SYSTEMS OF POLYNOMIALS CONNECTED WITH THE CHARLIER EXPANSIONS AND THE PEARSON DIFFERENTIAL AND DIFFERENCE EQUATIONS*

By

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INTRODUCTION

The problem of fitting mathematical curves to statistical data has commanded the attention of statisticians and mathematicians for many years. The curves referred to the most by English-speaking biometricians and mathematicians are perhaps those developed by Pearson from 1895-1916.[1] He showed that a series of curves could be obtained by assigning various values to the parameters in a certain first order differential equation. A few years later, Charlier[2], attacking the same question from a differ-

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*A dissertation submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in the University of Michigan—August, 1931.

[1]Karl Pearson, "Mathematical Contributions to the Theory of Evolution," Philosophical Transactions, A, Vol. 186 (1895), pp. 343-414; also "Supplement to a Memoir on Skew Variation," Phil. Trans., Vol. 197 (1901), pp. 443-456; also "Second Supplement to a Memoir on Skew Variation," Phil. Trans., A, Vol. 216 (1916), pp. 429-457.

[2]C. V. L. Charlier, "Ueber das Fehlergesetz," Arkiv for Matematik, Astronomi och Fysik, Vol. 2, No. 8 (1905), pp. 1-9; also "Ueber die Darstellung willkuerlicher Funktionen," Arkiv for Matematik, Astronomi och Fysik, Vol. 2, No. 20 (1905), pp. 1-35.

ent angle, showed that any function could probably be approximated by using a certain function and its derivatives in the terms of the series:

$$F(x) = A_0 f(x) + A_1 f'(x) + A_2 f''(x) + \cdots$$

where the A_i are constants.

Charlier found that the constants A_n could be formally determined, the n th constant A_n being dependent on the moments of $F(x)$ of order not greater than n. He illustrated the method of procedure for the case where $y = f(x)$ was the equation of the normal curve of error, i. e. one of the Pearson curves. In fact, the successive derivatives of this particular function gave rise to a well known system of polynomials, namely the Hermite polynomials, and the coefficients are dependent upon these polynomials also.

In recent years, Romanovsky[1] has succeeded in obtaining similar results for the case in which some of the other of the Pearson curves are used as the $f(x)$ in the Gram-Charlier series. The successive derivatives of these other special Pearson type curve functions also result in systems of polynomials which bear fundamental relations to each other.

It is the object of this investigation to show:

(1) That the constants obtained by Charlier for his Type A series can be much more readily obtained by making use of certain existing biorthogonality conditions;

(2) That if the Type A series be generalized to the form:

$$F(x) = C_0 Q(x) f_0(x) + C_1 \frac{d}{dx} Q(x) f_1(x) + C_2 \frac{d^2}{dx^2} Q(x) f_2(x) + \cdots$$

[1]V. Romanovsky, "Generalization of some types of the frequency curves of Professor Pearson," Biometrika, Vol. 16 (1924), pp. 106-117; also "Sur quelques classes nouvelles de Polynomes orthogonaux," Comptes Rendus de L'Academie des Sciences, Vol. 188 (1929), pp. 1023-1025.

where $f_n(x)$ is a polynomial of degree n in x, then the C_n can also be formally determined and depend upon the moments of $F(x)$ of order at most n;

(3) That the form of the polynomials obtained by Charlier and Romanovsky for certain solutions of the Pearson differential equation can be found for any solution of this equation and that the relations existing between polynomials of the same system can also be generalized for the general solution and for the most part obtained without having the explicit form of the solution;

(4) That results analogous to those obtained in (1) and (3) can be derived for the Charlier Type B series and the analogue of Pearson's differential equation, finite differences replacing the derivative.

The writer wishes to particularly express his appreciation to Prof. H. C. Carver for the valuable aid he has given both in the stimulating instruction characterized by frankness in indicating unsolved problems in his classes and through direct suggestions in the preparation of this paper.

CHAPTER I

Polynomials Connected with the Gram-Charlier Series

1. In the articles entitled "Ueber das Fehlergesetz" and "Ueber die Darstellung willkürlicher Funktionen"[1] Charlier proves the following well known theorem:

Charlier's theorem for series of type A—*If $F(x)$ is any real valued function of x, which has finite moments of all orders, then $F(x)$ may be formally expressed in terms of another function $f(x)$ and its derivatives as follows:*

$$\text{(A)} \quad F(x) = A_0 f(x) + A_1 f'(x) + A_2 f''(x) + \cdots + A_n f^{(n)}(x) + \cdots$$

where $f(x)$ has the following properties:

(a) *$f(x)$ and its derivatives are continuous for all real values of x,*

(b) *$f(x)$ and its derivatives vanish for $x = +\infty$ and $-\infty$*

(c) $\lim\limits_{x \to \pm\infty} x^m f^{(n)}(x) \quad 0$ *for all m and n,*

(d) $\int_{-\infty}^{+\infty} f(x)\,dx \neq 0.$

The conditions (c) and (d) are not given in Charlier's articles, but an eaxmination of the proof shows that he assumes implicitly that they are satisfied. $f(x) = \frac{x}{1+x^2}$ satisfies (a) and (b) without satisfying (c) and (d).

In the first section of the latter paper, Charlier determi :s the constants $A_0, A_1, A_2, \ldots, A_n, \ldots$ He takes e series (A), multiplies it successively by 1, x, x^2, . . ., and integrates each result between the limits $-\infty$ to $+\infty$. The fol-

[1] C. V. L. Charlier, loc. cit.

lowing equations result:

$$\int_{-\infty}^{+\infty} F(x)dx = A_0 \int_{-\infty}^{+\infty} f(x)dx$$

$$\int_{-\infty}^{+\infty} x\, F(x)dx = A_0 \int_{-\infty}^{+\infty} x\, f(x)dx + A_1 \int_{-\infty}^{+\infty} x\, f'(x)\, dx$$

$$\int_{-\infty}^{+\infty} x^2 F(x)dx = A_0 \int_{-\infty}^{+\infty} x^2 f(x)dx + A_1 \int_{-\infty}^{+\infty} x^2 f'(x)dx + A_2 \int_{-\infty}^{+\infty} x^2 f''(x)dx$$

Each of these equations contain a finite number of terms and the constants $A_0, A_1, A_2, \ldots$ may readily be determined by solving them. In fact we find that any constant A_n may be expressed as

$$A_n = \int_{-\infty}^{+\infty} P_n(x)\, F(x)dx$$

where $P_n(x)$ is a polynomial in x of degree not greater than n. An analysis of the underlying facts reveals that what Charlier has actually done is to show that under the conditions listed in the theorem there exists a uniquely determined set of polynomials $P_0(x), P_1(x), \ldots, P_n(x), \ldots, P_n(x)$ at most of degree n, biorthogonal to the set of derivatives or functions of $f(x)$, i. e. satisfy the biorthogonality conditions:

$$\int_{-\infty}^{+\infty} P_n(x) f^{(m)}(x)dx = 0 \text{ for } m \neq n$$

$$= 1 \text{ for } m = n$$

Further a study of the coefficients of these polynomials shows that

$$\frac{d\, P_n(x)}{dx} = -P_{n-1}(x),$$

i. e. we have the following theorem:

THEOREM: *If* $f(x)$ *satisfy the conditions* (a), (b), (c), *and* (d) *of Charlier's theorem for series* (A) *and if* $P_0(x)$, $P_1(x)$, . . . $P_n(x)$ . . . *is the system of polynomials in* x, $P_n(x)$ *of degree at most* n, *which is biorthogonal to* $f(x)$ *and its derivatives, i. e. satisfies the conditions*

$$\int_{-\infty}^{+\infty} P_n(x)\, f^{(m)}(x)\,dx = 0 \text{ for } m \neq n$$

$$= 1 \text{ for } m = n$$

then

$$\frac{dP_n(x)}{dx} = -P_{n-1}(x)$$

This can readily be shown to be true directly from a use of the biorthogonal property. For integrating by parts we obtain:

$$\int_{-\infty}^{+\infty} P_n(x) f^{(m)}(x)\,dx = P_n(x) f^{(m-1)}(x) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} P_n'(x)\, f^{(m-1)}(x)\,dx.$$

The first half of the right hand side of this equation vanishes due to condition (c) of Charlier's theorem for series (A). For the second half we have

$$-\int_{-\infty}^{+\infty} P_n'(x) f^{(m-1)}(x)\,dx = 0 \text{ for } m \neq n$$

$$= 1 \text{ for } m = n$$

But we know that

$$+\int_{-\infty}^{+\infty} P_{n-1}(x)\, f^{(m-1)}(x)\,dx = 0 \text{ for } m \neq n$$

$$= 1 \text{ for } m = n$$

determines uniquely the polynomials $P_{n-1}(x)$. It follows that

$$dP_n(x)/dx = -P_{n-1}(x)$$

A corollary to this last theorem may be stated as follows:
COROLLARY:

$$\text{If} \quad \int_{-\infty}^{+\infty} P_n(x) f^{(m)}(x)\, dx = 0 \qquad \text{for } m \neq n$$

$$= a_n \qquad \text{for } m = n$$

$a_i \neq 0 \qquad (i = 0, 1, 2, \cdots\cdots), \textit{ then}$

$$dP_n(x)/dx = -\frac{a_n}{a_{n-1}} P_{n-1}(x)$$

The proof is similar to the one just given. Integration by parts gives the following result:

$$-\int_{-\infty}^{+\infty} f^{(m-1)}(x)\, P_n'(x)\,dx = 0 \qquad \text{for } m \neq n$$

$$= a_n \qquad \text{for } m = n$$

But we know that

$$\int_{-\infty}^{+\infty} f^{(m-1)}(x)\, P_{n-1}(x)\,dx = 0 \qquad \text{for } m \neq n$$

$$= a_{n-1} \qquad \text{for } m = n$$

Therefore we may conclude that

$$-\frac{1}{a_n}\, \frac{dP_n(x)}{dx} = \frac{1}{a_{n-1}} P_{n-1}(x)$$

or

$$\frac{dP_n(x)}{dx} = -\frac{a_n}{a_{n-1}} P_{n-1}(x)$$

An illustration of this corollary is the case of the well known Hermite polynomials which are involved in Charlier's first paper.[1] These satisfy the conditions

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[1]C. V. L. Charlier, loc. cit. Charlier uses as $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-b)^2}{2\sigma^2}}$. In this paper we shall use the simpler basic function e^{-x^2}.

$$\int_{-\infty}^{+\infty} H_m(x) H_n(x) e^{-x^2} dx = 0 \qquad \text{for } m \neq n$$

$$= 2^n n! \sqrt{\pi} \qquad \text{for } m = n$$

and

$$H_n(x) e^{-x^2} = (-1)^n d^n (e^{-x^2})/dx^n.$$

Hence

$$\int_{-\infty}^{+\infty} H_m(x) d^n(e^{-x^2})/dx^n \quad dx = 0 \qquad \text{for } m \neq n$$

$$= (-2)^n n! \sqrt{\pi} \qquad \text{for } m = n$$

If then $f(x) = e^{-x^2}$ and $a_n = (-2)^n n! \sqrt{\pi}$ our corollary applies, i. e. we have

$$dH_n(x)/dx = 2nH_{n-1}(x)$$

We might further observe that if $a_n = (-1)^n n!$ then the polynomials $P_n(x)$ form a system of Appell polynomials[1] satisfying the relation

$$dP_n(x)/dx = nP_{n-1}(x)$$

the n th polynomial being the coefficient of $h^n/n!$ in the expansion of $a(h)e^{hx}$ where

$$a(h) = \alpha_0 + \frac{h}{1!}\alpha_1 + \frac{h^2}{2!}\alpha_2 + \cdots + \frac{h^n}{n!}\alpha_n + \cdots$$

The fact that differentiation of the n th polynomial results in the negative of the $(n-1)$th polynomial, shows that the n th polynomial may be obtained by integrating the $(n-1)$ th one,

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[1]M. P. Appell, "Sur une classe de Polynomes," Annales Scientifiques de L'Ecole Normale Superioure, Vol. IX, series 2 (1880), pp. 119-120.

which will consequently determine all of the terms of the n th polynomial except the constant. This constant may be found from any of the conditions of biorthogonality. The simplest of these conditions is:

$$\int_{-\infty}^{+\infty} P_n(x) f(x)\, dx = 0$$

Setting

$$P_n(x) = -\int_0^x P_{n-1}(x)\, dx + c$$

gives
$$\int_{-\infty}^{+\infty} \left[-\int_0^x P_{n-1}(x)\, dx + c \right] f(x)\, dx = 0$$

and so
$$c = \frac{\int_{-\infty}^{+\infty} \left[\int_0^x P_{n-1}(x)\, dx \right] f(x)\, dx}{\int_{-\infty}^{+\infty} f(x) dx}$$

so that
$$P_n(x) = -\int_0^x P_{n-1}(x) dx + \frac{\int_{-\infty}^{+\infty} \left[\int_0^x P_{n-1}(x)\, dx \right] f(x) dx}{\int_{-\infty}^{+\infty} f(x) dx}$$

This gives a very simple and elegant method of writing down successively the polynomials associated with any function $f(x)$ satisfying the conditions of the theorem.

Using the Charlier notation

$$\lambda_n = \frac{\int_{-\infty}^{\infty} x^n f(x) dx}{n!}$$

and observing that $P_0(x) = 1/\lambda_0$, we obtain the following polynomials:

$$
\begin{aligned}
P_1(x) &= -\int_0^x P_0(x)dx + \frac{\int_{-\infty}^{+\infty}\left[\int_0^x P_0(x)dx\right] f(x)dx}{\int_{-\infty}^{+\infty} f(x)dx} \\
&= -\frac{x}{\lambda_0} + \frac{\lambda_1}{\lambda_0^2} \; ,
\end{aligned}
$$

$$
\begin{aligned}
P_2(x) &= -\int_0^x P_1(x)dx + \frac{\int_{-\infty}^{+\infty}\left[\int_0^x P_1(x)dx\right] f(x)dx}{\int_{-\infty}^{+\infty} f(x)dx} \\
&= \frac{x^2}{\underline{|2}\,\lambda_0} - \frac{\lambda_1 x}{\lambda_0^2} + \frac{\lambda_1^2}{\lambda_0^3} - \frac{\lambda_2}{\lambda_0^2} \; , \\
P_3(x) &= -\int_0^x P_2(x)dx + \frac{\int_{-\infty}^{\infty}\left[\int_0^x P_2(x)dx\right] f(x)dx}{\int_{-\infty}^{+\infty} f(x)dx} \\
&= -\frac{x^3}{\underline{|3}\,\lambda_0} + \frac{\lambda_1 x^2}{\underline{|2}\,\lambda_0^2} - \frac{\lambda_1^2 x}{\lambda_0^3} + \frac{\lambda_2 x}{\lambda_0^2} + \frac{\lambda_1^3}{\lambda_0^4} - \frac{2\lambda_2\lambda_1}{\lambda_0^3} + \frac{\lambda_3}{\lambda_0^2} \; ,
\end{aligned}
$$

. . . .

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2. Just as the Hermite polynomials, based as they are on the derivatives of e^{-x^2}, are the starting point for expansions of the Gram-Charlier type and for the theorem just considered, so the Laguerre polynomials defined by $d^n(a+bx)^n e^{-x}/dx^n$ suggest an expansion of the type

$$
F(x) = C_0 f_0(x)\varphi(x) + C_1 \frac{d}{dx} f_1(x)\varphi(x) \div C_2 \frac{d^2}{dx^2} f_2(x)\varphi(x) + \cdots
$$

where $f_n(x)$ is a polynomial in x. As a matter of fact we can state the following theorem:

THEOREM: If $\varphi(x)$ *is a function such that*

(1) $\varphi(x)$ *and all its derivatives are continuous for all real values of* x,

(2) $\varphi(x)$ *and its derivatives are zero at* $x = +\infty$ *and* $-\infty$,

(3) $\lim_{x \to \pm\infty} x^n \varphi^m(x) = 0$,

(4) $\{f_n(x)\}$ *is a sequence of polynomials in* x *such that* $\int_{-\infty}^{+\infty} f_n(x)\varphi(x) \neq 0$,

then there exists a unique sequence of polynomials $P_m(x)$, $P_m(x)$ *at most of degree* m, *such that*

$$\int_{-\infty}^{+\infty} P_m(x) \frac{d^n}{dx^n} f_n(x)\varphi(x)\,dx = 0 \qquad \text{for } m \neq n$$
$$= 1 \qquad \text{for } m = n$$

If $f_n(x)$ *is at most of degree* n, *then the determination of* $P_m(x)$ *depends at most upon the moments of* φ *of order* n.[1]

The method of proof is modelled on Charlier's proof for the preceding case. By substituting in the n th integration by parts formula

$$\int u(x)v^{n+1}(x)dx = uv^{(n)} - u^1 v^{(n-1)}$$
$$+ u''v^{(n-2)} - \ldots + (-1)^n u^{(n)} v$$
$$+ (-1)^{n+1} \int u^{(n+1)}(x)\, v(x)\, dx,$$

we have

[1]The Laguerre polynomials are not a special case of this because there the interval of integration is $-a/b$ to $+\infty$.

$$\int_{-\infty}^{+\infty} P_m(x)\frac{d^n}{dx^n} f_n(x)\phi(x)dx = \Big\{P_m(x)\frac{d^{n-1}}{dx^{n-1}} f_n(x)\phi(x) - \frac{dP_m(x)}{dx}\frac{d^{n-2}}{dx^{n-2}} f_n(x)\phi(x)$$

$$+\frac{d^2}{dx^2}P_m(x)\left[\frac{d^{n-3}}{dx^{n-3}} f_n(x)\phi(x)\right]$$

$$+\cdots+(-1)^{n-1}\left[\frac{d^{n-1}}{dx^{n-1}}P_m(x)\right]f_n\phi(x)\Big\}_{-\infty}^{+\infty}$$

$$+(-1)^n\int_{-\infty}^{+\infty}\left[\frac{d^n}{dx^n}P_m(x)\right] f_n(x)\phi(x)dx$$

$$=(-1)^n\int_{-\infty}^{+\infty}\left[\frac{d^n}{dx^n}P_m(x)\right] f_n(x)\phi(x)dx$$

because of conditions (2) and (3) on $\phi(x)$. As a consequence, if $n > m$ then $\frac{d^n}{dx^n} P_m(x) = 0$, so that for $n > m$

$$\int_{-\infty}^{+\infty} P_m(x)\frac{d^n}{dx^n} f_n(x)\phi(x)dx = 0$$

that is to say $P_m(x)$ is orthogonal to $\frac{d^n}{dx^n} f_n(x)\phi(x)$ provided $n > m$ Hence $P_n(x)$ must satisfy only the following $n+1$ equations:

$$\int_{-\infty}^{+\infty} P_m(x)\frac{d^n}{dx^n} f_n(x)\phi(x)dx = 0$$

$$\int_{-\infty}^{+\infty} P_n(x)\frac{d}{dx} f_1(x)\phi(x)dx = (-1)\int_{-\infty}^{+\infty}\frac{dP_n(x)}{dx} f_1(x)\phi(x)dx = 0$$

$$\int_{-\infty}^{+\infty} P_n(x)\frac{d^2}{dx^2} f_2(x)\phi(x)dx = (-1)^2\int_{-\infty}^{+\infty}\frac{d^2P_n(x)}{dx^2} f_2(x)\phi(x)dx = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\int_{-\infty}^{+\infty} P_n(x)\frac{d^n}{dx^n} f_n(x)\phi(x)dx = (-1)^n\int_{-\infty}^{+\infty}\frac{d^n}{dx^n} P_n(x) f_n(x)\phi(x)dx = 1$$

Replacing now $P_n(x)$ by $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$ gives us the system of algebraic equations to be satisfied by $a_0, a_1, \cdots\cdots, a_n$, viz.:

$$a_0\int_{-\infty}^{+\infty} f_0(x)\phi(x)dx + a_1\int_{-\infty}^{+\infty} x f_0(x)\phi(x)dx$$

$$+ a_2\int_{-\infty}^{+\infty} x^2 f_0(x)\phi(x)dx + \cdots\cdots + a_n\int_{-\infty}^{+\infty} x^n f_0(x)\phi(x)dx = 0$$

$$a_1\int_{-\infty}^{+\infty} f_1(x)\phi(x)dx + 2a_2\int_{-\infty}^{+\infty} x f_1(x)\phi(x)dx + \cdots + na_n\int_{-\infty}^{+\infty} x^{n-1} f_1(x)\phi(x)dx = 0$$

$$2a_2\int_{-\infty}^{+\infty} f_2(x)\phi(x)dx + \cdots + n(n-1)a_n\int_{-\infty}^{+\infty} x^{n-2} f_2(x)\phi(x)dx = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(n-2)!a_{n-2}\int_{-\infty}^{+\infty} f_{n-2}(x)\phi(x)dx + \frac{(n-1)!}{1!}a_{n-1}\int_{-\infty}^{+\infty} x f_{n-2}(x)\phi(x)dx$$

$$+\frac{n!}{2!}a_n\int_{-\infty}^{+\infty} x^2 f_{n-2}(x)\phi(x)dx = 0$$

$$(n-1)!a_{n-1}\int_{-\infty}^{+\infty} f_{n-1}(x)\phi(x)dx + \frac{n!}{1!}a_n\int_{-\infty}^{+\infty} x f_{n-1}(x)\phi(x)dx = 0$$

$$(-1)^n n!a_n\int_{-\infty}^{+\infty} f_n(x)\phi(x)dx = 1$$

We have here a unique determination of a_n if the determinant of the coefficients is $\neq 0$. This is true since the determinant $\Delta = (-1)^n (\int f_0\phi)(\int f_1\phi)\ldots(\int f_n\phi)$ is $\neq 0$ because of the condition (4) on ϕ. If $f_n(x)$ is at most of degree n, it is obvious that the determination of the $P_n(x)$ resulting from the coefficients a_n depends at most upon the moments of ϕ of order n.

The first three polynomials of the type considered in the last theorem have the following form, the limits of integration being $-\infty$ and $+\infty$ in each case.

$$P_1(x) = \frac{\int x\phi(x)dx}{\int f_1(x)\phi(x)dx\int\phi(x)dx} - \frac{x}{\int f_1(x)\phi(x)dx}$$

$$= \frac{1}{\int f_1(x)\phi(x)dx}\left[\frac{\int x\phi(x)dx}{\int\phi(x)dx} - x\right],$$

$$P_2(x) = \frac{\int x f_1(x)\phi(x)dx\int x\phi(x)dx}{\int f_2(x)\phi(x)dx\int f_1(x)\phi(x)dx\int\phi(x)dx} - \frac{\int x^2\phi(x)dx}{2!\int f_2(x)\phi(x)dx\int\phi(x)dx}$$

$$- \frac{x\int x f_1(x)\,\phi(x)\,dx}{\int f_2(x)\,\phi(x)\,dx \int f_1(x)\,\phi(x)\,dx} + \frac{x^2}{2!\int f_2(x)\,\phi(x)\,dx} ,$$

$$P_3(x) = \frac{\int x f_2(x)\phi(x)dx \int x f_1(x)\phi(x)dx \int x\phi(x)dx}{\int f_3(x)\phi(x)dx \int f_2(x)\phi(x)dx \int f_1(x)\phi(x)dx \int \phi(x)dx}$$

$$- \frac{\int x^2 f_1(x)\phi(x)dx \int x\phi(x)dx}{2!\int f_3(x)\phi(x)dx \int f_1(x)\phi(x)dx \int \phi(x)dx}$$

$$- \frac{\int x f_2(x)\phi(x)dx \int x^2\phi(x)dx}{2!\int f_3(x)\phi(x)dx \int f_2(x)\phi(x)dx \int \phi(x)dx}$$

$$+ \frac{\int x^3\phi(x)dx}{3!\int f_3(x)\,\phi(x)dx \int \phi(x)dx} - \frac{x\int x f_2(x)\phi(x)dx \int x f_1(x)\phi(x)\,dx}{\int f_3(x)\phi(x)dx \int f_2(x)\phi(x)dx \int f_1(x)\phi(x)\,dx}$$

$$+ \frac{x\int x^2 f_1(x)\phi(x)dx}{2!\int f_3(x)\phi(x)dx \int f_1(x)\,\phi(x)dx} + \frac{x^2\int x f_2(x)\phi(x)dx}{2!\int f_3(x)\phi(x)dx \int f_2(x)\phi(x)dx}$$

$$- \frac{x^3}{3!\int f_3(x)\,\phi(x)dx} .$$

CHAPTER II

Polynomials Connected with Pearson's Differential Equation

1. In the work in mathematical statistics a large number of the problems that require study involve data properly classified into groups and about which further information is sought. This data is often classified to form a frequency distribution. The frequency distribution when grouped may appear to lie on a certain curve. If it can be shown that this curve is a mathematical curve, i. e. one for which we are able to set up an equation, then this frequency distribution can be readily examined and studied.

There are very few frequency distributions which actually conform to known mathematical equations. However, there are certain curves which seem to lend themselves much better to statistical manipulations than others. Among the most commonly used of these are the so-called Pearson type curves. Pearson[1] showed in a series of three articles how he obtained the equations of twelve distinct curves and this was done by considering the differential equation

$$\frac{1}{y}\frac{dy}{dx} = \frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2}$$

and solving it, after assigning particular values to the parameters a_0, a_1, b_0, b_1, and b_2. The equations of these curves and the differential equations from which they were derived are as follows:

[1]Karl Pearson, loc. cit.

Type	Equation	Differential Equation
I	$y = y_0\left(1+\frac{x}{a}\right)^{\nu a}\left(1-\frac{x}{b}\right)^{\nu b}$	$\frac{dy}{dx} = \frac{\nu(a+b)x}{(a+x)(b-x)}\,y$
II	$y = y_0\left(1-\frac{x^2}{a^2}\right)^{m}$	$\frac{dy}{dx} = \frac{-2mx}{a^2-x^2}\,y$
III	$y = y_0\left(1+\frac{x}{a}\right)^{\nu a} e^{-\nu x}$	$\frac{dy}{dx} = \frac{-\nu x}{a+x}\,y$
IV	$y = y_0\left(1+\frac{x^2}{a^2}\right)^{-m} e^{-\nu \arctan \frac{x}{a}}$	$\frac{dy}{dx} = \frac{-2mx-\nu a}{a^2+x^2}\,y$
V	$y = y_0\, x^{-h} e^{-\frac{\nu}{x}}$	$\frac{dy}{dx} = \frac{\nu-hx}{x^2}\,y$
VI	$y = y_0\,(x-a)^q x^{-h}$	$\frac{dy}{dx} = \frac{ha+(q-h)x}{x^2-ax}\,y$
VII	$y = y_0\, e^{-\frac{x^2}{2\sigma^2}}$	$\frac{dy}{dx} = \frac{-x}{\sigma^2}\,y$
VIII	$y = y_0\left(1+\frac{x}{a}\right)^{-m}$	$\frac{dy}{dx} = \frac{-m}{a+x}\,y$
IX	$y = y_0\left(1+\frac{x}{a}\right)^{m}$	$\frac{dy}{dx} = \frac{m}{a+x}\,y$
X	$y = \frac{n}{\sigma}\, e^{\pm\frac{x}{\sigma}}$	$\frac{dy}{dx} = \pm\frac{1}{\sigma}\,y$
XI	$y = y_0\, x^{-m}$	$\frac{dy}{dx} = \frac{-m}{x}\,y$
XII	$y = y_0\left(\frac{a_1+x}{a_2-x}\right)^{p}$	$\frac{dy}{dx} = \frac{h(a_1+a_2+2x)}{(a_1+x)(a_2+x)}\,y$

The curves most widely used are the normal curve of error, which Pearson calls Type VII, and the Type III curve.

Suppose a Pearson curve $f(x)$ has been found which seems to fit a given distribution fairly well. The question may well be asked: Is it possible by means of analytic methods to approach even nearer to the given distribution? For example, would it be possible to use this approximate function as the $f(x)$ in the Charlier series (A) and thus obtain a closer approximation to the observed frequency function.

Charlier in his paper "Ueber die Darstellung willkürlicher Functionen"[1] considered this question for $\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-b)^2}{2\sigma^2}}$, i. e. the normal curve of error. He showed that using this $\varphi(x)$ reduced the series (A) to the form:

$$(A') \quad F(x) = a_0\varphi(x) + a_3\varphi^{(3)}(x) + a_4\varphi^{(4)}(x) + \cdots + a_n\varphi^{(n)}(x) + \cdots$$

the first and second derivative terms vanishing due to the proper choice of constants. This series (A′) is frequently referred to as the Gram-Charlier Type A series. It is worthwhile to note that this $\varphi(x)$ is the same one whose derivatives we found in the first chapter resulted in the Hermite polynomials. These polynomials have the following interesting properties[2]:

(1) $dH_n(x)/dx = 2nH_{n-1}(x)$

(2) $H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$

(3) $H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$

The first of these relations shows that the derivative of any Hermite polynomial corresponds to the preceding polynomial multi-

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[1] C. V. L. Charlier, loc. cit.

[2] R. Courant and D. Hilbert, Methoden der Mathematischen Physik, 1, pp. 76.

plied by $2n$. The second equation is a recurrence relation between the $(n+1)$ th, n th and $(n-1)$ th polynomials, while the third relation is a differential equation of the second order involving only the n th polynomial.

The use of the equations of the other Pearson type curves as the $f(x)$ in the original Charlier series has in recent years been studied by Romanovsky. In the first[1] of two articles, he discusses the Pearson Type I, II and III curves as well as the Type VII—the normal curve referred to in the last paragraph. Just as the normal curve of error requires the use of the Hermite polynomials, he found that the Type I curve and Type II, which is a special case of Type I, involved the Jacobi polynomials

$$G_n(h,q,x)=\frac{x^{1-q}(1-x)^{q-h}}{q(q+1)\cdots(q+n-1)}\frac{d^n}{dx^n}\left[x^{q+n-1}(1-x)^{h+n-q}\right].$$

The n 'th Jacobi polynomial satisfies the second order differential equation[2].

$$x(1-x)G_n''(x)+[q-(p+1)x]G_n'(x)+(p+n)nG_n(x)=0$$

which corresponds to property (3) mentioned for the Hermite polynomials above. The Type III curve involves the Laguerre polynomials[3] defined by

$$L_n(x)=e^x\frac{d^n}{dx^n}(x^ne^{-x})$$

and these in turn satisfy the recurrence relation

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[1]V. Romanovsky: "Generalization of some types of the frequency curves of Professor Pearson." op. at pp. 106-117.

[2]R. Courant and D. Hilbert, op. cit., Vol. I, p. 75.

[3]R. Courant and D. Hilbert, op. cit., pp. 77-78.

$$L_{n+1}(x)-(2n+1-x)L_n(x)+n^2L_{n-1}(x)=0$$

and the differential equation

$$L'_n(x) - nL'_{n-1}(x) = -nL_{n-1}(x)$$

In the second article[2], Romanovsky reviews the cases of the Type IV, V and VI curves. The generalization of the Type IV curve gives the polynomial

$$P_n(m,x)=(a^2+x^2)^m e^{\nu\theta}\frac{d^n}{dx^n}\left[(a^2+x^2)^{-m+n}e^{-\nu\theta}\right]$$

where $\theta=\text{arc tan } x/a$. These polynomials possess properties similar to the other polynomials mentioned, viz.:

$$P_{n+1}(n+1,x)=\left[2(n+1-m)x-\nu a\right]P_n(n,x)$$

$$+2n\left[n+1-m\right](a^2+x^2)P_{n-1}(n,x)$$

and

$$(a^2+x^2)P''_n(n,x)+\left[2(1-m)x-\nu a\right]$$

$$P'_n(n,x)-n(n+1-2m)P_n(n,x)=0$$

Similarly for the Type V curve he finds the polynomials

$$P_n(h,x)=x^h e^{\frac{\nu}{x}}\frac{d^n}{dx^n}(x^{-h+2n}e^{-\frac{\nu}{x}}).$$

Also the relations

[2]V. Romanovsky, "Sur quelques Classes nouvells de Polynomes othogonaux," loc. cit.

$$P_{n+1}(n+1)x) = \left[(2n+2-p)x+\gamma\right]P_n(n,x) + n(2n+2-p)x^2 P_{n-1}(n,x)$$

and

$$x^2 P_n''(n,x) + \left[x(2-p)+\nu\right]P_n'(n,x) - n(n+1-p)\,P_n(n,x) = 0$$

hold.

Finally for the Type VI curve Romanovsky gets the polynomials:

$$P_n(-h,q,x) = (x-a)^{-q} x^h \frac{d^n}{dx^n}\left[(x-a)^{q+n} x^{-h+n}\right]$$

and the relations:

$$P_{n+1}(n+1,x) = \left[(-p+1)(x-a)+(q+1)x\right]P_n(n,x) + x(x-a)P_n'(n,x),$$

$$x(x-a)P_n''(n,x) + \left[(-p+1)(x-a)+(q+1)x\right]P_n'(n,x) - n(n+1+q-p)P_n(n,x) = 0.$$

We note, therefore, that if a solution of the Pearson differential equation is used as the generating function $f(x)$ in the Gram-Charlier series, that a distinct set of polynomials results in each case and that these polynomials satisfy certain recurrence relations and differential equations. These properties are not found in the case of functions such as sech x and sech $^m x$, which were discussed as generating functions by Charlier[1] and by Roa[2] respectively. The successive derivatives of the

[1] C. V. L. Charlier, "Ueber die Darstellung willkürlicher Funktionen," loc. cit., pp. 18-22.

[2] Emeterio Roa, "A Number of new generating Functions with Applications to Statistics," Doctor's Thesis, University of Michigan, 1923.

sech x do not result in polynomials such as the Hermite or Jacobi ones.

Since the generalization of the solutions of the Pearson curves leads to distinct sets of polynomials and since these polynomials satisfy certain fundamental relations, we are led to inquire whether these polynomials are not special cases of a general polynomial and may be obtained from it by specializing the coefficients and further whether such general polynomials, if they do exist, will satisfy certain recurrence relations and differential equations. These problems are among those which we shall consider in this chapter.

2. In order that we may develop the generalized polynomials, let us consider the Pearson differential equation where the numerator is of the first and the denominator of the second degree, i. e.

$$\frac{1}{y}\frac{dy}{dx} = \frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2}$$

For convenience we shall denote the numerator by N and the denominator by D. We then have the following theorem:

THEOREM: *If y is a non-identically zero solution of*

$$\frac{dy}{dx} = \frac{N}{D}y \tag{1}$$

then $\frac{D^n}{y}\frac{d^n y}{dx^n}$ is a polynomial of degree at most n.

The proof will proceed by mathematical induction. It is obvious that the theorem holds for $n = 1$, $P_1(x)$ being N. Since it is true that

$$D\frac{dy}{dx} = Ny$$

we obtain by differentiation

$$D\frac{d^2y}{dx^2} + D'\frac{dy}{dx} = N\frac{dy}{dx} + N'y,$$

or using (1) and multiplying the equation through by D we get

$$D^2 \frac{d^2y}{dx^2} = (N^2 - ND' + N'D)y$$

Since D' is linear and N' is a constant, it is obvious that $(N^2 - ND + N'D)$ is at most of degree 2.

Assume then that the statement holds for $m \leq n$ and we have

$$D^n \frac{d^n y}{dx^n} = P_n(x) y . \tag{2}$$

Differentiation gives

$$nD^{n-1} D' \frac{d^n y}{dx^n} + D^n \frac{d^{n+1}}{dx^{n+1}} = P_n(x) \frac{dy}{dx} + \frac{dP_n(x)}{dx} y.$$

Multiplying through by D we get

$$D^{n+1} \frac{d^{n+1}}{dx^{n+1}} = DP_n(x) \frac{dy}{dx} - nD^n D' \frac{d^n y}{dx^n} + D \frac{dP_n(x)}{dx} y,$$

and using (1) and (2), we have

$$\begin{aligned} P_{n+1}(x) \cdot y &= NP_n(x)y - nD'P_n(x)y + D\frac{dP_n(x)}{dx} y \\ &= \left[NP_n(x) - nD'P_n(x) + D \frac{dP_n(x)}{dx} \right] y. \end{aligned}$$

The coefficient of y is obviously a polynomial of degree at most $n+1$. Incidentally we have derived the relation:

$$P_{n+1}(x) = P_n(x)(N - nD') + D \frac{dP_n(x)}{dx} \tag{I}$$

an equation which gives the $(n+1)$ th polynomial in terms of the n th polynomial and its first derivative $P_n'(x)$.

3. More generally we have:

THEOREM: *If y is a non-identically zero solution of* (1), *then*

$$\frac{1}{y} D^{n-k} \frac{d^n}{dx^n} D^k y$$

is a polynomial $P_n(k, x)$, $P_n(k, x)$ *is at most of degree* n *in* x. *In particular if* $k = n$, *we have that*

$$\frac{1}{y}\frac{d^n}{dx^n} D^n y$$

is a polynomial in x *of degree at most* n.

This theorem can be proved directly following the lines of the preceding theorem, but it is simpler to obtain it as an immediate consequence of this theorem and the following lemma:

LEMMA: *If* y *satisfy the differential equation* (1) *then* $D^k y$, *where* k *is any real number, satisfies a differential equation of the same type, viz.*:

$$\frac{d}{dx}(D^k y) = \frac{N + kD'}{D} D^k y$$

Let $u = D^k y$

Then logarithmic differentiation gives at once

$$\frac{1}{u}\frac{du}{dx} = k\frac{D'}{D} + \frac{1}{y}\frac{dy}{dx} = \frac{N + kD'}{D}$$

It follows from this lemma that any result which we derive concerning the polynomials $P_n(x) = \frac{1}{y} D^n \frac{d^n y}{dx^n}$ where y satisfies $D\, dy/dx = Ny$, is immediately extensible to the polynomials $P_n(k, x) = \frac{1}{y} D^{n-k} \frac{d^n}{dx^n} D^k y$ by replacing N by $N + kD'$. In particular relation (I) becomes

$$(\mathrm{I}_k) \quad P_{n+1}(k+1, x) = [N + (k-n+1)D'] P_n(k+1, x) + D \cdot \frac{dP_n(k+1, x)}{dx}$$

which for $k = n$ reduces to

$$(\mathrm{I}_n) \quad P_{n+1}(n+1, x) = (N + D') P_n(n+1, x) + D \frac{dP_n(n+1, x)}{dx} .$$

We single out the case $k = n$ because of the fact that this case parallels most closely the Charlier or Hermite polynomial case. For in this latter case the n'th derivative of the generating function e^{-x^2} is the product of the generating function and a polynomial of degree n. *So in the case of any solution* y *of a Pearson differential equation, the* n *th derivative of* $D^n y$ *is the product of the generating function* y *and a polynomial of degree at most* n.

By means of relation (I), we can write down the successive polynomials $P_1(x)$, $P_2(x)$, ... The first five polynomials may be written as follows:

$$P_1(x) = N,$$

$$P_2(x) = (N - D')P_1(x) + D\frac{dP_1(x)}{dx} = N^2 - ND' + N'D,$$

$$P_3(x) = (N - 2D')P_2(x) + D\cdot\frac{dP_2(x)}{dx}$$

$$= N^3 - 3N^2D' + 3NN'D + 2ND'^2 - 2N'D'D - NDD'',$$

$$P_4(x) = (N - 3D')P_3(x) + D\frac{dP_3(x)}{dx}$$

$$= N^4 - 6N^3D' + 6N^2N'D + 11N^2D'^2 - 14NN'DD' - 4N^2DD'',$$

$$= -6ND'^3 + 6N'DD'^2 + 6NDD'D'' + 3N'^2D^2 - 3N'D^2D'',$$

$$P_5(x) = (N - 4D')P_4(x) + D\frac{dP_4(x)}{dx}$$

$$= N^5 - 10N^4D' + 10N^3N'D + 35N^3D'^2 - 50N^2N'DD'$$

$$- 10N^3DD'' - 50N^2D'^3 + 70NN'DD'^2 - 40N^2DD'D''$$

$$+ 15NN'^2D^2 - 25NN'D^2D'' + 24ND'^4 - 24N'DD'^3$$

$$- 36NDD'^2D'' - 20N'^2D^2D' + 24N'D^2D'D'' + 6ND^2D''^2$$

4. Following the analogy with Hermite polynomials, we obtain next a recurrence relation involving the ($n+1$) th, n'th and ($n-1$) th polynomials.

Starting with the original differential equation

$$D\frac{dy}{dx} = Ny$$

we take the n th derivative of both sides, which by Leibnitz's theorem on the derivative of a product gives us, since $\frac{d^3D}{dx^3} = 0$,

$$D\frac{d^{n+1}y}{dx^{n+1}} + nD'\frac{d^n y}{dx^n} + \frac{n(n-1)}{2!}D''\frac{d^{n-1}y}{dx^{n-1}} = N\frac{d^n y}{dx^n} + nN'\frac{d^{n-1}y}{dx^{n-1}}$$

Multiplying this last expression by D^n and collecting terms, we get:

$$D^{n+1}\frac{d^{n+1}y}{dx^{n+1}} + D^n(nD'-N)\frac{d^n y}{dx^n} + D^n\left[\frac{n(n-1)}{2!}D'' - nN'\right]\frac{d^{n-1}y}{dx^{n-1}} = 0$$

Replacing now $D^n \frac{d^n y}{dx^n}$ by $P_n(x)\,y$ and dividing through by y, we get the recurrence relation

$$\text{(II)}\quad P_{n+1}(x) + (nD'-N)P_n(x) + n\left[\frac{(n-1)}{2!}D'' - N'\right]D\,P_{n-1}(x) = 0$$

We note that the coefficients of $P_{n+1}(x)$ and $P_n(x)$ are the same as in relation (I) which we found to be

$$P_{n+1}(x) + P_n(x)(nD'-N) = D\,\frac{dP_n(x)}{dx}$$

Hence

$$\text{(III)}\qquad \frac{dP_n(x)}{dx} = n\left[N' - \frac{(n-1)}{2}D''\right]P_{n-1}(x)$$

or replacing n by $n+1$ we write:

$$\frac{dP_{n+1}(x)}{dx}=(n+1)(N'-\frac{n}{2}D'')P_n(x)=(n+1)(a_1-nb_2)\ P_n(x).$$

This equation is the generalized form of the one for Hermite polynomials, viz.:

$$\frac{dH_n(x)}{dx}=2n\,H_{n-1}(x)$$

5. Relations (I) and (III) may now be used to obtain a second order differential equation. Differentiating (I), we get:

$$P'_{n+1}(x)+(nD''-N')P_n(x)+(nD'-N)P_n'(x)$$
$$-D'P_n'(x)-DP_n''(x)=0.$$

Substitution of the value $dP_{n+1}(x)/dx$ from (III) gives us:

$$\text{(IV)}\qquad \begin{aligned}&DP_n''(x)+\left[N-(n-1)D'\right]P_n'(x)\\ &-n\left[N'-\frac{(n-1)D''}{2}\right]P_n(x)=0\end{aligned}$$

We readily see that the relation found for the Hermite polynomials

$$H_n''(x)-2xH_n'(x)+2nH_n(x)=0$$

is a special case of (IV).

Using the lemma previously proved and replacing N by $N+kD'$ we can write (IV) for the polynomials $P_n(k,x)$ and $P_n(n,x)$:

$$(\text{IV}_k)\qquad \begin{aligned}&DP_n''(k,x)+\left[N-(n-k-1)D'\right]P_n'(k,x)\\ &-n\left[N'-\frac{(n-2k-1)}{2}D''\right]P_n(k,x)=0,\end{aligned}$$

$$(\mathrm{IV}_n)^1 \quad \begin{aligned} &DP_n''(n,x)+(N+D')P_n'(n,x) \\ &-n\left[N'+\frac{(n+1)}{2}D''\right]P_n(n,x)=0 \end{aligned}$$

We recognize the second order differential equations mentioned earlier in this chapter for the polynomials of the Pearson Type

[1]Since D is any expression of the second degree and N is any expression of the first degree, it is obvious that $P_n(x)$ satisfies a linear equation of the second order of the form:

$$(A_0+A_1x+A_2x^2)y''+(B_0+B_1x)y'+Cy=0$$

where $C=-n\left[(n-1)A_2+B_1\right]$. It may be shown that if a differential equation of the form considered has as one solution a polynomial of degree n then C must be of the form specified. For suppose $Q_n(x)$ satisfies the above differential equation for y. Taking the n'th derivative of this equation we get

$$\frac{n(n-1)}{2!}\cdot 2\,A_2(n!a_0)+nB_1(n!a_0)+C(n!a_0)=0$$

and solving for C that:

$$C=-n\left[(n-1)A_2+B_1\right].$$

It follows from our work that if a differential equation has the form

$$\begin{aligned} &(A_0+A_1x+A_2x^2)y''+(B_0+B_1x)y' \\ &-n\left[(n-1)A_2+B_1\right]y=0 \end{aligned}$$

then one solution of this differential equation is a polynomial of degree at most n obtained by finding the solution y of the Pearson differential equation

$$\frac{dy}{dx}=\frac{B_0+B_1x-(A_1+2A_2x)}{A_0+A_1x+A_2x^2}\,y$$

and determining the polynomial

$$P_n(n,x)=\frac{1}{y}\frac{d^n}{dx^n}\left\{\left[A_0+A_1x+A_2x^2\right]^n y\right\}.$$

IV, V and VI as well as the Jacobi and Laguerre polynomials as special cases of formula (IV_n). Some further illustrations of (IV_n) are the Tschebycheff[1] and Legendre[2] polynomials. The Tschebycheff polynomials are developed from the differential equation

$$\frac{dy}{dx} = \frac{x}{1-x^2} y$$

and in this case formula (IV_n) becomes:

$$(1-x^2)P_n''(n,x) - xP_n'(n,x) + n^2 P_n(n,x) = 0$$

The Legendre polynomials

$$P_n(x) = \frac{1}{2^n n!} \quad \frac{d^n (x^2-1)^n}{dx^n}$$

have as a corresponding differential equation

$$\frac{dy}{dx} = \frac{0 \cdot y}{x^2-1}$$

and in turn formula (IV_n) is written:

$$(x^2-1)P_n''(n,x) + 2xP_n'(n,x) - n(n+1)P_n(n,x) = 0$$

6. Just as in formula (II) we established a recurrence relation for the polynomials $P_n(x)$, let us now obtain one for the polynomials $P_n(n,x)$.

Consider once more the first derivative of $D^k y$, i. e.

$$\frac{d}{dx}(D^{k+1}y) = (K+1)D' \cdot D^k y + D^{k+1}y'$$

$$= \left[N + (K+1)D'\right] D^k y$$

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[1]R. Courant and D. Hilbert, op. cit., pp. 73-74.

[2]Ibid, pp. 66-69.

Taking the n th derivative of both sides of the equation we get:

$$\frac{d^{n+1}}{dx^{n+1}}(D^{k+1}y) = \left[N+(k+1)D'\right]\frac{d^n}{dx^n}D^k y$$

$$+n\left[N'+(k+1)D''\right]\frac{d^{n-1}}{dx^{n-1}}D^k y \;.$$

Multiplying both sides of the equation by D^{n-k} and replacing $D^{n-k}\dfrac{d^n}{dx^n}D^k y$ by $P_n(k,x)y$, we have

$$(\mathrm{V}_k)\qquad \begin{aligned} P_{n+1}(k+1,x) &= \left[N+(k+1)D'\right]P_n(k,x) \\ &\quad + n\left[N'+(k+1)D''\right]D\cdot P_{n-1}(k,x). \end{aligned}$$

In case we set $k=n$, we may write

$$(\mathrm{V}_n)\qquad \begin{aligned} P_{n+1}(n+1,x) &= \left[N+(n+1)D'\right]P_n(n,x) \\ &\quad + n\left[N'+(n+1)D''\right]D\cdot P_{n-1}(n,x), \end{aligned}$$

a recurrence relation similar to (II) and involving the polynomials $P_{n+1}(n+1,x)$, $P_n(n,x)$ and $P_{n-1}(n,x)$.

7. Formula (V_n) may be written in still another form corresponding to formula (I), i. e. a relation consisting of the same terms as (V_n) except that the $(n-1)$ th polynomial $P_{n-1}(n,x)$ is replaced by the first derivative of the n th polynomial $P_n(n,x)$.

In order to obtain this relation we return to formula (III),

$$\frac{dP_n(x)}{dx} = n\left[N' - \frac{(n-1)}{2}D''\right]P_{n-1}(x)$$

and substitute for N the value $N+kD'$ and obtain

$$(\mathrm{III}_n)\qquad \frac{dP_n(n,x)}{dx} = n\left[N' + \frac{(n+1)}{2}D''\right]P_{n-1}(n,x)$$

or

$$P_{n-1}(n,x) = \frac{1}{n\left[N' + \left(\frac{n+1}{2}\right)D''\right]} \frac{dP_n(n,x)}{dx}$$

Substituting the value for $P_{n-1}(n,x)$ we thus obtain:

$$\text{(VI)} \qquad P_{n+1}(n+1,x) = \left[N + (n+1)D'\right]P_n(n,x) + \frac{N' + (n+1)D''}{N' + \left(\frac{n+1}{2}\right)D''} \cdot D \cdot \frac{P_n(n,x)}{dx}$$

From symmetry we might expect the fractional coefficient of the derivative $P_n'(n,x)$ to be unity, but unfortunately this is not the case.

8. In looking over the relations existing for the Laguerre polynomials we find one consisting of the first derivatives of the n th and $(n-1)$ th polynomials, and the $(n-1)$ th polynomial,[1] i. e.

$$P_n'(n,x) - nP_{n-1}'(n,x) = -n P_{n-1}(n-1,x)$$

This relation is a special case of another form of formula (VI) which we obtain in the following manner:
Differentiation of (VI) gives us:

$$\frac{dP_{n+1}(n+1,x)}{dx} = \left[N' + (n+1)D''\right]P_n(n,x) + \left[N + (n+1)D'\right]\frac{dP_n(n,x)}{dx}$$

$$+ \frac{N' + (n+1)D''}{N' + \frac{(n+1)}{2}D''} \cdot D' \cdot \frac{dP_n(n,x)}{dx} + \frac{N' + (n+1)D''}{N' + \frac{(n+1)}{2}D''} \cdot D \cdot \frac{d^2P_n(n,x)}{dx^2}$$

Substituting the value for $d^2P_n(n,x)/dx^2$ found in (V) changes this last expression to the form:

[1] R. Courant and D. Hilbert, op. cit., pp. 77-79.

$$\frac{dP_{n+1}(n+1,x)}{dx}=\left[N'+(n+1)D''\right]P_n(n,x)+\left[N+(n+1)D'\right]\frac{dP_n(n,x)}{dx}$$

$$+\frac{N'+(n+1)D''}{N'+\frac{(n+1)}{2}D''}\cdot D'\cdot\frac{dP_n(n,x)}{dx}+\frac{N'+(n+1)D''}{N'+\left(\frac{n+1}{2}\right)D''}$$

$$\left[-(N+D')\frac{dP_n(n,x)}{dx}+n\left(\frac{n+1}{2}D''+N'\right)P_n(n,x)\right],$$

which reduces to

$$\frac{dP_{n+1}(n+1,x)}{dx}=(n+1)\left[N'+(n+1)D''\right]P_n(n,x)$$

(VII)

$$+\left\{\left[N+(n+1)D'\right]-\frac{N'+(n+1)D''}{N'+\frac{(n+1)}{2}D''}\cdot N\right\}\frac{dP_n(n,x)}{dx}\,.$$

The special equation mentioned for the Laguerre polynomials will be recognized as a special case of formula (VII) if we recall that for the Laguerre polynomials the differential equation is of the form

$$\frac{dy}{dx}=\frac{p-x}{x}\,y$$

Substitution of x for D and $(p-x)$ for N reduces (VII) to

$$P'_{n+1}(n+1,x)=-(n+1)P_n(n,x)+(n+1)P'_n(n,x)\,.$$

9. In this chapter we have defined two general types of

polynomials $$P_n(x)=\frac{D^n}{y}\frac{d^ny}{dx^n}$$

and $$P_n(k,x)=\frac{D^{n-k}}{y}\frac{d^n}{dx^n}D^ky\,.$$

The relationships for these polynomials $P_n(x)$ and $P_n(k,x)$ were derived without using the form of the solution of the differential equation. Two fundamental formulas were derived, for $P_n(x)$:

(I) $$P_{n+1}(x)=(N-nD')P_n(x)+D\frac{dP_n(x)}{dx}$$

and for $P_n(n, x)$ the corresponding formula:

$$\text{(VI)}\quad P_{n+1}(n+1,x) = \left[N+(n+1)D'\right]P_n(n,x) + \frac{N'+(n+1)D''}{N'+\frac{(n+1)}{2}D''}D\frac{dP_n(n,x)}{dx}.$$

Two successive polynomials were shown to be related by the relations, for $P_n(x)$:

$$\text{(III)}\qquad \frac{dP_n(x)}{dx} = n\left[N' - \frac{(n-1)}{2}D''\right]P_{n-1}(x)$$

and for $P_n(n, x)$:

$$\text{(III}_n\text{)}\qquad \frac{dP_n(n,x)}{dx} = n\left[N' + \frac{(n+1)}{2}D''\right]P_{n-1}(n,x)$$

In addition we found that it was possible to set up recurrence relations involving the $(n+1)$th, nth and $(n-1)$th polynomials and found these to be, for $P_n(x)$:

$$\text{(II)}\quad P_{n+1}(x) + (nD'-N)P_n(x) + n\left[\frac{n-1}{2!}D'' - N'\right]D\cdot P_{n-1}(x) = 0$$

and for $P_n(n, x)$:

$$\text{(V}_n\text{)}\quad P_{n+1}(n+1,x) = \left[N+(n+1)D'\right]P_n(n,x) + n\left[N'+(n+1)D''\right]\cdot D\,P_{n-1}(n,x)$$

We further succeeded in developing a second order differential equation for the n'th polynomial $P_n(x)$:

$$\text{(IV)}\quad DP_n''(x) + \left[N-(n-1)D'\right]P_n'(x) - n\left[N' - \frac{(n-1)}{2}D''\right]P_n(x) = 0$$

and for $P_n(n,x)$:

$$\text{(IV}_n\text{)}\quad DP_n''(n,x) + (N+D')P_n'(n,x) - n\left[N' + \frac{(n+1)}{2}D''\right]P_n(n,x) = 0$$

We also showed that we could derive a relation between the derivatives of the polynomials $P_{n+1}(n+1, x)$, $P_n(n, x)$ and the polynomial $P_n(n, x)$:

$$\text{(VII)} \quad \frac{d P_{n+1}(n+1, x)}{dx} = (n+1)\left[N' + (n+1)D''\right] P_n(n, x) +$$

$$\left\{\left[N + (n+1)D'\right] - \frac{N' + (n+1)D''}{N' + \left(\frac{n+1}{2}\right)D''} \cdot N\right\} \frac{dP_n(n, x)}{dx}.$$

Finally, we noted that all of these formulas and relations apply to the Hermite, Jacobi, Tschebycheff and Legendre polynomials as well as the polynomials derived for the Pearson Type IV, V and VI curves by Romanovsky.

CHAPTER III

1. So far the discussion in this paper has been limited to the treatment of the Gram-Charlier series where the constants $A_0, A_1, A_2, \ldots, A_n, \ldots$ depend upon polynomials in x which are independent of the function $F(x)$, and the generating function $f(x)$ is a solution of the Pearson differential equation, the functions $F(x)$ and $f(x)$ being defined as continuous functions. The work in mathematical statistics involves not only the use of the continuous variate and the continuous function but also the case of the discrete variate and the discontinuous function where this function is defined for equally spaced values.

In dealing with the continuous variate we make use of the theory of the differential and integral calculus, or the calculus of limits, as it is sometimes called. On the other hand, for the discrete variate we turn to the theory of the calculus of finite differences. Further, it usually happens that there exists a parallelism between results based on the derivative and integral and those based on the finite differences and summations. As a consequence, it seems natural to attempt to derive results for the finite difference case paralleling those contained in the first half of this paper. The second part of this paper is devoted to this purpose. The first of the two following chapters considers matters pertaining to Charlier's Type B series which is the finite difference parallel to the Type A series, while the next chapter is devoted to the polynomials connected with the finite difference parallel of the Pearson differential equation.

Charlier in the second half of his article[1] "Ueber die Dar-

*C. V. L. Charlier, op. cit., pp. 23-35.

stellung willkürlicher Funktionen" considers a real valued function $F(x)$ and asserts that it may be formally expanded in terms of another function and its successive differences. Stated as a theorem, this may be written as follows:

CHARLIER'S THEOREM FOR SERIES B: *Any real valued function $F(x)$ which vanishes for $x = \infty$ and $-\infty$, may be formally expanded in terms of another function $g(x)$ and its successive differences in the form*

$$(B)\ F(x) = B_0 g(x) + B_1 \Delta g(x) + B_2 \Delta^2 g(x) + \cdots + B_n \Delta^n g(x) + \cdots$$

where $g(x)$ possesses the properties:

(a) $g(x)$ *and its differences are defined for all real values of x,*

(b) $g(x)$ *and its differences vanish for $x = +\infty$ and $-\infty$,*

(c) $x^m \Delta^n g(x) \Big|_{-\infty}^{+\infty} = 0$ *for all real values of m and n.*

(d) $\Delta^{-1} g(x) \Big|_{-\infty}^{+\infty} \neq 0.$

Paralleling the theory of the first half of his paper, Charlier determines the constants $B_0, B_1, B_2, \ldots, B_n, \ldots$ and finds that they may be expressed by the equation

$$B_n = \sum_{-\infty}^{+\infty} Q_n(x) F(x) = \Delta^{-1} Q_n(x) F(x) \Big|_{-\infty}^{+\infty}$$

where $Q_n(x)$ is a polynomial in x of degree not greater than n. Analyzing the answers that he obtains for $Q_n(x)$, we find that these polynomials form a uniquely determined set of polynomials $Q_0(x), Q_1(x), \ldots, Q_2(x), \ldots, Q_n(x), \ldots, Q_n(x)$ at most of degree n, biorthogonal in the sum sense to the successive differences of the function $g(x)$, i. e. they satisfy the biorthogonality conditions for the inverse of differences:

$$\Delta^{-1} Q_n(x)\, \Delta^m g(x) \Big|_{-\infty}^{+\infty} \begin{matrix} = 0 & \text{for} & n \neq m \\ 1 & \text{for} & n = m. \end{matrix}$$

Charlier does not observe that the polynomials $Q_n(x)$ bear a definite relation to one another, i. e.

$$\Delta Q_n(x) = - Q_{n-1}(x+1),$$

a relation similar to the one found for the polynomials $P_n(x)$ in Chapter I. We may state these facts in the following theorem:

THEOREM: *If* $g(x)$ *satisfy the conditions* (a), (b), (c), *and* (d) *of Charlier's Theorem for series* B *and if* $Q_0(x)$, $Q_1(x)$, . . ., $Q_n(x)$, . . . *is the system of polynomials in* x, $Q_n(x)$ *of degree at most* n, *which is biorthogonal to* $f(x)$ *and its differences, i. e. satisfies the conditions*

$$\Delta^{-1} Q_n(x) \Delta^m g(x) \Big|_{-\infty}^{+\infty} \begin{matrix} = 0 \text{ for } n \neq m \\ = 1 \text{ for } n = m \end{matrix}$$

then

$$\Delta Q_n(x) = - Q_{n-1}(x+1).$$

The proof requires the use of the finite integration by parts formula:

$$\Delta^{-1} u_x v_x = u_x \Delta^{-1} v_x - \Delta^{-1}\left[\Delta u_x \cdot \Delta^{-1} v_{x+1}\right].$$

Applying this formula we get

$$\Delta^{-1} Q_n(x) \Delta^m g(x) \Big|_{-\infty}^{+\infty} = Q_n(x) \cdot \Delta^{m-1} g(x) \Big|_{-\infty}^{+\infty}$$

$$- \Delta^{-1}\left[\Delta Q_n(x)\, \Delta^{m-1} g(x+1)\right]_{-\infty}^{+\infty}$$

The first term on the right hand side vanishes due to condition (c) of the theorem of Charlier. Comparing the term which remains, i. e.

$$-\Delta^{-1}\left[\Delta Q_n(x)\Delta^{m-1}g(x+1)\right]_{-\infty}^{+\infty} \begin{array}{ll} = 0 & \text{for } n \neq m \\ = 1 & \text{for } n = m \end{array}$$

with the biorthogonality condition

$$\Delta^{-1}\left[Q_{n-1}(x+1)\Delta^{m-1}g(x+1)\right]_{-\infty}^{+\infty} \begin{array}{ll} = 0 & \text{for } n \neq m \\ = 1 & \text{for } n = m \end{array}$$

we conclude that

$$\Delta Q_n(x) = -Q_{n-1}(x+1)$$

This theorem enables us to find the terms of the n th polynomial by taking the negative of the integral of the $(n-1)$ th polynomial, except for the constant of integration. Following the suggestion in our first chapter, we may also determine this constant. We have

$$Q_n(x) = -\Delta^{-1}Q_{n-1}(x+1)\Big|_0^x + C$$

and the simple biorthogonality condition

$$\Delta^{-1}Q_n(x)g(x)\Big|_{-\infty}^{+\infty} = 0.$$

It follows that

$$\Delta^{-1}\left[-\Delta^{-1}Q_{n-1}(x+1)+C\right]_0^x g(x)\Big|_{-\infty}^{+\infty} = 0$$

and solving for C we get

$$C = \frac{\Delta^{-1}\left[\Delta^{-1}Q_{n-1}(x+1)\right]_0^x g(x)\Big|_{-\infty}^{+\infty}}{\Delta^{-1}g(x)\Big|_{-\infty}^{+\infty}}$$

We may therefore determine the polynomials $Q_n(x)$ from the polynomials next preceding by the formula

$$Q_n(x) = -\Delta^{-1}Q_{n-1}(x+1)\Big|_0^x + \frac{\Delta^{-1}\left[\Delta^{-1}Q_{n-1}(x+1)\right]_0^x g(x)\Big|_{-\infty}^{+\infty}}{\Delta^{-1}g(x)\Big|_{-\infty}^{+\infty}}$$

If we adopt the Charlier notation

$$\varepsilon_m = \sum_{-\infty}^{+\infty} x^m g(x) = \Delta^{-1} x^m g(x) \Big|_{-\infty}^{+\infty}$$

and the common notation $x^{(m)} = x(x-1)(x-2) \cdot \cdot \cdot \cdot \cdot (x-m+1)$ and observe that $Q_0(x) = 1/\varepsilon_0$ and that

$$\Delta^{-1} x^{(m)} = \frac{x^{(m+1)}}{m+1}$$

we may obtain the polynomials $Q_1(x)$, $Q_2(x)$, ... without much computation as follows:

$$Q_1(x) = -\Delta^{-1} Q_0(x+1) \Big|_0^x + \frac{\Delta^{-1}\left[\Delta^{-1} Q_0(x+1)\right]_0^x g(x) \Big|_{-\infty}^{+\infty}}{\Delta^{-1} g(x) \Big|_{-\infty}^{+\infty}}$$

$$= -\frac{x}{\varepsilon_0} + \frac{\varepsilon_1}{\varepsilon_0^2}$$

$$Q_2(x) = -\Delta^{-1} Q_1(x+1) \,{}_0^x + \frac{\Delta^{-1}\left[\Delta^{-1} Q_1(x+1)\right]_0^x g(x) \Big|_{-\infty}^{+\infty}}{\Delta^{-1} g(x) \Big|_{-\infty}^{+\infty}}$$

$$= \frac{(x+1)^{(2)}}{|\underline{2}\, \varepsilon_0} - \frac{\varepsilon_1(x+1)}{\varepsilon_0^2} + \frac{2\varepsilon_1^2 + \varepsilon_1\varepsilon_0 - \varepsilon_2\varepsilon_0}{|\underline{2}\, \varepsilon_0^3}$$

$$\text{or } |\underline{2}\, \varepsilon_0^3 Q_2(x) = \varepsilon_0^2 x^2 - \varepsilon_0 x(2\varepsilon_1 - \varepsilon_0) + 2\varepsilon_1^2 - \varepsilon_2\varepsilon_0 - \varepsilon_1\varepsilon_0$$

$$Q_3(x) = -\Delta^{-1} Q_2(x+1) \Big|_0^x + \frac{\Delta^{-1}\left[\Delta^{-1} Q_2(x+1)\right]_0^x g(x) \Big|_{-\infty}^{+\infty}}{\Delta^{-1} g(x) \Big|_{-\infty}^{\infty}}$$

$$= -\frac{(x+2)^{(3)}}{|\underline{3}\, \varepsilon_0} + \frac{\varepsilon_1(x+2)^{(2)}}{|\underline{2}\, \varepsilon_0^2} - \frac{(2\varepsilon_1^2 + \varepsilon_1\varepsilon_0 - \varepsilon_2\varepsilon_0)(x+2)}{|\underline{2}\, \varepsilon_0^3}$$

$$+ \frac{\varepsilon_3\varepsilon_0^2 - 3\varepsilon_2\varepsilon_0^2 - 6\varepsilon_2\varepsilon_1\varepsilon_0 + 6\varepsilon_1^3 + 6\varepsilon_1^2\varepsilon_0 + 2\varepsilon_1\varepsilon_0^2}{|\underline{3}\, \varepsilon_0^4}$$

$$\text{or } |\underline{3}\, \varepsilon_0^4 Q_3(x) = -\varepsilon_0^3 x^3 + 3\varepsilon_0^2 x^2(\varepsilon_1 - \varepsilon_0)$$

$$- \varepsilon_0 x(2\varepsilon_0^2 - 6\varepsilon_1\varepsilon_0 - 3\varepsilon_2\varepsilon_0 + 6\varepsilon_1^2) + \varepsilon_0^2\varepsilon_3 + 3\varepsilon_2\varepsilon_0^2 + 2\varepsilon_1\varepsilon_0^2$$

$$- 6\varepsilon_2\varepsilon_1\varepsilon_0 - 6\varepsilon_1^2\varepsilon_0 + 6\varepsilon_1^3$$

. . . .

These results differ slightly from those obtained by Charlier in his article. This is due to the definition for differences used by Charlier, viz.:

$$\Delta g(x) = g(x) - g(x-1)$$

while we have used the definition

$$\Delta g(x) = g(x+1) - g(x) .$$

Denoting the difference

$$g(x) - g(x-1) \text{ by } \delta g(x)$$

Charlier determines a set of polynomials $T_n(x)$ satisfying the conditions ,

$$\delta^{-1}\left[T_n(x)\,\delta^m g(x)\right] \begin{array}{ll} = 0 & \text{for } m \neq n \\ = 1 & \text{for } m = n \end{array}$$

As a consequence by paralleling the reasoning above one proves easily that the $T_n(x)$ satisfy the recurrence relation

$$T_n(x+1) - T_n(x) = -T_{n-1}(x) .$$

By using this relation and the fact that

$$\delta^n g(x+n) = \Delta^n g(x)$$

it can be shown without much difficulty that

$$T_n(x+n-1) = Q_n(x)$$

The theorem proved in Ch. 1, par. 2, could no doubt be paralleled by using finite difference theory. Since the method of procedure is obvious there seems to be no need of taking it up in detail.

We have succeeded in showing in this chapter that the problem of determining the constants for the Charlier Type B series closely parallels the work of the first chapter and that these constants are readily obtained by using the biorthogonality conditions for finite differences.

CHAPTER IV

Polynomials Connected with the Pearson Difference Equation

1. In Chapter II we referred to certain solutions $f(x)$ of the Pearson differential equation and noted that graphically, these functions represented types of curves used in statistical work. Paralleling this work, we would expect to find that a difference equation similar in composition to the Pearson differential equation would have as solutions functions $g(x)$ which could be used to represent data consisting of discrete variates.

Carver, in an article in the "Handbook of Mathematical Statistics,"* suggests the use of a difference equation corresponding to the Pearson differential equation, i. e.:

$$\Delta u_x = \frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots} u_x ,$$

a difference equation with a numerator of the first and denominator of any desired degree in x. If we confine our work to a denominator of degree at most of the second in x, we should be able to obtain results comparing very favorably with those obtained in the second chapter.

An illustration of a solution of this difference equation found in Charlier's article "Ueber die Darstellung willkürlicher Funktionen,"[2] is the well known Poisson exponential function

$$\psi(x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

[1]H. C. Carver, "Frequency Curves," Handbook of Mathematical Statistics (H. L. Rietz, Editor), Chapter VII, pp. 111-114.

[2]C. V. L. Charlier, op. cit. p. 33.

This function satisfies the difference equation

$$\Delta u_x = \frac{\lambda - x - 1}{x+1}\, u_x$$

and this equation is recognized as a special form of the Pearson difference equation. If we take the successive differences of this Poisson exponential function, we find that these give rise to a unique set of polynomials. These polynomials may be written in the following form:

$$Q_1(x) = \lambda - (x+1)\,,$$

$$Q_2(x) = \lambda^2 - 2\lambda(x+2) + (x+2)(x+1)\,,$$

or making use of the usual difference notation for $x^{(m)} = x(x-1)(x-2)\cdot\cdots\cdot(x-m+1)$, we write

$$Q_2(x) = \lambda^2 - 2\lambda(x+2) + (x+2)^{(2)},$$

$$Q_3(x) = \lambda^3 - 3\lambda^2(x+3) + 3\lambda(x+3)^{(2)} - (x+3)^{(3)},$$

or $$Q_3(x-3) = \lambda^3 - 3\lambda^2 x + 3\lambda x^{(2)} - x^{(3)},$$

. . . .

$$Q_n(x) = \lambda^n - {}_nC_1\lambda^{n-1}(x+n) + {}_nC_2\lambda^{n-2}(x+n)^{(2)} + \cdots + (-1)^n(x+n)^{(n)}$$

or $$Q_n(x-n) = \lambda^n - {}_nC_1\lambda^{n-1}x + {}_nC_2\lambda^{n-2}x^{(2)} + \cdots + (-1)^n x^{(n)}.$$

These polynomials have the same form as that for the binomial expansion $(\lambda - x)^n$, particularly if we use the difference notation for representing powers of x. In other words, we might look upon the n th polynomial as being defined as

$$Q_n(x-n) = \left[\lambda - x^{(1)}\right]^n$$

A careful examination of these polynomials brings out the fact that consecutive ones are related to each other, viz., that we have,

$$\Delta Q_n(x) = -nQ_{n-1}(x+1) .$$

This relation is similar to the one found for Hermite polynomials.

The fact that the Charlier Type A. series in Chapter II consisted of successive derivatives and that the derivatives of the solutions of the Pearson differential equation led to a system of polynomials definitely related to one another, gave rise to the theory developed in that chapter. We found that it was not necessary in this theory to consider the form of the solution of the equation, but that a set of general polynomials could be set up which satisfied all the properties of the special polynomials. The Charlier Type B series consists of successive differences of a function $g(x)$ and it is quite natural for us to suspect that we can develop for the solutions of the Pearson difference equation a corresponding theory on polynomials.

This question of obtaining a system of polynomials from the solutions of the Pearson difference equation

$$(1) \qquad \Delta u_x = \frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2} u_x ,$$

numerator of the first degree and denominator of the second degree, will concern us in this chapter. We shall further show that these polynomials are related to one another by means of first and second order difference relations and by means of recurrence relations involving the $(n+1)$ th, n th and $(n-1)$ th polynomials, and shall illustrate these equations with the Poisson exponential function.

2. For convenience denote the numerator $(a_0 + a_1 x)$ in equation (1) by N_x and the denominator $(b_0 + b_1 x + b_2 x^2)$ by D_x . We may then define a set of polynomials by the following theorem:

THEOREM: *If* u_x *is a non-identically zero solution of*

$$\Delta u_x = \frac{N_x}{D_x} u_x$$

then $\frac{1}{u_x} D_x D_{x+1} D_{x+2} \cdots D_{x+n-1} \Delta^n u_x$ *is a polynomial of degree at most* n, i. e. $Q_n(x)$.

The proof will proceed by mathematical induction. If we recall the formula for the difference of a product

$$\Delta [u_x v_x] = v_x \Delta u_x + u_{x+1} \Delta v_x = v_x \Delta u_x + (u_x + \Delta u_x) \Delta v_x ,$$

we obtain by differencing

$$D_x \Delta u_x = N_x u_x = Q_1(x) u_x$$

the equation

$$D_{x+1} \Delta^2 u_x + \Delta u_x \Delta D_x = [Q_1(x) + \Delta Q_1(x)] \Delta u_x + u_x \cdot \Delta Q_1(x).$$

Using the value for Δu_x from the original difference equation and multiplying the equation through by D_x, we obtain:

$$D_x D_{x+1} \Delta^2 u_x = [N_x Q_1(x) + D_x \Delta Q_1(x) + N_x \Delta Q_1(x) + N_x \Delta D_x] u_x$$

Since the coefficient of u_x is a polynomial of degree at most 2 in x, we write

$$D_x D_{x+1} \Delta^2 u_x = Q_2(x) u_x .$$

Let us now assume that the statement holds for $m \leq n$, i. e.

$$D_x D_{x+1} D_{x+2} \cdots D_{x+n-1} \Delta^n u_x = Q_n(x) u_x .$$

Differencing both sides of this equation gives us

$$D_x D_{x+1} D_{x+2} \cdots D_{x+n-1} \Delta^{n+1} u_x + (\Delta^n u_x + \Delta^{n+1} u_x)(\Delta D_x D_{x+1} D_{x+2} \cdots D_{x+n-1})$$

$$= Q_n(x) u_x + (u_x + \Delta u_x) \Delta Q_n(x).$$

Now

$$\Delta D_x D_{x+1} \cdots D_{x+n-1} = D_{x+1} D_{x+2} \cdots D_{x+n} - D_x D_{x+1} \cdots D_{x+n-1} =$$

$$= (D_{x+1} \cdots D_{x+n-1})(D_{x+n} - D_x).$$

Hence by the definition of $Q_n(x)$

$$\Delta\left[D_x D_{x+1} \cdots D_{x+n-1} \Delta^n u_x\right] = \frac{D_{x+n} - D_x}{D_x} Q_n(x) u_x .$$

Substituting these values in the above equation as well as the value for Δu_x from (1) and multiplying by D_x, the equation reduces to

$$D_x D_{x+1} D_{x+2} \cdots D_{x+n} \Delta^{n+1} u_x = \Big[N_x Q_n(x) + D_x \Delta Q_n(x)$$

$$+ N_x \Delta Q_n(x) - D_{x+n} Q_n(x) + D_x Q_n(x) \Big] u_x .$$

The coefficient of u_x on the right hand side is a polynomial of degree at most n in x. We therefore conclude that

$$D_x D_{x+1} \cdots D_{x+n} \Delta^{n+1} u_x = Q_{n+1}(x) u_x .$$

We have also succeded in deriving a relation similar to relation (I) of Chapter II, i. e.

$$\text{(XI)} \qquad \begin{aligned} Q_{n+1}(x) &= (N_x + D_x - D_{x+n}) Q_n(x) \\ &\quad + (N_x + D_x) \Delta Q_n(x), \end{aligned}$$

a relation which shows that the ($n+1$) th polynomial is made up of the n th polynomial and the difference of the n th polynomial. This relation differs from relation (I) in the fact that the coefficient of $\Delta Q_n(x)$ is $N_x + D_x$ instead of D_x . This change seems to be connected with the fact that the original difference equation

$$D_x \Delta u_x = N_x u_x$$

can also be written

$$u_{x+1} = \frac{N_x + D_x}{D_x} u_x .$$

Formula (XI) may also be written

$$\text{(XI)} \quad Q_{n+1}(x) = (N_x + D_x) Q_n(x+1) - D_{x+n} Q_n(x)$$

since $\quad Q_n(x) + \Delta Q_n(x) = Q_{n+1}(x).$

It seems advisable to adopt a notation for the term

$$D_x D_{x+1} D_{x+2} \cdots D_{x+n-1}$$

since it will continue to be involved in the work that is to follow. The difference notation $x^{(m)} = x(x-1)(x-2) \cdots (x-m+1)$ suggests that we use the symbol $D_x^{(n)}$, i. e.

$$D_x^{(n)} = D_x D_{x-1} D_{x-2} \cdots D_{x-n+1}.$$

Then we will have

$$D_x D_{x+1} D_{x+2} \cdots D_{x+n-1} = D_{x+n-1}^{(n)}$$

and

$$\Delta D_{x+n-1}^{(n)} = D_{x+n} D_{x+n-1} \cdots D_{x+2} D_{x+1}$$

$$- D_{x+n-1} D_{x+n-2} \cdots D_{x+1} D_x = (D_{x+n} - D_x) \cdot D_{x+n-1}^{(n-1)}.$$

3. We may also define the general polynomials $Q_n(m, x)$ where m is any integer, by means of a theorem as follows:

THEOREM: *If* u_x *is a non-identically zero solution of the difference equation* (1), *then*

$$\frac{D_{x-m+n-1}^{(n)} \; \Delta^n \left[D_{x-1}^{(m)} \; u_x \right]}{D_{x-1}^{(m)} \; u_x}$$

is a polynomial $Q_n(m, x)$, *and* $Q_n(m, x)$ *is at most of degree* n *in* x. *In particular if* $m = n$, *we have*

$$\frac{1}{u_x} \Delta^n \left[D_{x-1}^{(n)} \; u_x \right]$$

is a polynomial in x *of degree at most* n.

This theorem may be proved by using the following lemma:

LEMMA: *If* u_x *satisfy the difference equation* (1), *then* $D_{x-1}^{(m)} u_x$, *where* m *is any positive integer, satisfies a difference equation of the same type, viz.*:

$$\Delta \left[D_{x-1}^{(m)} u_x \right] = \frac{D_{x-1}^{(m)} u_x \left[N_x + D_x - D_{x-m} \right]}{n_{x-m}}.$$

The proof proceeds easily by mathematical induction.
For $m = 1$ we have

$$\Delta\left[D_{x-1}u_x\right] = D_x \Delta u_x + u_x \Delta D_{x-1}$$

$$= N_x u_x + \Delta D_{x-1} u_x$$

$$= D_{x-1}u_x\left[\frac{N_x + D_x - D_{x-1}}{D_{x-1}}\right],$$

For $m = 2$, we get

$$\Delta\left[D_{x-2}D_{x-1}u_x\right] = D_{x-1}\Delta\left[D_{x-1}u_x\right] + D_{x-1}u_x \Delta D_{x-2}$$

$$= D_{x-2}D_{x-1}u_x\left[\frac{N_x + \Delta D_{x-1} + \Delta D_{x-2}}{D_{x-2}}\right],$$

or
$$\Delta\left[D_{x-1}^{(2)}u_x\right] = D_{x-1}^{(2)}u_x\left[\frac{N_x + D_x - D_{x-2}}{D_{x-2}}\right].$$

Let us assume that it holds for the mth case, i. e.

$$\Delta\left[D_{x-1}^{(m)}u_x\right] = D_{x-1}^{(m)}u_x\left[\frac{N_x + D_x - D_{x-m}}{D_{x-m}}\right].$$

Then

$$\Delta\left[D_{x-m-1}D_{x-1}^{(m)}u_x\right] = D_{x-m}\,\Delta\left[D_{x-1}^{(m)}u_x\right] + D_{x-1}^{(m)}u_x\,\Delta D_{x-m-1}$$

$$= D_{x-1}^{(m)}u_x\left[N_x + D_x - D_{x-m} + D_{x-m} - D_{x-m-1}\right]$$

$$= D_{x-1}^{(m+1)}u_x\left[\frac{N_x + D_x - D_{x-m-1}}{D_{x-m-1}}\right].$$

Making use of this lemma in proving the last theorem, we note that

$$\Delta^2\left[D_{x-1}^{(m)}u_x\right] = D_{x-1}^{(m)}u_x\ \frac{\left[N_x + D_x - D_{x-m}\right]}{D_{x-m}\,D_{x-m+1}}$$

or
$$D_{x-m}D_{x-m+1}\,\Delta^2\left[D_{x-1}^{(m)}u_x\right] = D_{x-1}^{(m)}u_x\ \left[N_x + D_x - D_{x-m}\right]$$

and in general that

$$\Delta^n\left[D_{x-1}^{(m)} u_x\right] = \frac{D_{x-1}^{(m)} u_x \left[Q_n(m,x) \right]}{D_{x-m+n-1}^{(n)}}$$

or $$D_{x-m+n-1}^{(n)} \Delta^n \left[D_{x-1}^{(m)} u_x\right] = D_{x-1}^{(m)} u_x \quad Q_n(m,x).$$

In particular, if $m = n$, we define the polynomials $Q_n(n,x)$ as $\Delta^n [D_{x-1}^{(n)} u_x] = Q_n(n,x)\, u_x$ which relation is of interest because the Δ^n has no D_x as multiplier. Any result derived for the polynomials $Q_n(x) = \frac{1}{u_x} D_{x+n-1}^{(n)} \Delta^n u_x$ where u_x is a solution of the difference equation (1) can now be extended to the polynomials $Q_n(m,x) = \dfrac{\Delta^n [D_{x-1}^{(m)} u_x]}{D_{x-1}^{(m-n)} u_x}$ by replacing N_x by $(N_x + D_x - D_{x-m})$ and D_x by D_{x-m}. For example, relation (XI) becomes

$$(\mathrm{XI}_m) \quad \begin{aligned} Q_{n+1}(m+1,x) &= (N_x + D_x - D_{x-m+n-1})\, Q_n(m+1), x) \\ &+ (N_x + D_x)\, \Delta Q_n(m+1,x) \end{aligned}$$

and when $m = n$, this relation reduces to

$$(\mathrm{XI}_n) \quad \begin{aligned} Q_{n+1}(n+1,x) &= (N_x + D_x - D_{x-1}) Q_n(n+1,x) + (N_x + D_x) \Delta Q_n(n+1,x) \\ &= (N_x + \Delta D_{x-1}) Q_n(n+1,x) + (N_x + D_x) \Delta Q_n(n+1,x). \end{aligned}$$

4. In analogy with the work of chapter II, we next proceed to find a recurrence relation involving the $(n+1)$ th, n th and $(n-1)$ th of the polynomials $Q(x)$. We take the n th difference of both sides of the equation

$$D_x \Delta u_x = N_x u_x$$

by making use of the formula for the n th difference of a product

$$\Delta^n [u_x v_x] = v_x \Delta^n u_x + n \Delta v_x \Delta^{n-1} u_{x+1}$$

$$+ \frac{n(n-1)}{2!} \Delta^2 v_x \Delta^{n-2} u_{x+2} + \cdots .$$

We then obtain the equation

$$D_x \Delta^{n+1} u_x + n \Delta D_x \Delta^n u_{x+1} + \frac{n(n-1)}{2!} \Delta^2 D_x \Delta^{n-1} u_{x+2} =$$

$$N_x \Delta^n u_x + n \Delta N_x \Delta^{n-1} u_{x+1} ,$$

$\Delta^3 D_x$ and $\Delta^2 N_x$ being equal to zero. Multiplying through by $D_{x+n}^{(n)}$ we get

$$D_{x+n}^{(n)} \Delta^{n+1} u_x + n D_{x+n}^{(n)} \Delta D_x \Delta^n u_{x+1}$$

$$+ \frac{n(n-1)}{2!} D_{x+n}^{(n)} \Delta^2 D_x \Delta^{n-1} u_{x+2}$$

$$= N_x D_{x+n}^{(n)} \Delta^n u_x + n D_{x+n}^{(n)} \Delta N_x \Delta^{n-1} u_{x+1}$$

But $u_{x+1} = u_x + \Delta u_x$ and $u_{x+2} = u_x + 2\Delta u_x + \Delta^2 u_x$.

Substituting these values in the last equation and using the defiin- ition for the polynomials $Q_n(x)$, we obtain:

$$Q_{n+1}(x) u_x + \frac{n \Delta D_x}{D_x} \left[D_{x+n} Q_n(x) + Q_{n+1}(x) \right] u_x$$

$$+ \frac{n(n-1)}{2!} \frac{D_{x+1}}{D_{x+1}} \frac{\Delta^2 D_x}{D_x} \left[D_{x+n}^{(2)} Q_{n-1}^{(x)} + 2 D_{x+n} Q_n(x) + Q_{n+1}(x) \right] u_x$$

$$= \frac{N_x}{D_x} D_{x+n} Q_n(x) u_x + \frac{n D_{x+n} \Delta N_x}{D_x} \left[D_{x+n-1} Q_{n-1}(x) + Q_n(x) \right] u_x$$

Dividing through by u_x and collecting like terms, this expression reduces to

$$\left[1+\frac{n\Delta D_x}{D_x}+\frac{n(n-1)}{2}\frac{\Delta^2 D_x}{D_x}\right]Q_{n+1}(x)+\left[\frac{nD_{x+n}\Delta D_x}{D_x}+\frac{n(n-1)\Delta^2 D_x D_{x+n}}{D_x}\right.$$

$$\left.-\frac{N_x D_{x+n}}{D_x}-\frac{nD_{x+n}\Delta N_x}{D_x}\right]Q_n(x)+\left[\frac{n(n-1)}{2!D_x}D_{x+n-1}D_{x+n}\Delta^2 D_x\right.$$

$$\left.-\frac{nD_{x+n-1}D_{x+n}\Delta N_x}{D_x}\right]Q_{n-1}(x)=0$$

Now we know that

$$u_{x+n}=u_x+n\Delta u_x+\frac{n(n-1)}{2!}\Delta^2 u_x+----$$

and so we may write D_{x+n} and N_{x+n} in this same form, i. e.

$$D_{x+n}=D_x+n\Delta D_x+\frac{n(n-1)}{2!}\Delta^2 D_x,$$

$$\Delta D_{x+n}=\Delta D_x+n\Delta^2 D_x,$$

and $$N_{x+n}=N_x+n\Delta N_x$$

the third and higher differences of D_x and the second and higher differences of N_x being equal to zero. The coefficient of $Q_{n+1}(x)$ reduces to $\frac{2D_{x+n}}{D_x}$ and the coefficient of $Q_n(x)$ also reduces to a simpler form. Dividing through by $\frac{2D_{x+n}}{D_x}$ we finally get the recurrence relation:

$$\text{(XII)}\quad \begin{aligned}&Q_{n+1}(x)+(n\Delta D_{x+n-1}-N_{x+n})Q_n(x)\\ &+nD_{x+n-1}\left[\frac{(n-1)}{2}\Delta^2 D_x-\Delta N_x\right]Q_{n-1}(x)=0\end{aligned}$$

i. e. the ($n+1$) th polynomial may be obtained from the n th and ($n-1$) th polynomials.

In Chapter II we found that relations (I) and (II) were identical for the first two terms, and as a consequence we equated the third terms and obtained a relation between the derivative of a polynomial $P_n(x)$ and the polynomial preceding it. In order that we may obtain a similar expression for the difference polynomials, we must change the appearance of formula (XII).

By lowering the degree in formula (XI) from n to $n-1$ and solving for $D_{x+n-1}\,Q_{n-1}(x)$ we find that

$$D_{x+n-1}\,Q_{n-1}(x) = (N_x + D_x)\,Q_{n-1}(x+1) - Q_n(x).$$

Substitution of this relation in formula XII gives

$$Q_{n+1}(x) = (N_{x+n} - n\Delta D_{x+n-1})\,Q_n(x)$$

$$+\, n\left[\Delta N_x - \frac{(n-1)}{2}\Delta^2 D_x\right]\left[(N_x + D_x)\,Q_{n-1}(x+1) - Q_n(x)\right]$$

or $$Q_{n+1}(x) = \left[N_{x+n} - n\Delta N_x - n\Delta D_{x+n-1} + \frac{n(n-1)}{2}\Delta^2 D_x\right]Q_n(x)$$

$$+\, n\left[\Delta N_x - \frac{(n-1)}{2}\Delta^2 D_x\right](N_x + D_x)\,Q_{n-1}(x+1).$$

Just as in Chapter IV, paragraph 3, the coefficient of $Q_n(x)$ reduces and becomes the same as the coefficient of $Q_n(x)$ in formula (XI) and we have

$$Q_{n+1}(x) = (N_x + D_x - D_{x+n})\,Q_n(x)$$

(XII')

$$+\, n\left[\Delta N_x - \frac{(n-1)}{2}\Delta^2 D_x\right](N_x + D_x)\,Q_{n-1}(x+1).$$

We therefore conclude that

(XIII) $$\Delta Q_n(x) = n\left[\Delta N_x - \frac{(n-1)}{2}\Delta^2 D_x\right]Q_{n-1}(x+1),$$

a relation expressing the difference of a polynomial $Q_n(x)$ in terms of the next preceding polynomial in $(x+1)$, i. e.

$Q_{n-1}\,(x+1)$. For the polynomial $Q_n(n,x)$, formula (XII) may be written in the form

$$(\mathrm{XIII}_n)\quad \Delta Q_n(n,x) = n\left[\Delta N_x + \frac{(n+1)}{2}\Delta^2 D_x\right] Q_{n-1}(n,x+1),$$

this relation being obtained by replacing N_x by $(N_x + D_x - D_{x-n})$ and D_x by D_{x-n}.

Formula XIII which was just derived is the general form of the relation we found to hold for the Poisson exponential function polynomials, i. e.

$$\Delta Q_n(x) = -\,n\,Q_{n-1}(x+1).$$

We find further that these polynomials satisfy a special form of (XI), i. e.

$$Q_{n+1}(x) + (x+n+1-\lambda)Q_n(x) - \lambda\Delta Q_n(x) = 0$$

and for formula (XII) we get the special form

$$Q_{n+1}(x) + (x+2n+1-\lambda)Q_n(x) + n(x+n)Q_{n-1}(x) = 0$$

This recurrence relation is also similar to the one given for Laguerre polynomials.

5. Turning now to the problem of obtaining a second order difference relation for the polynomials $Q_n(x)$, we proceed to difference formula (XI), i. e.

$$Q_{n+1}(x) = (N_x + D_x \;\; - D_{x+n})Q_n(x) + (N_x + D_x)\Delta Q_n(x)$$

and get

$$\Delta Q_{n+1}(x) = (\Delta N_x + \Delta D_x - \Delta D_{x+n}) Q_n(x)$$

$$+ (N_{x+1} + D_{x+1} - D_{x+n+1}) \Delta Q_n(x)$$

$$+ (\Delta N_x + \Delta D_x) \Delta Q_n(x) + (N_{x+1} + D_{x+1}) \Delta^2 Q_n(x).$$

Substituting for $\Delta Q_{n+1}(x)$ the value

$$(n+1)\left[\Delta N_x - \frac{n}{2}\Delta^2 D_x\right]\left[Q_n(x) + \Delta Q_n(x)\right]$$

found in formula (XIII), gives us

$$(n+1)\left[\Delta N_x - \frac{n}{2}\Delta^2 D_x\right]\left[Q_n(x) + \Delta Q_n(x)\right] =$$

$$\left[\Delta N_x + \Delta D_x - \Delta D_{x+n}\right] Q_n(x) + \left[N_{x+1} + D_{x+1} - D_{x+n+1}\right] \Delta Q_n(x)$$

$$+ \left[\Delta N_x + \Delta D_x\right] \Delta Q_n(x) + \left[N_{x+1} + D_{x+1}\right] \Delta^2 Q_n(x).$$

Collecting the coefficients of like terms and simplifying them, we finally get

$$(N_{x+1} + D_{x+1}) \Delta^2 Q_n(x) + \left[N_{x-n+1} - (n-1)\Delta D_x\right] \Delta Q_n(x)$$

(XIV)

$$- n\left[\Delta N_x - \frac{(n-1)}{2}\Delta^2 D_x\right] Q_n(x) = 0,$$

a relation very similar in form to formula (IV) and consisting of the first and second differences of the polynomial $Q_n(x)$. This relation when applied to the Poisson exponential function gives

$$\lambda \Delta^2 Q_n(x) + (\lambda - x - 1)\Delta Q_n(x) + n Q_n(x) = 0,$$

an equation which can be checked by substituting the value of the general Poisson polynomial in it.

The extension of formula (XIV) to the polynomials $Q_n(m, x)$ and $Q_n(n, x)$ by making the proper substitutions for N_x

and D_x results in the following expressions:

$$(N_{x+1}+D_{x+1})\Delta^2 Q_n(m,x)$$

$$(\mathrm{XIV}_m) \quad +\left[N_{x-n+1}+D_{x-n+1}-D_{x-m-n+1}-(n-1)\Delta D_{x-m}\right]\Delta Q_n(m,x)$$

$$-n\left[\Delta N_x+\Delta D_x-\Delta D_{x-m}-\left(\tfrac{n-1}{2}\right)\Delta^2 D_{x-m}\right]Q_n(m,x)=0,$$

which may also be written as:

$$(N_{x+1}+D_{x+1})\Delta^2 Q_n(m,x)$$

$$+\left[N_{x-n+1}+(m-n+1)\Delta D_x-\frac{m(m+1)}{2}\Delta^2 D_x\right]\Delta Q_n(m,x)$$

$$-n\left[\Delta N_x-\frac{n-2m-1}{2}\Delta^2 D_x\right]Q_n(m,x)=0$$

In particular if $m=n$ we have:

$$(N_{x+1}+D_{x+1})\Delta^2 Q_n(n,x)+$$

$$(\mathrm{XIV}_n) \quad +\left[N_{x-n+1}+\Delta D_x-\frac{n(n+1)}{2}\Delta^2 D_x\right]\Delta Q_n(n,x)$$

$$-n\left[\Delta N_x+\frac{(n+1)}{2}\Delta^2 D_x\right]Q_n(n,x)=0.$$

6. The next set of relations we shall derive are recurrence relations for the polynomials $Q_n(m,x)$ and $Q_n(n,x)$ In the lemma proved in this chapter we found that

$$\Delta\left(D_{x-1}^{(m+1)}u_x\right)=\left[D_{x-1}^{(m)}u_x\right]\left[N_x+D_x-D_{x-m-1}\right].$$

Taking the n th difference of both sides of the equation gives:

$$\Delta^{n+1}\left(D_{x-1}^{(m+1)}u_x\right)=(N_x+D_x-D_{x-m-1})\Delta^n\left[D_{x-1}^{(m)}u_x\right]$$

$$+n(\Delta N_x-\Delta D_x-\Delta D_{x-m-1})\Delta^{n-1}\left[D_x^{(m)}u_{x+1}\right],$$

the second difference of the trinomial $(N_x+D_x-D_{x-m-1})$ being equal to zero. Multiplying this last expression through by

$D^{(n+1)}_{x-m+n-1}$ and substituting for $D^{(n+1)}_{x-m+n-1}\Delta^{n+1}D^{(m+1)}_{x-1}u_x$ the value $D^{(m+1)}_{x-1}Q_{n+1}(m,x)u_x$, we get

$$D^{(m+1)}_{x-1}Q_{n+1}(m+1,x)u_x = (N_x+D_x-D_{x-m-1})D^{(m+1)}_{x-1}Q_n(m,x)u_x$$
$$+n(\Delta N_x+\Delta D_x-\Delta D_{x-m-1})D^{(m+2)}_x\left(\frac{N_x+D_x}{D_x}\right)Q_{n-1}(m,x+1)u_x.$$

Dividing through by $D^{(m+1)}_{x-1}u_x$ we get a recurrence relation involving the polynomials $Q_{n+1}(m+1,x)$, $Q_n(m,x)$ and $Q_{n-1}(m,x+1)$, i. e.

$$Q_{n+1}(m+1,x)=(N_x+D_x-D_{x-m-1})Q_n(m,x)$$

(XV$_m$)

$$+n\left[\Delta N_x+(m+1)\Delta^2 D_x\right](N_x+D_x)Q_{n-1}(m,x+1).$$

For $m=n$, this expression reduces to:

$$Q_{n+1}(n+1,x)=(N_x+D_x-D_{x-n-1})Q_n(n,x)$$

(XV$_n$)

$$+n\left[\Delta N_x+(n+1)\Delta^2 D_x\right](N_x+D_x)Q_{n-1}(n,x+1).$$

7. Another form of this relation is obtained by substituting the value found in (XIII$_n$) for $Q_{n-1}(n,x+1)$, i. e.

$$Q_{n-1}(n,x+1)=\frac{1}{n\left[\Delta N_x+\frac{n+1}{2}\Delta^2 D_x\right]}\Delta Q_n(n,x),$$

in formula (XV$_n$), which gives

$$Q_{n+1}(n+1,x)=(N_x+D_x-D_{x-n-1})Q_n(n,x)$$

(XVI)

$$+(N_x+D_x)\frac{\Delta N_x+(n+1)\Delta^2 D_x}{\Delta N_x+\frac{(n+1)}{2}\Delta^2 D_x}\Delta Q_n(n,x),$$

a relation very similar to formula (VI).

8. There remains one more formula in Chapter II for which we have not yet found a parallel in this chapter, i. e. formula VII. To obtain this parallel expression, we difference formula (XVI), thereby obtaining:

$$\Delta Q_{n+1}(n+1,x) = (\Delta N_x + \Delta D_x - \Delta D_{x-n-1})\, Q_n(n,x)$$

$$+ (N_{x+1} + D_{x+1} - D_{x-n})\, \Delta Q_n(n,x)$$

$$+ \frac{\Delta N_x + (n+1)\Delta^2 D_x}{\Delta N_x + \frac{(n+1)}{2}\Delta^2 D_x}\left[(\Delta N_x + \Delta D_x)\Delta Q_n(n,x) + (N_{x+1} + D_{x+1})\Delta^2 Q_n(n,x)\right].$$

In formula (XIV_n) we found a value for

$$(N_{x+1} + D_{x+1})\Delta^2 Q_n(n,x)$$

which when substituted in this last expression gives us:

$$\Delta Q_{n+1}(n+1,x) = (\Delta N_x + \Delta D_x - \Delta D_{x-n+1}) Q_n(n,x) + (N_{x+1} + D_{x+1} - D_{x-n})\Delta Q_n(n,x)$$

$$+ \frac{\Delta N_x + (n+1)\Delta^2 D_x}{\Delta N_x + \frac{(n+1)}{2}\Delta^2 D_x}\left[\Delta N_x + \Delta D_x + \frac{n(n+1)}{2}\Delta^2 D_x - N_{x-n+1} - \Delta D_x\right]\Delta Q_n(n,x)$$

$$+ n\,\frac{\Delta N_x + (n+1)\Delta^2 D_x}{\Delta N_x + \frac{(n+1)}{2}\Delta^2 D_x}\left[\Delta N_x + \frac{(n+1)}{2}\Delta^2 D_x\right] Q_n(n,x).$$

Collecting coefficients we get

$$\Delta Q_{n+1}(n+1,x) = \left[\Delta N_x + n\Delta N_x + \Delta D_x - \Delta D_{x-n-1} + n(n+1)\Delta^2 D_x\right] Q_n(n,x)$$

$$+ \left[N_{x+1} + D_{x+1} - D_{x-n}\right] Q_n(n,x)$$

$$+ \frac{\Delta N_x + (n+1)\Delta^2 D_x}{\Delta N_x + \frac{(n+1)}{2}\Delta^2 D_x}\left[\Delta N_x - N_{x-n+1} + \frac{n(n+1)}{2}\Delta^2 D_x\right]\Delta Q_n(n,x)$$

and by simplifying the coefficients this expression finally reduces to the formula

$$\Delta Q_{n+1}(n+1,x)=(n+1)\left[\Delta N_x+(n+1)\Delta^2 D_x\right] Q_n(n,x)$$

(XVII)
$$+\left\{N_{x+1}+(n+1)\Delta D_x-\frac{n(n+1)}{2}\Delta^2 D_x - \left[\frac{\Delta N_x+(n+1)\Delta^2 D_x}{\Delta N_x+\left(\frac{n+1}{2}\right)\Delta^2 D_x}\right]\left[N_{x-n}-\frac{n(n+1)}{2}\Delta^2 D_x\right]\right\}\Delta Q_n(n,x)$$

a relation which is also similar in form to formula VII.

Before concluding this chapter, we might examine the character of the polynomials $Q_n(n,x)$ when the original function is the Poisson exponential function $\psi(x)=\frac{e^{-\lambda}\lambda^x}{x!}$.

We find these polynomials to have the following form:

$$\frac{1}{\psi(x)}\Delta\frac{xe^{-\lambda}\lambda^x}{x!}=Q_1(1,x)=\lambda-x,$$

$$\frac{1}{\psi(x)}\Delta^2\frac{x^{(2)}e^{-\lambda}\lambda^x}{x!}=Q_2(2,x)=\lambda^2-2\lambda x+x^{(2)}$$

$$\frac{1}{\psi(x)}\Delta^3\frac{x^{(3)}e^{-\lambda}\lambda^x}{x!}=Q_3(3,x)=\lambda^3-3\lambda^2 x+3\lambda x^{(2)}-x^{(3)},$$

$$\frac{1}{\psi(x)}\Delta^n\frac{x^{(n)}e^{-\lambda}\lambda^x}{x!}=Q_n(n,x)=\lambda^n-n\lambda^{n-1}x+\frac{n(n-1)}{2!}\lambda^{n-2}x^{(2)}+\cdots+(-1)^n x^{(n)},$$

$$=\left[\lambda-x^{(1)}\right]^n=Q_n(x-n).$$

Substituting the proper values for N_x and D_x in formula (XIV_n) we get

$$\lambda\Delta^2 Q_n(n,x)+(\lambda-x+n-1)\Delta Q_n(n,x)+nQ_n(n,x)=0$$

In the same way we find for formula (XI_n) the relation

$$Q_{n+1}(n+1,x) = (\lambda - x + n)Q_n(n,x) + \lambda\Delta Q_n(n,x)$$

and for formula (XVII), the reduced relation

$$\Delta Q_{n+1}(n+1,x) = -(n+1)Q_n(n,x),$$

which is somewhat like the relation obtained for ($XIII_n$).
We might call attention to the fact that these polynomials are identical with the polynomials obtained by Charlier[1] satisfying the relations

$$\delta^{-1}\left[T_n(x)\,\delta^m \frac{e^{-\lambda}\lambda^x}{x!}\right]_{-\infty}^{+\infty} \begin{array}{l} = 0 \text{ for } m \neq n \\ = 1 \text{ for } m = n \end{array}$$

9. Summarizing the results of this chapter, we have found that if the general solution $g(x)$ of the difference equation

$$\Delta u_x = \frac{a_0 + a_1 x}{b_0 + b_1 x + b_2 x^2}\, u_x$$

is used as the generating function $g(x)$ in the Charlier Type B series, that the successive differences give rise to two general types of polynomials which we defined as follows:

$$Q_n(x) = \frac{1}{u_x} D_x^{(n)} \Delta^{n+1} u_x$$

and

$$Q_n(n,x) = \frac{1}{u_x} \Delta^n D_{x-n}^{(n)}\, u_x\,.$$

With the aid of the properties of the Δ operator, we derived a set of relations and equations for these polynomials of the following form:

[1] C. V. L. Charlier: "Ueber die Darstellung willkürlicher Funktionen," p. 34.

(XI) $Q_{n+1}(x) = (N_x + D_x - D_{x+n}) Q_n(x) + (N_x + D_x) \Delta Q_n(x),$

(XI$_n$) $Q_{n+1}(n+1, x) = (N_x + \Delta D_{x-1}) Q_n(n+1, x) + (N_x + D_x) \Delta Q_n(n, x),$

(XII)
$$Q_{n+1}(x) = (N_{x+n} - n \Delta D_{x+n-1}) Q_n(x) + n D_{x+n-1} \left[\Delta N_x - \frac{(n-1)}{2} \Delta^2 D_x \right] Q_{n-1}(x),$$

(XII$'$)
$$Q_{n+1}(x) = (N_x + D_x - D_{x+n}) Q_n(x) + n \left[\Delta N_x - \frac{(n-1)}{2} \Delta^2 D_x \right] (N_x + D_x) Q_{n-1}(x+1),$$

(XIII) $\Delta Q_n(x) = n \left[\Delta N_x - \frac{(n-1)}{2} \Delta^2 D_x \right] Q_{n-1}(x+1),$

(XIII$_n$) $\Delta Q_n(n, x) = n \left[\Delta N_x + \frac{n+1}{2} \Delta^2 D_x \right] Q_{n-1}(n, x+1),$

(XIV)
$$(N_{x+1} + D_{x+1}) \Delta^2 Q_n(x) + \left[N_{x-n+1} - (n-1) \Delta D_x \right] \Delta Q_n(x) - n \left[\Delta N_x - \frac{(n-1)}{2} \Delta^2 D_x \right] Q_n(x) = 0,$$

(XIV$_n$)
$$(N_{x+1} + D_{x+1}) \Delta^2 Q_n(n, x) + \left[N_{x-n+1} + \Delta D_x - \frac{n(n+1)}{2} \Delta^2 D_x \right] \Delta Q_n(n, x) - n \left[\Delta N_x + \frac{n+1}{2} \Delta^2 D_x \right] Q_n(n, x) = 0,$$

(XV$_n$)
$$Q_{n+1}(n+1, x) = (N_x + D_x - D_{x-n-1}) Q_n(n, x) + n \left[\Delta N_x + (n+1) \Delta^2 D_x \right] (N_x + D_x) Q_{n-1}(n, x+1),$$

(XVI)
$$Q_{n+1}(n+1, x) = (N_x + D_x - D_{x-n-1}) Q_n(n, x) + (N_x + D_x) \frac{\Delta N_x + (n+1) \Delta^2 D_x}{\Delta N_x + \frac{(n+1)}{2} \Delta^2 D_x} \Delta Q_n(n, x),$$

$$
\text{(XVII)}\quad \Delta Q_{n+1}(n+1,x) = (n+1)\left[\Delta N_x + (n+1)\Delta^2 D_x\right] Q_n(n,x)
$$
$$
+ \left\{ N_{x+1} + (n+1)\Delta D_x - \frac{n(n+1)}{2}\Delta^2 D_x \right.
$$
$$
\left. - \left[\frac{\Delta N_x + (n+1)\Delta^2 D_x}{\Delta N_x + \frac{(n+1)}{2}\Delta^2 D_x}\right]\left[N_{x-n} - \frac{n(n+1)}{2}\Delta^2 D_x\right]\right\} \Delta Q_n(n,x).
$$

Each of these formulas corresponds and is similar to a formula found in Chapter II. In fact, it seems probable that if we developed the formulas in this present chapter from the equation

$$
\frac{\Delta u_x}{\Delta x} = \frac{N_x}{D_x} u_x
$$

and permitted the Δ_x to approach zero as a limit, the formulas of Chapter II would result, the above formulas being the case where $\Delta_x = 1$.

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