

way. In what follows the empirical side will be treated exclusively, and it will be treated on a general plan, investigating not the particular way in which statistical, chemical, physical, and astronomical observations are made, but the common rules according to which they are all submitted to computation.

II. LAWS OF ERRORS.

§ 5. Every observation is supposed to contain information, partly as to the phenomenon in which we are particularly interested, partly as to all the circumstances, connected with it, which are regarded as essential. In comparing several observations, it makes a very great difference, whether such essential circumstances have remained unchanged, or whether one or several of them have changed between one observation and another. The treatment of the former case, that of *repetitions*, is far simpler than that of the latter, and is therefore more particularly the subject of our investigations; nevertheless, we must try to master also the more difficult general case in its simplest forms, which force themselves upon us in most of the empirical sciences.

By *repetitions* then we understand those observations, in which all the essential circumstances remain unchanged, in which therefore the results or phenomena should agree, if all the operative causes had been included among our essential circumstances. Furthermore, we can without hesitation treat as repetitions those observations, in which we assume that no essential circumstance has changed, but do not know for certain that there has been no such change. Strictly speaking, this would furnish an example of observations with systematic errors; but provided there has been no change in the care with which the essential circumstances have been determined or checked, it is permissible to employ the simpler treatment applicable to the case of repetitions. This would not however be permissible, if, for instance, the observer during the repetitions has perceived any uncertainty in the records of a circumstance, and therefore paid greater attention to the following repetitions.

§ 6. The special features of the observations, and in particular their degree of accuracy, depend on causes which have been left out as unessential circumstances, or on some overlooked uncertainty in the statement of the essential circumstances. Consequently no speculation can indicate to us the accuracy and particularities of observations. These must be estimated by comparison of the observations with each other, but only in the case of repetitions can this estimate be undertaken directly and without some preliminary work. The phrase *law of errors* is used as a general name for any mathematical expression representing the distribution of the varying results of repetitions.

Laws of actual errors are such as correspond to repetitions actually carried out. But observations yet unmade may also be erroneous, and where we have to speak hypothetically about observations, or have to do with the prediction of results of future repetitions, we are generally obliged to employ the idea of "laws of errors". In order to prevent any misunderstanding we then call this idea "*laws of presumptive errors*". The two kinds of laws of errors cannot generally be quite the same thing. Every variation in the number of repetitions must entail some variations in the corresponding law of errors; and if we compare two laws of actual errors obtained from repetitions of the same kind in equal number, we almost always observe great differences in every detail. In passing from actual repetitions to future repetitions, such differences at least are to be expected. Moreover, whilst any collection of observations, which can at all be regarded as repetitions, will on examination give us its law of actual errors, it is not every series of repetitions that can be used for predictions as to future observations. If, for instance, in repeated measurements of an angle, the results of our first measurements all fell within the first quadrant, while the following repetitions still more frequently, and at last exclusively, fell within the second quadrant, and even commenced to pass into the third, it would evidently be wrong to predict that the future repetitions would repeat the law of actual errors for the totality of these observations. In similar cases the observations must be rejected as bad or misconceived, and no law of presumptive errors can be directly based upon them.

§ 7. Suppose, however, that on comparing repetitions of some observation we have several times determined the law of actual errors in precisely the same way, employing at first small numbers of repetitions, then larger and still larger numbers for each law. If then, on comparing these laws of actual errors with one another, we remark that they become more alike in proportion as the numbers of repetitions grow greater, and that the agreements extend successively to all those details of the law which are not by necessity bound to vary with the number of repetitions, then we cannot have any hesitation in using the law of actual errors, deduced from the largest possible number of repetitions, for predictions concerning future observations, made under essentially the same circumstances.

This, however, is wholly legitimate only, when it is to be expected that... *if we could obtain repetitions in indefinitely increasing numbers, the law of errors would then approach a single definite form, namely the law of presumptive errors itself, and would not oscillate between several forms, or become altogether or partly indeterminate.* (Note the analogy with the difference between converging and oscillating infinite series). We must therefore distinguish between good and bad observations, and only the good ones, that is those which satisfy the above mentioned condition, *the law of large numbers*, yield laws of presumptive errors and afford a basis for prediction.

As we cannot repeat a thing indefinitely often, we can never be quite certain that

a given method of observation may be called good. Nevertheless, we shall always rely on laws of actual errors, deduced from very large numbers of concordant repetitions, as sufficiently accurate approximations to the law of presumptive errors.

And, moreover, the purely hypothetical assumption of the existence of a law of presumptive errors may yield some special criteria for the right behaviour of the laws of actual errors, corresponding to the increasing number of the repetitions, and establish the conditions necessary to justify their use for purposes of prediction.

We must here notice that, when a series of repetitions by such a test proves bad and inapplicable, we shall nevertheless often be able, sometimes by a theoretical criticism of the method, and sometimes by watching the peculiarities in the irregularities of the laws of errors, to find out the reason why the given method of observation is not as good as others, and to change it so that the checks will at least show that it has been improved. In the case mentioned in the preceding paragraph, for instance, the remedy is obvious. The time of observation is there to be reckoned among the essential circumstances.

And if we do not attain our object, but should fail in many attempts at throwing light upon some phenomenon by means of good observations, it may be said even at this stage, before we have been made acquainted with the various means that may be employed, and the various forms taken by the laws of errors, that absolute abandonment of the law of large numbers, as quite inapplicable to any given refractory phenomenon, will generally be out of the question. After repeated failures we may for a time give up the whole matter in despair; but even the most thorough sceptic may catch himself speculating on what may be the cause of his failure, and, in doing so, he must acknowledge that the error is never to be looked for in the objective nature of the conditions, but in an insufficient development of the methods employed. From this point of view then the law of large numbers has the character of a belief. There is in all external conditions such a harmony with human thought that we, sooner or later, by the use of due sagacity, particularly with regard to the essential subordinate circumstances of the case, will be able to give the observations such a form that the laws of actual errors, with respect to repetitions in increasing numbers, will show an approach towards a definite form, which may be considered valid as the law of presumptive errors and used for predictions.

§ 8. Four different means of representing the law of errors must be described, and their respective merits considered, namely:

- Tabular arrangements,
- Curves of Errors,
- Functional Laws of Errors,
- Symmetric Functions of the Repetitions.

In comparing these means of representing the laws of errors, we must take into

consideration which of them is the easiest to employ, and neither this nor the description of the forms of the laws of errors demands any higher qualification than an elementary knowledge of mathematics. But we must take into account also, how far the different forms are calculated to emphasise the important features of the laws of errors, i. e. those which may be transferred from the laws of actual errors to the laws of presumptive errors. On this single point, certainly, a more thorough knowledge of mathematics would be desirable than that which may be expected from the majority of those students who are obliged to occupy themselves with observations. As the definition of the law of presumptive errors presupposes the determination of limiting values to infinitely numerous approximations, some propositions from the differential calculus would, strictly speaking, be necessary.

III. TABULAR ARRANGEMENTS.

§ 9. In stating the results of all the several repetitions we give the law of errors in its simplest form. Identical results will of course be noted by stating the number of the observations which give them.

The table of errors, when arranged, will state all the various results and the frequency of each of them.

The table of errors is certainly improved, when we include in it the *relative frequencies* of the several results, that is, the ratio which each absolute frequency bears to the total number of repetitions. It must be the *relative frequencies* which, according to the law of large numbers, are, as the number of observations is increased, to approach the constant values of the law of presumptive errors. Long usage gives us a special word to denote this transition in our ideas: *probability* is the relative frequency in a law of presumptive errors, the proportion of the number of coincident results to the total number, on the supposition of infinitely numerous repetitions. There can be no objection to considering the *relative frequency* of the law of actual errors as an approximation to the corresponding *probability* of the law of presumptive errors, and the doubt whether the *relative frequency* itself is the best approximation that can be got from the results of the given repetitions, is rather of theoretical than practical interest. Compare § 73.

It makes some difference in several other respects — as well as in the one just mentioned — if the phenomenon is such that the results of the repetitions show qualitative differences or only differences of magnitude.

§ 10. In the former case, in which no transition occurs, but where there are such abrupt differences that none of the results are more closely connected with one another than with the rest, the tabular form will be the only possible one, in which the law of errors can