true in all but the most pathological cases that the finiteness of $\int_0^\infty \langle \alpha(T_t f)^2, \mu\rangle\, dt$ for some $0 \neq f \in C_c^+(D)$ is equivalent to the finiteness of it for all $f \in C_c^+(D)$.

In order to present the rest of our results, we need to recall a few facts about positive harmonic functions. We noted earlier that if μ is invariant for the semigroup T_t, then it possesses a density $\mu = \mu(x) \in C_{\tilde{L}}(D)$. On the other hand, it is not necessarily true that all the elements in $C_{\tilde{L}}(D)$ are invariant densities for T_t (see the second paragraph of Remark 8 below). Let $C_{\tilde{L}}^{\mathrm{inv}}(D) \subset C_{\tilde{L}}(D)$ denote the subcone of invariant densities. We now apply the Martin boundary theory [12] to this subcone. An element $\mu \in C_{\tilde{L}}^{\mathrm{inv}}(D)$ is called *minimal* if the relations $\mu_1 \leq \mu$ and $\mu_1 \in C_{\tilde{L}}^{\mathrm{inv}}(D)$ imply that $\mu_1 = c\mu$ for some constant c. By the Martin boundary theory, there exists an index set $\tilde{\Lambda}_0^{\mathrm{inv}}$, which is the part of the minimal Martin boundary corresponding to the invariant functions, and a collection of minimal elements $\{\mu_\rho\}_{\rho \in \tilde{\Lambda}_0^{\mathrm{inv}}} \in C_{\tilde{L}}^{\mathrm{inv}}(D)$ (normalized, say, by $\mu_\rho(x_0) = 1$, for some $x_0 \in D$), which we shall term *minimal invariant densities*, such that every element $\mu \in C_{\tilde{L}}^{\mathrm{inv}}(D)$ can be represented in the form

$$\mu = \int_{\tilde{\Lambda}_0^{\mathrm{inv}}} \mu_\rho m_\mu(d\rho) \tag{1.9}$$

for a unique finite measure m_μ on $\tilde{\Lambda}_0^{\mathrm{inv}}$. Conversely, every finite measure m_μ gives rise to an element of $C_{\tilde{L}}^{\mathrm{inv}}(D)$ via the representation in (1.9).

Theorem 3 below gives for each $\mu \in C_{\tilde{L}}^{\mathrm{inv}}(D)$ the general structure of the mean measure π_μ corresponding to the invariant distribution $\zeta_\infty^{(\mu)}$.

THEOREM 3. (i) *For each $\rho \in \tilde{\Lambda}_0^{\mathrm{inv}}$, there exists a constant $c_\rho = c_\rho(\alpha) \in [0, 1]$ such that the mean measure of $\zeta_\infty^{(\mu_\rho)}$ satisfies $\pi_{\zeta_\infty^{(\mu_\rho)}} = c_\rho \mu_\rho$. Furthermore, $c_\rho(\alpha)$ is nonincreasing in α.*

(ii) *Let $\mu \in C_{\tilde{L}}^{\mathrm{inv}}(D)$ with corresponding measure m_μ as in* (1.9). *Then*

$$\pi_{\zeta_\infty^{(\mu)}} = \int_{\tilde{\Lambda}_0^{\mathrm{inv}}} c_\rho \mu_\rho m_\mu(d\rho). \tag{1.10}$$

(iii) *Let $\mu_1, \mu_2 \in C_{\tilde{L}}^{(\mathrm{inv})}(D)$. Then $\zeta_\infty^{(\mu_1)} = \zeta_\infty^{(\mu_2)}$ if and only if $\pi_{\zeta_\infty^{(\mu_1)}} = \pi_{\zeta_\infty^{(\mu_2)}}$.*

Let

$$\tilde{\Lambda}_0^{\mathrm{inv},\alpha} = \{\rho \in \tilde{\Lambda}_0^{\mathrm{inv}} \colon c_\rho(\alpha) > 0\},$$

where c_ρ is as in Theorem 3(i), and let $C_{\tilde{L}}^{\mathrm{inv},\,\alpha}(D) \subset C_{\tilde{L}}^{\mathrm{inv}}(D)$ be the subcone generated by the elements of $\tilde{\Lambda}_0^{\mathrm{inv},\,\alpha}$; that is, analogous to (1.9), $\mu \in C_{\tilde{L}}^{\mathrm{inv},\,\alpha}(D)$ if and only if there exists a finite measure m_μ on $\tilde{\Lambda}_0^{\mathrm{inv},\,\alpha}$ such that $\mu = \int_{\tilde{\Lambda}_0^{\mathrm{inv},\,\alpha}} \mu_\rho m_\mu(d\rho)$. The following result is an immediate corollary of Theorem 3.

COROLLARY 1. *The set of nontrivial invariant distributions* $\{\zeta_\infty^{(\mu)}\colon \mu \in C_{\tilde{L}}^{\mathrm{inv}}(D)\}$, *which was obtained by the procedure of Proposition* 2(ii), *is in one-to-one correspondence with* $C_{\tilde{L}}^{\mathrm{inv},\,\alpha}(D)$ *via the map* $\mu \to \pi_{\zeta_\infty^{(\mu)}}$ *and is nonincreasing in its dependence on* α.

REMARK 4. If we were to set $\alpha \equiv 0$, then starting from $\zeta_0 = \delta_\mu$, with $\mu \in \mathscr{M}(D)$, the measure-valued diffusion would reduce to a deterministic measure-valued path; namely, $\zeta_t = \delta_{\mu T_t}$. Thus, the set of invariant distributions would be equal to $\{\delta_\mu\colon \mu \in C_{\tilde{L}}^{\mathrm{inv}}(D)\}$.

Theorem 3 shows that for a minimal invariant density $k\mu_\rho$, $k > 0$, the mean measure $\pi_{\zeta_\infty^{(k\mu_\rho)}}$ must always be a multiple of μ_ρ. On the other hand, if for example, $\mu = \mu_{\rho_1} + \mu_{\rho_2}$ with $c_{\rho_1} > 0$ and $c_{\rho_2} = 0$, then the mean measure for $\zeta_\infty^{(\mu)}$ will be $c_{\rho_1}\mu_1$ which is not a multiple of μ. For a class of specific examples, see Corollary 3(iii).

We conjecture that c_ρ appearing in Theorem 3(i) can only take two possible values—0 and 1 (see Problem 2 at the end of this section). If μ_ρ is a minimal invariant density for T_t and $c_\rho = c_\rho(\alpha) \in (0,1)$, then for $f \in C_c^+(D)$, it follows from Theorem 1 that $\mathrm{Var}_{\zeta_\infty^{(\mu)}}(f) < \infty$ if and only if $\int_0^\infty < \alpha(T_t f)^2, \mu > dt < \infty$. Yet, according to Proposition 2(iii), the finiteness of this integral for all $f \in C_c^+(D)$ guarantees that $c_\rho = 1$. This gives the following corollary in the case $c_\rho \in (0,1)$.

COROLLARY 2. *Let* μ_ρ *be a minimal density for* T_t, *and consider the invariant distribution* $\zeta_\infty^{(\mu)}$. *Let* $c_\rho = c_\rho(\alpha)$ *be as in Theorem* 3 *and assume that* $c_\rho \in (0,1)$. *Then* $\mathrm{Var}_{\zeta_\infty^{(\mu)}}(f) = \infty$, *for some* $0 \neq f \in C_c^+(D)$.

In light of Theorem 3 and Corollary 1, our strategy for the rest of the paper will be as follows. We fix a linear operator $L = L_0 + \beta$ on a domain $D \subset R^d$ for which the cone $C_{\tilde{L}}^{\mathrm{inv}}(D)$ is nonempty. For each minimal element $\mu_\rho \in C_{\tilde{L}}^{\mathrm{inv}}(D)$, we wish to find an explicit growth condition on α which holds for sufficiently small α and such that whenever that condition is satisfied, (1.6) will hold. It will then follow that the invariant distribution $\zeta_\infty^{(\mu_\rho)}$ satisfies $\pi_{\zeta_\infty^{(\mu_\rho)}} = \mu_\rho$. Similarly, we wish to find an explicit growth condition on α which holds for sufficiently large α and such that whenever that condition is satisfied, (1.8) will hold. It will then follow that $\zeta_\infty^{(\mu_\rho)} = \delta_0$. The first task is considerably easier than the second one because in place of (1.6), we may avail ourselves of the sufficient condition (1.7) which involves only a solution of the linear heat equation, whereas condition (1.8) depends on the behavior of a solution to a nonlinear equation. At least with regard to the case of super-Brownian motion, the use of (1.7) in place of (1.6) sacrifices little, if any, precision; see Remark 5 after Theorem 4.

We begin with the case of d-dimensional, critical, super-Brownian motion. We have $L_0 = \frac{1}{2}\Delta$, $\beta = 0$ and $D = R^d$. In this case $\tilde{L} = L = \frac{1}{2}\Delta$ and $C_{\tilde{L}}^{\mathrm{inv}}(D) = C_{(1/2)\Delta}^{\mathrm{inv}}(R^d)$ is the cone of positive constants. Thus, if an invariant distribution has a σ-finite mean measure, that mean measure must be a multiple of Lebesgue measure.

THEOREM 4. *Consider critical, super-Brownian motion with a variable variance parameter, corresponding to the semilinear equation* $u_t = \frac{1}{2}\Delta u - \alpha u^2$ *on* R^d, $d \geq 1$. *Let* l *denote Lebesgue measure on* R^d.

(i) *If*

$$\int_1^\infty \frac{1}{t^d}\left(\int_{R^d} \exp\left(-\frac{|x|^2}{t}\right)\alpha(x)\,dx\right)dt < \infty, \tag{1.11}$$

then the invariant distribution $\zeta_\infty^{(l)}$ *satisfies* $\pi_{\zeta_\infty^{(l)}} = l$.
In particular, if $d \geq 2$ *and* $\alpha(x) \leq c(1+|x|)^\gamma$ *with* $\gamma < d-2$, *then* (1.11) *will be satisfied*.

(ii) *If for some* $k > 0$,

$$\int_{|x|<t^{1/2}(\log t)^k} \frac{1}{\alpha(x)}\,dx = o(t\log t) \quad \text{as } t \to \infty, \tag{1.12}$$

then $\zeta_\infty^{(l)} = \delta_0$.
In particular, if $d \geq 1$ *and* $\alpha(x) \geq c(1+|x|)^\gamma$ *with* $\gamma \geq d-2$, *then* (1.12) *will be satisfied*.

(iii) *If* $d = 1$, *then for any* $\alpha > 0$, $\zeta_\infty^{(l)} = \delta_0$.

REMARK 5. There is no problem defining super-Brownian motion when $\alpha \gneqq 0$. The proof of Theorem 4(iii) actually shows that the result continues to hold when $\alpha \gneqq 0$. Note that for $d \geq 2$, the critical exponent for α with regard to the existence of an invariant probability distribution is $d-2$; however, for $d = 1$, there is never an invariant probability distribution even if α is compactly supported. The proof of part (i) of the theorem utilizes the sufficient condition (1.7) in place of the necessary and sufficient condition (1.6). The fact that the exact value $d-2$ of the critical exponent was obtained shows that little if any information was lost by this approximation.

We need a few facts about symmetric operators for the next theorem (see [12], Section 4.10). The underlying diffusion process on $D \subset R^d$ is reversible when the corresponding operator L_0 is symmetric with respect to some reference density. In this case L_0 takes the form

$$L_0 = \tfrac{1}{2}\nabla \cdot a\nabla + a\nabla Q\nabla = \tfrac{1}{2}\exp(-2Q)\nabla \cdot \exp(2Q)a\nabla. \tag{1.13}$$

We will assume that $a_{i,j} \in C^1(D)$ and $Q \in C^2(D)$. The operator $L = L_0 + \beta$ is symmetric with respect to the density

$$m_{\mathrm{sym}} = \exp(2Q).$$

We call m_{sym} or any of its multiples a *reversible density* for the operator L. The adjoint operator (with respect to Lebesgue measure) is $\tilde{L} = \frac{1}{2}\nabla \cdot a\nabla - a\nabla Q\nabla - \nabla \cdot (a\nabla Q) + \beta$. Analogous to an invariant density (i.e., a function μ satisfying $\mu T_t = \mu$), we define an invariant function to be a function h satisfying $T_t h = h$. An invariant function is a positive harmonic function for L on D; that is, $h \in C_L(D)$. Let $C_L^{\mathrm{inv}}(D) \subset C_L(D)$ denote the subcone of invariant functions. It is easy to check that in the symmetric case, a density μ is invariant if and only if it is of the form $\mu = hm_{\mathrm{sym}}$, where h is an invariant function; that is,

$$C_{\tilde{L}}^{\mathrm{inv}}(D) = \{hm_{\mathrm{sym}} \colon h \in C_L^{\mathrm{inv}}(D)\}. \tag{1.14}$$

The next result gives a sufficient condition for $\zeta_\infty^{(\mu)}$ to have mean measure μ in the case of symmetric operators.

THEOREM 5. *Let $L = L_0 + \beta$ be a symmetric operator with L_0 as in* (1.13) *and assume that L is subcritical. Let hm_{sym} be an invariant density as in* (1.14). *If $\alpha \le \frac{c}{h}$, for some $c > 0$, then the mean measure of the invariant distribution $\zeta_\infty^{(hm_{\mathrm{sym}})}$ satisfies $\pi_{\zeta_\infty^{(hm_{\mathrm{sym}})}} = hm_{\mathrm{sym}}$.*

The next theorem treats certain one-dimensional processes and goes in the opposite direction to Theorem 5. It gives a sufficient condition for $\zeta_\infty^{(\mu)}$ to be equal to δ_0. Before we can state the theorem we need to recall some facts ([12], Section 5.1). If

$$L_0 = \frac{1}{2}a\frac{d^2}{dx^2} + b\frac{d}{dx}$$

on $D = (-\infty, \infty)$, then letting $Q(x) = \int_0^x \frac{b}{a}(y)\,dy - \frac{1}{2}\log a$, one can write $L_0 = \frac{1}{2}\exp(-2Q)\frac{d}{dx}(\exp(2Q)a\frac{d}{dx})$; thus, L_0 is symmetric with respect to the density

$$m_{\mathrm{sym}}(x) = \exp\left(2Q(x)\right) = \frac{1}{a(x)}\exp\left(\int_0^x \frac{2b}{a}(y)\,dy\right). \tag{1.15}$$

Assume now that L_0 corresponds to a transient diffusion; that is, L_0 on $(-\infty, \infty)$ is subcritical. This is equivalent to the integrability of

$$\exp\left(-\int_0^x \frac{2b}{a}(y)\,dy\right)$$

at either $+\infty$ or $-\infty$. The space of all (not necessarily positive) solutions w of $L_0 w = 0$ in R is two-dimensional and is spanned by the functions 1 and $\int_0^x dz \exp(-\int_0^z \frac{2b}{a}(y)\,dy)$. From this it is easy to see that in the transient case the cone $C_{L_0}(D)$ of positive harmonic functions is always two-dimensional, and it is easy to exhibit its minimal elements. However, we need to consider two cases separately.

CASE 1 ($\int_{-\infty}^{\infty} dx \exp(-\int_0^x \frac{2b}{a}(y)\,dy) < \infty$). In this case the minimal elements of $C_{L_0}(D)$ are

$$h_1(x) = \int_{-\infty}^{x} dz \exp\biggl(-\int_0^z \frac{2b}{a}(y)\,dy\biggr),$$
$$h_2(x) = \int_{x}^{\infty} dz \exp\biggl(-\int_0^z \frac{2b}{a}(y)\,dy\biggr). \tag{1.16}$$

CASE 2 ($\int_{-\infty}^{\infty} dx \exp(-\int_0^x \frac{2b}{a}(y)\,dy) = \infty$). In this case we will assume without loss of generality that $\int_{-\infty}^{0} dx \exp(-\int_0^x \frac{2b}{a}(y)\,dy) < \infty$. Then the minimal elements of $C_{L_0}(D)$ are

$$h_1(x) = \int_{-\infty}^{x} dz \exp\biggl(-\int_0^z \frac{2b}{a}(y)\,dy\biggr),$$
$$h_2(x) = 1. \tag{1.17}$$

We now make the additional assumption that $C_{L_0}^{\mathrm{inv}}(D) = C_{L_0}(D)$. (In case 1 above, this is equivalent to the assumption that the diffusion corresponding to L_0 is conservative, that is, that its semigroup T_t satisfies $T_t 1 = 1$. Thus the coefficients a and b must satisfy Feller's well-known integral criterion for nonexplosion [12]. In Case 2, the diffusion needs to be conservative, but in addition, a growth condition on the inward drift from $+\infty$ must be assumed.)

From the above discussion and (1.14), it follows that the cone $C_{\tilde{L}_0}^{\mathrm{inv}}(D)$ of invariant densities for the diffusion process takes the form

$$C_{\tilde{L}_0}^{\mathrm{inv}}(D) = \bigl\{(c_1 h_1 + c_2 h_2) m_{\mathrm{sym}}\colon c_1, c_2 \geq 0, c_1 + c_2 > 0\bigr\}, \tag{1.18}$$

where m_{sym} is as in (1.15) and h_1, h_2 are as in (1.16) or (1.17).

For the next theorem, we make the following assumption. (See the remark following the theorem for a discussion concerning the relaxing of this assumption.)

ASSUMPTION 1. (i) $a(x) = c_1 x^{l_1}$ for $x \gg 1$ and $a(x) = c_2 |x|^{l_2}$ for $x \ll -1$, where $c_1, c_2 > 0$.

(ii) $b(x) = d_1 x^{k_1}$ for $x \gg 1$ and $b(x) = -d_2 |x|^{k_2}$ for $x \ll -1$, where $d_1, d_2 \neq 0$ and at least one of d_1, d_2 is positive.

(iii) $k_1,\ k_2 < 1$ and $k_i > l_i - 1$ for some i for which $d_i > 0$.

Under Assumption 1, L_0 corresponds to a transient diffusion and it can be verified that $C^{\mathrm{inv}}_{L_0}(D) = C_{L_0}(D)$; thus $C^{\mathrm{inv}}_{\tilde{L}_0}(D)$ is given by (1.18).

THEOREM 6. *Let* $L_0 = \frac{1}{2}a\frac{d^2}{dx^2} + b\frac{d}{dx}$ *on* $D = (-\infty, \infty)$ *satisfy Assumption* 1 *and let* $\beta = 0$. *Let* $h_1 m_{\mathrm{sym}}, h_2 m_{\mathrm{sym}}$ *be the minimal invariant densities, where* h_1, h_2 *are as in* (1.16) *or* (1.17) *and* m_{sym} *is as in* (1.15). *If* $\alpha \geq \frac{c}{h_i}$, *for some* $i \in \{1, 2\}$ *and some* $c > 0$, *then for* $j \neq i$, $\zeta_\infty^{(h_j m_{\mathrm{sym}})} = \delta_0$.

REMARK 6. We suspect that the result in Theorem 6 holds in general for subcritical one-dimensional operators $L = L_0 + \beta$ for which the corresponding two-dimensional cone $C_L(D)$ satisfies $C_L(D) = C^{\mathrm{inv}}_L(D)$. In fact, the proof we give works in somewhat more generality than is stated in Theorem 6, as will be pointed out in the remark following the completion of its proof.

Combining Theorems 3, 5 and 6, we have the following corollary.

COROLLARY 3. *Let the underlying motion of the measure-valued diffusion be given by an operator* L_0 *satisfying Assumption* 1 *and let* $\beta = 0$.

(i) *If* $\alpha \leq C\min(\frac{1}{h_1}, \frac{1}{h_2})$, $C > 0$, *then* $\pi_{\zeta_\infty^{(\mu)}} = \mu$, *for any invariant density* $\mu = (c_1 h_1 + c_2 h_2) m_{\mathrm{sym}}$.

(ii) *If* $\alpha \geq C\max(\frac{1}{h_1}, \frac{1}{h_2})$, $C > 0$, *then* $\zeta_\infty^{(\mu)} = \delta_0$, *for any invariant density* $\mu = (c_1 h_1 + c_2 h_2) m_{\mathrm{sym}}$.

(iii) *If* $\frac{C_1}{h_j} \leq \alpha \leq \frac{C_2}{h_j}$, *for some* $j \in \{1, 2\}$ *and* $C_1, C_2 > 0$, *then*

$$\pi_{\zeta_\infty^{((c_1 h_1 + c_2 h_2) m_{\mathrm{sym}})}} = c_j h_j m_{\mathrm{sym}}.$$

REMARK 7. Consider Corollary 3 in the case $\alpha = 1$, so that the corresponding semilinear equation is $u_t = L_0 u - u^2$. If the operator L_0 satisfies (1.16), then case (i) of the corollary holds, whereas if L_0 satisfies (1.17), then case (iii) of the corollary holds with $j = 2$. In particular then, when (1.17) holds, the mean measure of any nontrivial invariant distribution $\zeta_\infty^{(\mu)}$ will be a reversible density for L_0.

In light of Theorems 5 and 6, it is natural to wonder what occurs in general when the underlying motion is a multidimensional symmetric diffusion, especially when $C^{\mathrm{inv}}_L(D)$ is more than two-dimensional. That is, if h_1 and h_2 are minimal elements in $C^{\mathrm{inv}}_L(D)$ and $\alpha = \frac{1}{h_1}$, what can be said about $\pi_{\zeta_\infty^{(h_2 m_{\mathrm{sym}})}}$? We don't know the answer in general but the following example is illuminating.

Let $L_0 = \frac{1}{2}\Delta + b \cdot \nabla$ on $D = R^d$, $d \geq 2$, where b is a nonzero constant vector, and let $\beta = 0$. Then $m_{\mathrm{sym}}(x) = \exp(2b \cdot x)$ and $C^{(\mathrm{inv})}_L(D) = C_L(D) = \{\exp(\nu \cdot x) : |\nu + b| = |b|\}$. Thus, μ is an invariant density if and only if it is of the form $\mu(x) = \exp((\nu + 2b) \cdot x)$ with ν on the sphere $S \equiv \{\nu \in R^d : |\nu + b| = |b|\}$. Let $h_\nu(x) = \exp(\nu \cdot x)$, for $\nu \in R^d$. By Theorem 5, for $\nu \in S$, the invariant distribution $\zeta_\infty^{(h_\nu m_{\rm sym})}$ has mean measure $h_\nu(x)m_{\rm sym}(x) = \exp((\nu+2b)\cdot x)$ if $\alpha(x) \le \frac{c}{h_\nu(x)} = c\exp(-\nu\cdot x)$. What is the mean measure of $\zeta_\infty^{(h_\nu m_{\rm sym})}$ if $\alpha(x) = \frac{c}{h_\eta}(x)$ with $\eta \in S$ and $\eta \ne \nu$? The answer is given by the following theorem.

THEOREM 7. *Use the notation in the above example and let $\nu \in S$.*

(i) *If $\eta \in S - \{-\nu - 2b\}$, then $\zeta_\infty^{(h_\nu m_{\rm sym})}$ has mean measure $h_\nu m_{\rm sym}$ whenever $\alpha \le \frac{c}{h_\eta}$.*

(ii) *If $\eta = -\nu - 2b$ and $d = 1$ or 2, then $\zeta_\infty^{(h_\nu m_{\rm sym})} = \delta_0$ whenever $\alpha \ge \frac{c}{h_\eta}$.*

(iii) *If $\eta = -\nu - 2b$ and $d \ge 3$, then $\zeta_\infty^{(h_\nu m_{\rm sym})}$ has mean measure $h_\nu m_{\rm sym}$ whenever $\alpha \le \frac{c}{h_\eta}$.*

(iv) *If $\eta = -\nu - 2b - s(\nu+b)$ for $s > 0$ (in which case $\eta \notin S$), then $\zeta_\infty^{(h_\nu m_{\rm sym})} = \delta_0$ whenever $\alpha \ge \frac{c}{h_\eta}$.*

Having completed the presentation of the results, we now point out in a series of three remarks a number of significant facts that follow from Proposition 2.

REMARK 8. Recall the definition from criticality theory, Definition 1 above, in the paragraph following (1.4). It follows from Proposition 2(i) that if L is supercritical or if L is subcritical and none of the functions in $C_{\tilde L}(D)$ are invariant, then the measure-valued diffusion does not possess an invariant distribution with σ-finite mean measure, regardless of the variance parameter $\alpha > 0$. [In the critical case, $C_{\tilde L}(D)$ is one-dimensional and its unique element (up to constant multiples) is necessarily invariant ([12], Theorem 4.8.6).]

In particular, for example, if L_0 on D corresponds to a recurrent diffusion and $\beta \gneq 0$, then L on D is supercritical ([12], Theorem 4.6.3). As an example of subcriticality with no invariant measures, consider $L_0 = \frac{1}{2}\frac{d^2}{dx^2}$ on $D = (0, 1)$ with $\beta = 0$. Then $C_{\tilde L}(D)$ is generated by the functions x and $1 - x$ and no element of $C_{\tilde L}(D)$ is invariant ([12], page 217). The same thing is true for the multidimensional analog, namely, Brownian motion in the unit ball, in which case $C_{\tilde L}(D)$ is generated by the Poisson kernel functions $u_\rho(x) = \frac{1-|x|^2}{|\rho - x|^d}$, indexed by $\rho \in S^{d-1}$.

REMARK 9. The measure-valued process is said to exhibit *local extinction* if for every compact set B contained in D, there exists an almost surely finite random time τ_B such that $P(X(t, B) = 0 \text{ for all } t > \tau_B) = 1$. In [13, 8], it was shown that starting from a finite measure, the measure-valued diffusion exhibits local extinction if and only if L is not supercritical on D. Recall from Defintion 1 that L is supercritical on D if and only if $C_{\tilde L}(D)$ is empty. Thus, if L is supercritical on D, then $C_{\tilde L}^{\rm inv}(D)$ is empty. Combining this with Proposition 2(i), it follows that *the exhibition of local extinction when starting from a finite measure is a necessary condition for the existence of an invariant* *distribution with σ-finite mean measure.* In particular, it then follows that the support of an invariant distribution does not include finite measures. As an application of this, we point out that if L_0 corresponds to a positive recurrent diffusion and $\beta = 0$, then the measure-valued diffusion cannot possess an invariant distribution with σ-finite mean measure. Indeed, for positive recurrent diffusions there is an invariant probability measure, and every invariant measure is a constant multiple of this probability measure. Thus, it follows from Proposition 2(i) that if ν were an invariant distribution with σ-finite mean measure, then its mean measure would have to be a multiple of the invariant probability measure for the positive recurrent diffusion. From this it would follow that the support of ν is contained in the space of finite measures.

REMARK 10. From the log-Laplace equation (1.1), it follows easily that

$$\nu = \int_{C_{\tilde{L}}^{\mathrm{inv}}(D)} \zeta_\infty^{(\mu)} Q(d\mu) \tag{1.20}$$

is an invariant distribution for any probability measure Q on $C_{\tilde{L}}^{\mathrm{inv}}(D)$. In particular, it follows that if the collection of invariant distributions in Corollary 1 contains a nonzero element, then there exist invariant distributions whose mean measures are infinite. Indeed, if $c_\rho(\alpha) > 0$, then choosing a probability distribution F on $(0, \infty)$ with infinite expectation, it follows from Theorem 3 that $\nu \equiv \int_0^\infty \zeta_\infty^{(\theta\mu_\rho)} F(d\theta)$ has mean measure $\pi_\nu = (c_\rho(\alpha) \int_0^\infty \theta F(d\theta))\mu_\rho = \infty$.

We now provide a little intuition to explain why for each $\mu \in C_{\tilde{L}}^{\mathrm{inv}}(D)$, the veracity of the equality $\pi_{\zeta_\infty^{(\mu)}} = \mu$ depends monotonically on α. The proof of Proposition 2(i) in the next section [see (2.7)] shows that if $X(0) = \mu$ a.s., that is, the initial distribution is given by $\zeta_0 = \delta_\mu$, then for all $t \in [0, \infty)$, the mean measure for the distribution ζ_t of $X(t)$ is equal to μ. Equivalently, for all $f \in C_c^+(D)$,

$$\int_{\mathscr{M}(D)} \langle f, \eta \rangle \, d\zeta_t(\eta) = \langle f, \mu \rangle \quad \text{for all } t \in [0, \infty). \tag{1.21}$$

However, since the functional $\eta \to \langle f, \eta \rangle$ is unbounded, the weak convergence of ζ_t to $\zeta_\infty^{(\mu)}$ does not guarantee that the mean measure of $\zeta_\infty^{(\mu)}$ will also be equal to μ. Defining cutoff functions Ψ_N: $[0, \infty) \to [0, \infty)$ by $\Psi_N(x) = x$, $x \in [0, N]$, $\Psi_N(x) = N$, $x > N$, the functionals $\eta \to \Psi_N(\langle f, \eta \rangle)$ are bounded and continuous on $\mathscr{M}(D)$; thus

$$\begin{aligned} &\lim_{t \to \infty} \int_{\mathscr{M}(D)} \Psi_N(\langle f, \eta \rangle) \, d\zeta_t(\eta) \\ &\quad = \int_{\mathscr{M}(D)} \Psi_N(\langle f, \eta \rangle) \, d\zeta_\infty^{(\mu)}(\eta), \qquad N = 1, 2, \ldots. \end{aligned} \tag{1.22}$$

From (1.21) and (1.22) it follows that if the mean measure of $\zeta_\infty^{(\mu)}$ is not equal to μ, then for some $f \in C_c^+(D)$, one has

$$\lim_{N\to\infty} \limsup_{t\to\infty} \int_{\{\eta:\ \langle f,\eta\rangle > N\}} \langle f, \eta\rangle \zeta_t(\eta) > 0. \tag{1.23}$$

The phenomenon described by (1.23) is known as *clustering*. By (1.21), mass must be conserved on the average for every finite time t, but as $t \to \infty$ this conservation may be achieved by clustering. Clustered mass is lost in the weak limit; thus if clustering occurs, then the invariant distribution $\zeta_\infty^{(\mu)}$ will have mean measure less than μ. Now in the course of the proof of Theorem 1, it is shown that the variance of $\langle f, \eta\rangle$ under ζ_t is given by

$$\int_{\mathscr{M}(D)} \langle f, \eta\rangle^2 \, d\zeta_t(\eta) - \left(\int_{\mathscr{M}(D)} \langle f, \eta\rangle \, d\zeta_t\right)^2 = 2\int_0^t \langle \alpha(T_s f)^2, \mu\rangle \, ds.$$

[Use (2.34) and (1.21).] We see then that this variance is monotonic in α. If the variance grows sufficiently fast in t, then the average mass preservation is achieved by clustering, while if it grows sufficiently slowly, then the average mass preservation is achieved without clustering.

We conclude this section with a number of interesting open problems suggested by the above results.

PROBLEM 1. The construction in Proposition 2(ii) produces the collection $\{\zeta_\infty^{(\mu)}: \mu \in C_{\tilde{L}}^{\mathrm{inv}}(D)\}$ of invariant distributions which, by Corollary 1, is in one-to-one correspondence with the cone $C_{\tilde{L}}^{\mathrm{inv},\alpha}(D)$. This collection of invariant distributions can be expanded to invariant distributions of the form (1.20). Show that every invariant distribution for the measure-valued process is of the form (1.20).

REMARK 11. In Remarks 7 and 8 it was noted that for any $\alpha > 0$, there is no invariant distribution with σ-finite mean in either of the following two cases:

(i) $C_{\tilde{L}}^{\mathrm{inv}}(D)$ is empty, which occurs in particular if L is supercritical.

(ii) The average mass creation parameter $\beta = 0$ and L_0 corresponds to a positive recurrent diffusion. Thus, it would follow from Problem 1 that no invariant distribution, even with infinite mean measure, exists in the above two cases.

PROBLEM 2. Show that if μ_ρ is a minimal invariant density, then $\pi_{\zeta_\infty^{(\mu_\rho)}} = 0$ or μ_ρ. Equivalently, in the notation of Theorem 3, show that $c_\rho(\alpha) = 0$ or 1.

PROBLEM 3. Is it true that for certain classes of measure-valued diffusions, every invariant measure must necessarily have a finite second moment operator? If such were the case, then by Theorem 1, it would follow that for those classes of measure-valued diffusions, (1.7) with μ replaced by μ_ρ would be a necessary and sufficient condition for $\zeta_\infty^{(\mu_\rho)}$ to be nontrivial, and in the nontrivial case, the mean measure would always be μ_ρ. This would also show that $c_\rho(\alpha) = 0$ or 1 for those classes of measure-valued diffusions. For example, if it turned out that an invariant measure for critical super-Brownian motion with a space dependent variance parameter (i.e., the case treated in Theorem 4) must have a finite second moment operator, then (1.11) would turn out to be the necessary and sufficient condition for $\zeta_\infty^{(l)}$ to be a nontrivial invariant distribution. Indeed, the proof of Theorem 4 shows that (1.11) is a necessary and sufficient condition for (1.7) to hold with μ replaced by Lebesgue measure.

PROBLEM 4. Show that the result of Theorem 6 holds for general one-dimensional, subcritical operators L for which $C_{\tilde{L}}^{\mathrm{inv}}(D)$ is two-dimensional.

PROBLEM 5. In the case that L is critical, $C_{\tilde{L}}^{\mathrm{inv}}(D) = C_{\tilde{L}}(D)$ and is one-dimensional. If in addition, L is symmetric with respect to the measure m_{sym}, then the unique invariant density (up to constant multiples) will be of the form hm_{sym} where h is the unique function (up to constant multiples) in $C_L(D)$. In contrast to Theorem 5 for subcritical operators, show that if $\alpha \geq \frac{c}{h}$, then $\zeta_\infty^{(hm_{\mathrm{sym}})} = \delta_0$.

REMARK 12. In the case $\beta = 0$ (in which case $h = 1$) and α bounded between two positive constants, Problem 5 has been carried out in [9] under certain assumptions on the semigroup T_t. In that paper, the author treats the case that the underlying diffusion corresponding to L_0 on D is null recurrent. (As Remark 9 shows, for $\beta = 0$ and any $\alpha > 0$, there will never be a nontrivial invariant distribution with σ-finite mean measure if L_0 is positive recurrent.) Without assuming that L_0 is symmetric, but under three regularity conditions on the semigroup T_t, the author proves that if $\zeta_0 = \delta_\mu$, where μ is the unique (up to constant multiples) invariant σ-finite measure for the diffusion corresponding to L_0 on D, then $\zeta_\infty^{(\mu)} \equiv w - \lim \zeta_t = \delta_0$. The author, citing [3], claims that her regularity conditions on the semigroup hold whenever L_0 is uniformly elliptic with bounded, continuous coefficients. In fact, as will be discussed momentarily, considerably more stringent restrictions must be placed on the coefficients to guarantee that her conditions hold. We should also point out that in the introduction to the same article, the author claims that in the case that $\beta = 0$, α is constant, and L_0 is transient, it follows from [6] that there exists an invariant distribution for the measure-valued process. However, the result on the existence of invariant distributions in [6] is not proved in complete generality in the transient case, but rather under the assumption that there exists an invariant measure μ for the semigroup T_t, that (1.6) holds and that the so-called *F-property* holds for entrance laws at $-\infty$. The existence of an invariant distribution in the transient case is probably not true in complete generality. Indeed, as was shown in Remark 8, if $L_0 = \frac{1}{2}\frac{d^2}{dx^2}$ on $D = (0,1)$ with $\beta = 0$, then L_0 corresponds to a transient diffusion and the cone $C_{\tilde{L}}^{\mathrm{inv}}(D)\}$ is empty; thus the recipe of Proposition 2 does not produce an invariant distribution for any $\alpha > 0$. We expect that there are in fact no invariant distributions in this case. (This would follow from Problem 1.) The question of whether there will always be an invariant distribution in the transient case under the assumption that there does exist an invariant measure and with α constant and $\beta = 0$, but without the additional assumption (1.6), seems to be open.

Returning to the three regularity conditions imposed on the semigroup T_t in [9], the least complicated of these conditions is that $\lim_{t\to\infty} \frac{1}{h(t)} \int_0^t T_s\phi\, ds = \int_D \phi\, d\mu$, for $\phi \in C_c^+(D)$, where $h(t) = t^\rho l(t)$ for some $\rho \in [0, 1]$ and $l(t)$ is slowly varying. Since [3] only gives *upper* bounds on T_t for *symmetric* operators, the above condition on T_t cannot follow from those results. It is not hard to see, as has been pointed out to me by S. R. S. Varadhan, that if $\{\gamma_n\}_{n=1}^{\infty}$ is a sufficiently rapidly increasing sequence of positive numbers and $a(x) = 1$, for $\gamma_{2n-1} + 1 \le |x| \le \gamma_{2n}$ and $a(x) = 2$ for $\gamma_{2n} + 1 \le |x| \le \gamma_{2n+1}$, and $L = \frac{1}{2} a \frac{d^2}{dx^2}$ or $L = \frac{1}{2} \frac{d}{dx}(a \frac{d}{dx})$ on $(-\infty, \infty)$, then T_t will not satisfy the above condition.

The rest of this paper is organized as follows. In Section 2, we prove Propositions 1 and 2 and Theorems 1 and 2; in Section 3, we prove Theorems 3 and 5; in Section 4 we prove Theorems 4 and 7; and in Section 5 we prove Theorem 6.

From now on, we will denote the probability measure and the corresponding expectation for the measure-valued process starting with initial distribution $\zeta_0 = \nu \in \mathscr{M}_1(\mathscr{M}(D))$ by P_ν and E_ν, respectively.

2. Proofs of Propositions 1 and 2 and Theorems 1 and 2.

PROOF OF PROPOSITION 1. In [8], we proved the existence of a finite measure-valued process satisfying (1.1) when ν is a finite measure. From (1.1), it follows that if ν_1 and ν_2 are finite measures, then the measure-valued process starting from $\nu_1 + \nu_2$ may be obtained as the independent sum of measure-valued processes starting from ν_1 and from ν_2. Since a σ-finite measure ν may always be expressed in the form $\nu = \sum_{i=1}^{\infty} \nu_i$, where the ν_i are finite measures, a measure-valued process starting from a σ-finite measure ν and satisfying (1.1) can always be constructed as an independent sum of finite measure-valued processes. Using the conditional expectation (1.1) then allows for the construction of a measure-valued process starting from any initial distribution $\zeta_0 \in \mathscr{M}_1(\mathscr{M}(D))$. Since $P_\nu(\langle f, X(t)\rangle < \infty) = \lim_{\lambda\to 0} E_\nu \exp(-\langle \lambda f, X(t)\rangle)$, it follows from (1.1) that $X(t)$ will almost surely take values in $\mathscr{M}(D)$ if and only if (1.3b) holds for all $f \in C_0^+(D)$. Clearly, (1.3a) is necessary for (1.3b) to hold. By the maximum principle, $u_{\lambda f}$ is increasing in λ and $u_{\lambda f} \le \lambda T_t f$. Thus, $\lim_{\lambda\to 0} u_{\lambda f}(\cdot, t) = 0$, and it follows from the dominated convergence theorem that (1.3a) is sufficient for (1.3b) to hold. The final statement of the proposition follows from the inequality $u_{\lambda f} \le \lambda T_t f$. $\square$

PROOF OF PROPOSITION 2. (i) Let $\nu \in \mathscr{M}_1(\mathscr{M}(D))$ be any initial distribution satisfying (1.3). From (1.1), we have for $\lambda > 0$ and $f \in C_c^+(D)$,

$$E_\nu \exp\bigl(-\langle \lambda f, X(t)\rangle\bigr) = \int_{\mathscr{M}(D)} \exp\bigl(-\langle u_{\lambda f}(\cdot, t), \eta\rangle\bigr)\, d\nu(\eta). \tag{2.1}$$

Consider first the left-hand side of (2.1). Since $X(t)$ is almost surely σ-finite, we have by the mean value theorem and the dominated convergence theorem that

$$\frac{1}{\lambda}\bigl(1 - \exp(-\langle \lambda f, X(t)\rangle)\bigr) = M\bigl(\lambda, X(t)\bigr)\langle f, X(t)\rangle,$$

where $0 < M(\lambda, X(t)) < 1$ and $\lim_{\lambda\to 0} M(\lambda, X(t)) = 1$ a.s. P_ν. It then follows that

$$\lim_{\lambda\to 0} \frac{1}{\lambda}\bigl(1 - E_\nu \exp(-\langle \lambda f, X(t)\rangle)\bigr) = \langle f, \pi_{\zeta_t}\rangle, \tag{2.2}$$

where the right-hand side of (2.2) may be infinite.

Consider now the right-hand side of (2.1). By Taylor's theorem, $1 - \exp(-x) = N(x)x$, where $\lim_{x\to 0} N(x) = 1$. Thus, it follows from (1.3) that

$$\frac{1}{\lambda}\bigl(1 - \exp(-\langle u_{\lambda f}(\cdot, t), \eta\rangle)\bigr) = N(\lambda, \eta)\Bigl\langle \frac{u_{\lambda f}(\cdot, t)}{\lambda}, \eta\Bigr\rangle, \tag{2.3}$$

where $0 < N(\lambda, \eta) < 1$ and $\lim_{\lambda\to 0} N(\lambda, \eta) = 1$ a.s. P_ν. The function $v_\lambda \equiv \frac{u_{\lambda f}}{\lambda}$ satisfies the equation

$$\begin{aligned} v_t &= L_0 v + \beta v - \lambda\alpha v^2 \quad \text{in } D \times (0, \infty), \\ v(x, 0) &= f(x) \quad \text{in } D. \end{aligned} \tag{2.4}$$

By the maximum principle, v_λ is decreasing in λ. We will show below that

$$\lim_{\lambda\to 0} v_\lambda(x, t) = T_t f(x). \tag{2.5}$$

Thus, $N(\lambda, \eta)\langle \frac{u_{\lambda f}}{\lambda}(\cdot, t), \eta\rangle \le \langle T_t f, \eta\rangle$ and $\lim_{\lambda\to 0} N(\lambda, \eta)\langle \frac{u_{\lambda f}}{\lambda}(\cdot, t), \eta\rangle = \langle T_t f, \eta\rangle$. From Fatou's lemma it then follows that

$$\lim_{\lambda\to 0} \frac{1}{\lambda}\Bigl(1 - \int_{\mathscr{M}(D)} \exp\bigl(-\langle u_{\lambda f}(\cdot, t), \eta\rangle\bigr)\, d\nu(\eta)\Bigr) = \langle T_t f, \pi_\nu\rangle, \tag{2.6}$$

where the right-hand side of (2.6) may be infinite. From (2.1), (2.2) and (2.6) we obtain

$$\langle f, \pi_{\zeta_t}\rangle = \langle T_t f, \pi_\nu\rangle. \tag{2.7}$$

If ν is an invariant distribution with σ-finite mean measure, then $\pi_{\zeta_t} = \pi_{\zeta_0} = \pi_\nu$ and $\langle f, \pi_\nu\rangle < \infty$; thus we obtain from (2.7) that $\langle f, \pi_\nu\rangle = \langle T_t f, \pi_\nu\rangle < \infty$, proving that π_ν is invariant for the semigroup T_t.

We now return to prove (2.5). By the Feynman–Kac formula we have

$$v_\lambda(x, t) = \mathscr{E}_x\biggl(\exp\biggl(\int_0^t (\beta - \lambda\alpha v_\lambda)(Y(r), t - r)\, dr\biggr) f(Y(t)); \tau_D > t\biggr).$$

By the maximum principle, $v_\lambda(x, t) \le T_t f(x)$. Thus,

$$\begin{aligned}(2.8)\qquad &\mathscr{E}_x\bigg(\exp\bigg(\int_0^t (\beta - \lambda\alpha T_{t-r} f)(Y(r))\,dr\bigg) f(Y(t)); \tau_D > t\bigg)\\ &\quad \le v_\lambda(x, t) \le T_t f(x).\end{aligned}$$

Letting $\lambda \to 0$ in (2.8), and using the dominated convergence theorem and the Feynman–Kac representation for T_t, we obtain (2.5).

(ii) By standard results on weak convergence, ζ_t will converge weakly if and only if $\lim_{t\to\infty} \int_{\mathscr{M}(D)} \exp(-\langle f, \eta\rangle)\,d\zeta_t(\eta)$ exists for all $f \in C_c^+(D)$.

Note that the solution u_f to the semilinear evolution equation (1.2) satisfies the following integral equation:

$$(2.9)\qquad u_f(x, t) = T_t f(x) - \int_0^t T_{t-s}(\alpha u_f^2(\cdot, s))(x)\,ds.$$

First let $\zeta_0 = \delta_\mu$. From (1.1), (2.9), and the fact that μ is invariant, we have

$$\begin{aligned}&\int_{\mathscr{M}(D)} \exp(-\langle f, \eta\rangle)\,d\zeta_t(\eta)\\ &\qquad = E_{\delta_\mu} \exp(-\langle f, X(t)\rangle) = \exp(-\langle u_f(\cdot, t), \mu\rangle)\\ &\qquad = \exp\bigg(-\langle T_t f, \mu\rangle + \int_0^t \langle T_{t-s}(\alpha u_f^2(\cdot, s)), \mu\rangle\,ds\bigg)\\ &\qquad = \exp\bigg(-\langle f, \mu\rangle + \int_0^t \langle \alpha u_f^2(\cdot, s), \mu\rangle\,ds\bigg).\end{aligned}$$

It then follows that

$$\lim_{t\to\infty} \int_{\mathscr{M}(D)} \exp\big(-\langle f, \eta\rangle\big)\,d\zeta_t(\eta) = \exp\bigg(-\langle f, \mu\rangle + \int_0^\infty \langle \alpha u_f^2(\cdot, t), \mu\rangle\,dt\bigg).$$

This proves the weak convergence and gives (1.5).

To show that $\zeta_\infty^{(\mu)}$ is an invariant distribution, we use the Markov property to obtain

$$(2.10)\qquad \int_{\mathscr{M}(D)} \exp\big(-\langle f, \eta\rangle\big)\,d\zeta_{t+s}(\eta) = \int_{\mathscr{M}(D)} E_\eta \exp(-\langle f, X(s)\rangle)\,d\zeta_t(\eta).$$

The measure-valued process $X(t)$ is Feller; thus $E_\eta \exp(-\langle f, X(s)\rangle)$ is continuous in η. Letting $t \to \infty$ in (2.10) gives

$$\int_{\mathscr{M}(D)} \exp(-\langle f, \eta\rangle)\,d\zeta_\infty^{(\mu)}(\eta) = \int_{\mathscr{M}(D)} E_\eta \exp(-\langle f, X(s)\rangle)\,d\zeta_\infty^{(\mu)}(\eta),$$

which shows that $\zeta_\infty^{(\mu)}$ is invariant.

Now let $\zeta_0 = \mathrm{Poiss}_\mu$ and assume that $\lim_{t\to\infty} \sup_{x\in D} T_t f(x) = 0$. Then

$$\begin{aligned}E_{\mathrm{Poiss}_\mu} \exp(-\langle f, X(t)\rangle) &= \int_{\mathscr{M}(D)} \exp(-\langle u_f(\cdot, t), \eta\rangle)\,d\mathrm{Poiss}_\mu(\eta)\\ &= \exp\big(-\langle 1 - \exp(-u_f(\cdot, t)), \mu\rangle\big).\end{aligned}$$

Since $u_f \le T_t f$, it follows that

$$\begin{aligned} &\left|\langle 1-\exp(-u_f(\cdot,t)),\mu\rangle - \langle u_f(\cdot,t),\mu\rangle\right| \\ &\quad \le \tfrac{1}{2}\langle u_f^2(\cdot,t),\mu\rangle \le \tfrac{1}{2}\sup_{x\in D} T_t f(x)\langle T_t f,\mu\rangle = \tfrac{1}{2}\sup_{x\in D} T_t f(x)\langle f,\mu\rangle. \end{aligned}$$

We conclude that $\lim_{t\to\infty}\langle 1-\exp(-u_f(\cdot,t)),\mu\rangle = \lim_{t\to\infty}\langle u_f(\cdot,t),\mu\rangle$, and the rest of the proof is as in the previous case.

(iii) Replacing f by λf in (1.5), and using the type of argument used in the proof of part (i), one finds that $\langle f, \pi_{\zeta_\infty^{(\mu)}}\rangle \le \langle f,\mu\rangle$ and that equality holds if and only if (1.6) holds.

We now show that (1.7) implies (1.6). Since $u_{\lambda f} \le \lambda T_t f$, we have

$$\alpha\frac{u_{\lambda f}^2}{\lambda} \le \lambda\alpha(T_t f)^2. \tag{2.11}$$

From (2.11) and two applications of the dominated convergence theorem, it follows that (1.7) implies (1.6).

(iv) From (1.1) and part (ii), it follows that

$$\int_{\mathscr{M}(D)} \exp\bigl(-\langle f,\eta\rangle\bigr)\,d\zeta_\infty^{(\mu)}(\eta) = \lim_{t\to\infty}\exp\bigl(-\langle u_f(\cdot,t),\mu\rangle\bigr).$$

Part (iv) is now immediate. $\square$

PROOF OF THEOREM 1. Assume first that $\int_0^\infty\langle\alpha(T_t f)^2,\mu\rangle\,dt < \infty$. Using (1.1) and the fact that $\zeta_\infty^{(\mu)}$ is an invariant distribution, we have for any $0 \le f \in C_c(D)$, $\lambda \ge 0$, and $t > 0$,

$$\begin{aligned} \int_{\mathscr{M}(D)} \exp(-\langle\lambda f,\eta\rangle)\,d\zeta_\infty^{(\mu)}(\eta) &= \int_{\mathscr{M}(D)} E_\eta\exp\bigl(-\langle\lambda f, X(t)\rangle\bigr)\,d\zeta_\infty^{(\mu)}(\eta) \\ &= \int_{\mathscr{M}(D)} \exp\bigl(-\langle u_{\lambda f}(\cdot,t),\eta\rangle\bigr)\,d\zeta_\infty^{(\mu)}(\eta). \end{aligned} \tag{2.12}$$

Substituting λf for f in (1.2), formally differentiating twice and using the notation $u = u(x,t;\lambda) \equiv u_{\lambda f}(x,t)$, we obtain the following differential equations for $u^{(1)}(x,t;\lambda) \equiv \frac{\partial u_{\lambda f}}{\partial\lambda}(x,t,\lambda)$ and for $u^{(2)}(x,t;\lambda) \equiv \frac{\partial^2 u_{\lambda f}}{\partial\lambda^2}(x,t,\lambda)$:

$$\begin{aligned} u_t^{(1)} &= L_0 u^{(1)} + \beta u^{(1)} - 2\alpha u u^{(1)} \quad \text{in } D\times(0,\infty) \\ u^{(1)}(x,0) &= f(x) \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} u_t^{(2)} &= L_0 u^{(2)} + \beta u^{(2)} - 2\alpha(u^{(1)})_1^2 - 2\alpha u u^{(2)} \text{ in } D\times(0,\infty), \\ u^{(2)}(x,0) &= 0. \end{aligned} \tag{2.14}$$

In particular, letting $p(t, x, y)$ denote the kernel for the semigroup T_t, we have

$$(2.15)\qquad \begin{aligned} &u^{(1)}(x, t; 0) = T_t f(x) \quad \text{and} \\ &u^{(2)}(x, t; 0) = -2\int_0^t ds \int_D p(t-s, x, y)\alpha(y)(T_s f)^2(y)\, dy. \end{aligned}$$

We will justify these formal differentiations upon completion of the proof of the theorem.

By the mean value theorem, we have for $\lambda \geq 0$, $\lambda_1 > 0$ and $\lambda \neq \lambda_1$,

$$\begin{aligned} &\int_{\mathscr{M}(D)} \frac{1}{\lambda_1 - \lambda}\big(\exp(-\langle \lambda_1 f, \eta\rangle) - \exp(-\langle \lambda f, \eta\rangle)\big)\, d\zeta_\infty^{(\mu)}(\eta) \\ &\qquad = \int_{\mathscr{M}(D)} \langle f, \eta\rangle \exp\big(-\lambda^*(\eta)\langle f, \eta\rangle\big)\, d\zeta_\infty^{(\mu)}(\eta), \end{aligned}$$

where $\lambda^*(\eta)$ is between λ and λ_1. Using the bounded convergence theorem in the case $\lambda > 0$ and Fatou's lemma in the case $\lambda = 0$, we conclude that

$$(2.16)\qquad \begin{aligned} &\lim_{\lambda_1 \to \lambda} \int_{\mathscr{M}(D)} \frac{1}{\lambda_1 - \lambda}\big(\exp(-\langle \lambda_1 f, \eta\rangle) - \exp\big(-\langle \lambda f, \eta\rangle\big)\big)\, d\zeta_\infty^{(\mu)}(\eta) \\ &\qquad = \int_{\mathscr{M}(D)} \langle f, \eta\rangle \exp(-\langle \lambda f, \eta\rangle)\, d\zeta_\infty^{(\mu)}(\eta). \end{aligned}$$

A proof similar to that of Proposition 2(i) shows that

$$(2.17)\qquad \begin{aligned} &\lim_{\lambda_1 \to \lambda} \int_{\mathscr{M}(D)} \frac{1}{\lambda - \lambda_1}\big(\exp(-\langle u(\cdot, t, \lambda), \eta\rangle) - \exp(-\langle u(\cdot, t, \lambda_1), \eta\rangle)\big)\, d\zeta_\infty^{(\mu)}(\eta) \\ &\qquad = \int_{\mathscr{M}(D)} \langle u^{(1)}(\cdot, t, \lambda), \eta\rangle \exp(-\langle u(\cdot, t, \lambda), \eta\rangle)\, d\zeta_\infty^{(\mu)}(\eta). \end{aligned}$$

From (2.12), (2.16) and (2.17) we conclude that

$$(2.18)\qquad \begin{aligned} &\int_{\mathscr{M}(D)} \langle f, \eta\rangle \exp(-\langle \lambda f, \eta\rangle)\, d\zeta_\infty^{(\mu)}(\eta) \\ &\qquad = \textstyle\int_{\mathscr{M}(D)} \langle u^{(1)}(\cdot, t, \lambda), \eta\rangle \exp\big(-\langle u(\cdot, t, \lambda), \eta\rangle\big)\, d\zeta_\infty^{(\mu)}(\eta). \end{aligned}$$

We now calculate the second derivative at $\lambda = 0$ for each side of (2.18). Varying the left-hand side of (2.18) and using the mean value theorem, we have for $\lambda > 0$,

$$\begin{aligned} &\frac{1}{\lambda}\int_{\mathscr{M}(D)} \langle f, \eta\rangle (1 - \exp(-\langle \lambda f, \eta\rangle))\, d\zeta_\infty^{(\mu)}(\eta) \\ &\qquad = \int_{\mathscr{M}(D)} \langle f, \eta\rangle^2 \exp(-\lambda(\eta)\langle f, \eta\rangle)\, d\zeta_\infty^{(\mu)}(\eta), \end{aligned}$$

where $\lambda(\eta) \in (0, \lambda)$. Letting $\lambda \to 0$ and using Fatou's lemma shows that

$$(2.19)\qquad \begin{aligned} &\lim_{\lambda \to 0} \frac{1}{\lambda}\int_{\mathscr{M}(D)} \langle f, \eta\rangle \big(1 - \exp(-\langle \lambda f, \eta\rangle)\big)\, d\zeta_\infty^{(\mu)}(\eta) \\ &\qquad = \int_{\mathscr{M}(D)} \langle f, \eta\rangle^2\, d\zeta_\infty^{(\mu)}(\eta). \end{aligned}$$

We now vary the right-hand side of (2.18). We write

$$
\begin{aligned}
&\langle u^{(1)}(\cdot, t, 0), \eta\rangle - \langle u^{(1)}(\cdot, t, \lambda), \eta\rangle \exp\bigl(-\langle u(\cdot, t, \lambda), \eta\rangle\bigr) \\
&\quad = \langle u^{(1)}(\cdot, t, 0), \eta\rangle - \langle u^{(1)}(\cdot, t, \lambda), \eta\rangle + \langle u^{(1)}(\cdot, t, \lambda), \eta\rangle \\
&\qquad \times \bigl(1 - \exp(-\langle u(\cdot, t, \lambda), \eta\rangle)\bigr).
\end{aligned}
\tag{2.20}
$$

Recalling that $u(\cdot, t, 0) \equiv 0$, letting $\lambda > 0$ and using (2.20), we obtain

$$
\begin{aligned}
&\frac{1}{\lambda}\biggl(\int_{\mathscr{M}(D)} \langle u^{(1)}(\cdot, t, 0), \eta\rangle \, d\zeta_\infty^{(\mu)}(\eta) \\
&\quad - \int_{\mathscr{M}(D)} \langle u^{(1)}(\cdot, t, \lambda), \eta\rangle \exp\bigl(-\langle u(\cdot, t, \lambda), \eta\rangle\bigr) \, d\zeta_\infty^{(\mu)}(\eta)\biggr) \\
&\quad \geq \int_{\mathscr{M}(D)} \biggl\langle \frac{u^{(1)}(\cdot, t, 0) - u^{(1)}(\cdot, t, \lambda)}{\lambda}, \eta \biggr\rangle d\zeta_\infty^{(\mu)}(\eta) \\
&\qquad + \int_{\mathscr{M}(D)} \langle u^{(1)}(\cdot, t, \lambda), \eta\rangle \biggl(\frac{1 - \exp\bigl(-\langle u(\cdot, t, \lambda), \eta\rangle\bigr)}{\lambda}\biggr) d\zeta_\infty^{\mu}(\eta).
\end{aligned}
\tag{2.21}
$$

By (2.13) and the maximum principle, $u^{(1)}(\cdot, t, 0) \geq u^{(1)}(\cdot, t, \lambda) \geq 0$. Thus, it follows from Fatou's lemma that

$$
-\langle u^{(2)}(\cdot, t, 0), \eta\rangle \leq \liminf_{\lambda \to 0} \biggl\langle \frac{u^{(1)}(\cdot, t, 0) - u^{(1)}(\cdot, t, \lambda)}{\lambda}, \eta \biggr\rangle.
$$

The proof of Proposition 2(i) showed that

$$
\lim_{\lambda \to 0} \frac{1 - \exp(-\langle u(\cdot, t, \lambda), \eta\rangle)}{\lambda} = \langle T_t f, \eta\rangle,
$$

and (2.15), Fatou's lemma and the fact that $u^{(1)}(\cdot, t, 0) \geq u^{(1)}(\cdot, t, \lambda)$ give $\lim_{\lambda \to 0} \langle u^{(1)}(\cdot, t, \lambda), \eta\rangle = \langle T_t f, \eta\rangle$. Using these facts along with (2.21) and applying Fatou's lemma again gives

$$
\begin{aligned}
&\liminf_{\lambda \to 0} \frac{1}{\lambda}\biggl(\int_{\mathscr{M}(D)} \langle u^{(1)}(\cdot, t, 0), \eta\rangle \, d\zeta_\infty^{(\mu)}(\eta) \\
&\qquad - \int_{\mathscr{M}(D)} \langle u^{(1)}(\cdot, t, \lambda), \eta\rangle \exp\bigl(-\langle u(\cdot, t, \lambda), \eta\rangle\bigr) \, d\zeta_\infty^{(\mu)}(\eta)\biggr) \\
&\quad \geq -\int_{\mathscr{M}(D)} \langle u^{(2)}(\cdot, t, 0), \eta\rangle \, d\zeta_\infty^{(\mu)}(\eta) + \int_{\mathscr{M}(D)} \langle T_t f, \eta\rangle^2 \, d\zeta_\infty^{(\mu)}(\eta).
\end{aligned}
\tag{2.22}
$$

By Proposition 2(iii) and the assumption that $\int_0^\infty \langle \alpha (T_t f)^2, \mu\rangle \, dt < \infty$, if follows that $\pi_{\zeta_\infty^{(\mu)}} = \mu$. Using this along with the invariance of μ for the semigroup T_t, it follows from (2.15) that

$$
-\int_{\mathscr{M}(D)} \bigl\langle u^{(2)}(\cdot, t, 0), \eta\bigr\rangle \, d\zeta_\infty^{(\mu)}(\eta) = 2\int_0^t \langle \alpha (T_s f)^2, \mu\rangle \, ds.
\tag{2.23}
$$

From (2.18), (2.19), (2.22) and (2.23), we have

$$\begin{aligned}(2.24)\qquad &\int_{\mathscr{M}(D)} \langle f, \eta\rangle^2 \, d\zeta_\infty^{(\mu)}(\eta) \\ &\quad \geq 2\int_0^t \langle \alpha(T_s f)^2, \mu\rangle \, ds + \int_{\mathscr{M}(D)} \langle T_t f, \eta\rangle^2 \, d\zeta_\infty^{(\mu)}(\eta).\end{aligned}$$

By Jensen's inequality and the invariance of μ,

$$(2.25)\qquad \int_{\mathscr{M}(D)} \langle T_t f, \eta\rangle^2 \, d\zeta_\infty^{(\mu)}(\eta) \geq \left(\int_{\mathscr{M}(D)} \langle T_t f, \eta\rangle \, d\zeta_\infty^{(\mu)}(\eta)\right)^2 = \langle f, \mu\rangle^2.$$

Using (2.25) in (2.24) and letting $t \to \infty$ gives

$$(2.26)\qquad \mathrm{Var}_{\zeta_\infty^{(\mu)}}(f) \geq 2\int_0^\infty \langle \alpha(T_t f)^2, \mu\rangle \, dt.$$

We now prove the reverse inequality. By (1.1), we have $E_{\delta_\mu} \exp(-\langle \lambda f, X(t)\rangle) = \exp(-\langle u(\cdot, t, \lambda), \mu\rangle)$. Calculations very similar to those that led from (2.12) to (2.18) show that

$$(2.27)\qquad E_{\delta_\mu}\langle f, X(t)\rangle \exp(-\langle \lambda f, X(t)\rangle) = \langle u^{(1)}(\cdot, t, \lambda), \mu\rangle \exp\big(-\langle u(\cdot, t, \lambda), \mu\rangle\big).$$

Varying the left-hand side of (2.27), we obtain similar to (2.19),

$$(2.28)\qquad \lim_{\lambda\to 0} \frac{1}{\lambda} E_\mu \langle f, X(t)\rangle (1 - \exp(-\langle \lambda f, X(t)\rangle)) = E_\mu \langle f, X(t)\rangle^2.$$

Varying the right-hand side of (2.27), we have

$$\begin{aligned}(2.29)\qquad &\frac{1}{\lambda}\big(\langle u^{(1)}(\cdot, t, 0), \mu\rangle - \big\langle u^{(1)}(\cdot, t, \lambda), \mu\big\rangle \exp\big(-\langle u(\cdot, t, \lambda), \mu\rangle\big)\big) \\ &\quad = \left\langle \frac{u^{(1)}(\cdot, t, 0) - u^{(1)}(\cdot, t, \lambda)}{\lambda}, \mu \right\rangle \\ &\qquad + u^{(1)}(\cdot, t, \lambda), \mu > \left(\frac{1 - \exp(-\langle u(\cdot, t, \lambda), \mu\rangle)}{\lambda}\right).\end{aligned}$$

From (2.13), it follows that the function $w = w(x, t, \lambda) \equiv \frac{u^{(1)}(\cdot, t, 0) - u^{(1)}(\cdot, t, \lambda)}{\lambda}$ satisfies the equation

$$\begin{aligned}(2.30)\qquad &w_t = L_0 w + \beta w - 2\alpha u(\cdot, \cdot; \lambda) w + 2\alpha \tfrac{u(\cdot, \cdot; \lambda)}{\lambda} u^{(1)}(\cdot, \cdot; \lambda) \\ &w(x, 0, \lambda) = 0.\end{aligned}$$

The function $\frac{u(\cdot, \cdot; \lambda)}{\lambda}$ appearing on the right-hand side of (2.30) satisfies (2.4) and, by the maximum principle, is decreasing in λ. Since $u(\cdot, \cdot, \lambda)$ is increasing in λ, it follows from (2.13) that $u^{(1)}(\cdot, \cdot, \lambda)$ is nonnegative and decreasing in λ. Using all this, we conclude from (2.30) and the maximum principle that $w(\cdot, \cdot, \lambda)$ is nonnegative and decreasing in λ. Thus it follows from the monotone convergence theorem that

$$\lim_{\lambda\to 0} \left\langle \frac{u^{(1)}(\cdot, t, 0) - u^{(1)}(\cdot, t, \lambda)}{\lambda}, \mu \right\rangle = -\big\langle u^{(2)}(\cdot, t, 0), \mu\big\rangle.$$

Using this along with (2.15) and the invariance of μ gives

$$\lim_{\lambda\to 0}\left\langle \frac{u^{(1)}(\cdot,t,0)-u^{(1)}(\cdot,t,\lambda)}{\lambda},\mu\right\rangle = 2\int_0^t \langle \alpha(T_s f)^2,\mu\rangle\,ds. \tag{2.31}$$

The invariance of μ and the explanation appearing in the next to the last sentence above (2.22) give

$$\lim_{\lambda\to 0^+}\langle u^{(1)}(\cdot,t,\lambda),\mu\rangle = \langle T_t f,\mu\rangle = \langle f,\mu\rangle \tag{2.32}$$

and

$$\lim_{\lambda\to 0}\frac{1-\exp\bigl(-\langle u(\cdot,t,\lambda),\mu\rangle\bigr)}{\lambda} = \langle T_t f,\mu\rangle = \langle f,\mu\rangle. \tag{2.33}$$

From (2.27)–(2.29) and (2.31)–(2.33) we obtain

$$E_\mu\langle f,X(t)\rangle^2 = 2\int_0^t \langle \alpha(T_s f)^2,\mu\rangle\,ds + \langle f,\mu\rangle^2. \tag{2.34}$$

The left-hand side of (2.34) can be written as $\int_{\mathscr{M}(D)}\langle f,\eta\rangle^2 d\zeta_t(\eta)$, where ζ_t is the distribution of $X(t)$ under P_{δ_μ}. By Proposition 2(ii), ζ_t converges weakly to $\zeta_\infty^{(\mu)}$. Thus, it follows from (2.34) and a standard property of weak convergence that

$$\int_{\mathscr{M}(D)}\langle f,\eta\rangle^2 d\zeta_\infty^{(\mu)} \le 2\int_0^\infty \langle \alpha(T_t f)^2,\mu\rangle dt + \langle f,\mu\rangle^2. \tag{2.35}$$

Thus,

$$\operatorname{Var}_{\zeta_\infty^{(\mu)}}(f) \le 2\int_0^\infty \langle \alpha(T_t f)^2,\mu\rangle dt. \tag{2.36}$$

The first statement of the theorem now follows from (2.26) and (2.36).

We now assume that $\int_0^\infty \langle \alpha(T_t f)^2,\pi_{\zeta_\infty^{(\mu)}}\rangle dt = \infty$. We follow the above proof up through (2.22). In the line that follows (2.22), it is noted that $\pi_{\zeta_\infty^{(\mu)}} = \mu$. In the present case, noting that by Proposition 2(i), $\pi_{\zeta_\infty^{(\mu)}}$ is invariant for T_t, we can simply replace $\langle \alpha(T_t f)^2,\mu\rangle$ by $\langle \alpha(T_t f)^2,\pi_{\zeta_\infty^{(\mu)}}\rangle$ on the right hand side of (2.23), (2.24) and (2.26) and $\langle f,\mu\rangle$ by $\langle f,\pi_{\zeta_\infty^{(\mu)}}\rangle$ on the right hand side of (2.25). It then follows that $\operatorname{Var}_{\zeta_\infty^{(\mu)}}(f) = \infty$.

We now justify the formal differentiation which led to (2.13) and (2.14). The argument for the proof of (2.13) is similar to that used in the proof of Proposition 2; thus, we will only consider (2.14). Let $0 \le \lambda_0 < \lambda$ and define $w_{\lambda_0} = w_{\lambda_0}(x,t;\lambda) \equiv \frac{u^{(1)}(x,t;\lambda)-u^{(1)}(x,t;\lambda_0)}{\lambda-\lambda_0}$. From (2.13), we obtain

$$(w_{\lambda_0})_t = Lw_{\lambda_0} + \beta w_{\lambda_0} - 2\alpha u(\cdot,\cdot;\lambda)w_{\lambda_0} - 2\alpha\frac{u(\cdot,\cdot;\lambda)-u(\cdot,\cdot;\lambda_0)}{\lambda-\lambda_0}u^{(1)}(\cdot,\cdot;\lambda_0),$$

$$w_{\lambda_0}(x,0,\lambda) = 0.$$

Using the Feynman–Kac formula, we can represent w_{λ_0} as

$$\begin{aligned} w_{\lambda_0}(x,t;\lambda) = -\mathscr{E}_x \int_0^t \biggl(2\alpha(X(s)) &\frac{u(X(s),t-s;\lambda) - u(X(s),t-s;\lambda_0)}{\lambda-\lambda_0} \\ &u^{(1)}(X(s),t-s;\lambda_0)\biggr) \\ \times \exp\biggl(\int_0^s (\beta - 2\alpha u)&(X(r),s-r;\lambda)\,dr\biggr)\,ds. \end{aligned} \tag{2.37}$$

Since $u^{(1)}$ is decreasing in λ, it follows that for some $\lambda^*(x,t) \in (\lambda_0,\lambda)$ we have $0 \le \frac{u(x,t;\lambda)-u(x,t;\lambda_0)}{\lambda-\lambda_0} = u^{(1)}(x,t;\lambda^*(x,t)) \le u^{(1)}(x,t;\lambda_0)$. Using these facts along with Fatou's lemma and the dominated convergence theorem in (2.37) gives

$$\begin{aligned} &\lim_{\lambda\to\lambda_0^+} w_{\lambda_0}(x,t;\lambda) \\ &\quad = -\mathscr{E}_x \int_0^t (2\alpha(u^{(1)})^2)(X(s),t-s;\lambda_0) \\ &\qquad \times \exp\biggl(\int_0^s (\beta-2\alpha u)(X(r),s-r;\lambda_0)\,dr\biggr)\,ds. \end{aligned} \tag{2.38}$$

The corresponding calculation can also be made with $\lambda < \lambda_0$; for this case, one dominates the integrand in (2.37) with λ close to λ_0 by $2\alpha(X(s))(u^{(1)})^2(X(s), t-s;\lambda_1)\exp(\int_0^s(\beta-2\alpha u)(X(r), s-r;\lambda_1)\,dr)$ for some $\lambda_1 < \lambda_0$. Thus, we conclude that (2.38) holds with the left-hand side replaced by $u^{(2)}(x,t;\lambda_0)$, which is just the integrated form of (2.14). □

PROOF OF THEOREM 2. The proof is embedded in the proof of Theorem 1. Indeed, comparing (2.24) and (2.35), recalling that by assumption $\int_0^\infty \langle \alpha(T_t f)^2, \mu\rangle\,dt < \infty$ for $f \in C_c^+(D)$, and letting $t\to\infty$ shows that

$$\limsup_{t\to\infty} \int_{\mathscr{M}(D)} \langle T_t f, \eta\rangle^2 \, d\zeta_\infty^{(\mu)}(\eta) \le \langle f,\mu\rangle^2. \tag{2.39}$$

But then from (2.25) and (2.39) we conclude that

$$\lim_{t\to\infty} \int_{\mathscr{M}(D)} \langle T_t f, \eta\rangle^2 \, d\zeta_\infty^{(\mu)}(\eta) = \langle f,\mu\rangle^2. \tag{2.40}$$

Using the notation established preceding the statement of Theorem 2, we can write the term $\langle T_t f,\eta\rangle$ in (2.40) as $\langle f, T_t^*\eta\rangle$. Since $\int_{\mathscr{M}(D)}\langle f, T_t^*\eta\rangle\,d\zeta_\infty^{(\mu)}(\eta) = \langle f,\mu\rangle$, it follows from (2.40) that

$$\lim_{t\to\infty} \int_{\mathscr{M}(D)} \bigl(\langle f, T_t^*\eta\rangle - \langle f,\mu\rangle\bigr)^2 \, d\zeta_\infty^{(\mu)}(\eta) = 0. \tag{2.41}$$

By the definition of $T_t^*\zeta_\infty^{(\mu)}$, (2.41) can be written as

$$\lim_{t\to\infty} \int_{\mathscr{M}(D)} \bigl(\langle f, \eta\rangle - \langle f,\mu\rangle\bigr)^2 \, dT_t^*\zeta_\infty^{(\mu)}(\eta) = 0. \tag{2.42}$$

Since (2.42) holds for all $f \in C_c^+(D)$, it follows that $w-\lim_{t\to\infty} T_t^* \zeta_\infty^{(\mu)} = \delta_\mu$. $\square$

3. Proofs of Theorems 3 and 5.

PROOF OF THEOREM 3. (i) By Proposition 2(iii), $\pi_{\zeta_\infty^{(\mu_\rho)}} \le \mu_\rho$, and by Proposition 2(i), $\pi_{\zeta_\infty^{(\mu_\rho)}} \in C_{\tilde L}^{\mathrm{inv}}(D)$. Since μ_ρ is minimal in $C_{\tilde L}^{\mathrm{inv}}(D)$, it follows that there exists a $c_\rho \in [0,1]$ such that $\pi_{\zeta_\infty^{(\mu_\rho)}} = c_\rho \mu_\rho$.

We now show that $c_\rho(\alpha)$ is nonincreasing in α. From (1.1) and the definition of $\zeta_\infty^{(\mu_\rho)}$, we have

$$\int_{\mathscr{M}(D)} \exp(-\langle \lambda f, \eta\rangle)\, d\zeta_\infty^{(\mu_\rho)}(\eta) = \lim_{t\to\infty} \exp\big(-\langle u_{\lambda f}(\cdot,t), \mu_\rho\rangle\big). \tag{3.1}$$

Denote the right-hand side of of (3.1) by $H(\lambda,\alpha)$. By the maximum principle, $u_{\lambda f}$ is decreasing in its dependence on α; thus $H(\lambda,\alpha)$ is nondecreasing in its dependence on α. Similar to the proof of Proposition 2(i), we have

$$\langle f, \pi_{\zeta_\infty^{(\mu_\rho)}}\rangle = \lim_{\lambda\to 0} \frac{1-H(\lambda,\alpha)}{\lambda}.$$

Thus, $\pi_{\zeta_\infty^{(\mu_\rho)}}$ is nonincreasing in α. Since $\pi_{\zeta_\infty^{(\mu_\rho)}} = c_\rho(\alpha)\mu_\rho$, it follows that $c_\rho(\alpha)$ is nonincreasing in its dependence on α.

(ii) Let μ satisfy (1.9). From (1.1) and (2.9) we have

$$\begin{aligned}
&E_{\delta_\mu} \exp\big(-\langle f, X(t)\rangle\big)\\
&\quad= \exp\big(-\langle u_f(\cdot,t), \mu\rangle\big)\\
&\quad= \exp\bigg(-\bigg\langle u_f(\cdot,t), \int_{\tilde\Lambda_0^{\mathrm{inv}}} \mu_\rho m_\mu(d\rho)\bigg\rangle\bigg)\\
&\quad= \exp\bigg(-\int_{\tilde\Lambda_0^{\mathrm{inv}}} \langle u_f(\cdot,t), \mu_\rho\rangle m_\mu(d\rho)\bigg)\\
&\quad= \exp\bigg(-\int_{\tilde\Lambda_0^{\mathrm{inv}}} \bigg\langle T_t f - \int_0^t T_{t-s}(\alpha u_f^2(\cdot,s))ds, \mu_\rho\bigg\rangle m_\mu(d\rho)\bigg)\\
&\quad= \exp\bigg(-\int_{\tilde\Lambda_0^{\mathrm{inv}}} \bigg\langle f - \int_0^t \alpha u_f^2(\cdot,s)\,ds, \mu_\rho\bigg\rangle m_\mu(d\rho)).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{\mathscr{M}(D)} \exp(-\langle f, \eta\rangle)\, d\zeta_\infty^{(\mu)}(\eta)\\
&\qquad= \lim_{t\to\infty} E_{\delta_\mu} \exp(-\langle f, X(t)\rangle)\\
&\qquad= \exp\bigg(-\int_{\tilde\Lambda_0^{\mathrm{inv}}} \bigg\langle f - \int_0^\infty \alpha u_f^2(\cdot,s)\,ds, \mu_\rho\bigg\rangle m_\mu(d\rho)).
\end{aligned} \tag{3.2}$$

The proof of Proposition 2(iii) showed that

$$\langle f, \pi_{\mu_\rho}\rangle = \langle f, \mu_\rho\rangle - \lim_{\lambda\to 0}\frac{1}{\lambda}\int_0^\infty \langle \alpha u_{\lambda f}^2(\cdot, t), \mu_\rho\rangle\, dt.$$

Combining this with part (i) of the present theorem, we obtain

$$c_\rho(\alpha)\langle f, \mu_\rho\rangle = \langle f, \mu_\rho\rangle - \lim_{\lambda\to 0}\frac{1}{\lambda}\int_0^\infty \langle \alpha u_{\lambda f}^2(\cdot, t), \mu_\rho\rangle\, dt. \tag{3.3}$$

Thus, substituting (3.3) in the right hand side of (3.2), replacing f by λf in (3.2), differentiating in λ and setting $\lambda = 0$, it follows that (1.10) holds.

(iii) One direction is trivial. For the other direction, assume that $\pi_{\zeta_\infty^{(\mu_1)}} = \pi_{\zeta_\infty^{(\mu_2)}}$. By (1.10) and the uniqueness of the Martin representation, it follows that the Martin measures m_{μ_1} and m_{μ_2} coincide on the set $\{\rho: c_\rho > 0\}$. On the other hand, if $c_\rho = 0$, then by (i), $\zeta_\infty^{(\mu_\rho)} = \delta_0$ and thus from (1.5),

$$\exp\left(-\left\langle f - \int_0^\infty \alpha u_f^2(\cdot, t)\, dt, \mu_\rho\right\rangle\right) = 1.$$

Using these two facts and applying (3.2) to μ_1 and μ_2 shows that

$$\int_{\mathscr{M}(D)} \exp(-\langle f, \eta\rangle)\, d\zeta_\infty^{(\mu_1)}(\eta) = \int_{\mathscr{M}(D)} \exp\Big(-\langle f, \eta\rangle\Big)\, d\zeta_\infty^{(\mu_2)}(\eta),$$

and thus $\zeta_\infty^{(\mu_1)} = \zeta_\infty^{(\mu_2)}$. □

PROOF OF THEOREM 5. Applying (1.7) with $\alpha = \frac{1}{h}$ and $\mu = hm_{\text{sym}}$, it is enough to show that

$$\int_0^\infty \big\langle (T_t f)^2, m_{\text{sym}}\big\rangle\, dt < \infty,$$

for $f \in C_c^+(D)$. Denote the kernel for the semigroup T_t by $p(t, x, y)$. The symmetry assumption gives $p(t, x, y)m_{\text{sym}}(x) = p(t, y, x)m_{\text{sym}}(y)$. By assumption, L on D is subcritical; let $G(x, y) \equiv \int_0^\infty p(t, x, y)\, dt$ denote its Green's function. We have

$$\begin{aligned}
&\int_0^\infty \langle (T_t f)^2, m_{\text{sym}}\rangle\, dt\\
&\quad = \int_0^\infty dt \int_D m_{\text{sym}}(x)\, dx \int_D dy\ p(t, x, y) f(y) \int_D dz\ p(t, x, z) f(z)\\
&\quad = \int_0^\infty dt \int_D dx \int_D dy\, p(t, y, x) f(y) m_{\text{sym}}(y) \int_D dz\ p(t, x, z) f(z)\\
&\quad = \int_0^\infty dt \int_D dy \int_D dz\, p(2t, y, z) f(y) m_{\text{sym}}(y) f(z)\\
&\quad = \tfrac{1}{2}\int_D dy \int_D dz\, G(y, z) f(y) m_{\text{sym}}(y) f(z) < \infty,
\end{aligned}$$

where the last inequality follows from the fact that f is compactly supported. □

4. Proofs of Theorems 4 and 7.

PROOF OF THEOREM 4. (i) Using (1.7), it suffices to prove that if (1.11) holds, then

$$\int_0^\infty dt \int_{R^d} dx\ \alpha(x)(T_t f)^2(x) < \infty.$$

[Actually, we will show that the finiteness of the above integral is equivalent to (1.11). The equivalence is useful with regard to Problem 3.] We have

$$\begin{aligned} &\int_{R^d} \alpha(x)(T_t f)^2(x)\,dx \\ &\quad = (2\pi t)^{-d} \int_{R^d} dx \int_{R^d} dy \int_{R^d} dz \exp\left(-\frac{|y-x|^2+|z-x|^2}{2t}\right) \\ &\qquad\qquad \times f(y)f(z)\alpha(x). \end{aligned} \tag{4.1}$$

From the inequality $2|x\cdot y| \le \frac{|x|^2}{2} + 2|y|^2$ and the fact that f is compactly supported, it follows that there exists a $C > 0$ such that

$$\begin{aligned} \frac{1}{C}\exp\left(-\frac{|x|^2}{2t}\right) f(y)f(z) &\le \exp\left(-\frac{|y-x|^2+|z-x|^2}{2t}\right) f(y)f(z) \\ &\le C\exp\left(-\frac{|x|^2}{2t}\right) f(y)f(z). \end{aligned} \tag{4.2}$$

From (4.1), (4.2) and the fact that f is compactly supported, it follows that there exists a $K > 0$, depending on f, such that

$$\begin{aligned} &\frac{1}{Kt^d}\int_{R^d} \exp\left(-\frac{|x|^2}{2t}\right)\alpha(x)\,dx \\ &\quad \le \int_{R^d} \alpha(x)(T_t f)^2(x)\,dx \le \frac{K}{t^d}\int_{R^d}\exp\left(-\frac{|x|^2}{2t}\right)\alpha(x)\,dx. \end{aligned} \tag{4.3}$$

Integrating both sides of (4.3) in t and making the change of variables $s = 2t$ on the right-hand side, it follows that $\int_0^\infty dt \int_{R^d} dx\ \alpha(x)(T_t f)^2(x) < \infty$ if and only if (1.11) holds. The final statement of part (i) follows after a standard calculation using (1.11).

(ii) Using (1.8), it suffices to prove that if (1.12) holds, then

$$\lim_{t\to\infty} \int_{R^d} u_f(x,t)\,dx = 0. \tag{4.4}$$

Integrating (2.9) gives

$$\int_{R^d} u_f(x,t)dx = \int_{R^d} f(x)\,dx - \int_0^t \int_{R^d} \alpha(x)u_f^2(x,s)\,dx\,ds. \tag{4.5}$$

It follows that $\int_{R^d} u_f(x,t)\,dx$ is monotone decreasing in t. Thus, to prove (4.4), it is enough to show that there exists a sequence $\{t_n\}$ satisfying $\lim_{n\to\infty} t_n = \infty$

and such that

$$\lim_{n\to\infty}\int_{R^d} u_f(x,t_n)\,dx = 0. \tag{4.6}$$

Let $k > 0$ be as in (1.12) and let

$$r(t) = t^{1/2}(\log t)^k \quad \text{and} \quad D(t) = \{x \in R^d\colon |x| < r(t)\}.$$

Since $u_f \le T_t f$, (4.6) will hold if we show that

$$\lim_{t\to\infty}\int_{R^d - D(t)} T_t f(x)\,dx = 0 \tag{4.7}$$

and that

$$\lim_{n\to\infty}\int_{D(t_n)} u_f(x,t_n)\,dx = 0. \tag{4.8}$$

We begin with (4.7). Again using the inequality $2|x\cdot y| \le \frac{1}{2}|x|^2 + 2|y|^2$ and the fact that f is compactly supported, we have

$$\begin{aligned}
&\int_{R^d - D(t)} T_t f(x)\,dx\\
&\quad = \int_{|x|\ge r(t)}\int_{R^d}(2\pi t)^{-d/2}\exp\biggl(-\frac{|y-x|}{2t}\biggr) f(y)\,dy\,dx\\
&\quad \le Ct^{-d/2}\int_{|x|\ge r(t)}\exp\biggl(-\frac{|x|^2}{4t}\biggr)dx = C_1 t^{-d/2}\int_{r(t)}^{\infty}\exp\biggl(-\frac{r^2}{4t}\biggr) r^{d-1}\,dr\\
&\quad = C_1\int_{t^{-1/2}r(t)}^{\infty}\exp\biggl(-\frac{u^2}{4}\biggr)u^{d-1}\,du.
\end{aligned} \tag{4.9}$$

Since $r(t) = t^{1/2}(\log t)^k$, it follows from (4.9) that (4.7) holds.

We now prove (4.8). From (4.5), it follows that $\int_0^\infty\int_{R^d}\alpha(x)u_f^2(x,t)\,dx\,dt < \infty$. Thus, there exists a sequence $\{t_n\}$ satisfying $\lim_{n\to\infty} t_n = \infty$ and such that

$$\int_{R^d}\alpha(x)u_f^2(x,t_n)\,dx \le \frac{1}{t_n\log t_n}. \tag{4.10}$$

[For otherwise, we would have $\int_{R^d}\alpha(x)u_f^2(x,t)\,dx > \frac{1}{t\log t}$, for all large t, and this would contradict the fact that $\int_0^\infty\int_{R^d}\alpha(x)u_f^2(x,t)\,dx\,dt < \infty$.] Using the Schwarz inequality along with (4.10), we have

$$\begin{aligned}
\int_{D(t_n)} u_f(x,t_n)\,dx &\le \biggl(\int_{D(t_n)}\alpha(x)u_f^2(x,t_n)\,dx\biggr)^{1/2}\biggl(\int_{D(t_n)}\frac{1}{\alpha(x)}\,dx\biggr)^{1/2}\\
&\le \biggl(\frac{1}{t_n\log t_n}\int_{D(t_n)}\frac{1}{\alpha(x)}\,dx\biggr)^{1/2}.
\end{aligned} \tag{4.11}$$

Now (4.8) follows from (1.12) and (4.11). The final statement of part (ii) follows by a straightforward calculation.

(iii) We need to show that (4.4) holds. By the maximum principle, u_f is monotone nonincreasing in its dependence on α. Thus, it is enough to prove

(4.4) in the case that α is compactly supported. The proof we give is just a slight variation of the proof of Lemma 2.4.1 in [5] which treats the case of a single point catalyst; that is, the case that α is a delta-function. Let $p(t,x,y) = (2\pi t)^{-1/2}\exp(-\frac{(y-x)^2}{2t})$. By self-similarity, we have $kp(k^2t,kx,ky) = p(t,x,y)$. Using this and (2.9), we have

$$t^{1/2}u_f(t^{1/2}x,t) = \int_R p(1,x,t^{-1/2}y)f(y)\,dy - \int_0^t ds\int_R dy\, p\Big(1-\frac{s}{t},x,t^{-1/2}y\Big)\alpha(y)u_f^2(y,s). \tag{4.12}$$

Replacing x by x_t in (4.12) and assuming that $\lim_{t\to\infty} x_t = x \in R$, it follows from the dominated convergence theorem that the first term on the right-hand side of (4.12) satisfies

$$\lim_{t\to\infty}\int_R p(1,x_t,t^{-1/2}y)f(y)\,dy = p(1,x,0)\|f\|_1. \tag{4.13}$$

We break the second term on the right-hand side of (4.12) into two parts. Fix $\eta\in(0,1)$ and note that $u_f(x,s)\le\int_R p(s,x,y)f(y)\,dy \le (2\pi\eta t)^{-\frac{1}{2}}\|f\|_1$, for $s\ge\eta t$. Using this, we have

$$\begin{aligned}&\int_{\eta t}^t ds\int_R dy\, p\Big(1-\frac{s}{t},x_t,t^{-1/2}y\Big)\alpha(y)u_f^2(y,s)\\ &\quad\le \|f\|_1^2\int_{\eta t}^t ds\int_R dy\frac{t^{1/2}}{(2\pi)^{3/2}\eta t(t-s)^{1/2}}\alpha(y)\\ &\quad= \frac{2\|f\|_1^2\|\alpha\|_1}{(2\pi)^{3/2}}\frac{(1-\eta)^{1/2}}{\eta}.\end{aligned} \tag{4.14}$$

On the other hand, from (4.5) we have $\int_0^\infty dt\int_R dy\,\alpha(y)u_f^2(y,t)<\infty$, and thus the dominated convergence theorem gives

$$\begin{aligned}&\lim_{t\to\infty}\int_0^{\eta t} ds\int_R dy\, p\Big(1-\frac{s}{t},x_t,t^{-1/2}y\Big)\alpha(y)u_f^2(y,s)\\ &\quad= p(1,x,0)\int_0^\infty dt\int_R dy\,\alpha(y)u_f^2(y,t).\end{aligned} \tag{4.15}$$

Since the right-hand side of (4.14) goes to 0 as $\eta\to 1$, we conclude from (4.12)–(4.15) that

$$\lim_{t\to\infty} t^{1/2}u_f(t^{1/2}x_t,t) = p(1,x,0)\Big(\|f\|_1 - \int_0^\infty dt\int_R dy\,\alpha(y)u_f^2(y,t)\Big). \tag{4.16}$$

It is easy to check that the above convergence is uniform over all paths $\{x_t\}_{t\ge0}$ which satisfy $\lim_{t\to\infty}x_t = x$. Thus, choosing $x_t = \frac{z}{t^{1/2}}$, for $z\in R$, squaring both sides of (4.16), multiplying by $\alpha(z)$ and then integrating over z in R and using the fact that α is compactly supported, we obtain

$$\begin{aligned}\lim_{t\to\infty} t\int_R \alpha(z)u_f^2(z,t)\,dz \\ = \|\alpha\|_1 p^2(1,0,0)\biggl(||f||_1 - \int_0^\infty dt\int_R dy\,\alpha(y)u_f^2(y,t)\biggr)^2.\end{aligned} \tag{4.17}$$

Since $\int_0^\infty dt\int_R dy\,\alpha(y)u_f^2(y,t) < \infty$, it follows that $\liminf_{t\to\infty} t\int_R \alpha(y) \times u_f^2(y,t)\,dy = 0$. [For otherwise, we would have $\int_R \alpha(y)u_f^2(y,t)\,dy \geq \frac{c}{t}$, for some $c > 0$ and all large t, and this would contradict the fact that $\int_0^\infty dt\int_R dy\,\alpha(y)u_f^2(y,t) < \infty$.] Thus, the right-hand side of (4.17) must equal 0; that is,

$$\|f\|_1 = \int_0^\infty dt\int_R dy\,\alpha(y)u_f^2(y,t). \tag{4.18}$$

We now conclude from (4.18) and (4.5) that $\lim_{t\to\infty}\int_R u_f(x,t)\,dx = 0$. $\square$

PROOF OF THEOREM 7. We may dispense with the inequalities on α and assume that $\alpha(x) = \frac{1}{h_\eta(x)} = \exp(-\eta\cdot x)$. Let $\mu(x) = h_\nu(x)m_{\text{sym}}(x) = \exp((\nu + 2b)\cdot x)$, where $\nu \in S$. To prove the theorem, we will show that (1.7) holds if η and d are as in part (i) or part (iii) of the proposition, and that (1.8) holds if η and d are as in part (ii) or part (iv).

We begin with parts (i) and (iii). The diffusion process $Y(t)$ corresponding to $L_0 = \frac{1}{2}\Delta + b\cdot\nabla$ can be represented as $Y(t) = B(t) + bt$, where $B(t)$ is a d-dimensional Brownian motion. Thus, $T_tf(x) = \int_{R^d}(2\pi t)^{-d/2}\exp(-\frac{|y-x-bt|^2}{2t}) \times f(y)\,dy$. We have

$$\begin{aligned}&\int_0^\infty \langle\alpha(T_tf)^2,\mu\rangle\,dt \\ &\quad= \int_{R^d\times R^d} dy\,dz\,f(y)f(z)\int_0^\infty dt\int_{R^d} dx(2\pi t)^{-d} \\ &\qquad\times\exp\biggl(-\frac{|y-x-bt|^2}{2t} - \frac{|z-x-bt|^2}{2t}\biggr) \\ &\qquad\times\exp\bigl((\nu+2b-\eta)\cdot x\bigr).\end{aligned} \tag{4.19}$$

Simplifying the expression in the exponential above, we have

$$\begin{aligned}&-\frac{|y-x-bt|^2}{2t} - \frac{|z-x-bt|^2}{2t} + (\nu+2b-\eta)\cdot x \\ &\quad= -\frac{1}{t}\biggl|x - \biggl(\frac{y+z+\nu t-\eta t}{2}\biggr)\biggr|^2 + \frac{|y+z+\nu t-\eta t|^2}{4t} \\ &\qquad - \frac{|y-bt|^2}{2t} - \frac{|z-bt|^2}{2t}.\end{aligned} \tag{4.20}$$

Using (4.20), we have

$$
\begin{aligned}
&\int_{R^d} dx(2\pi t)^{-d}\exp\biggl(-\frac{|y-x-bt|^2}{2t}-\frac{|z-x-bt|^2}{2t}\biggr)\exp((\nu+2b-\eta)\cdot x)\\
&\quad=2^{-d}(\pi t)^{-d/2}\exp\biggl(\frac{|y+z+\nu t-\eta t|^2}{4t}-\frac{|y-bt|^2}{2t}-\frac{|z-bt|^2}{2t}\biggr)\\
&\quad=2^{-d}(\pi t)^{-d/2}\exp\biggl(\biggl(\frac{|\nu-\eta|^2}{4}-|b|^2\biggr)t-\frac{|y-z|^2}{4t}\\
&\qquad\qquad+\frac{1}{2}(y+z,\nu-\eta)+(y+z,b)\biggr).
\end{aligned}
\tag{4.21}
$$

Since the diameter of S is $2|b|$, it follows that if $\eta\in S$, then $\frac{|\nu-\eta|^2}{4}-|b|^2\le 0$ and the inequality is strict if and only if $\eta\ne -\nu-2b$. Using this fact along with (4.19), (4.21) and the fact that f has compact support, if follows that $\int_0^\infty\langle\alpha(T_t f)^2,\ \mu\rangle dt<\infty$ if the conditions in part (i) or part (iii) of the proposition hold.

We now turn to parts (ii) and (iv). We use the method and notation employed above to prove part (ii) of Theorem 4. For any vector $w\in R^d$, define $D_w(t)=D(t)+wt$, where $D(t)$ is as defined between (4.6) and (4.7). Analogous to (4.7) and (4.8), it suffices to show for an appropriate choice of w that

$$
\lim_{t\to\infty}\int_{R^d-D_w(t)} T_t f(x)h_\nu(x)m_{\mathrm{sym}}(x)\,dx=0 \tag{4.22}
$$

and that

$$
\lim_{n\to\infty}\int_{D_w(t_n)} u_f(x,t_n)h_\nu(x)m_{\mathrm{sym}}(x)\,dx=0. \tag{4.23}
$$

Let $p(t,x,y)=(2\pi t)^{-d/2}\exp(-\frac{|y-x-bt|^2}{2t})$ denote the transition probability density for T_t. We have

$$
\begin{aligned}
&\int_{R^d-D_w(t)} T_t f(x)h_\nu(x)m_{\mathrm{sym}}(x)\,dx\\
&\qquad=\int_{R^d}dy\int_{R^d-D_w(t)}dx\; p(t,x,y)f(y)h_\nu(x)m_{\mathrm{sym}}(x)\\
&\qquad=\int_{R^d}dy\int_{R^d-D_w(t)}dx\; p(t,y,x)f(y)h_\nu(x)m_{\mathrm{sym}}(y)\\
&\qquad=\int_{R^d}dy\int_{R^d-D_w(t)}dx\; p^{h_\nu}(t,y,x)f(y)h_\nu(y)m_{\mathrm{sym}}(y)\\
&\qquad=\int_{R^d}dy\; f(y)h_\nu(y)m_{\mathrm{sym}}(y)\mathscr{P}_y^{h_\nu}(Y(t)\in R^d-D_w(t)),
\end{aligned}
\tag{4.24}
$$

where $p^{h_\nu}(t,x,y)=\frac{1}{h_\nu(x)}p(t,x,y)h_\nu(y)$ is the transition probability function for the h-transformed operator $L^{h_\nu}=L+\frac{\nabla h_\nu}{h_\nu}\cdot\nabla=\frac{1}{2}\Delta+(b+\nu)\cdot\nabla$ and $\mathscr{P}_y^{h_\nu}$ is the solution to the martingale problem for L^{h_ν}. Since the diffusion process $Y(t)$ corresponding to L^{h_ν} can be represented as $Y(t)=B(t)+(b+\nu)t$, where $B(t)$

is a Brownian motion, it follows that $\mathscr{P}_y^{h^\nu}(Y(t) \in R^d - D_w(t)) = \mathscr{P}_y^B(Y(t) \in R^d - D_{w-b-\nu}(t))$, where $\mathscr{P}_y^B$ denotes Wiener measure starting from y. Thus, if we choose $w = b + \nu$, then we have from (4.24) that

$$\begin{aligned}
&\int_{R^d - D_w(t)} T_t f(x) h_\nu(x) m_{\text{sym}}(x)\,dx \\
&\qquad = \int_{R^d} dy\ f(y) h_\nu(y) m_{\text{sym}}(y) \mathscr{P}_y^B(Y(t) \in R^d - D(t)).
\end{aligned} \tag{4.25}$$

Now by symmetry, it follows that the quantity $\int_{R^d - D(t)} T_t f(x)\,dx$ appearing on the left-hand side of (4.7) can be written as $\int_{R^d} f(y) \mathscr{P}_y^B(Y(t) \in R^d - D(t))\,dy$. Using this along with (4.25) and the fact that f has compact support, it follows that (4.7) and (4.22) are equivalent when $w = b+\nu$. Since we've already proved (4.7), this proves (4.22) with $w = b + \nu$.

We now show that (4.23) holds for $w = b + \nu$ Assume first that (ii) holds so that $d = 1$ or 2, $\eta = -\nu - 2b$ and $\alpha(x) = \exp(-\eta \cdot x)$. [In this case, we will actually show that (4.23) holds for any w.] Note then that $\frac{1}{\alpha} h_\nu m_{\text{sym}} \equiv 1$. Recall that $|D_w(t)| = |D(t)| = c(r(t))^d = ct^{\frac{d}{2}}(\log t)^{dk}$. Arguing as in the proof of Theorem 4, but using $h_\nu m_{\text{sym}}\,dx$ in place of dx, it follows analogous to (4.10) that there exists a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that

$$\int_{R^d} \alpha(x) u_f^2(x, t_n) h_\nu(x) m_{\text{sym}}(x) dx \le \frac{1}{t_n \log t_n}. \tag{4.26}$$

Using the Schwarz inequality along with (4.26) gives

$$\begin{aligned}
&\int_{D_w(t)} u_f(x, t_n) h_\nu(x) m_{\text{sym}}(x)\,dx \\
&\quad \le \left(\frac{1}{t_n \log t_n} \int_{D_w(t_n)} \frac{1}{\alpha(x)} h_\nu(x) m_{\text{sym}}(x)\,dx \right)^{1/2} \\
&\quad = \left(\frac{1}{t_n \log t_n} |D_w(t_n)| \right)^{1/2} = \left(\frac{c}{t_n \log t_n} t_n^{d/2} (\log t_n)^{dk} \right)^{1/2}.
\end{aligned} \tag{4.27}$$

The parameter $k > 0$ is arbitrary; thus choosing $k < \frac{1}{2}$ and using the fact that $d \le 2$, it follows from (4.27) that (4.23) holds.

Now assume that (iv) holds so that $\eta = -\nu - 2b - s(\nu + b)$, for some $s > 0$ and $\alpha(x) = \exp(-\eta \cdot x)$. Unlike in the case above, in this case we will need the fact that $w = b + \nu$. Note that $(\frac{1}{\alpha} h_\nu m_{\text{sym}})(x) = \exp(-s(\nu + b) \cdot x)$. Then analogous to (4.27) we have

$$
\begin{aligned}
&\int_{D_w(t)} u_f(x,t_n)h_\nu(x)m_{\mathrm{sym}}(x)\,dx \\
&\quad\le \left(\frac{1}{t_n\log t_n}\int_{D_w(t_n)}\exp(-s(\nu+b)\cdot x)\,dx\right)^{1/2} \\
&\quad= \left(\frac{1}{t_n\log t_n}\exp(-st_n|\nu+b|^2)\int_{D(t_n)}\exp(-s(\nu+b)\cdot x)\,dx\right)^{1/2} \\
&\quad\le \left(c\frac{t_n^{d/2}(\log t_n)^{dk}}{t_n\log t_n}\exp(-st_n|\nu+b|^2)\exp(s|\nu+b|t_n^{1/2}(\log t_n)^k)\right)^{1/2}.
\end{aligned}
\tag{4.28}
$$

Since $\nu\in S$, we have $|\nu+b|=|b|>0$. Thus, (4.23) follows from (4.28). □

5. Proof of Theorem 6. We will prove the theorem under the assumption that (1.16) holds. Then we will show how to modify it in the case that (1.17) holds. We will show that $\zeta_\infty^{(h_1 m_{\mathrm{sym}})}=\delta_0$ if $\alpha\ge\frac{1}{h_2}$. An identical argument of course works with the roles of h_1 and h_2 switched. By (1.8), it is enough to show that $\lim_{t\to\infty}\int_{-\infty}^{\infty}u_f(x,t)h_1(x)m_{\mathrm{sym}}(x)\,dx=0$. Integrating (2.9) against the invariant measure h_1m_{sym} gives (4.5) with dx replaced by $h_1m_{\mathrm{sym}}\,dx$ and thus shows that $\int_{-\infty}^{\infty}u_f(x,t)h_1(x)m_{\mathrm{sym}}(x)\,dx$ is decreasing in t. Thus it suffices to show that $\lim_{n\to\infty}\int_{-\infty}^{\infty}u_f(x,t_n)h_1(x)m_{\mathrm{sym}}(x)\,dx=0$ for some sequence $\{t_n\}$ with $\lim_{n\to\infty}t_n=\infty$.

Since $u_f\le T_tf$, it suffices to prove that for each $\varepsilon>0$ there exists a time dependent interval $I_\varepsilon(t)=[c_\varepsilon(t),d_\varepsilon(t)]$ and a sequence $\{t_n\}$ with $\lim_{n\to\infty}t_n=\infty$ such that

$$
\lim_{n\to\infty}\int_{I_\varepsilon(t_n)}u_f(x,t_n)h_1(x)m_{\mathrm{sym}}(x)\,dx=0
\tag{5.1}
$$

and

$$
\lim_{t\to\infty}\int_{R-I_\varepsilon(t)}T_tf(x)h_1(x)m_{\mathrm{sym}}(x)\,dx\le\varepsilon.
\tag{5.2}
$$

We begin by estimating the integral in (5.2). Let $p(t,x,y)$ denote the transition probability density for the semigroup T_t. We have

$$
\begin{aligned}
&\int_{R-I_\varepsilon(t)}T_tf(x)h_1(x)m_{\mathrm{sym}}(x)\,dx \\
&\quad=\int_{R-I_\varepsilon(t)}dx\int_{-\infty}^{\infty}dy\;p(t,x,y)f(y)h_1(x)m_{\mathrm{sym}}(x) \\
&\quad=\int_{R-I_\varepsilon(t)}dx\int_{-\infty}^{\infty}dy\;p(t,y,x)f(y)h_1(x)m_{\mathrm{sym}}(y) \\
&\quad=\int_{R-I_\varepsilon(t)}dx\int_{-\infty}^{\infty}dy\;p^{h_1}(t,y,x)f(y)h_1(y)m_{\mathrm{sym}}(y),
\end{aligned}
\tag{5.3}
$$

where $p^{h_1}(t, x, y) = \frac{1}{h_1(x)} p(t, x, y) h_1(y)$ is the transition probability density corresponding to the h-transformed operator $L_0^{h_1} = L_0 + a\frac{h_1'}{h_1}\frac{d}{dx}$ (see [12], Section 4.1).

Let $\mathscr{P}_{\bullet}^{h_1}$ denote the solution to the generalized martingale problem for the operator $L_0^{h_1}$. Since $h_1(x) = \int_{-\infty}^{x} dz \exp(-\int_0^z \frac{2b}{a}(y)\,dy) = c\mathscr{P}_x(\lim_{t\to\infty} Y(t) = \infty)$, where $c = \frac{1}{h_1(\infty)}$ ([12], Section 5.1), it follows ([12], Section 7.2) that

$$\mathscr{P}_{\bullet}^{h_1}(\ ) = \mathscr{P}_{\bullet}\left(\,\Big|\, \lim_{t\to\infty} Y(t) = \infty\right). \tag{5.4}$$

From (5.3) we have

$$\begin{aligned} &\int_{R-I_\varepsilon(t)} T_t f(x) h_1(x) m_{\mathrm{sym}}(x)\,dx \\ &\qquad = \int_{-\infty}^{\infty} f(y) h_1(y) m_{\mathrm{sym}}(y) \mathscr{P}_y^{h_1}(Y(t) \in R - I_\varepsilon(t))\,dy. \end{aligned} \tag{5.5}$$

Let $\gamma_f = |\mathrm{supp}(f)| \sup_y (f h_1 m_{\mathrm{sym}})(y)$. We want to choose $I_\varepsilon(t)$ so that

$$\lim_{t\to\infty} \mathscr{P}_y^{h_1}(Y(t) \in I_\varepsilon(t)) > 1 - \frac{\varepsilon}{\gamma_f}, \text{ uniformly over } \mathrm{supp}(f). \tag{5.6}$$

If (5.6) holds, then (5.2) will follow from (5.5). Since by (5.4); $\mathscr{P}_{\bullet}^{h_1}(\lim_{t\to\infty} Y(t) = \infty) = 1$, we may assume that $\lim_{t\to\infty} c_\varepsilon(t) = \infty$.

Now consider the integral in (5.1). Applying the Schwarz inequality gives

$$\begin{aligned} &\int_{I_\varepsilon(t_n)} u_f(x, t) h_1(x) m_{\mathrm{sym}}(x)\,dx \\ &\qquad \le \left(\int_{I_\varepsilon(t_n)} \alpha u_f^2(x, t) h_1(x) m_{\mathrm{sym}}(x)\,dx \right)^{1/2} \\ &\qquad\quad \times \left(\int_{I_\varepsilon(t_n)} \frac{1}{\alpha(x)} h_1(x) m_{\mathrm{sym}}(x)\,dx \right)^{1/2}. \end{aligned} \tag{5.7}$$

Arguing as in the proof of Theorem 4, but using $h_1 m_{\mathrm{sym}}\,dx$ in place of dx, it follows analogous to (4.10) that there exists a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that

$$\int_R \alpha(x) u_f^2(x, t_n) h_1(x) m_{\mathrm{sym}}(x)\,dx \le \frac{1}{t_n \log t_n}. \tag{5.8}$$

This gives an upper bound along a sequence $\{t_n\}$ for the first term on the right-hand side of (5.7). For the second term, we use the bound $\alpha \ge \frac{1}{h_2}$ to obtain

$$\begin{aligned} \int_{I_\varepsilon(t)} \frac{1}{\alpha(x)} h_1(x) m_{\mathrm{sym}}(x)\,dx &\le \int_{I_\varepsilon(t)} h_1(x) h_2(x) m_{\mathrm{sym}}(x)\,dx \\ &\le c \int_{I_\varepsilon(t)} h_2(x) m_{\mathrm{sym}}(x)\,dx. \end{aligned}$$

By (1.15) and (1.16), we have

$$h_2(x)m_{\mathrm{sym}}(x) = \frac{1}{a(x)}\exp\biggl(\int_0^x \frac{2b}{a}(y)\,dy\biggr)\int_x^\infty \exp\biggl(-\int_0^y \frac{2b}{a}(z)\,dz\biggr)\,dy.$$

One can check, using Assumption 1, that

$$\lim_{x\to\infty}\frac{(\int_x^\infty \exp(-\int_0^y \frac{2b}{a}(z)dz)dy)'}{\Bigl(\frac{a}{2b}(x)\exp(-\int_0^x \frac{2b}{a}(y)dy)\Bigr)'} = 1;$$

thus, by l'Hôpital's rule,

$$\lim_{x\to\infty} 2b(x)h_2(x)m_{\mathrm{sym}}(x) = 1. \tag{5.9}$$

We want to choose $I_\varepsilon(t)$ so that

$$\int_{I_\varepsilon(t)} \frac{1}{b(s)}\,ds = o(t\log t) \quad \text{as } t\to\infty. \tag{5.10}$$

If (5.10) holds, then (5.1) will follow from (5.7)–(5.9) along with the fact that $c_\varepsilon(t)$, the left-hand endpoint of $R - I_\varepsilon(t)$, satisfies $\lim_{t\to\infty} c_\varepsilon(t) = \infty$.

Thus, to complete the proof, we must choose $I_\varepsilon(t)$ such that (5.6) and (5.10) hold. We use a result in [11] which studies the asymptotic behavior of certain one-dimensional diffusions. It is here that we make essential use of Assumption 1. We are assuming that (1.16) is in effect; thus, it follows from Assumption 1 that $b(x) = d_1x^{k_1}$ and $a(x) = c_1x^{l_1}$ for $x \gg 1$, where $l_1 - 1 < k_1 < 1$ and d_1, $c_1 > 0$. Choose $x_0 > 0$ such that the above equalities hold for $x \ge x_0$. Let $\mu_x(t) = (x^{1-k_1} + d_1(1-k_1)t)^{1/(1-k_1)}$, for $t \ge x_0$ and note that μ_x solves the equation $\mu'(t) = b(\mu(t))$ and $\mu(0) = x$. Let $\psi(t) = \int_{x_0}^t \frac{a}{b^3}(x)\,dx$, for $t \ge x_0$. Note from Assumption 1 that $\psi(\infty) < \infty$ if and only if $l_1 - 3k_1 + 1 < 0$. Under Assumption 1, the conditions required for Theorem 3-ii-a or 3-ii-b in [11] to hold are met. This theorem concerns the behavior of the diffusion corresponding to $\mathscr{P}_\bullet$ on the event $\{\lim_{t\to\infty} Y(t) = \infty\}$, which by (5.4) is equivalent to the behavior of the diffusion corresponding to $\mathscr{P}_\bullet^{h_1}$. Translating this theorem into our notation, it follows that

$$\begin{aligned}&\text{if } \psi(\infty) < \infty,\ \text{then } \tfrac{Y(t)-\mu_x(t)}{b(\mu_x(t))} \text{ converges a.s. } \mathscr{P}_x^{h_1} \text{ as } t\to\infty\\ &\text{to a nondegenerate limit for each } \ x \ge x_0\end{aligned} \tag{5.11}$$

and

$$\begin{aligned}&\text{if } \psi(\infty) = \infty,\ \text{then } \tfrac{Y(t)-\mu_x(t)}{b(\mu_x(t))} = B(\psi(\mu_x(t))) + o(\psi^{1/2}(\mu_x(t)))\\ &\text{as } t\to\infty, \text{in } \mathscr{P}_x^{h_1} \text{ probability, for each } x \ge x_0,\ \text{where } B(t)\\ &\text{is a standard Brownian motion.}\end{aligned} \tag{5.12}$$

If (5.11) holds, then for each $\varepsilon > 0$ and each $x \ge x_0$, there exists an $N_\varepsilon(x)$ such that

$$\lim_{t\to\infty} \mathscr{P}_x^{h_1}\bigl(|Y(t) - \mu_x(t)| \le N_\varepsilon(x)b(\mu_x(t))\bigr) \ge 1 - \frac{\varepsilon}{\gamma_f}.$$

Let $x_1 = \max(\sup(\mathrm{supp}(f)), x_0)$. By the strong Markov property and the fact that the diffusion is transient to $+\infty$ under $\mathscr{P}^{h_1}_{\bullet}$, it then follows that for each $\varepsilon > 0$ there exists an M_ε such that

$$\lim_{t\to\infty} \mathscr{P}^{h_1}_x\big(|Y(t) - \mu_{x_1}(t)| \le M_\varepsilon b(\mu_{x_1}(t))\big) \ge 1 - \frac{\varepsilon}{\gamma_f} \text{ for all } x \in \mathrm{supp}(f). \tag{5.13}$$

If, on the other hand, (5.12) holds, then since the law of the iterated logarithm guarantees that a Brownian motion $B(t)$ almost surely satisfies $B(t) \le (t \log t)^{1/2}$ for sufficiently large t, if follows that for each $x \ge x_0$,

$$\lim_{t\to\infty} \mathscr{P}^{h_1}_x\big(|Y(t) - \mu_x(t)| \le b(\mu_x(t))\psi^{1/2}(\mu_x(t)) \log \psi(\mu_x(t))\big) = 1.$$

As above, an application of the strong Markov property allows us to conclude that for each $\varepsilon > 0$ there exists an m_ε such that

$$\begin{aligned}&\lim_{t\to\infty} \mathscr{P}^{h_1}_x\Big(|Y(t) - \mu_{x_1}(t)| \le m_\varepsilon b(\mu_{x_1}(t))\psi^{1/2}(\mu_{x_1}(t)) \log \psi(\mu_{x_1}(t))\Big)\\ &\qquad \ge 1 - \frac{\varepsilon}{\gamma_f}, \text{ for all } x \in \mathrm{supp}(f).\end{aligned} \tag{5.14}$$

Recall the definitions of a, b, μ_x and ψ in the paragraph following (5.10). We have for some $A > 0$,

$$\begin{aligned}&\big(d_1(1-k_1)t\big)^{1/(1-k_1)}\\ &\qquad \le \mu_{x_1}(t) \le (d_1(1-k_1)t)^{1/(1-k_1)} + At^{k_1/(1-k_1)} \quad \text{for } t \ge 1.\end{aligned} \tag{5.15}$$

Also, for some $B_1, B_2 > 0$,

$$B_1 t^{k_1/(1-k_1)} \le b(\mu_{x_1}(t)) \le B_2 t^{k_1/(1-k_1)} \quad \text{for } t \ge 1. \tag{5.16}$$

In the case that $\psi(\infty) = \infty$, we have

$$\psi(t) = \begin{cases} \dfrac{c_1}{d_1^3(l_1 - 3k_1 + 1)}\big(t^{l_1-3k_1+1} - x_0^{l_1-3k_1+1}\big), & \text{for } l_1 - 3k_1 + 1 > 0,\\[2ex] \dfrac{c_1}{d_1^3}\log\dfrac{t}{x_0}, & \text{for } l_1 - 3k_1 + 1 = 0.\end{cases}$$

Thus, for some $C > 0$,

$$\psi(\mu_{x_1}(t)) \le \begin{cases} Ct^{(l_1-3k_1+1)/1-k_1}, & \text{if } l_1 - 3k_1 + 1 > 0,\\ C\log t, & \text{if } l_1 - 3k_1 + 1 = 0.\end{cases} \tag{5.17}$$

We now prescribe $I_\varepsilon(t)$. For $t > 3$ and some $D_\varepsilon > 0$, define

$$(5.18)\qquad I_\varepsilon(t)=\begin{cases}\left\{y\colon \left|(d_1(1-k_1)t)^{1/(1-k_1)}-y\right|\le D_\varepsilon t^{k_1/(1-k_1)}\right\}, & \\ \hfill \text{if } l_1-3k_1+1<0, & \\ \left\{y\colon \left|(d_1(1-k_1)t)^{1/(1-k_1)}-y\right|\le D_\varepsilon t^{k_1/(1-k_1)}(\log t)^{1/2}\log\log t\right\}, & \\ \hfill \text{if } l_1-3k_1+1=0, & \\ \left\{y\colon \left|(d_1(1-k_1)t)^{1/(1-k_1)}-y\right|\le D_\varepsilon t^{(l_1-k_1+1)/(2(1-k_1))}\log t\right\}, & \\ \hfill \text{if } l_1-3k_1+1>0. & \end{cases}$$

It then follows from (5.13)–(5.18) that if D_ε is sufficently large, then (5.6) holds.

We now show that (5.10) also holds with this choice of I_ε. Recall that $b(x) = d_1 x^{k_1}$ for $x \ge x_0$. Using (5.18) and doing a little calculation, one finds that there exists a $c_\varepsilon > 0$ such that for $t > 3$,

$$(5.19)\qquad \int_{I_\varepsilon(t)} \frac{1}{b(s)}\,ds \le \begin{cases} c_\varepsilon, & \text{if } l_1-3k_1+1<0,\\ c_\varepsilon(\log t)^{1/2}\log\log t, & \text{if } l_1-3k_1+1=0,\\ c_\varepsilon t^{(l_1-3k_1+1)/(2(1-k_1))}\log t, & \text{if } l_1-3k_1+1>0.\end{cases}$$

The inequality $\frac{l_1-3k_1+1}{2(1-k_1)} < 1$ is equivalent to $k_1 > l_1 - 1$, and this latter inequality is contained in Assumption 1. Thus, (5.10) follows from (5.19).

We now describe the changes to be made in the above proof when (1.17) is assumed to hold instead of (1.16). There are two cases. In the case that $\zeta_0 = h_2 m_{\mathrm{sym}}$ and $\alpha \ge \frac{1}{h_1}$, the proof is exactly the same as the above proof except that the roles of h_1 and h_2 have to be switched. Now consider the case that $\zeta_0 = h_1 m_{\mathrm{sym}}$ and $\alpha \ge \frac{1}{h_2}$. One follows the proof as above up through the paragraph containing (5.10), the only thing to point out being that in applying l'Hôpital's rule to get (5.9), the indeterminate form will now be $\frac{\infty}{\infty}$ rather than $\frac{0}{0}$.

At that point in the proof, we appealed to a result in [11]. We noted that the conditions on the coefficients of the operator L_0 met the requirements of a theorem in [11] which gives the asymptotic behavior of the diffusion $Y(t)$ under $\mathcal{P}_\cdot$ (that is, the diffusion corresponding to L_0) on the event $\{\lim_{t\to\infty} Y(t) = \infty\}$. Since $\mathcal{P}_\cdot^{h_1}$ is related to $\mathcal{P}_\cdot$ by (5.4) and since in the case treated above $\mathcal{P}_\cdot(\lim_{t\to\infty} Y(t) = \infty) > 0$, this theorem then also gave the asymptotic behavior of $Y(t)$ under $\mathcal{P}_\cdot^{h_1}$, and it is this latter behavior that we needed. In the present case, $\mathcal{P}_\cdot(\lim_{t\to\infty} Y(t) = \infty) = 0$. Thus, we must apply the theorem in [11] directly to the $\mathcal{P}_\cdot^{h_1}$-diffusion corresponding to the operator $L_0^{h_1}$. We have

$$L_0^{h_1} = \frac{1}{2}a\frac{d^2}{dx^2} + \tilde{b}\frac{d}{dx} \quad \text{where } \tilde{b} = b + a\frac{h_1'}{h_1}.$$

Using Assumption 1, one can check that the conditions needed for Theorem 5 in [11] to hold are met for the operator $L_0^{h_1}$. One now needs to calculate the quantities $\mu_x(t)$ and $\psi(t)$ for this operator. The l'Hôpital's rule argument alluded to above shows that

$$\lim_{t\to\infty} \frac{a}{2b}(t)\frac{h_1'}{h_1}(t) = -1.$$

By Assumption 1, $b(x) = d_1 x^{k_1}$ for $x \gg 1$ where $d_1 < 0$ [if $d_1 > 0$, then case (1.16) would hold]. Thus, the operator L^{h_1} has diffusion coefficient $a(x) = c_1 x^{l_1}$ for $x \gg 1$ and drift coefficient $\tilde{b}$ satisfying $\lim_{t\to\infty} \frac{\tilde{b}(t)}{|d_1|t^{k_1}} = 1$. In light of this, we obtain the same order asymptotics for $\mu_x(t)$ and $\psi(t)$ as in the previous case and the proof goes through in a similar fashion. $\square$

REMARK. After some preliminary estimates, the proof of Theorem 6 came down to finding intervals $I_\varepsilon(t)$ which satisfy (5.6) and (5.10). We used Theorem 5-ii-a and 3-ii-b in [11] to find appropriate $I_\varepsilon(t)$'s so that (5.6) would hold and then showed that these $I_\varepsilon(t)$'s also work for (5.10). Now the conditions required in [11] for Theorem 3-ii-a and 3-ii-b to hold are considerably more generic than the conditions in our Assumption 1; thus, with our choice of $I_\varepsilon(t)$'s, (5.6) will hold under the more general conditions in [11]. These more generic conditions pose a problem, however, when it comes to verifying (5.10). A Taylor series expansion shows that everything works out nicely up to first-order terms, but we could not see how to control the second-order terms in a satisfactory manner unless we assumed that the diffusion coefficent $a(x)$ and the drift $b(x)$ were asymptotically equivalent to powers of x. Thus, our proof of Theorem 6 holds when $a(x)$ and $b(x)$ are asymptotically equivalent to powers of x. In Assumption 1, we have assumed that $a(x)$ and $b(x)$ are exactly powers of x for large $|x|$ only because this simplifies considerably some of the calculations.

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